# Polynomial-time right-ideal morphisms and congruences 

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#### Abstract

We continue with the functional approach to the P -versus-NP problem, begun in [4, 3). We previously constructed a monoid $\mathcal{R} \mathcal{M}^{P}$ that is non-regular iff NP $\neq \mathrm{P}$. We now construct homomorphic images of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ with interesting properties. In particular, the homomorphic image $\mathcal{M}_{\text {poly }}^{P}$ of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is finitely generated and $\mathcal{J}^{0}$-simple, and is non-regular iff $\mathrm{P} \neq \mathrm{NP}$. The group of units of $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is the famous Richard Thompson group $V$.


## 1 Introduction

In [4] we defined the monoids fP and $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. The monoid fP consists of the partial functions $A^{*} \rightarrow A^{*}$ that are computable by deterministic Turing machines in polynomial time, and that have polynomial I/O-balance (defined below). In [4] it was proved that fP is finitely generated. The submonoid $\mathcal{R} \mathcal{M}^{P}$ consists of the elements of fP that are right-ideal morphisms of $A^{*}$ (defined below). This monoid was studied further in [3], where we proved that $\mathcal{R} \mathcal{M}^{\mathbf{P}}$ is not finitely generated. We saw that the one-way functions (in the sense of worst-case time-complexity) are exactly the non-regular element of fP , and that $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular in fP iff $f$ is regular in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. So, one-way functions exist iff fP is non-regular, iff $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is non-regular. It is well-known that one-way functions (according to worst-case time-complexity) exist iff NP $\neq \mathrm{P}$. For P -vs.-NP, see e.g. [13, 14, 10, 12, 16]; for worst-case one-way functions, see e.g. [10, 12]. For definitions related to semigroups and monoids, see e.g. [11].

In this paper we define some congruences on $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ that are algebraic forms of the padding argument. The padding argument is often used in computational complexity in order to decrease the complexity of a problem by lengthening the inputs (since complexity is measured as a function of the input-length, lengthening the input reduces complexity). We use the padding argument in the proof of finite generation of fP in [4]; for another use, see Lemma [2.22. These congruences lead to infinitely many quotient monoids (i.e., homomorphic images of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ ), some of which have interesting and unique properties:

- $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is regular iff $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular (iff $N \mathrm{P}=\mathrm{P}$ ); moreover, $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is finitely generated, and its group of units is the well-known finitely presented infinite simple group $V$ of Richard Thompson [9];
- $\mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$ is a homomorphic image of $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ and is regular;
- $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ is a homomorphic image of $\mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$, acts faithfully on the Cantor space $A^{\omega}$, and has just two non-zero $\mathcal{D}$-classes;
- $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is a homomorphic image of $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$, is congruence-simple, and has just one non-zero $\mathcal{D}$-class. More details about these motivations are given at the end of this Introduction.

We now give some definitions. A function $f: A^{*} \rightarrow A^{*}$ is polynomially balanced iff there exists a polynomial $p$ such that for all $x \in \operatorname{Dom}(f):|f(x)| \leq p(|x|)$ and $|x| \leq p(|f(x)|)$.

Here we always use $A=\{0,1\}$ as our alphabet. In [4, $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ was called $\mathcal{R} \mathcal{M}_{2}^{\mathrm{P}}$, where the subscript 2 indicated the size of $A$; but since here the size of $A$ will always be 2 , we drop the subscript 2 .

For an alphabet $A$, the set of all words over $A$ is denoted by $A^{*}$; this includes the empty string $\varepsilon$. By a "word" or "string" we will always mean a finite word. The set of all non-empty words over
$A$ is denoted by $A^{+}\left(=A^{*}-\{\varepsilon\}\right)$. The length of a word $x \in A^{*}$ is denoted by $|x|$. For $n \geq 0$ we let $A^{n}=\left\{x \in A^{*}:|x|=n\right\}$, and $A^{\leq n}=\left\{x \in A^{*}:|x| \leq n\right\}$.

For two strings $v, w \in A^{*}$, when $v$ is a prefix of $w$ we write $v \leq_{\text {pref }} w$; i.e., there exists $x \in A^{*}$ such that $v x=w$. The relation $\leq_{\text {pref }}$ is a partial order on $A^{*}$, and is called the prefix order. We write $v<_{\text {pref }} w$ when $v \leq_{\text {pref }} w$ and $v \neq u$ (strict prefix order). We write $v \|_{\text {pref }} w$ when $v \leq_{\text {pref }} w$ or $w \leq_{\text {pref }} v$, and then we say that $v$ and $w$ are prefix-comparable. One easily proves that $w \|_{\text {pref }} v$ iff there exist $x_{1}, x_{2} \in A^{*}$ such that $w x_{1}=v x_{2}$. A set $P \subset A^{*}$ is a prefix code iff no two elements of $P$ are prefix-comparable. A set $R \subseteq A^{*}$ is a right ideal iff $R A^{*}=R$. It is easy to prove (see e.g. (4) that for every right ideal $R$ there exists a unique prefix code $P$ such that $R=P A^{*}$. For prefix codes and related concepts, see e.g. 2].

For a partial function $f: A^{*} \rightarrow A^{*}$, the domain is $\operatorname{Dom}(f)=\left\{x \in A^{*}: f(x)\right.$ is defined $\}$, and the image is $\operatorname{Im}(f)=f\left(A^{*}\right)=f(\operatorname{Dom}(f))$. When we say "function", we mean partial function. When $\operatorname{Dom}(f)=A^{*}, f$ is a called a total function. The restriction of $f$ to a set $S \subseteq A^{*}$ is denoted by $\left.f\right|_{S}$. The identity map on $A^{*}$ is denoted by $\mathbf{1}$ or $\mathbf{1}_{A^{*}}$, and its restriction to $S$ is denoted by $\mathbf{1}_{S}$.

A function $h: A^{*} \rightarrow A^{*}$ is a right-ideal morphism iff $\operatorname{Dom}(h)$ is a right ideal, and all $x \in \operatorname{Dom}(h)$ and all $w \in A^{*}: \quad h(x w)=h(x) w$. In that case, $\operatorname{Im}(h)$ is also a right ideal. For a right-ideal morphism $h$, let $\operatorname{domC}(h)$ (called the domain code) be the prefix code that generates Dom $(h)$ as a right ideal. Similarly, let $\operatorname{imC}(h)$, called the image code, be the prefix code that generates $\operatorname{Im}(h)$ as a right ideal. In general, $\operatorname{imC}(h) \subseteq h(\operatorname{domC}(h))$, and it can happen that $\operatorname{imC}(h) \neq h(\operatorname{domC}(h))$.

It will often be useful to represent any set $S \subseteq\{0,1, \#\}^{*}$ as a prefix code. We choose one way to do that, as follows. Let $P=\{00,01,11\}$; this is obviously a prefix code. We define code $(0)=$ $00, \operatorname{code}(0)=01, \operatorname{code}(\#)=11$. For $w=a_{1} \ldots a_{1} \ldots a_{n} \in\{0,1, \#\}^{*}$ we define $\operatorname{code}(w)=$ $\operatorname{code}\left(a_{1}\right) \ldots \operatorname{code}\left(a_{i}\right) \ldots \operatorname{code}\left(a_{n}\right)$. Then for every $L \subseteq\{0,1\}^{*}, \operatorname{code}(L) 11$ (defined to be $\{\operatorname{code}(x) 11:$ $x \in L\}$ ) is a prefix code. We also encode any function $f: A^{*} \rightarrow A^{*}$ into a right-ideal morphism $f^{C}$ : $A^{*} \rightarrow A^{*}$, defined by $\operatorname{domC}\left(f^{C}\right)=\operatorname{code}(\operatorname{Dom}(f)) 11$, and $f^{C}(\operatorname{code}(x) 11)=\operatorname{code}(f(x)) 11$ (for all $x \in \operatorname{Dom}(f)$ ). The right-ideal morphisms $f^{C}$, for $f \in \mathrm{fP}$, have the following important normality property (see Def. 5.6): $f^{C}\left(\operatorname{domC}\left(f^{C}\right)\right)=\operatorname{imC}\left(f^{C}\right)$. This follows immediately from the fact that $f^{C}\left(\operatorname{domC}\left(f^{C}\right)\right)=\{\operatorname{code}(f(x)) 11: x \in \operatorname{Dom}(f)\}$ is a prefix code.

We define

$$
\mathcal{R} \mathcal{M}^{\mathrm{P}}=\left\{f \in \mathrm{fP}: f \text { is a right-ideal morphism of } A^{*}\right\} .
$$

By Prop. 2.6 in [4], if $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular in fP then $f$ is regular in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. Hence: The monoid $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular iff $\mathrm{P}=\mathrm{NP}$.

Since $\mathrm{P} \neq \mathrm{NP}$ is equivalent to the non-regularity of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, we are interested in approaches towards proving non-regularity or regularity of this monoid. In this paper we study some congruences on $\mathcal{R} \mathcal{M}^{\mathrm{P}}$; these provide us with infinitely many homomorphic images of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. Four of these are of particular interest; they form a chain $\mathcal{R} \mathcal{M}^{\mathrm{P}} \rightarrow \mathcal{M}_{\text {poly }}^{\mathrm{P}} \rightarrow \mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}} \rightarrow \mathcal{M}_{\mathrm{bd}}^{\mathrm{P}} \rightarrow \mathcal{M}_{\text {end }}^{\mathrm{P}}$. The last three are regular monoids. Moreover, we find a submonoid $\mathcal{R} \mathcal{M}^{n+o(n)}$ of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ which is non-regular, and which maps homomorphically onto $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$; in addition, $\mathcal{R} \mathcal{M}^{n+o(n)}$ is $\equiv_{\text {poly }}$-equivalent to $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ (see the Remark at the end of the paper for details). Thus we have the following monoid homomorphisms (where $\nearrow$ is injective):

where $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is regular iff $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular (Theorem [5.16), and $\mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$ (hence $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ and $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ ) is regular. On the other hand, $\mathcal{R} \mathcal{M}^{n+o(n)}$ is non-regular (Prop. 6.2). The triangle of maps starting at $\mathcal{R} \mathcal{M}^{n+o(n)}$ is a commutative diagram. These monoids have other interesting properties:

- $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ (hence its homomorphic images) is finitely generated (Theorem 4.7); on the other hand, $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is not finitely generated (see [3).
- $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is congruence-simple and has only one non-zero $\mathcal{D}$-class (Theorems 2.24 and 2.23); so $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is the end of the chain. A priori, it was not obvious that $\mathcal{R} \mathcal{M}^{P}$ should have a coarsest non-trivial congruence at all.
- $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ has exactly two non-zero $\mathcal{D}$-classes (Theorem 3.22), and acts faithfully on $A^{\omega}$; in fact, $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ is the monoid of the action of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ on $A^{\omega}$ (Prop. 3.8).
- The group of units of $\mathcal{M}_{\text {poly }}^{\mathrm{P}}, \mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$, and $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$, is the famous Richard Thompson group $V$, alias $G_{2,1}$ (Theorem 3.16(2)), whereas the group of units of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is trivial (4] Prop. 2.12).

In the above homomorphism chain, the monoid $\mathcal{M}_{\text {poly }}^{P}$ (which is regular iff $P=N P$ ) is placed between a monoid that is proved to be non-regular, and a monoid that is proved to be regular. Whether all this brings us closer to an answer to the P-vs.-NP question remains open.

## 2 End-equivalence

We start out with the most basic congruence on $\mathcal{R} \mathcal{M}^{P}$, which turns out to be maximal (i.e., it is not contained in any other congruence, except the trivial congruence).

Definition 2.1 Two sets $L_{1}, L_{2} \subseteq A^{*}$ are end-equivalent (denoted by $L_{1} \equiv_{\text {end }} L_{2}$ ) iff the right ideals $L_{1} A^{*}$ and $L_{2} A^{*}$ intersect the same right ideals of $A^{*}$ (i.e., for every right ideal $R \subseteq A^{*}: R \cap L_{1} A^{*} \neq \varnothing$ iff $R \cap L_{2} A^{*} \neq \varnothing$ ).

Here we say that two sets $S_{1}$ and $S_{2}$ intersect iff $S_{1} \cap S_{2} \neq \varnothing$. Note that for $\equiv_{\text {end }}$ it is intersection with $L_{i} A^{*}$ that matters, not just intersection with $L_{i}$ (unless $L_{i}$ is already a right ideal). The empty set is only end-equivalent to itself. In the above definition it is sufficient to use intersections with monogenic right ideals (a monogenic right ideal is of the form $w A^{*}$ with $w \in A^{*}$ ):

Lemma $2.2 L_{1} \equiv_{\text {end }} L_{2}$ iff $L_{1} A^{*}$ and $L_{2} A^{*}$ intersect the same monogenic right ideals.
Proof. Let $R$ be any right ideal $R$ that intersects $L_{1} A^{*}$, and let $x \in R \cap L_{1} A^{*}$. Then $x A^{*} \subseteq R$; and $x A^{*}$ intersects $L_{1} A^{*}$, hence (by intersection with the same monogenic right ideals), $x A^{*}$ intersects $L_{2} A^{*}$. So, since $x A^{*} \subseteq R, R$ also intersects $L_{2} A^{*}$. In the same way one proves that every right ideal that intersects $L_{2} A^{*}$ also intersects $L_{1} A^{*}$.

Note that if $L_{1} \equiv_{\text {end }} L_{2}$ and $L_{1} \neq \varnothing$, then $L_{1} A^{*} \cap L_{2} A^{*} \neq \varnothing$. Indeed, $L_{1} A^{*}$ has a non-empty intersection with itself, so by end-equivalence, $L_{1} A^{*}$ intersects $L_{2} A^{*}$ non-emptily too. However, if $L_{1} \equiv_{\text {end }} L_{2}$ it could happen that $L_{1} \cap L_{2}=\varnothing$; e.g., let $L_{1}=\{1\}$ and $L_{2}=\{10,11\}$.

The next Lemma 2.3(1) implies that $\equiv_{\text {end }}$ is definable in the first-order logic of $A^{*}$ with concatenation.

Lemma 2.3 Let $L_{1}, L_{2} \subseteq A^{*}$.
(1) $L_{1} \equiv_{\text {end }} L_{2}$ iff
$\left(\forall x_{1} \in L_{1}, w_{1} \in A^{*}\right)\left(\exists x_{2} \in L_{2}\right)\left[x_{1} w_{1} \|_{\text {pref }} x_{2}\right]$ and
$\left(\forall x_{2} \in L_{2}, w_{2} \in A^{*}\right)\left(\exists x_{1} \in L_{1}\right)\left[x_{1} \|_{\text {pref }} x_{2} w_{2}\right]$.
(2) If $L_{1} \equiv_{\text {end }} L_{2}$ then $L_{1} \equiv_{\text {end }} L_{2} \equiv_{\text {end }} L_{1} A^{*} \cap L_{2} A^{*} \equiv_{\text {end }} L_{1} \cup L_{2}$.

Proof. (1) If $L_{1} A^{*}$ and $L_{2} A^{*}$ intersect the same right ideals then for every $x_{1} \in L_{1}$ and $w_{1} \in A^{*}$, the right ideal $x_{1} w_{1} A^{*}$ intersects $L_{2} A^{*}$; hence there exists $x_{2} \in L_{2}$ and $u_{1}, u_{2} \in A^{*}$ such that $x_{1} w_{1} u_{1}=$ $x_{2} u_{2}$. Hence, $x_{1} w_{1}$ and $x_{2}$ are prefix comparable. Similarly, for every $x_{2} \in L_{2}$ and $w_{2} \in A^{*}$ there exists $x_{1} \in L_{1}$ such that $x_{1}$ and $x_{2} w_{2}$ are prefix comparable.

In the other direction, let us assume the prefix comparability condition, and let $R$ be a right ideal that intersects $L_{1} A^{*}$. We want to show that $R$ also intersects $L_{2} A^{*}$. Since $R$ intersects $L_{1} A^{*}$, there exists $x_{1} w_{1} \in R$ such that $x_{1} \in L_{1}$ and $w_{1} \in A^{*}$. Then let $x_{2} \in L_{2}$ be prefix comparable with $x_{1} w_{1}$. If $x_{1} w_{1}=x_{2} z$ for some $z \in A^{*}$ then $x_{2} z=x_{1} w_{1} \in R$, so $R$ intersects $L_{2} A^{*}$. If $x_{2}=x_{1} w_{1} z$ for some $z \in A^{*}$ then $x_{2} \in R$ (since $x_{1} w_{1} \in R$, and $R$ is a right ideal), so $R$ intersects $L_{2}$ (and $L_{2} A^{*}$ ).
(2) Every right ideal that intersects $L_{1} A^{*} \cap L_{2} A^{*}$ obviously intersects $L_{1} A^{*}$ and $L_{2} A^{*}$. If a right ideal $R$ intersects $L_{1} A^{*}$, let $x_{1} w_{1} \in L_{1} A^{*} \cap R$. Then by end-equivalence, the right ideal $x_{1} w_{1} A^{*}$ intersects $L_{2} A^{*}$, i.e., $x_{1} w_{1} z \in L_{2} A^{*}$ for some $z \in A^{*}$. But $x_{1} w_{1} z$ also belongs to $R$, so $R$ intersects $L_{1} A^{*} \cap L_{2} A^{*}$. So, $L_{1} A^{*}$ and $L_{1} A^{*} \cap L_{2} A^{*}$ are end-equivalent.

If a right ideal $R$ intersects $\left(L_{1} \cup L_{2}\right) A^{*}=L_{1} A^{*} \cup L_{2} A^{*}$ then $R$ intersects $L_{1} A^{*}$ or $L_{2} A^{*}$. If $R$ intersects $L_{1} A^{*}$, then since $L_{1} \equiv_{\text {end }} L_{2}$, it intersects $L_{2} A^{*}$ too; so in any case, $R$ intersects $L_{2} A^{*}$. Hence, $L_{1} \cup L_{2} \equiv_{\text {end }} L_{2}$.

Lemma 2.4 (1) If $f$ is a right-ideal morphism, and if $L_{1}, L_{2} \subseteq \operatorname{Dom}(f)$ are sets such that $L_{1} \equiv{ }_{\text {end }} L_{2}$, then $f\left(L_{1}\right) \equiv_{\text {end }} f\left(L_{2}\right)$.
(2) For any right-ideal morphism $f, f(\operatorname{domC}(f)) \equiv_{\text {end }} \operatorname{imC}(f)$.

Proof. (1) Let $R$ be any right ideal that intersects $f\left(L_{1}\right) A^{*}$, and let $y_{1} \in R \cap f\left(L_{1}\right) A^{*}$. So, since $L_{1} \subseteq \operatorname{Dom}(f)$, we have $y_{1}=f\left(x_{1}\right) u_{1}$ for some $x_{1} \in L_{1}, u_{1} \in A^{*}$. And since $y_{1} \in R$ we have also $x_{1} u_{1} \in f^{-1}\left(y_{1}\right) \subseteq f^{-1}(R)$. Thus, $x_{1} u_{1} \in f^{-1}(R) \cap L_{1} A^{*}$. Since $f^{-1}(R)$ is a right ideal and $L_{1} \equiv_{\text {end }} L_{2}$, there exists $x_{2} u_{2} \in f^{-1}(R) \cap L_{2} A^{*}$ (with $x_{2} \in L_{2}, u_{2} \in A^{*}$ ). Hence, $f\left(x_{2}\right) u_{2} \in f\left(f^{-1}(R) \cap L_{2} A^{*}\right) \subseteq$ $R \cap f\left(L_{2}\right) A^{*}$; here we use $f\left(f^{-1}(R)\right) \subseteq R$ and $L_{2} \subseteq \operatorname{Dom}(f)$. So, $R$ intersects $f\left(L_{2}\right) A^{*}$. Similarly, any right ideal that intersects $f\left(L_{2}\right) A^{*}$ intersects $f\left(L_{1}\right) A^{*}$. Thus $f\left(L_{1}\right) \equiv_{\text {end }} f\left(L_{2}\right)$.
(2) Obviously, $L \equiv_{\text {end }} L A^{*}$ for any set $L \subseteq A^{*}$. We also have $f(\operatorname{domC}(f)) A^{*}=f\left(\operatorname{domC}(f) A^{*}\right)=$ $\operatorname{imC}(f) A^{*}$. Hence $f(\operatorname{domC}(f)) \equiv_{\text {end }} \operatorname{imC}(f)$.

Let $A^{\omega}$ be the set of all $\omega$-sequences of elements of $A$; we call the elements of $A^{\omega}$ ends. See e.g. [15] for the study of infinite words. For a set $L \subseteq A^{*}$, the set $L A^{\omega}$ is called the ends of $L$, and denoted by ends $(L)$; equivalently, ends $(L)$ is the set of ends that have at least one prefix in $L$, so ends $(L)=\operatorname{ends}\left(L A^{*}\right)$. The Cantor set topology on $A^{\omega}$ is described by $\left\{L A^{\omega}: L \subseteq A^{*}\right\}$ as set of open sets. Similarly, $A^{*}$ is a topological space too, with the set of right ideals as set of open sets; we call this the right-ideal topology of $A^{*}$.

Notation: For $S \subseteq A^{\omega}$, the closure is denoted by $\mathrm{cl}(S)$, and the interior by in $(S)$.
In any topological space, two sets intersect the same open sets iff they have the same closure. Hence we have the following topological characterization of end-equivalence.

Proposition 2.5 For all $L_{1}, L_{2} \subseteq A^{*}: \quad L_{1} \equiv_{\text {end }} L_{2}$ iff $\mathrm{cl}\left(\operatorname{ends}\left(L_{1}\right)\right)=\mathrm{cl}\left(\operatorname{ends}\left(L_{2}\right)\right)$ in the Cantor space $A^{\omega}$, iff $\mathrm{cl}\left(L_{1}\right)=\mathrm{cl}\left(L_{2}\right)$ in the right-ideal topology of $A^{*}$.

The following example illustrates the importance of closure in the above Proposition. Let $L_{1}=0^{*} 1$, and $L_{2}=\{\varepsilon\}$. Then $0^{*} 1 \equiv_{\text {end }}\{\varepsilon\}$, and $\operatorname{cl}\left(0^{*} 1\{0,1\}^{\omega}\right)=\{0,1\}^{\omega}=\operatorname{cl}\left(\{\varepsilon\}\{0,1\}^{\omega}\right)$. But ends $\left(0^{*} 1\right) \neq$ ends $(\{\varepsilon\})$, since ends $(\{\varepsilon\})=\{0,1\}^{\omega}$, while ends $\left(0^{*} 1\right)=0^{*} 1\{0,1\}^{\omega}=\{0,1\}^{\omega}-\left\{0^{\omega}\right\}$.

The equivalence relation $\equiv_{\text {end }}$ can be generalized to a pre-order: For $L_{1}, L_{2} \subseteq A^{*}$ we define $L_{1} \subseteq_{\text {end }} L_{2}$ iff every right ideal that intersects $L_{1} A^{*}$ also intersects $L_{2} A^{*}$. Obviously, $L_{1} \equiv_{\text {end }} L_{2}$ iff $L_{1} \subseteq_{\text {end }} L_{2}$ and $L_{2} \subseteq_{\text {end }} L_{2}$. And we have: $L_{1} \subseteq_{\text {end }} L_{2}$ iff cl(ends $\left.\left(L_{1}\right)\right) \subseteq \mathrm{cl}\left(\operatorname{ends}\left(L_{2}\right)\right)$.

The following Lemma shows that a one-point change in ends $(P)$ does not change end-equivalence.
Lemma 2.6 For any prefix code $P \subset A^{*}$ and any end $v \in \operatorname{ends}(P)$ there exists a prefix code, called $P(-v)$, such that ends $(P(-v))=\operatorname{ends}(P)-\{v\}$ and $P(-v) \equiv_{\text {end }} P$.

Proof. Let $v=v_{1} \ldots v_{i} \ldots \in \operatorname{ends}(P)$, where $v_{i} \in A$ for all $i \geq 1$. Since $v \in$ ends $(P)$, there exists $i_{0} \geq 0$ such that $v_{1} \ldots v_{i_{0}} \in P$. We now define

$$
P(-v)=\left(P-\left\{v_{1} \ldots v_{i_{0}}\right\}\right) \cup\left\{v_{1} \ldots v_{i_{0}} \ldots v_{j} \bar{v}_{j+1}: j \geq i_{0}\right\}
$$

The set $\left\{v_{1} \ldots v_{i_{0}} \ldots v_{j} \bar{v}_{j+1}: j \geq i_{0}\right\}$ is the border of the end $v$ from $v_{1} \ldots v_{i_{0}}$ onwards; indeed, $v_{1} \ldots v_{i_{0}} \ldots v_{j} \bar{v}_{j+1}$ is the sibling node of $v_{1} \ldots v_{i_{0}} \ldots v_{j} v_{j+1}$, and the latter is a prefix of $v$. Clearly, $P(-v)$ is a prefix code and ends $(P(-v))=$ ends $(P)-\{v\}$.

To show that $P(-v) \equiv_{\text {end }} P$, it is obvious that every right ideal that intersects $P(-v) A^{*}$ also intersects $P A^{*}$ (since ends $(P(-v)) \subseteq$ ends $(P)$ ). Conversely, let $R \subseteq A^{*}$ be any right ideal that intersects $P A^{*}$ (at say $\left.u \in R \cap P A^{*}\right)$; we want to show that $R$ also intersects $P(-v) A^{*}$. If $u \in$ $\left(P-\left\{v_{1} \ldots v_{i_{0}}\right\}\right) A^{*}$ then $u \in P(-v) A^{*}$. Alternatively, $v_{1} \ldots v_{i_{0}}$ is a prefix of $u$, i.e., $u=v_{1} \ldots v_{i_{0}} z$ for some $z \in A^{*}$. Let $v_{1} \ldots v_{i_{0}} \ldots v_{j}$ be the longest prefix of $u$ that is also a prefix of the end $v$. If $u=$ $v_{1} \ldots v_{i_{0}} \ldots v_{j}$, then $u \bar{v}_{j+1}=v_{1} \ldots v_{i_{0}} \ldots v_{j} \bar{v}_{j+1} \in R \cap P(-v) A^{*}$; if, on the other hand, $|u|>j$ then $u$ is of the form $u=v_{1} \ldots v_{i_{0}} \ldots v_{j} \bar{v}_{j+1} \ldots$ (the only alternative would be that $u=v_{1} \ldots v_{i_{0}} \ldots v_{j} v_{j+1} \ldots$, but that would contradict the maximality of $j$ ). Thus, $u=v_{1} \ldots v_{i_{0}} \ldots v_{j} \bar{v}_{j+1} \ldots$; this belongs to $P(-v) A^{*}$, since $v_{1} \ldots v_{i_{0}} \ldots v_{j} \bar{v}_{j+1} \in P(-v)$.

Proposition 2.7 For any prefix code $P \subset A^{*}$, we have:

$$
\bigcup_{Q \equiv_{\text {end }} P} \operatorname{ends}(Q)=\operatorname{in}(\operatorname{cl}(\operatorname{ends}(P))), \quad \text { and } \quad \bigcup_{Q \equiv_{\text {end }} P} Q A^{*} \equiv_{\text {end }} P .
$$

Moreover,

$$
\bigcap_{Q \equiv_{\text {end }} P} \operatorname{ends}(Q)=\varnothing=\bigcap_{Q \equiv_{\text {end }} P} Q A^{*}
$$

Proof. Concerning the union of sets of ends:
$[\subseteq]$ : If $Q \equiv$ end $P$, then (by Prop. 2.5$)$, ends $(Q) \subseteq \mathrm{cl}(\operatorname{ends}(P))$. And ends $(Q)$ is open, hence ends $(Q) \subseteq$ in( $\mathrm{cl}($ ends $(P))$.
[ొ]: If $x \in A^{*}$ is such that ends $(\{x\}) \subseteq \operatorname{in}(\operatorname{cl}(\operatorname{ends}(P)))$, then ends $(\{x\}) \subseteq \operatorname{cl}(\operatorname{ends}(P))$, hence $\mathrm{cl}($ ends $(\{x\}) \subseteq \mathrm{cl}($ ends $(P))$, hence $\mathrm{cl}($ ends $(\{x\} \cup P))=\operatorname{cl}($ ends $(P))$. Therefore, $\{x\} \cup P \equiv$ end $P$ (by Prop. 2.51). Thus, ends $(\{x\}) \subseteq \bigcup_{Q \equiv_{\text {end }} P}$ ends $(Q)$ for every ends $(\{x\}) \subseteq \operatorname{in}(\operatorname{cl}($ ends $(P)))$; thus $\operatorname{in}(\operatorname{cl}($ ends $(P))) \subseteq \bigcup_{Q \equiv_{\text {end }} P}$ ends $(Q)$.

Concerning the union of the $Q A^{*}$ : Let $R$ be any right ideal that intersects $\bigcup_{Q \equiv_{\text {end }} P} Q A^{*}$. Then (by the definition of union) there exists $Q \equiv_{\text {end }} P$ such that $R$ intersects $Q A^{*}$. Since $Q \equiv$ end $P$, every right ideal intersecting $Q A^{*}$ intersects $P A^{*}$, so $R$ intersects $P A^{*}$. Conversely, it is obvious that every right ideal that intersects $P$ also intersects $\bigcup_{Q \equiv_{\text {end }} P} Q A^{*}$. Thus, $\bigcup_{Q \equiv_{\text {end }} P} Q A^{*} \equiv_{\text {end }} P$.

Concerning the intersection of the $Q A^{*}$, we observe that for any $n>0, P A^{n} \equiv_{\text {end }} P$. Moreover, $\bigcap_{n>0} P A^{n} A^{*}=\varnothing$ (since for any length $n$, this intersection contains no word of length $<n$ ). Hence $\bigcap_{Q \equiv_{\text {end }} P} Q A^{*}$ (which is a subset of $\bigcap_{n>0} P A^{n} A^{*}$ ) is $\varnothing$.

Concerning the intersection of the ends $(Q)$ : By Lemma 2.6 we have $\bigcap_{v \in \operatorname{ends}(P)}$ ends $(P(-v))=\varnothing$. The result follows, since $P(-v) \equiv{ }_{\text {end }} P$.

End-equivalence can also be defined for right-ideal morphisms:
Definition 2.8 Two right-ideal morphisms $f_{1}, f_{2}$ are end-equivalent (denoted by $f_{1} \equiv_{\text {end }} f_{2}$ ) iff $\operatorname{Dom}\left(f_{1}\right) \equiv$ end $\operatorname{Dom}\left(f_{2}\right)$, and the restrictions of $f_{1}$ and $f_{2}$ to $\operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right)$ are equal.

Notation: By $[f]_{\text {end }}$ we denote the $\equiv_{\text {end }}$-equivalence class of $f$ in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. So for $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ we have $[f]_{\text {end }}=\left\{g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}: g \equiv_{\text {end }} f\right\} \subset \mathcal{R} \mathcal{M}^{\mathrm{P}}$. Note that although $\equiv_{\text {end }}$ is defined for all right-ideal morphisms, we define $[f]_{\text {end }}$ to contain only elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.

Proposition 2.9 (1) Let $P_{1}, P_{2} \subset A^{*}$ be prefix codes such that $P_{1} \equiv_{\text {end }} P_{2}$, let $P_{\cap}$ be the prefix code that generates the right ideal $P_{1} A^{*} \cap P_{2} A^{*}$, and let $P_{\cup}$ be the prefix code that generates the right ideal $P_{1} A^{*} \cup P_{2} A^{*}$. Then $P_{1} \equiv_{\text {end }} P_{2} \equiv_{\text {end }} P_{\cap} \equiv_{\text {end }} P_{\cup}$.
(2) Let $f_{1}, f_{2}$ be right-ideal morphisms such that $f_{1} \equiv_{\text {end }} f_{2}$. Then $f_{1} \cap f_{2}$ and $f_{1} \cup f_{2}$ are right-ideal morphisms, and $f_{1} \equiv_{\text {end }} f_{2} \equiv_{\text {end }} f_{1} \cap f_{2} \equiv_{\text {end }} f_{1} \cup f_{2}$.
(3) If $f_{1}, f_{2} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ and $f_{1} \equiv_{\text {end }} f_{2}$, then $f_{1} \cap f_{2}, f_{1} \cup f_{2} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$.

Proof. (1) This follows from Lemma 2.3(2), and the fact that $L A^{*} \equiv_{\text {end }} L$ for all sets $L \subseteq A^{*}$.
(2) By (1) we have $\operatorname{Dom}\left(f_{1}\right) \equiv_{\text {end }} \operatorname{Dom}\left(f_{2}\right) \equiv_{\text {end }} \operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right) \equiv_{\text {end }} \operatorname{Dom}\left(f_{1}\right) \cup \operatorname{Dom}\left(f_{2}\right)$. Also, $\operatorname{Dom}\left(f_{1} \cap f_{2}\right)=\operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right)$. And since $f_{1}=f_{2}$ on $\operatorname{Dom}\left(f_{1} \cap f_{2}\right)$ we have $f_{1}=f_{1} \cap f_{2}$ on $\operatorname{Dom}\left(f_{1} \cap f_{2}\right)$. Hence $f_{1} \equiv_{\text {end }} f_{1} \cap f_{2}$, and similarly for $f_{2}$. Also, $\operatorname{Dom}\left(f_{1} \cup f_{2}\right)=\operatorname{Dom}\left(f_{1}\right) \cup \operatorname{Dom}\left(f_{2}\right)$. And $f_{1}=f_{1} \cup f_{2}$ on $\operatorname{Dom}\left(f_{1}\right)$, and $f_{2}=f_{1} \cup f_{2}$ on $\operatorname{Dom}\left(f_{2}\right)$, hence $f_{1} \equiv$ end $f_{1} \cup f_{2} \equiv$ end $f_{2}$.
(3) Since the complexity class P is closed under $\cap$ and $\cup \operatorname{Dom}\left(f_{1} \cap f_{2}\right)$ and $\operatorname{Dom}\left(f_{1} \cup f_{2}\right)$ belong to P . Since $f_{1} \cap f_{2}$ is the restriction of $f_{1}$ to $\operatorname{Dom}\left(f_{1} \cap f_{2}\right)$, we have $f_{1} \cap f_{2} \in \mathcal{R} \mathcal{M}_{2}^{\mathrm{P}}$. To compute $\left(f_{1} \cup f_{2}\right)(x)$ in polynomial time, check whether $x \in \operatorname{Dom}\left(f_{1}\right)$, and if so, compute $f_{1}(x)$; otherwise, check whether $x \in \operatorname{Dom}\left(f_{2}\right)$, and compute $f_{2}(x)$. Polynomial balance of $f_{1} \cap f_{2}$ and $f_{1} \cup f_{2}$ follows from the polynomial balance of $f_{1}$ and $f_{2}$.

Corollary 2.10 Every $\equiv_{\text {end }}$-class (within $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ or within the monoid of all right-ideal morphisms) is a lattice under $\subseteq, \cup$ and $\cap$. In particular, $[f]_{\text {end }}$ is a lattice, for every $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$.

Proof. The lattice property follows from Prop. 2.9.

## Proposition 2.11 (preservation of injectiveness under $\equiv_{\text {end }}$ ).

If $f, g$ are right-ideal morphisms such that $g \equiv_{\text {end }} f$, and if $f$ is injective, then $g$ is injective.
Proof. From $g \equiv_{\text {end }} f$ it follows that $f$ and $g$ agree on $D=\operatorname{Dom}(g) \cap \operatorname{Dom}(f)$. So, $h=\left.g\right|_{D}=\left.f\right|_{D}$ is injective. Also, $h \equiv_{\text {end }} g$. To show that $g$ is injective, let $x_{1}, x_{2} \in \operatorname{Dom}(g)$ be such that $g\left(x_{1}\right)=g\left(x_{2}\right)$. From $D=\operatorname{Dom}(h) \equiv$ end $\operatorname{Dom}(g)$ it follows that $x_{1} A^{*}$ intersects $\operatorname{Dom}(h)$ at $x_{1} u$ (for some $u \in A^{*}$ ). Hence, since $g$ is a right-ideal morphism, $g\left(x_{1} u\right)=g\left(x_{2} u\right)$, where $x_{1} u \in \operatorname{Dom}(h)$ and $x_{2} u \in \operatorname{Dom}(g)$. Again, since $\operatorname{Dom}(h) \equiv_{\text {end }} \operatorname{Dom}(g)$ it follows that $x_{2} u A^{*}$ intersects $\operatorname{Dom}(h)$ at $x_{2} u v$ (for some $v \in A^{*}$ ). Then $g\left(x_{1} u v\right)=g\left(x_{2} u v\right)$, where $x_{1} u v \in \operatorname{Dom}(h)$ and $x_{2} u v \operatorname{Dom}(h)$. Since $h\left(x_{1} u v\right)=g\left(x_{1} u v\right)=$ $g\left(x_{2} u v\right)=h\left(x_{2} u v\right)$, injectiveness of $h$ implies $x_{1} u v=x_{2} u v$; hence $x_{1}=x_{2}$, so $g$ is injective.

Definition 2.12 (maximum extension). For any right-ideal morphism $f: A^{*} \rightarrow A^{*}$ we define $f_{\mathrm{e}, \max }=\bigcup\left\{g: g\right.$ is a right-ideal morphism with $\left.g \equiv_{\text {end }} f\right\}$.

It can happen that $f_{\mathrm{e}, \text { max }} \notin \mathcal{R} \mathcal{M}^{\mathrm{P}}$ when $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ (and that is in fact the "usual" case - see Prop. (3.9).

## Proposition 2.13.

(1) For every right-ideal morphism $f, f_{\mathrm{e}, \max }$ is a function, and a right-ideal morphism $A^{*} \rightarrow A^{*}$. It is the maximum extension of $f$ among all right-ideal morphisms that are $\equiv_{\text {end }} f . S o,_{\mathrm{e}, \text { max }} \equiv_{\text {end }} f$, and $f_{\mathrm{e}, \max }$ is the unique right-ideal morphism that is maximal (under $\subseteq$ ) in the set $\{g: g$ is a right-ideal morphism, and $\left.f \equiv_{\text {end }} g\right\}$.
(2) For any right-ideal morphisms $h, k: A^{*} \rightarrow A^{*}$ :

$$
h \equiv_{\text {end }} k \quad \text { iff } \quad h_{\mathrm{e}, \max }=k_{\mathrm{e}, \max } .
$$

Proof. (1) If $f_{\mathrm{e}, \max }$ were not a function there would exist right-ideal morphisms $g_{1}, g_{2}$ such that $g_{1} \equiv_{\text {end }} g_{2} \equiv_{\text {end }} f, f \subseteq g_{1}, f \subseteq g_{2}$, and for some $x \in \operatorname{Dom}\left(g_{1}\right) \cap \operatorname{Dom}\left(g_{2}\right): g_{1}(x) \neq g_{2}(x)$. But $f \subseteq g_{1} \cup g_{2}$, and by Prop. 2.9, $f \equiv_{\text {end }} g_{i} \equiv_{\text {end }} g_{1} \cup g_{2}$, and $g_{1} \cup g_{2}$ is a function. Hence $g_{1}(x)=g_{2}(x)$ for all $x \in \operatorname{Dom}\left(g_{1}\right) \cap \operatorname{Dom}\left(g_{2}\right)$. So, $f_{\mathrm{e}, \max }(x)$ has at most one value for all $x$. The facts that $f_{\mathrm{e}, \max }$ is a right-ideal morphism, and that it is maximum, are straightforward.

By Prop. 2.7, $\operatorname{Dom}(f) \equiv_{\text {end }} \bigcup_{g \equiv_{\text {end }} f} \operatorname{Dom}(g)$. Since $f_{\mathrm{e}, \max }$ agrees with $f$ on $\operatorname{Dom}(f)$ we conclude that $f \equiv_{\text {end }} f_{\mathrm{e}, \max }$.
(2) This follows immediately from $f \equiv_{\text {end }} f_{\mathrm{e}, \text { max }}$.

A right-ideal morphism $f: A^{*} \rightarrow A^{*}$ can be extended to the partial function $f: A^{\omega} \rightarrow A^{\omega}$, defined as follows:

$$
f(p w)=f(p) w, \text { for any } p \in \operatorname{domC}(f) \text { and } w \in A^{\omega} .
$$

Then $\operatorname{Dom}(f) \cap A^{\omega}=\operatorname{domC}(f) A^{\omega}=\operatorname{Dom}(f) A^{\omega}$; and $\operatorname{Im}(f) \cap A^{\omega}=\operatorname{imC}(f) A^{\omega}=\operatorname{Im}(f) A^{\omega}$. We use the same name $f$ for the extended function, and its restrictions to $A^{*}$ or to $A^{\omega}$; the context will always make it clear which function is being used.

Any right-ideal morphism $f: A^{*} \rightarrow A^{*}$ is continuous (with respect to the topology defined on $A^{*}$ by the right ideals); indeed, for every right ideal $R \subseteq A^{*}, f^{-1}(R)$ is a right ideal. Similarly, the extension of $f$ to $A^{\omega}$ is a continuous function in the Cantor space topology.

Lemma 2.14 There exist right-ideal morphisms $f, g: A^{*} \rightarrow A^{*}$ such that

$$
\begin{aligned}
& \operatorname{Dom}\left(f_{\mathrm{e}, \max }\right) A^{\omega} \varsubsetneqq \operatorname{in}\left(\mathrm{cl}\left(\operatorname{Dom}(f) A^{\omega}\right)\right), \quad \text { and } \\
& g_{\mathrm{e}, \max } \circ f_{\mathrm{e}, \max } \neq(g \circ f)_{\mathrm{e}, \max } .
\end{aligned}
$$

Proof. For example, let $\operatorname{domC}(f)=0^{*} 1$, and $f\left(0^{2 n} 1\right)=0^{2 n+1} 1$, and $f\left(0^{2 n+1} 1\right)=0^{2 n} 1$, for all $n \geq 0$. Then $f=f_{\mathrm{e}, \text { max }}$; indeed, any strict extension of $f$ would need to make $f\left(0^{m}\right)$ defined for some $m \geq 0$; but such an extension to a right-ideal morphism would not agree with $f$ on $\operatorname{Dom}(f)$ (since $f$ transposes $0^{2 n} 1$ and $\left.0^{2 n+1} 1\right)$. So, $\operatorname{Dom}\left(f_{e, \max }\right)\{0,1\}^{\omega}=0^{*} 1\{0,1\}^{\omega}$, whereas cl $\left(0^{*} 1\{0,1\}^{\omega}\right)=\{0,1\}^{\omega}$, and $\operatorname{in}\left(\{0,1\}^{\omega}\right)=\{0,1\}^{\omega}$.

Since $f$ in the above example is injective, we can let $g=f^{-1}$. We have $f=f_{\mathrm{e}, \max }$ and $f^{-1}=$ $\left(f^{-1}\right)_{\mathrm{e}, \text { max }}$. Then $g_{\mathrm{e}, \max } \circ f_{\mathrm{e}, \text { max }}$ is the identity restricted to $0^{*} 1\{0,1\}^{*}$, whose maximum end-equivalent extension is the full identity $\mathbf{1}_{A^{*}}$. So in this example, $g_{\mathrm{e}, \max } \circ f_{\mathrm{e}, \max } \neq(g \circ f)_{\mathrm{e}, \max }$.

Lemma 2.15 Let $f_{1}, f_{2}$ be right-ideal morphisms with $f_{1} \equiv_{\text {end }} f_{2}$, and let $x \in \operatorname{Dom}\left(f_{1}\right)$. Then there exists $v \in A^{*}$ such that $x v A^{*} \subseteq \operatorname{Dom}\left(f_{2}\right)$, and for all $w \in A^{*}: f_{2}(x v w)=f_{1}(x v w)$.

Proof. If $x \in \operatorname{Dom}\left(f_{1}\right)$ then the right ideal $x A^{*}$ intersects $\operatorname{Dom}\left(f_{1}\right)$, hence $x A^{*}$ intersects $\operatorname{Dom}\left(f_{2}\right)$ (since $\operatorname{Dom}\left(f_{1}\right) \equiv_{\text {end }} \operatorname{Dom}\left(f_{2}\right)$ ). Thus there exists $x v \in x A^{*}$ such that $x v \in \operatorname{Dom}\left(f_{2}\right)$. Hence $x v A^{*} \subseteq$ $\operatorname{Dom}\left(f_{2}\right)$ (since $\operatorname{Dom}\left(f_{2}\right)$ is a right ideal). We have $f_{2}(x v w)=f_{1}(x v w)$ because $f_{1}$ and $f_{2}$ agree where they are both defined.

Just as we saw for right-ideal morphisms in general, $f_{\mathrm{e}, \max }$ can be extended to $\operatorname{dom} \mathrm{C}\left(f_{\mathrm{e}, \max }\right) A^{\omega}$.
Proposition 2.16 For all right-ideal morphisms $f, g: A^{*} \rightarrow A^{*}$ we have:

$$
g \equiv_{\text {end }} f \quad \text { iff } \quad g_{\mathrm{e}, \max }=f_{\mathrm{e}, \max } \text { on } A^{\omega} \text {. }
$$

Proof. The implication " $\Rightarrow$ " is clear from the definitions. Conversely, if $g_{\mathrm{e}, \max }$ and $f_{\mathrm{e}, \max }$ act the same on $A^{\omega}$ then $\operatorname{Dom}\left(g_{\mathrm{e}, \max }\right) A^{\omega}=\operatorname{Dom}\left(f_{\mathrm{e}, \max }\right) A^{\omega}$, hence $\mathrm{cl}\left(\operatorname{Dom}\left(g_{\mathrm{e}, \max }\right) A^{\omega}\right)=\mathrm{cl}\left(\operatorname{Dom}\left(f_{\mathrm{e}, \max }\right) A^{\omega}\right)$. Hence (by Prop. [2.5), $\operatorname{Dom}\left(g_{\mathrm{e}, \max }\right) \equiv_{\text {end }} \operatorname{Dom}\left(f_{\mathrm{e}, \max }\right)$. Since $\operatorname{Dom}(h) \equiv_{\text {end }} \operatorname{Dom}\left(h_{\mathrm{e}, \max }\right)$ for any righideal morphism $h$, we conclude that $\operatorname{Dom}(g) \equiv_{\text {end }} \operatorname{Dom}(f)$.

Since for all $w \in A^{\omega}, g_{\mathrm{e}, \max }(w)=f_{\mathrm{e}, \max }(w)$, we have for all $x \in \operatorname{Dom}(g) \cap \operatorname{Dom}(f)$ and all $v \in A^{\omega}$ : $g_{\mathrm{e}, \max }(x v)=f_{\mathrm{e}, \max }(x v)$. Hence (since $g_{\mathrm{e}, \max }(x v)=g_{\mathrm{e}, \max }(x) v$, and similarly for $f$ ), $g_{\mathrm{e}, \max }(x) v=$ $f_{\mathrm{e}, \max }(x) v$, for all $v \in A^{\omega}$. Taking $v=10^{\omega}$ (for example) then implies $g_{\mathrm{e}, \max }(x)=f_{\mathrm{e}, \max }(x)$, for all $x \in \operatorname{Dom}(g) \cap \operatorname{Dom}(f)$. Since $g_{\mathrm{e}, \text { max }}$ agrees with $g$ on $\operatorname{Dom}(g)$ (and similarly for $f$ ), we conclude that $g(x)=f(x)$, for all $x \in \operatorname{Dom}(g) \cap \operatorname{Dom}(f)$. Hence, $g \equiv_{\text {end }} f$.

Remark. It is also true that $g \equiv_{\text {end }} f$ is equivalent to the following: $\operatorname{Dom}(g) \equiv_{\text {end }} \operatorname{Dom}(f)$, and $g$ and $f$ agree on ends $(\operatorname{Dom}(g)) \cap \operatorname{ends}(\operatorname{Dom}(f))$. However, $g \equiv_{\text {end }} f$ is not equivalent to the property that $g$ and $f$ agree on $A^{\omega}$; indeed, the actions of $g$ and $f$ on $A^{\omega}$ could have different domains (even if $g \equiv_{\text {end }} f$ ). In Cor. 3.8 we will see that $g$ and $f$ agree on $A^{\omega}$ iff $g \equiv_{\text {bd }} f$ (which is a different congruence than $\equiv_{\text {end }}$ ).

Proposition 2.17 The relation $\equiv_{\text {end }}$ is a congruence for right-ideal morphisms; i.e., for all right-ideal morphisms $f_{1}, f_{2}, g$ : if $f_{1} \equiv_{\text {end }} f_{2}$, then $f_{1} g \equiv_{\text {end }} f_{2} g$ and $g f_{1} \equiv_{\text {end }} g f_{2}$.

Proof. The result follows from the next four claims.
Claim 1. $\operatorname{Dom}\left(f_{1} g\right) \equiv_{\text {end }} \operatorname{Dom}\left(f_{2} g\right)$.
Proof. Let $R$ be a right-ideal that intersects $\operatorname{Dom}\left(f_{1} g\right)$, so there exists $x_{1} \in R$ such that $x_{1} \in \operatorname{Dom}\left(f_{1} g\right)$; equivalently, $g\left(x_{1}\right) \in \operatorname{Dom}\left(f_{1}\right)$. Thus, $g(R)$ intersects $\operatorname{Dom}\left(f_{1}\right)$, therefore (since $\operatorname{Dom}\left(f_{1}\right) \equiv_{\text {end }} \operatorname{Dom}\left(f_{2}\right)$, and $g(R)$ is a right ideal) $g(R)$ intersects $\operatorname{Dom}\left(f_{2}\right)$. So, for some $g\left(x_{2}\right) \in g(R)$ with $x_{2} \in R, g\left(x_{2}\right) \in$ $\operatorname{Dom}\left(f_{2}\right)$. The latter is equivalent to $x_{2} \in \operatorname{Dom}\left(f_{2} g\right)$; so $R$ intersects $\operatorname{Dom}\left(f_{2} g\right)$. [This proves Claim 1.]
Claim 2. $f_{1} g$ and $f_{2} g$ agree on $\operatorname{Dom}\left(f_{1} g\right) \cap \operatorname{Dom}\left(f_{2} g\right)$.
Proof. Suppose $x \in \operatorname{Dom}\left(f_{1} g\right) \cap \operatorname{Dom}\left(f_{2} g\right)$. Then $f_{1} g(x)$ and $f_{2} g(x)$ are defined, so $g(x) \in \operatorname{Dom}\left(f_{1}\right) \cap$ $\operatorname{Dom}\left(f_{2}\right)$. Hence, since $f_{1}$ and $f_{2}$ agree on $\operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right)$ (because $f_{1} \equiv_{\text {end }} f_{2}$ ), we have $f_{1} g(x)=$ $f_{2} g(x)$. [This proves Claim 2.]
Claim 3. $\operatorname{Dom}\left(g f_{1}\right) \equiv_{\text {end }} \operatorname{Dom}\left(g f_{2}\right)$.
Proof. Let $R$ be a right ideal that intersects $\operatorname{Dom}\left(g f_{1}\right)$, so there exists $x \in R$ such that $g f_{1}(x)$ is defined, hence $f_{1}(x)$ is defined. Hence, by Lemma 2.15, there exists $v \in A^{*}$ such that $x v \in \operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right)$, and $f_{1}(x v)=f_{2}(x v)$. Thus, $g f_{1}(x v)=g f_{2}(x v)$. So, $x v \in \operatorname{Dom}\left(g f_{2}\right)$. Since $R$ is a right ideal, $x v \in R$, hence $R$ intersects $\operatorname{Dom}\left(g f_{2}\right)$.

In a similar way one proves that every right ideal that intersects $\operatorname{Dom}\left(g f_{2}\right)$ also intersects $\operatorname{Dom}\left(g f_{1}\right)$. [This proves Claim 3.]
Claim 4. $g f_{1}$ and $g f_{2}$ agree on $\operatorname{Dom}\left(g f_{1}\right) \cap \operatorname{Dom}\left(g f_{2}\right)$.
Proof. Since $f_{1} \equiv$ end $f_{2}, f_{1}$ and $f_{2}$ agree on $\operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right)$. Moreover, $\operatorname{Dom}\left(g f_{i}\right) \subseteq \operatorname{Dom}\left(f_{i}\right)$ (for $i=1,2)$, so the Claim holds. [This proves Claim 4.]

Corollary 2.18 For all right-ideal morphisms $f, g: A^{*} \rightarrow A^{*},\left(g_{\mathrm{e}, \max } \circ f_{\mathrm{e}, \max }\right)_{\mathrm{e}, \max }=(g \circ f)_{\mathrm{e}, \max }$.
Proof. By Prop. $2.13(1), g_{\mathrm{e}, \max } \equiv_{\text {end }} g$ and $f_{\mathrm{e}, \max } \equiv_{\text {end }} f$. Hence by Prop. 2.17, $g_{\mathrm{e}, \max } \circ f_{\mathrm{e}, \max } \equiv_{\text {end }} g \circ f$.

Definition 2.19 (ends monoid $\mathcal{M}_{\text {end }}^{P}$ ). The ends monoid consists of the $\equiv_{\text {end }}$-classes of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$; the multiplication is the multiplication of $\equiv_{\text {end }}$-classes. It is denoted by $\mathcal{M}_{\text {end }}^{\mathrm{P}}$, or by $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\text {end }}$.

As a set, $\mathcal{M}_{\text {end }}^{\mathrm{P}}=\left\{[f]_{\text {end }}: f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}\right\}$. The multiplication in $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is well-defined since $\equiv_{\text {end }}$ is a congruence, by Prop. 2.17, Hence, $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is a monoid which is a homomorphic image of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. The monoid version $M_{2,1}$ of the Richard Thompson group $V$ (a.k.a. $G_{2,1}$ ) is a submonoid of $\mathcal{M}_{\text {end }}^{\mathrm{P}} ; M_{2,1}$ is defined in [6]; see [9] for more information on the Thompson group.

There is a one-to-one correspondence between $\equiv_{\text {end }}$-classes and maximum end-extensions of elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ (by Prop. 2.16). So, $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ can also be defined as

$$
\mathcal{M}_{\mathrm{end}}^{\mathrm{P}}=\left(\left\{f_{\mathrm{e}, \text { max }}: f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}\right\}, \cdot\right)
$$

with multiplication "." defined by

$$
g_{\mathrm{e}, \max } \cdot f_{\mathrm{e}, \max }=\left(g_{\mathrm{e}, \max } \circ f_{\mathrm{e}, \max }\right)_{\mathrm{e}, \max } \quad\left(=(g \circ f)_{\mathrm{e}, \max }\right) .
$$

Here we used Cor. 2.18,

For the remainder of this section we need some definitions.
For any monoid $M$, the $\mathcal{L}$-order (denoted by $\leq_{\mathcal{L}}$ ) and the $\mathcal{R}$-order (denoted by $\leq_{\mathcal{R}}$ ), are defined (for any $s, t, u, v \in M$ ) by $t \leq_{\mathcal{L}} s$ iff there exists $m \in M$ such that $t=m s$; and $v \leq_{\mathcal{R}} u$ iff there exists $n \in M$ such that $v=u n$. The $\mathcal{D}$-relation on $M$ is defined (for $x, y \in M$ ) by $x \equiv_{\mathcal{D}} y$ iff there exists $z \in M$ such that $x \equiv_{\mathcal{R}} z \equiv_{\mathcal{L}} y$; equivalently, there exists $w \in M$ such that $x \equiv_{\mathcal{L}} w \equiv_{\mathcal{R}} y$. A monoid $M$ is called $\mathcal{D}^{0}$-simple if $M$ has only one $\mathcal{D}$-class, except possibly for a zero. These are well-known concepts in semigroup theory; see e.g., [11.

A right ideal $R \subseteq A^{*}$ is called essential iff $R \equiv_{\text {end }} A^{*}\left(\right.$ iff $\left.\mathrm{cl}(\operatorname{ends}(R))=A^{\omega}\right)$. Equivalently, the prefix code that generated $R$ (as a right ideal) is a maximal prefix code.

For any function $f: A^{*} \rightarrow A^{*}$, the relation $\bmod f$ is the equivalence relation defined on $\operatorname{Dom}(f)$ by $x_{1} \bmod f x_{2}$ iff $f\left(x_{1}\right)=f\left(x_{2}\right)$. The equivalence classes of $\bmod f$ are $\left\{f^{-1} f(x): x \in \operatorname{Dom}(f)\right\}$. For two partial functions $g, f: A^{*} \rightarrow A^{*}$, we say $\bmod f \leq \bmod g$ ("the relation $\bmod f$ is coarser than $\bmod g$ ", or " $\bmod g$ is finer than $\bmod f$ ") iff $\operatorname{Dom}(f) \subseteq \operatorname{Dom}(g)$, and for all $x \in \operatorname{Dom}(f): g^{-1} g(x) \subseteq f^{-1} f(x)$. Equivalently, $\bmod f \leq \bmod g$ iff every $\bmod f$-class is a union of $\bmod g$-classes.

A monoid $M$ is called congruence-simple iff $M$ is non-trivial and the only congruences on $M$ are the equality relation and the one-class congruence.

The length-lexicographic order on $\{0,1\}^{*}$ is a well-order, defined as follows for any $x_{1}, x_{2} \in\{0,1\}^{*}$ : $x_{1} \leq_{\ell \ell} x_{2}$ iff $\left|x_{1}\right|<\left|x_{2}\right|$, or $\left|x_{1}\right|=\left|x_{2}\right|$ and $x_{1}$ precedes $x_{2}$ in the dictionary order on $\{0,1\}^{*}$ (based on the alphabetic order $0<1$ ).

The next lemma is the $\mathcal{R} \mathcal{M}^{\mathrm{P}}$-version of Prop. 2.1 of (4).
Lemma 2.20 If $f, r \in \mathcal{R} \mathcal{M}^{P}$ and $r$ is regular with an inverse $r^{\prime} \in \mathcal{R} \mathcal{M}^{P}$ then:

$$
\begin{array}{lllll}
f \leq_{\mathcal{R}} r & \text { iff } & f=r r^{\prime} f & \text { iff } & \operatorname{lm}(f) \subseteq \operatorname{lm}(r) .  \tag{1}\\
f \leq_{\mathcal{L}} r & \text { iff } & f=f r^{\prime} r & \text { iff } & \bmod f \leq \bmod r .
\end{array}
$$

Proof. (1) $f \leq_{\mathcal{R}} r$ iff for some $u \in \mathcal{R} \mathcal{M}^{\mathrm{P}}: f=r u$. Then $f=r r^{\prime} r u=r r^{\prime} f$. Also, it is straightforward that $f=r u$ implies $\operatorname{Im}(f) \subseteq \operatorname{Im}(r)$.

Conversely, if $\operatorname{Im}(f) \subseteq \operatorname{Im}(r)$ then $\mathbf{1}_{\operatorname{Im}(f)}=\mathbf{1}_{\operatorname{Im}(r)} \circ \mathbf{1}_{\operatorname{Im}(f)}=\left.r \circ r^{\prime}\right|_{\operatorname{Im}(r)} \circ \mathbf{1}_{\operatorname{Im}(f)}$. Hence, $f=$ $\mathbf{1}_{\operatorname{Im}(f)} \circ f=\left.r \circ r^{\prime}\right|_{\operatorname{Im}(r)} \circ \mathbf{1}_{\operatorname{Im}(f)} \circ f=\left.r \circ r^{\prime}\right|_{\operatorname{Im}(r)} \circ f \leq_{\mathcal{R}} r$.
(2) $f \leq_{\mathcal{L}} r$ iff for some $v \in \mathcal{R} \mathcal{M}^{\mathrm{P}}: f=v r$. Then $f=v r r^{\prime} r=f r^{\prime} r$. And it is straightforward that $f=v r$ implies $\bmod f \leq \bmod r$.

Conversely, if $\bmod f \leq \bmod r$ then for all $x \in \operatorname{Dom}(f), r^{-1} r(x) \subseteq f^{-1} f(x)$. And for every $x \in \operatorname{Dom}(f),\{f(x)\}=f \circ f^{-1} \circ f(x)$. Moreover, $f \circ r^{-1} \circ r(x) \subseteq f \circ f^{-1} \circ f(x)=\{f(x)\}$, and since $r^{-1} \circ r(x) \neq \varnothing$, it follows that $f \circ r^{-1} \circ r(x)=\{f(x)\}$. So, $f=f \circ r^{-1} \circ r$. Moreover, $f \circ r^{\prime} \circ r(x) \in f \circ r^{-1} \circ r(x)=\{f(x)\}$, hence $f \circ r^{\prime} \circ r(x)=f(x)$. Hence, $f=f r^{\prime} r \leq_{\mathcal{L}} r$.

Definition 2.21 (rank function). For any set $S \subseteq A^{*}$ the rank function of $S$ is defined for all $x \in S$ by $\operatorname{rank}_{S}(x)=\left|\left\{z \in S: z \leq_{\ell \ell} x\right\}\right|$ (where $\leq_{\ell \ell}$ denotes the length-lexicographic order). When $x \notin S$, $\operatorname{rank}_{S}(x)$ is undefined.

We will use padding with a fully time-constructible function in order to turn any algorithm into a linear-time algorithms. A "Turing machine" will always mean a multi-tape Turing machine. By definition, a function $t: \mathbb{N} \rightarrow \mathbb{N}$ is fully time-constructible iff $t$ is total, and increasing, and there exists a deterministic Turing machine such that for some $n_{0} \in \mathbb{N}$, and for all $n \geq n_{0}$, and for every input of length $n$, the machine runs for time exactly $t(n)$.

For example, any polynomial function $n \mapsto c\left(n^{d}+1\right.$ ) (where $c, d$ are positive integers), and any exponential function $n \mapsto c d^{n}$ (where $c>0$ and $d \geq 2$ are integers) are known to be fully timeconstructible. The sum $t_{1}(n)+t_{2}(n)$, and the product $t_{1}(n) \cdot t_{2}(n)$ of two fully time-constructible functions are also fully time-constructible. See e.g. [13], [1], [16], for information about time-constructible functions.

Lemma 2.22 Let $f:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ be a partial recursive right-ideal morphism with decidable domain. Then we have:
(1) There exists a fully time-constructible functiont such that $f$ is computed by a Turing machine with time-complexity $\leq t$.
(2) Let $t$ be any fully time-constructible function such that $f$ is computed by a Turing machine with time-complexity $\leq t$. Let $F$ be the restriction of $f$ to $\bigcup_{x \in \operatorname{domC}(f)} x A^{|x| \cdot t(|x|)} A^{*}$. In other words, $\operatorname{domC}(F)=\bigcup_{x \in \operatorname{domC}(f)} x A^{|x| \cdot t(|x|)} ;$ and $F(x u v)=f(x)$ uv for all $x \in \operatorname{domC}(f), u \in A^{|x| \cdot t(|x|)}$, $v \in A^{*}$.

Then $F \equiv_{\text {ends }} f$, and $F$ has linear time-complexity and has linear balance (both bounded from above by the function $n \mapsto 3 n$ ).

Proof. (1) Let $M$ be a deterministic Turing machine that computes $f$ and that eventually halts on every input. We construct a new Turing machine $M^{\prime}$ for $f$ which has the same running time for on all inputs of length $\leq n$, for all $n$. On input $x \in A^{n}, M^{\prime}$ simulates $M$ on all inputs of length $n$, but only outputs $f(x)$. If $f(x)$ is not defined, $M^{\prime}$ produces no output, but since $\operatorname{Dom}(f)$ is decidable, $M^{\prime}$ nevertheless has a time-complexity for every input. Let $t$ be the time-complexity function of $M^{\prime}$. Then $t$ is fully time-constructible, and it is the running time of a Turing machine that that halts for all inputs and that computes $f$.
(2) Since each set $A^{|x| \cdot t(|x|)}$ is a maximal prefix code, we have $\operatorname{domC}(F) \equiv_{\text {ends }} \operatorname{domC}(f)$. Also, $F$ is a restriction of $f$. Hence, $F \equiv_{\text {ends }} f$.

To compute $F(w)$ in linear time, we first consider the case where $w \in \operatorname{Dom}(F)$, i.e., $w=x u v$ for some $x \in \operatorname{domC}(f), u \in A^{|x| \cdot t(|x|)}, v \in A^{*}$. We first run the Turing machine for $f$ on the prefixes of xuv until $x$ (the smallest prefix on which $f$ is defined) is found. In detail, each prefix of $w$ is considered in turn, and copied on a work tape; when the next prefix is considered, one more letter is added on the right, and the head of the work tape is moved back to the left end. So the copying of prefixes takes time $\leq|x|^{2}(\leq|x| \cdot t(|x|))$. Checking that the prefix belongs to $\operatorname{Dom}(f)$ takes time $\leq t(|x|)$, so check all the prefixes takes time $\leq|x| \cdot t(|x|)$. In total, the time to find $x$ and to compute $f(x)$ takes time $\leq 2|x| \cdot t(|x|)$.

Then we check that the rest of the input, namely $u v$, has length $|x| \cdot t(|x|)$. Since $|x| \cdot t(|x|)$ is time-constructible, this can be done in time $\leq|x| \cdot t(|x|)$. During this time, $u v$ is copied to the output tape; this takes time $|u v|=|x| \cdot t(|x|)+|v|$. So the total time is $\leq 2|x| t(|x|)+|x| t(|x|)+|v| \leq 3 \cdot|x u v|$ (since $|u|=|x| t(|x|)$ ). Thus, $F(x u v)$ is computed in time $\leq 3 \cdot|x u v|$.

The complexity bound implies that $|F(x u v)| \leq 3 \cdot|x u v|$. For the input balance we have: $|F(x u v)|=$ $|f(x)|+|u|+|v| \geq \frac{1}{2} \cdot|x| \cdot t(|x|)+\frac{1}{2} \cdot|u|+|v| \geq \frac{1}{2} \cdot|x u v|$.

To handle the case of an arbitrary input $w$ (not necessarily in $\operatorname{Dom}(F)$ ), we follow the same procedure as above, but we add a counter that stops the computation after time $3|w|$. The machine rejects, and produces no output, if $w$ has not been found to be in $\operatorname{Dom}(F)$ by that time.

Theorem 2.23 The monoid $\mathcal{M}_{\mathrm{end}}^{\mathrm{P}}$ is regular, $\mathcal{D}^{0}$-simple, and finitely generated. Moreover, every $\overline{\text { end }}$-class contains a regular element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.

Proof. An initial remark: Every $\mathcal{D}^{0}$-simple monoid is regular; but we prove regularity separately first because it will be used in the proof of $\mathcal{D}^{0}$-simplicity.

For every $f \in \mathcal{R} \mathcal{M}^{P}$ there exists an inverse function $f^{\prime}$ that is balanced, but that is not necessarily polynomial-time computable; balance is inherited from $f$ if we restrict the domain of $f^{\prime}$ to $\operatorname{Im}(f)$. Let $T($.$) be a fully time-constructible upper bound on the time-complexity of f^{\prime}$ (see Lemma [2.22). We can restrict both $f$ and $f^{\prime}$ in order to reduce the time-complexity (padding argument), while preserving end-equivalence, as in Lemma 2.22, Namely, we replace $\operatorname{domC}\left(f^{\prime}\right)$ by the prefix code $\bigcup_{y \in \operatorname{domC}\left(f^{\prime}\right)} y A^{|y| \cdot T(|y|)}$. Let $F^{\prime}$ be this restriction of $f^{\prime}$, and let the restriction of $f$ be $F=f \circ F^{\prime} \circ f=\operatorname{id}_{\operatorname{Dom}\left(F^{\prime}\right)} \circ f$. Then $F$ and $F^{\prime}$ have polynomial (in fact, linear) time-complexity,
so $F, F^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. Moreover, $F^{\prime}$ is an inverse of $F$, and $f \equiv_{\text {end }} F$, and $f^{\prime} \equiv_{\text {end }} F^{\prime}$. Therefore, $f \equiv \equiv_{\text {end }} F=F F^{\prime} F \in[f]_{\text {end }} \cdot\left[F^{\prime}\right]_{\text {end }} \cdot[f]_{\text {end }}$. Hence $[f]_{\text {end }}=[f]_{\text {end }} \cdot\left[F^{\prime}\right]_{\text {end }} \cdot[f]_{\text {end }}$, so $[f]_{\text {end }}$ is regular in $\mathcal{M}_{\text {end }}^{\mathrm{P}}$. And $F\left(\equiv_{\text {end }} f\right)$ is a regular element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.

Proof of $\mathcal{D}^{0}$-simplicity: For every non-empty $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ there exists $f_{0} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $f \equiv_{\text {end }} f_{0}$ and such that $\operatorname{imC}\left(f_{0}\right)$ is infinite. Indeed, let us pick some $x_{0} \in \operatorname{domC}(f)$ and define $f_{0}$ by

$$
\begin{aligned}
& \operatorname{domC}\left(f_{0}\right)=\left(\operatorname{domC}(f)-\left\{x_{0}\right\}\right) \cup x_{0} 0^{*} 1 ; \\
& f_{0}\left(x_{0} 0^{n} 1\right)=f\left(x_{0}\right) 0^{n} 1 \text { for all } n \geq 0, \text { and } \\
& f_{0}(x)=f(x) \text { for all } x \in \operatorname{domC}(f)-\left\{x_{0}\right\} .
\end{aligned}
$$

Then $\operatorname{imC}\left(f_{0}\right)$ contains $f\left(x_{0}\right) 0^{*} 1$, hence it is infinite. So, from now on we assume that for $f$ itself, $\operatorname{imC}(f)$ is infinite.
Claim: If $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ has infinite $\operatorname{imC}(f)$, then there exists a right-ideal morphism $g$ with the following properties: $g$ is partial recursive with decidable domain, $\operatorname{dom} \mathrm{C}(g)$ is a maximal prefix code, $g$ is injective, and $\operatorname{Im}(g)=\operatorname{Im}(f)$.
Proof of the Claim: We construct $g$ as follows. Let $\leq_{\ell \ell}$ denote the length-lexicographic order on $\{0,1\}^{*}$. For any $y \in \operatorname{imC}(f)$, we can compute $\operatorname{rank}(y)=\left|\left\{z \in \operatorname{imC}(f): z \leq_{\ell \ell} y\right\}\right|$; computability follows from the fact that $f$ is polynomially balanced. The function rank is injective; it is also onto $\mathbb{N}$ since $\operatorname{imC}(f)$ is infinite. We define $g\left(0^{n} 1\right)$ to be the element $y \in \operatorname{imC}(f)$ such that $\operatorname{rank}(y)=n$ (for any $n \geq 0$ ). So, $g$ is injective and $g^{-1}(y)=0^{\operatorname{rank}(y)} 1$ for all $y \in \operatorname{imC}(f)$. We have $\operatorname{domC}(g)=0^{*} 1$, which is a maximal prefix code; obviously, $0^{*} 1$ is a decidable language. Then $g$ is partial recursive with decidable domain, and injective, and $\operatorname{Im}(f)=\operatorname{Im}(g)$. This proves the Claim.

Let $t($.$) be a fully time-constructible upper bound on the time complexities of g, g^{-1}, f$ and $f^{\prime}$. Then, by padding $f$ and $g$ as in Lemma 2.22 we obtain functions $f_{1}, g_{1} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $f \equiv_{\text {end }} f_{1}$, $g \equiv$ end $g_{1}$, both $f_{1}$ and $g_{1}$ are computable in linear time, and both $f_{1}$ and $g_{1}$ are regular in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. Moreover, $\operatorname{domC}\left(g_{1}\right)\left(\equiv_{\text {end }} 0^{*} 1\right)$ is a maximal prefix code (equivalently, $\operatorname{Dom}\left(g_{1}\right)$ is an essential right ideal), $g_{1}$ is injective, and $\operatorname{Im}\left(f_{1}\right)=\operatorname{Im}\left(g_{1}\right)$.

Since $\operatorname{Im}\left(f_{1}\right)=\operatorname{Im}\left(g_{1}\right)$ and $f_{1}, g_{1}$ are regular elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, Lemma 2.20(1) implies $f_{1} \equiv_{\mathcal{R}} g_{1}$.
Since $g_{1}$ is injective, the relation $\bmod g_{1}$ is the equality relation on $\operatorname{Dom}\left(g_{1}\right)$. Hence, since $g_{1}$ is a regular element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, Lemma 2.20 (2) implies that $g_{1} \equiv_{\mathcal{L}} \mathbf{1}_{\text {Dom }\left(g_{1}\right)}$ (the identity map restricted to $\left.\operatorname{Dom}\left(g_{1}\right)\right)$. Since $\operatorname{domC}\left(g_{1}\right)$ is a maximal prefix code, $\operatorname{Dom}\left(g_{1}\right)$ is an essential right ideal; equivalently, $\mathbf{1}_{\text {Dom }\left(g_{1}\right)} \equiv_{\text {end }} 1$.

Overall we now have $f \equiv_{\text {end }} f_{1} \equiv_{\mathcal{R}} g_{1} \equiv_{\mathcal{L}} \mathbf{1}_{\text {Dom }\left(g_{1}\right)} \equiv_{\text {end }} \mathbf{1}$. Hence, in $\mathcal{M}_{\text {end }}^{\mathrm{P}},[f]_{\text {end }} \equiv \mathcal{D}[\mathbf{1}]_{\text {end }}$.
Proof of finite generation: The proof is based on the fact that $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ has evaluation maps for programs with bounded balance and time-complexity. This was described in detail in Section 4 of [4] and Section 2 of [3]. We briefly give the definition here: For a polynomial $q_{2}$ such that $q_{2}(n)=a\left(n^{2}+1\right)$ (for some fixed large constant $a$ ), we define an evaluation map $\operatorname{evR}_{q_{2}}^{C} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ by

$$
\operatorname{evR}_{q_{2}}^{C}(\operatorname{code}(w) 11 u v)=\operatorname{code}(w) 11 \phi_{w}(u) v
$$

for all Turing machine programs $w$ with balance and time-complexity $\leq q_{2}$, and all $u \in \operatorname{domC}\left(\phi_{w}\right)$, and $v \in A^{*}$. Here, $\phi_{w} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ denotes the function with program $w$. Then we have

$$
\phi_{w}=\pi_{|\operatorname{code}(w) 11|}^{\prime} \circ \operatorname{evR}_{q_{2}}^{C} \circ \pi_{\operatorname{code}(w) 11}
$$

where $\pi_{n}^{\prime}$ is defined by $\pi_{n}^{\prime}\left(x_{1} x_{2}\right)=x_{2}$ whenever $x_{1}, x_{2} \in A^{*}$ with $\left|x_{1}\right|=n$ (and $\pi_{n}^{\prime}$ is undefined on other arguments); and $\pi_{u}$ is defined by $\pi_{u}(x)=u x$ for all $u, x \in A^{*}$. See [3] for the proof that such a function $\operatorname{evR}_{q_{2}}^{C}$ exists.

In the proof of regularity of $\mathcal{M}_{\text {end }}^{P}$ above, we saw that every $\phi_{v} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ is end-equivalent to some $\phi_{w} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $\phi_{w}$ has linear time-complexity (in fact, it is $\leq 3 n$, by Lemma 2.22). We can obtain $\phi_{w}$ as $\phi_{w}=\phi_{v} \circ \mathbf{1}_{P_{w}}$, where

$$
P_{w}=\bigcup_{x \in \operatorname{domC}\left(\phi_{v}\right)} x A^{T(|x|)^{2}} ;
$$

here, $T($.$) is the time-complexity of \phi_{w}$. Since $T($.$) is a polynomial of the form c\left(n^{2}+1\right)$, the function $n \mapsto n \cdot T(n)^{2}$ is fully time-constructible. Since the time-complexity of $\phi_{w}$ is linear with coefficient $\leq 3$ (by Lemma (2.22), the evaluation map evR $\mathrm{R}_{q_{2}}^{C}$ can evaluate $\phi_{w}$ without any need for further padding; so we have

$$
\phi_{v} \equiv \equiv_{\text {end }} \quad \phi_{w}=\pi_{|\operatorname{code}(w) 11|}^{\prime} \circ \operatorname{evR}_{q_{2}}^{C} \circ \pi_{\operatorname{code}(w) 11} .
$$

So, $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is generated by $\left\{\left[\pi_{0}\right]_{\text {end }},\left[\pi_{1}\right]_{\text {end }},\left[\pi_{1}^{\prime}\right]_{\text {end }},\left[\operatorname{evR} R_{q_{2}}^{C}\right]_{\text {end }}\right\}$.
In the proof of Theorem 2.24 the concept of $\mathcal{J}^{0}$-simplicity is used. By definition, a monoid $M$ with a zero is $\mathcal{J}^{0}$-simple iff for all non-zero elements $a, b \in M$ there exist $x_{1}, x_{2}, x_{3}, x_{4} \in M$ such that $a=x_{1} b x_{2}$ and $b=x_{3} a x_{4}$. For more information on the $\mathcal{J}$-relation and the $\mathcal{J}$-preorder, see e.g. [11]. Obviously, $\mathcal{D}^{0}$-simplicity implies $\mathcal{J}^{0}$-simplicity.

Theorem $2.24 \mathcal{M}_{\mathrm{end}}^{\mathrm{P}}$ is congruence-simple.
Proof. The proof is similar to the proof of congruence-simplicity of the Thompson-Higman monoid $M_{2,1}$ in [7]. Let $\mathbf{0}$ be the $\equiv_{\text {end }}$-class of the empty map; this class consists only of the empty map 0 . When $\cong$ is any congruence on $\mathcal{M}_{\text {end }}^{P}$ that is not the equality relation, we will show that the whole monoid $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is congruent to 0 . We will make use of $\mathcal{J}^{0}$-simplicity of $\mathcal{M}_{\text {end }}^{\mathrm{P}}$, which follows from its $\mathcal{D}^{0}$-simplicity (and also from the $\mathcal{J}^{0}$-simplicity of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, Prop. 2.7 in [4]).

Since $\cong$ is a congruence on $\mathcal{M}_{\mathrm{end}}^{\mathrm{P}}$, and since $\equiv_{\text {end }}$ is a congruence on $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, it follows that $\cong$ can also be defined as a congruence on $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ that is coarser than $\equiv_{\text {end }}$. We will show that if $\cong$ on $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is not $\equiv_{\text {end }}$, then $\cong$ is the trivial (one-class) congruence.
Case (0): Assume that $\Phi \cong \mathbf{0}$ for some element $\Phi \neq \mathbf{0}$ in $\mathcal{M}_{\text {end }}^{\mathrm{P}}$. Then for all $\alpha, \beta \in \mathcal{M}_{\text {end }}^{\mathrm{P}}$ we have obviously $\alpha \Phi \beta \cong \mathbf{0}$. Moreover, by $\mathcal{J}^{0}$-simplicity, $\mathcal{M}_{\text {end }}^{\mathrm{P}}=\left\{\alpha \Phi \beta: \alpha, \beta \in \mathcal{M}_{\text {end }}^{\mathrm{P}}\right\}$, since $\Phi \neq \mathbf{0}$. Hence all elements are congruent to $\mathbf{0}$.

For the remainder of the proof we let $\varphi, \psi \in \mathcal{R} \mathcal{M}^{\mathrm{P}}-\{0\}$ be representatives of two different $\equiv_{\text {end }}{ }^{-}$ classes (i.e., $\varphi \not \equiv_{\text {end }} \psi$ ) such that $[\varphi]_{\text {end }} \cong[\psi]_{\text {end }}$. Notation: For any $u, v \in A^{*},(v \leftarrow u)$ denotes the right-ideal morphism $u w \mapsto v w$ (for all $w \in A^{*}$ ).
Case (1): $\operatorname{Dom}(\varphi) \not \equiv_{\text {end }} \operatorname{Dom}(\psi)$.
Then there exists $x_{1} \in A^{*}$ such that $x_{1} A^{*}$ intersects $\operatorname{Dom}(\varphi)$ (e.g., at $x_{0}$ ), but $x_{1} A^{*}$ does not intersect $\operatorname{Dom}(\psi)$. Then $x_{0} A^{*} \subseteq \operatorname{Dom}(\varphi)$, but $\operatorname{Dom}(\psi) \cap x_{0} A^{*}=\varnothing$. Or, vice versa, there exists $x_{0} \in A^{*}$ such that $x_{0} A^{*} \subseteq \operatorname{Dom}(\psi)$, but $\operatorname{Dom}(\varphi) \cap x_{0} A^{*}=\varnothing$. Let us assume the former.

Letting $\beta=\left(x_{0} \leftarrow x_{0}\right)$, we have $\varphi \beta()=.\left(\varphi\left(x_{0}\right) \leftarrow x_{0}\right)$. But $\psi \beta()=$.0 , since $x_{0} A^{*} \cap \operatorname{Dom}(\psi)=\varnothing$. So, $[\varphi \beta]_{\text {end }} \cong[\psi \beta]_{\text {end }}=\mathbf{0}$, but $[\varphi \beta]_{\text {end }} \neq \mathbf{0}$. Hence, applying case ( 0 ) to $\Phi=[\varphi \beta]_{\text {end }}$ we conclude that the entire monoid $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is congruent to $\mathbf{0}$.
Case (2.1): $\operatorname{Dom}(\varphi) \equiv_{\text {end }} \operatorname{Dom}(\psi)$ and $\operatorname{Im}(\varphi) \not \equiv_{\text {end }} \operatorname{Im}(\psi)$.
Then there exists $y_{0} \in A^{*}$ such that $y_{0} A^{*} \subseteq \operatorname{Im}(\varphi)$, but $\operatorname{Im}(\psi) \cap y_{0} A^{*}=\varnothing$; or, vice versa, $y_{0} A^{*} \subseteq \operatorname{Im}(\psi)$, but $\operatorname{Im}(\varphi) \cap y_{0} A^{*}=\varnothing$. Let us assume the former.

Let $x_{0} \in A^{*}$ be such that $y_{0}=\varphi\left(x_{0}\right)$. Then $\left(y_{0} \leftarrow y_{0}\right) \circ \varphi \circ\left(x_{0} \leftarrow x_{0}\right)=\left(y_{0} \leftarrow x_{0}\right)$. On the other hand, $\left(y_{0} \leftarrow y_{0}\right) \circ \psi \circ\left(x_{0} \leftarrow x_{0}\right)=\mathbf{0}$. Indeed, if $x_{0} A^{*} \cap \operatorname{Dom}(\psi)=\varnothing$ then for all $w \in A^{*}$ : $\psi \circ\left(x_{0} \leftarrow x_{0}\right)\left(x_{0} w\right)=\psi\left(x_{0} w\right)=\varnothing$. And if $x_{0} A^{*} \cap \operatorname{Dom}(\psi) \neq \varnothing$ then for those $w \in A^{*}$ such that $x_{0} w \in \operatorname{Dom}(\psi)$ we have $\left(y_{0} \leftarrow y_{0}\right) \circ \psi \circ\left(x_{0} \leftarrow x_{0}\right)\left(x_{0} w\right)=\left(y_{0} \leftarrow y_{0}\right)\left(\psi\left(x_{0} w\right)\right)=\varnothing$, since $\operatorname{Im}(\psi) \cap y_{0} A^{*}=\varnothing$. Now case (0) applies to $\mathbf{0} \neq \Phi=\left[\left(y_{0} \leftarrow y_{0}\right) \circ \varphi \circ\left(x_{0} \leftarrow x_{0}\right)\right]_{\text {end }} \cong \mathbf{0}$; hence all elements of $\mathcal{M}_{\text {end }}^{P}$ are congruent to $\mathbf{0}$.
Case (2.2): $\operatorname{Dom}(\varphi) \equiv_{\text {end }} \operatorname{Dom}(\psi)$ and $\operatorname{Im}(\varphi) \equiv_{\text {end }} \operatorname{Im}(\psi)$.
Then we can restrict $\varphi$ and $\psi$ to $\operatorname{Dom}(\varphi) \cap \operatorname{Dom}(\psi)\left(\equiv_{\text {end }} \operatorname{Dom}(\varphi) \equiv_{\text {end }} \operatorname{Dom}(\psi)\right)$, by choice of representatives in $[\varphi]_{\text {end }}$, respectively $[\psi]_{\text {end }}$; so now $\operatorname{domC}(\varphi)=\operatorname{domC}(\psi)$. Since $\varphi \neq \psi$, there exist $x_{0} \in \operatorname{domC}(\varphi)=\operatorname{domC}(\psi)$ and $y_{0} \in \operatorname{Im}(\varphi), y_{1} \in \operatorname{Im}(\psi)$, such that $\varphi\left(x_{0}\right)=y_{0} \neq y_{1}=\psi\left(x_{0}\right)$. We have two subcases.

Subcase (2.2.1): $y_{0}$ and $y_{1}$ are not prefix-comparable.
Then $\left(y_{0} \leftarrow y_{0}\right) \circ \varphi \circ\left(x_{0} \leftarrow x_{0}\right)=\left(y_{0} \leftarrow x_{0}\right)$.
On the other hand, $\left(y_{0} \leftarrow y_{0}\right) \circ \psi \circ\left(x_{0} \leftarrow x_{0}\right)\left(x_{0} w\right)=\left(y_{0} \leftarrow y_{0}\right)\left(y_{1} w\right)=\varnothing$ for all $w \in A^{*}$ (since $y_{0}$ and $y_{1}$ are not prefix-comparable). So, $\left(y_{0} \leftarrow y_{0}\right) \circ \psi \circ\left(x_{0} \leftarrow x_{0}\right)=\mathbf{0}$. Hence case ( 0 ) applies to $\mathbf{0} \neq \Phi=\left[\left(y_{0} \leftarrow y_{0}\right) \circ \varphi \circ\left(x_{0} \leftarrow x_{0}\right)\right]_{\text {end }} \cong \mathbf{0}$.
Subcase (2.2.2): $y_{0}$ is a prefix of $y_{1}$, and $y_{0} \neq y_{1}$. (The case where $y_{0}$ is a prefix of $y_{1}$ is similar.)
Then $y_{1}=y_{0} a u_{1}$ for some $a \in A, u_{1} \in A^{*}$. Letting $b \in A-\{a\}$, and $y_{2}=y_{0} b$, we obtain a string $y_{2}$ that is not prefix-comparable with $y_{1}$. Now, $\left(y_{2} \leftarrow y_{2}\right) \circ \varphi \circ\left(x_{0} \leftarrow x_{0}\right)\left(x_{0} b\right)=\left(y_{2} \leftarrow y_{2}\right)\left(y_{0} b\right)=y_{2}$; so, $\Phi=\left(y_{2} \leftarrow y_{2}\right) \circ \varphi \circ\left(x_{0} \leftarrow x_{0}\right) \neq \mathbf{0}$. But for all $w \in A^{*},\left(y_{2} \leftarrow y_{2}\right) \circ \psi \circ\left(x_{0} \leftarrow x_{0}\right)\left(x_{0} w\right)$ $=\left(y_{2} \leftarrow y_{2}\right)\left(y_{1} w\right)=\varnothing$, since $y_{2}$ and $y_{1}$ are not prefix-comparable; so, $\left(y_{2} \leftarrow y_{2}\right) \circ \psi \circ\left(x_{0} \leftarrow x_{0}\right)=\mathbf{0}$. Thus, case 0 applies to $\mathbf{0} \neq \Phi=\left(y_{2} \leftarrow y_{2}\right) \circ \varphi \circ\left(x_{0} \leftarrow x_{0}\right) \cong\left(y_{2} \leftarrow y_{2}\right) \circ \psi \circ\left(x_{0} \leftarrow x_{0}\right)=\mathbf{0}$.

Proposition 2.25 The group of units of $\mathcal{M}_{\mathrm{end}}^{\mathrm{P}}$ is $\left\{[f]_{\text {end }}: f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}\right.$ and $f$ is a bijection between two essential right ideals of $\left.A^{*}\right\}$.

Proof. If $f \in \mathcal{R M} \mathcal{M}^{\mathbf{P}}$ is a bijection between essential right ideals, then $f$ is also a bijection from $R_{1}=\operatorname{Dom}(f)$ onto $R_{2}=\operatorname{Im}(f)$; and $R_{1}$ and $R_{2}$ are decidable subsets of $A^{*}$ (since $R_{1} \in \mathrm{P}$ and $\left.R_{2} \in \mathrm{NP}\right)$. Hence $f^{-1}: R_{2} \rightarrow R_{1}$ is partial recursive, and has decidable domain and image. Also, $f \circ f^{-1}=\operatorname{id}_{R_{2}}$, and $f^{-1} \circ f=\operatorname{id}_{R_{1}}$. Since $R_{1}, R_{2}$ are essential right ideals, $\mathrm{id}_{R_{2}} \equiv_{\text {end }}$ id $\equiv_{\text {end }} \mathrm{id}_{R_{1}}$. So, $[f]_{\text {end }} \cdot\left[f^{-1}\right]_{\text {end }}=[\mathrm{id}]_{\text {end }}=\left[f^{-1}\right]_{\text {end }} \cdot[f]_{\text {end }}$. By Lemma 2.22, $f^{-1}$ is $\equiv_{\text {end }}$-equivalent to an element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ (with linear time-complexity and linear balance). Hence, $\left[f^{-1}\right]_{\text {end }} \in \mathcal{M}_{\text {end }}^{\mathrm{P}} ;$ so $[f]_{\text {end }}$ belongs to the group of units.

Conversely, suppose $[F]_{\text {end }} \equiv_{\mathcal{H}}[\mathrm{id}]_{\text {end }}$ in $\mathcal{M}_{\text {end }}^{\mathrm{P}}$, where $F \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. Then there exists $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $f \equiv_{\text {end }} F$, and $f$ regular in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ (by Theorem [2.23). Since $[f]_{\text {end }} \equiv_{\mathcal{L}}$ [id] $]_{\text {end }}$, there exist $f_{2}, \mathrm{id}_{R_{2}} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $f_{2} \circ f=\mathrm{id}_{R_{2}}$; and $\mathrm{id}_{R_{2}} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ implies $R_{2} \in \mathrm{P}$. Since $[f]_{\text {end }} \equiv \mathcal{R}$ [id $]_{\text {end }}$, there exist $f_{1}, \operatorname{id}_{R_{1}} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $f \circ f_{1}=\operatorname{id}_{R_{1}}$; and $\operatorname{id}_{R_{1}} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ implies $R_{1} \in \mathrm{P}$. Since $\mathrm{id}_{R_{2}} \equiv$ end $\mathrm{id} \equiv_{\text {end }} \mathrm{id}_{R_{1}}, R_{1}$ and $R_{2}$ are essential. Since $f$ and $\mathrm{id}_{R_{2}}$ are regular, Lemma 2.20(1) implies $\operatorname{Im}(f)=R_{2}$. Since $f$ and $\mathrm{id}_{R_{1}}$ are regular, Lemma 2.20(2) implies $f$ is injective and $\operatorname{Dom}(f)=R_{1}$. Hence, $f$ is a bijection from $R_{1}$ onto $R_{2}$.

We prove next that in the definition of $\mathcal{M}_{\text {end }}^{P}$ we can replace $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ by the monoid $\mathcal{R} \mathcal{M}^{\text {rec }}$, defined by $\mathcal{R} \mathcal{M}^{\text {rec }}=\left\{f: f\right.$ is a right-ideal morphism on $A^{*}$ that is partial recursive, $\operatorname{Dom}(f)$ is decidable, and $f$ has a total recursive input-output balance $\}$.
Recall that $[f]_{\text {end }} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ is defined by $[f]_{\text {end }}=\left\{F \in \mathcal{R} \mathcal{M}^{\mathrm{P}}: F \equiv_{\text {end }} f\right\}$.
Proposition 2.26 The monoid $\mathcal{R} \mathcal{M}^{\text {rec }} / \equiv_{\text {end }}$ is isomorphic to $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\text {end }} \quad\left(=\mathcal{M}_{\text {end }}^{\mathrm{P}}\right)$.
Proof. Let us show that the map $H:[f]_{\text {end }} \longmapsto\left\{F \in \mathcal{R} \mathcal{M}^{\text {rec }}: F \equiv_{\text {end }} f\right\}$ (for all $f \in \mathcal{R M}^{\mathrm{P}}$ ) is a bijection from $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ onto $\mathcal{R} \mathcal{M}^{\text {rec }} / \equiv_{\text {end }}$. The map $H$ is injective because different $\equiv_{\text {end }}$-classes are disjoint, in both $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ and $\mathcal{R} \mathcal{M}^{\text {rec }}$. The map is also surjective because for every $g \in \mathcal{R} \mathcal{M}^{\text {rec }}$ there exists $g_{\mathrm{pad}} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $g_{\mathrm{pad}} \equiv_{\text {end }} g$. We can take $g_{\mathrm{pad}}$ to be the restriction of $g$ to $\bigcup_{y \in \operatorname{domC(g)}}$ y $A^{|y| \cdot t(|y|)} A^{*}$, as in Lemma[2.22. Moreover, $H$ is a homomorphism since $\equiv_{\text {end }}$ is a congruence.

Question: We proved that $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ is a congruence-simple homomorphic image of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. Does $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ have other congruence-simple homomorphic images?

## 3 Bounded end-equivalence

Definition 3.1 (bounded end-equivalence of sets).
(1) Two sets $L_{1}, L_{2} \subset A^{*}$ are boundedly end-equivalent (denoted by $L_{1} \equiv_{\text {bd }} L_{2}$ ) iff $L_{1} \equiv_{\text {end }} L_{2}$, and there exists a total function $\beta: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $x_{1} \in L_{1}$ and $x_{2} \in L_{2}: x_{1} \|_{\text {pref }} x_{2}$ implies $\left|x_{1}\right| \leq \beta\left(\left|x_{2}\right|\right)$ and $\left|x_{2}\right| \leq \beta\left(\left|x_{1}\right|\right)$.
(2) More generally, let $\mathcal{T}$ be any non-empty family of total functions $\mathbb{N} \rightarrow \mathbb{N}$ such that:

- $\mathcal{T}$ contains upper bounds on sum and composition; this means that for all $\tau_{1}, \tau_{2} \in \mathcal{T}$ there exist $\tau_{3}, \tau_{4} \in \mathcal{T}$ such that for all $n \in \mathbb{N}: \tau_{1}(n)+\tau_{2}(n) \leq \tau_{3}(n)$, and $\tau_{1}\left(\tau_{2}(n)\right) \leq \tau_{4}(n)$.
- There exists $\tau \in \mathcal{T}$ such that $\tau$ is an increasing function, and $n \leq \tau(n)$ for all $n \in \mathbb{N}$.

Two sets $L_{1}, L_{2} \subset A^{*}$ are $\mathcal{T}$-end-equivalent (denoted by $L_{1} \equiv \mathcal{T} L_{2}$ ) iff $L_{1} \equiv_{\text {end }} L_{2}$, and there exists a function $\tau \in \mathcal{T}$ such that for all $x_{1} \in L_{1}$ and $x_{2} \in L_{2}: x_{1} \|_{\text {pref }} x_{2}$ implies $\left|x_{1}\right| \leq \tau\left(\left|x_{2}\right|\right)$ and $\left|x_{2}\right| \leq \tau\left(\left|x_{1}\right|\right)$.

Note that the bounding function $\beta$ or $\tau$ for $L_{1} \equiv_{\mathrm{bd}} L_{2}$ depends on $L_{1}$ and $L_{2}$. The only assumption on the function $\beta: \mathbb{N} \rightarrow \mathbb{N}$ is that it is total on $\mathbb{N}$; no computability assumptions are made.

## Examples and counter-examples:

For any prefix code $P \subset A^{*}$ and any total function $\beta: \mathbb{N} \rightarrow \mathbb{N}$ we have $\bigcup_{x \in P} x A^{\beta(|x|)} \equiv_{\mathrm{bd}} P$.
For the prefix codes $\{0,1\}$ and $0^{*} 1$ we have $\{0,1\} \equiv_{\text {end }} 0^{*} 1$, but $\{0,1\} \not \equiv \equiv_{\text {bd }} 0^{*} 1$.
When $P$ is a prefix code, $P \not \equiv_{\mathrm{bd}} P A^{*}$; in this, $\equiv_{\mathrm{bd}}$ differs from $\equiv_{\text {end }}$. And $A^{*}$ is not $\equiv_{\mathrm{bd}}$-equivalent to itself, so $\equiv_{b d}$ is not reflexive in general. When $L$ is a prefix code, $L$ is boundedly end-equivalent to itself. If $L$ is a union of two prefix codes, then $L$ might not be boundedly end-equivalent to itself (e.g., $\{0,1\} \cup\left\{0^{n} 1: n \geq 0\right\}$ ). From here on we will use $\equiv_{b d}$ only between prefix codes.

Closure of $\mathcal{T}$ under composition guarantees that $\equiv_{\mathcal{T}}$ is transitive. Typical examples of families $\mathcal{T}$ as above are the following (where we only take those functions that are increasing and satisfy $n \leq \tau(n)$ ): - $\mathbb{N}^{\mathbb{N}}$, i.e., the family of all total functions on $\mathbb{N}$; then $\equiv_{\mathcal{T}}$ is $\equiv_{\text {bd }}$.

- rec $=$ the family of all partial recursive functions $\mathbb{N} \rightarrow \mathbb{N}$ with decidable domain, i.e., the partial recursive functions that are extendable to total recursive functions.
- E3 = the family of all elementary recursive functions, i.e., level 3 of the Grzegorczyk hierarchy; these are the primitive recursive functions with size bounded by a constant iteration of exponentials. - poly $=$ the family of all polynomials with non-negative integer coefficients.
- lin $=$ the family of all affine functions of the form $n \mapsto a n+b$ (where $a \geq 1, b \geq 0$ ).

We have the following Cantor-space characterization of $\equiv_{b d}$ between prefix codes:
Proposition 3.2 For prefix codes $P_{1}, P_{2} \subset A^{*}$ we have $P_{1} \equiv{ }_{\mathrm{bd}} P_{2}$ iff ends $\left(P_{1}\right)=\operatorname{ends}\left(P_{2}\right)$.
Proof. [ $\Leftarrow]$ If ends $\left(P_{1}\right)=\operatorname{ends}\left(P_{2}\right)$ then by applying closure we obtain $P_{1} \equiv_{\text {end }} P_{2}$ (by Prop. 2.5). To prove boundedness of this end-equivalence, let $x_{1} \in P_{1}, x_{2} \in P_{2}$ be such that $x_{1} \|_{\text {pref }} x_{2}$. Let us assume $x_{1}$ is a prefix of $x_{2}$ (if $x_{2}$ is a prefix of $x_{1}$ the reasoning is symmetric). The existence of a total function $\beta: \mathbb{N} \rightarrow \mathbb{N}$ such that $\left|x_{2}\right| \leq \beta\left(\left|x_{1}\right|\right)$ for every $x_{2}$ that has $x_{1}$ as a prefix, is equivalent to the finiteness of $x_{1} A^{*} \cap P_{2}$ for every $x_{1} \in P_{1}$. Indeed, the lengths of the words in a set $S \subseteq A^{*}$ (over a finite alphabet $A$ ) are bounded iff that set $S$ is finite.

Since ends $\left(P_{1}\right)=\operatorname{ends}\left(P_{2}\right)$, every end that passes through $x_{1}$ (i.e., that belongs to the subtree $x_{1} A^{*}$ of $A^{*}$ ) intersects $P_{2}$. Hence, the tree $x_{1} A^{*}-P_{2} A^{+}=\left(x_{1} A^{*}-P_{2} A^{*}\right) \cup\left(P_{2} \cap x_{1} A^{*}\right)$ has no infinite path. By the König Infinity Lemma, this implies that this tree is finite. Hence $x_{1} A^{*} \cap P_{2}$, which is the set of leaves of this finite tree, is finite.
$\left[\Rightarrow\right.$ ] If $P_{1} \equiv_{\mathrm{bd}} P_{2}$ with bounding function $\beta: \mathbb{N} \rightarrow \mathbb{N}$, consider $x_{1} w \in \operatorname{ends}\left(P_{1}\right)$, with $x_{1} \in P_{1}$ and $w \in A^{\omega}$. If $x_{1}$ has a prefix $x_{2} \in P_{2}$ then obviously, $x_{1} w \in \operatorname{ends}\left(P_{2}\right)$.

Let us assume next that $x_{1}$ does not have a prefix in $P_{2}$; we want to show that in this case too, $x_{1} w \in \operatorname{ends}\left(P_{2}\right)$. For every $n \in \mathbb{N}$, let $w_{n}$ be the prefix of length $n$ of $w$. Since $P_{1} \equiv_{\text {end }} P_{2}$, every right ideal $x_{1} w_{n} A^{*}$ intersects $P_{2} A^{*}$; so, $x_{1} w_{n} u_{n}=x_{2, n} v_{n}$ for some $x_{2, n} \in P_{2}$ and some $u_{n}, v_{n} \in A^{*}$. It follows that $x_{1} \|_{\text {pref }} x_{2, n}$, hence $x_{1}$ is a prefix of $x_{2, n}$ (since we assumed that $x_{1}$ does not have a prefix in $P_{2}$ ). Hence, $\left|x_{2, n}\right| \leq \beta\left(\left|x_{1}\right|\right)$. It also follows from $x_{1} w_{n} u_{n}=x_{2, n} v_{n}$ that $x_{1} w_{n} \|_{\text {pref }} x_{2, n}$, so either $x_{1} w_{n}$ is a prefix of $x_{2, n}$, or $x_{2, n}$ is a prefix of $x_{1} w_{n}$ (while $x_{1}$ is also a prefix of $x_{2, n}$ ).

Case 1: $x_{1} w_{n}$ is a prefix of $x_{2, n}$. Then $\left|x_{2, n}\right| \geq\left|x_{1}\right|+n$. If we choose $n$ so that $n>\beta\left(\left|x_{1}\right|\right)$, this case is ruled out.

Case 2: $x_{2, n}$ is a prefix of $x_{1} w_{n}$. Then $x_{2, n}$ is a vertex on the end $x_{1} w$ (between $x_{1}$ and $x_{1} w_{n}$ ), hence $x_{1} w$ is equal to an end through $x_{2, n}$, so $x_{1} w \in \operatorname{ends}\left(P_{2}\right)$.

## Remarks:

(1) Prop. 3.2 was proved for prefix codes. When $P_{1}, P_{2} \subseteq A^{*}$ are not prefix codes, the proposition does not always hold. E.g., $A^{*} \not \equiv_{\mathrm{bd}} A^{*}$ (non-reflexivity, as we saw), but obviously ends $\left(A^{*}\right)=\operatorname{ends}\left(A^{*}\right)$.

One could argue that ends $\left(L_{1}\right)=\operatorname{ends}\left(L_{2}\right)$ is the more reasonable definition of " $L_{1} \equiv_{\text {bd }} L_{2}$ ". But our definition of $\equiv_{\text {bd }}$ (Def. 3.1) has the advantage of generalizing to $\equiv \mathcal{T}$.

In any case, since we will use $\equiv_{\text {bd }}$ only with prefix codes, the question doesn't matter in this paper.
(2) The relation $\equiv_{\text {bd }}$ can be generalized to a pre-order, denoted by $\subseteq_{\text {bd }}$ : For prefix codes $P_{1}, P_{2} \subset A^{*}$ we define $P_{1} \subseteq_{\text {bd }} P_{2}$ iff ends $\left(P_{1}\right) \subseteq \operatorname{ends}\left(P_{2}\right)$. Equivalently, $P_{1} \subseteq_{\text {bd }} P_{2}$ iff there exists $Q \subseteq P_{2}$ such that $P_{1} \equiv_{\text {bd }} Q$. Similarly, $\equiv_{\mathcal{T}}$ can be generalized by defining $P_{1} \subseteq_{\mathcal{T}} P_{2}$ iff there exists $Q \subseteq P_{2}$ such that $P_{1} \equiv_{\mathcal{T}} Q$.
Notation: For any right ideal $R \subseteq A^{*}$, the prefix code that generates $R$ (as a right ideal) is denoted by $\operatorname{prefC}(P)$.

Proposition 3.3 For any prefix code $P \subset A^{*}$, we have:
$\bigcap\left\{\operatorname{ends}(Q): Q\right.$ is a prefix code and $\left.Q \equiv_{\text {bd }} P\right\}=\operatorname{ends}(P)$,
$\bigcap\left\{Q A^{*}: Q\right.$ is a prefix code and $\left.Q \equiv_{\text {bd }} P\right\}=\varnothing$, $\operatorname{prefC}\left(\bigcup\left\{Q A^{*}: Q\right.\right.$ is a prefix code and $\left.\left.Q \equiv_{\mathrm{bd}} P\right\}\right) \equiv_{\mathrm{bd}} P$.

Proof. The first intersection is ends $(P)$ since ends $(Q)=\operatorname{ends}(P)$ when $Q \equiv_{\mathrm{bd}} P$; so this result is different than the corresponding result for $\equiv_{\text {end }}$ (in Prop. 2.7). For the second intersection result this is similar to the proof of Prop. 2.7.

For the union of the $Q A^{*}$ we have by Prop. [2.7, $\operatorname{prefC}\left(\bigcup_{Q \equiv_{\mathrm{bd} P} P} Q A^{*}\right) \equiv_{\text {end }} P$. Also, by Prop. 3.2, ends $\left(Q A^{*}\right)=\operatorname{ends}(P)$ for every prefix code $Q$ such that $Q \equiv \equiv_{\mathrm{bd}} P$; hence, ends $\left(\bigcup_{Q \equiv \mathrm{bd}} P A^{*}\right)=$ ends $(P)$. Then the result follows by Prop. 3.2.

Definition 3.4 (bounded end-equivalence of functions). Two right-ideal morphisms $f, g$ are boundedly end-equivalent (denoted by $f \equiv_{\text {bd }} g$ ) iff $\operatorname{domC}(f) \equiv_{\text {bd }} \operatorname{domC}(g)$, and $f(x)=g(x)$ for all $x \in \operatorname{Dom}(f) \cap \operatorname{Dom}(g)$. Equivalently, $f \equiv_{\text {bd }} g$ iff $f \equiv_{\text {end }} g$ and $\operatorname{domC}(f) \equiv_{\mathrm{bd}} \operatorname{domC}(g)$.

For any family $\mathcal{T}$ of total functions as in Def. 3.1 we define: $f \equiv \mathcal{T} g$ iff $f \equiv_{\text {end }} g$ and $\operatorname{domC}(f) \equiv \mathcal{T}$ $\operatorname{domC}(g)$.

Notation: When $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$, the $\equiv_{\text {bd }}$-class in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ of $f$ is denoted by $[f]_{\text {bd }}$. So, $[f]_{\text {bd }}=\left\{g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}\right.$ : $\left.g \equiv_{\text {bd }} f\right\}$. More generally, $[f]_{\mathcal{T}}$ denotes the $\equiv_{\mathcal{T}}$-class in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ of $f$. Note that we define $[f]_{\text {bd }}$ and $[f]_{\mathcal{T}}$ to only contain elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.

For the rest of this Section we study $\equiv_{\text {bd }}$. The relations $\equiv_{\text {poly }}$ and $\equiv_{\text {E3 }}$ will be investigated in the next Section.

Proposition 3.5 (1) Let $P_{1}, P_{2} \subset A^{*}$ be prefix codes such that $P_{1} \equiv_{\text {bd }} P_{2}$, let $P_{\cap}$ be the prefix code that generates the right ideal $P_{1} A^{*} \cap P_{2} A^{*}$, and let $P_{\cup}$ be the prefix code that generates the right ideal $P_{1} A^{*} \cup P_{2} A^{*}$. Then $P_{1} \equiv_{\text {bd }} P_{2} \equiv_{\mathrm{bd}} P_{\cap} \equiv_{\mathrm{bd}} P \cup$.
(2) Let $f_{1}, f_{2}$ be right-ideal morphisms such that $f_{1} \equiv_{\text {bd }} f_{2}$. Then $f_{1} \cap f_{2}$ and $f_{1} \cup f_{2}$ are right-ideal morphisms, and $f_{1} \equiv_{\text {bd }} f_{2} \equiv_{\text {bd }} f_{1} \cap f_{2} \equiv_{\text {bd }} f_{1} \cup f_{2}$.

Proof. (1) This follows from Prop. 3.2, since ends $\left(P_{1} A^{*} \cap P_{2} A^{*}\right)=\operatorname{ends}\left(P_{1}\right) \cap \operatorname{ends}\left(P_{2}\right)$ and ends $\left(P_{1} A^{*} \cup\right.$ $\left.P_{2} A^{*}\right)=\operatorname{ends}\left(P_{1}\right) \cup \operatorname{ends}\left(P_{2}\right)$.
For (2) the proof is similar to the proof of Prop. 2.9,
Just as for $\equiv_{\text {end }}$ (see Def. 2.12), we define a maximum extension within a $\equiv_{\text {bd }}$-class or a $\equiv_{\mathcal{T}}$-class.
Definition 3.6 For any right-ideal morphism $f: A^{*} \rightarrow A^{*}$ we define
$f_{\mathrm{b}, \max }=\bigcup\left\{g: g\right.$ is a right-ideal morphism with $\left.g \equiv_{\mathrm{bd}} f\right\}$.
For a family $\mathcal{T}$ of functions as in Def. 3.1 we define
$f_{\mathcal{T} \text {,max }}=\bigcup\left\{g: g\right.$ is a right-ideal morphism with $\left.g \equiv_{\mathcal{T}} f\right\}$.
Then, just as in Prop. 2.13, we have:

## Proposition 3.7.

(1) For every right-ideal morphism $f$, $f_{\mathrm{b}, \max }$ is a function, and a right-ideal morphism $A^{*} \rightarrow A^{*}$. Moreover, $f \equiv_{\text {bd }} f_{\mathrm{b}, \text { max }}$.
(2) For any right-ideal morphisms $f, g$ we have: $g \equiv_{\mathrm{bd}} f$ iff $g_{\mathrm{b}, \max }=f_{\mathrm{b}, \max }$.

Proof. The same proof as for Prop. 2.13 works here (using Prop. 3.5 and Prop. (3.3).
Recall the action of a right-ideal morphism $f: A^{*} \rightarrow A^{*}$ on $A^{\omega}$ : For any $p \in \operatorname{domC}(f)$ and $w \in A^{\omega}$, we define $f(p w)=f(p) w$. The domain of the action of $f$ on $A^{\omega}$ is $\operatorname{domC}(f) A^{\omega}$. Accordingly, $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ acts on $A^{\omega}$ (non-faithfully). We have the following characterization of $\equiv_{\mathrm{bd}}$ in terms of the Cantor space:

Corollary 3.8 Two right-deal morphisms $f, g: A^{*} \rightarrow A^{*}$ have the same action on $A^{\omega}$ iff $g \equiv_{\mathrm{bd}} f$. Hence on $A^{\omega}: g_{\mathrm{b}, \max } \circ f_{\mathrm{b}, \max }=(g \circ f)_{\mathrm{b}, \max }$.

The relation $\equiv_{\mathrm{bd}}$ is a congruence on the monoid of all right-ideal morphisms of $A^{*}$, and in particular on $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.

Proof. If $g \equiv_{\text {bd }} f$ then $\operatorname{domC}(g) \equiv_{\text {bd }} \operatorname{domC}(f)$, hence by Prop. 3.2; ends $(\operatorname{domC}(g))=\operatorname{ends}(\operatorname{domC}(f))$. So the actions of $f$ and $g$ on $A^{\omega}$ have the same domain. Since $g \equiv_{\text {bd }} f$, the functions $f$ and $g$ agree on their common domain in $A^{\omega}$, so they have the same action on $A^{\omega}$.

Conversely, if $f$ and $g$ act in the same way on $A^{\omega}$ then ends $(\operatorname{domC}(g))=\operatorname{ends}(\operatorname{domC}(f))$, so $\operatorname{domC}(g) \equiv_{b d} \operatorname{domC}(f)$ (by Prop. 3.2). Also, if $f$ and $g$ act in the same way on $A^{\omega}$ then for all $x \in \operatorname{Dom}(f) \cap \operatorname{Dom}(g)$ and for all $w \in A^{\omega}: \quad f(x w)=g(x w)$. Hence (since $x \in \operatorname{Dom}(f) \cap \operatorname{Dom}(g)$ ), $f(x)=g(x)$. So, $f$ and $g$ agree on $\operatorname{Dom}(f) \cap \operatorname{Dom}(g)$, hence $f \equiv_{\text {bd }} g$.

The rest of the corollary follows now.
Proposition 3.9 There exists $g \in \mathcal{R} \mathcal{M}^{\text {rec }}$ such that $\operatorname{Dom}\left(g_{\mathrm{b}, \max }\right)$ is undecidable; so $g_{\mathrm{b}, \max } \notin \mathcal{R}^{\text {rec }}$. Moreover, this function $g$ can be chosen so that in addition we have $g_{\mathrm{b}, \max }=g_{\mathrm{e}, \max }$.

Proof. Let $L \subset 0^{*}$ (over the one-letter alphabet $\{0\}$ ) be an r.e. language that is undecidable. We assume in addition that for all $0^{i}, 0^{j} \in L$ we have $|i-j|>2$ (if $i \neq j$ ). Let $M$ be a deterministic Turing machine that accepts $L$. Let $T\left(0^{n}\right)$ be the running time of $M$ on input $0^{n} \in L ; T(w)$ is undefined for $w \in\{0,1\}^{*}-L$. We define the right-ideal morphism $g$ by

$$
\operatorname{domC}(g)=\bigcup_{0^{n} \in L} 0^{n}\{0,1\} 1\{0,1\}^{T\left(0^{n}\right)}
$$

$$
g\left(0^{n} a 1 z\right)=0^{n} \bar{a} 1 z
$$

if $0^{n} \in L, a \in\{0,1\}$, and $z \in\{0,1\}^{T\left(0^{n}\right)}\{0,1\}^{*}$. Here $\bar{a}$ denotes the complement of $a$ (i.e., $\overline{0}=1$, $\overline{1}=0)$. The set $\operatorname{dom} C(g)$ is a prefix code because of the $|i-j|>2$ condition on $L$.

Membership in $\operatorname{Dom}(g)$ is decidable in linear time: for an input $0^{n} a 1 z$ with $a \in\{0,1\}$ and $z \in$ $\{0,1\}^{*}$, it suffices to run the machine $M$ for $\leq|z|$ steps on $0^{n}$. And $\left|g\left(0^{n} a 1 z\right)\right|=\left|0^{n} a 1 z\right|$, hence $g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$.

We consider the following right-ideal morphism $h$, which extends $g$ :

$$
\begin{aligned}
& \operatorname{domC}(h)=L\{0,1\} 1 \\
& h\left(0^{n} a 1\right)=0^{n} \bar{a} 1
\end{aligned}
$$

if $0^{n} \in L, a \in\{0,1\}$.
We claim that $h=g_{\mathrm{b}, \max }=g_{\mathrm{e}, \max }$. Indeed, we have $h \equiv_{\mathrm{bd}} g$, since ends $(\operatorname{domC}(g))=L\{0,1\} 1\{0,1\}^{\omega}$ $=$ ends $(\operatorname{domC}(h))$ ) (using Prop. 3.2). And $h$ cannot be further extended to a right-ideal morphism that is $\equiv_{\mathrm{bd}} h$ (because $h$ permutes $0^{n} 01,0^{n} 11$ ).

Also, cl(ends $(\operatorname{domC}(h)))=L\{0,1\} 1\{0,1\}^{\omega} \cup\left\{0^{\omega}\right\}$. But $h$ cannot be extended to any prefix $0^{i}(i \in \mathbb{N})$ of $0^{\omega}$ (again because $h$ permutes $0^{n} 01,0^{n} 11$ ). So, $h$ is also the maximal $\equiv_{\text {end }}$-equivalent extension of $g$.

Lemma 3.10 For every right-ideal morphisms $f$ and $g$ such that $f \subseteq g$ and $f \equiv_{\mathrm{bd}} g$ we have:
(1) For all $x \in \operatorname{domC}(g): x A^{*} \cap \operatorname{domC}(f)$ is finite.
(2) For any $x \in \operatorname{domC}(g), f$ can be extended to a right-ideal morphism whose domain code is ( $\left.\operatorname{domC}(f)-x A^{*}\right) \cup\{x\} \quad$ (this is called $a$ one-point extension of $f$ ). If $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ then this extension is also in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.
(3) For any finite subset $C \subset \operatorname{domC}(g), f$ can be extended to a right-ideal morphism whose domain code includes $C$ (this is called a finite extension of $f$ ).

This extension belongs to $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ if $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$.
(4) Items (1), (2), (3) hold in particular when $g=f_{\mathrm{b}, \max }$.
(5) There exist $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that for some $x \in \operatorname{domC}\left(f_{\mathrm{e}, \max }\right)$ the set $x A^{*} \cap \operatorname{domC}(f)$ is infinite.

Proof. (1) Let $\beta($.$) be the bounding function that corresponds to f \equiv_{\mathrm{bd}} g$. Then the tree
$T_{x, f}=\left\{z \in A^{*}: x\right.$ is a prefix of $z$ and $z$ is a prefix of some word in $\left.\operatorname{domC}(f)\right\}$
has $x$ as root and $x A^{*} \cap \operatorname{domC}(f)$ as set of leaves, and has depth $\leq \beta(|x|)$. Moreover, the degree of each vertex is $\leq 2$. Hence the tree $T_{x, f}$ and its set of leaves $x A^{*} \cap \operatorname{domC}(f)$ are finite.
(2) Since $f \equiv_{\mathrm{bd}} g$ and $f \subseteq g$, every end that starts at $x$ intersects domC $(f)$. Since $g$ and $f$ agree on $\operatorname{domC}(f), f$ can be extended from $\operatorname{Dom}(f)$ to $\left(x A^{*} \cap \operatorname{domC}(f)\right) \cup \operatorname{Dom}(f)$. For domain codes, the effect of this extension is to replace $x A^{*} \cap \operatorname{domC}(f)$ by $\{x\}$.

If $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ then the one-point extension is also in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, since $x A^{*} \cap \operatorname{domC}(f)$ is finite.
(3) follows from (2).
(4) Recall that $f \equiv_{\text {bd }} f_{\mathrm{b}, \max }$, by Prop. 3.7 , (1); hence the Lemma applies to the maximum $\equiv_{\mathrm{bd}}$-equivalent extension $f_{\mathrm{b}, \max }$.
(5) For $\equiv_{\text {end }}$ and $f_{\mathrm{e}, \text { max }}$ the situation is different. E.g., when $f=\left.\mathbf{1}\right|_{0^{*} 1}$, we have $f_{\mathrm{e}, \text { max }}=\mathbf{1}$ and $\operatorname{domC}\left(f_{\mathrm{e}, \max }\right)=\{\varepsilon\}$. Then for $x=\varepsilon$ we have $x\{0,1\}^{*} \cap \operatorname{domC}(f)=0^{*} 1$, which is infinite.

The precise definitions of one-point and finite extensions of a right-ideal morphism are as follows.

Definition 3.11 Let $f, g$ be right-ideal morphisms. We call $g$ a one-point extension of $f$ iff (1) $g$ is an extension of $f$, (2) $g \equiv_{\mathrm{bd}} f$, (3) there exists $x_{0} \in \operatorname{domC}(g)$ such that $\operatorname{Dom}(g)=x_{0} A^{*} \cup \operatorname{Dom}(f)$.

We call $g$ a finite extension of $f$ iff (1) $f \subseteq g$, and (2) $g \equiv_{\mathrm{bd}} f$, as above, and (3) there exists a finite prefix code $F \subseteq \operatorname{domC}(g)$ such that $\operatorname{Dom}(g)=F A^{*} \cup \operatorname{Dom}(f)$.

A one-point extension is a special case of a finite extension (when $F=\left\{x_{0}\right\}$ ), and a finite extension can be constructed by a finite sequence of one-point extensions.

It follows from the definition that for a finite extension, $\operatorname{domC}(g)=F \cup\left(\operatorname{domC}(f)-F A^{*}\right)$.
Moreover, $\operatorname{domC}(f) \cap F A^{*}$ is finite (by Lemma 3.10(1)). Since $\operatorname{domC}(g)=F \cup(\operatorname{domC}(f)-$ (domC $\left.(f) \cap F A^{*}\right)$ ), we conclude that the symmetric difference $\operatorname{domC}(g) \Delta \operatorname{domC}(f)$ is finite. Conversely, suppose that $f \subseteq g, g \equiv_{\mathrm{bd}} f$, and $\operatorname{domC}(g) \Delta \operatorname{domC}(f)$ is finite; then $g$ is a finite extension of $f$ (in the sense of Def. 3.11). Indeed, $F=\operatorname{domC}(g)-\operatorname{domC}(f)$ is finite, and satisfies $\operatorname{Dom}(g)=F A^{*} \cup \operatorname{Dom}(f)$. Thus, for right ideal morphisms $g, f$ that satisfy $f \subseteq g$ and $g \equiv_{\mathrm{bd}} f$ we have: $g$ is a finite extension of $f$ iff $\operatorname{domC}(g) \Delta \operatorname{domC}(f)$ is finite.

By Lemma 3.10, $f_{\mathrm{b}, \max }$ can be constructed from $f$ by an $\omega$-sequence of one-point extensions. On the other hand, the example $f=$ id $\left.\right|_{0^{*} 1}$ shows that a right-ideal morphism $f$ might be extendable (to id $\equiv_{\text {end }} f$ in this example), without having any finite extension. So, in general, $f_{\mathrm{e}, \text { max }}$ cannot be obtained from $f$ by an $\omega$-sequence of finite extensions.

The following consequence of the Lemma is a little surprising.
Proposition 3.12 Let $\mathcal{T}$ be any family of functions as in Def. 3.1 such that, in addition, poly $\subseteq \mathcal{T}$. Then for every right-ideal morphism $f, f_{\mathcal{T}, \max }=f_{\mathrm{b}, \max }$.

In particular, $f_{\text {poly }, \max }=f_{\mathrm{b}, \max }$.
Proof. Since $\equiv_{\mathcal{T}}$ implies $\equiv_{\mathrm{bd}}, f_{\mathcal{T}, \text { max }} \subseteq f_{\mathrm{b}, \text { max }}$.
On the other hand let $x \in \operatorname{domC}\left(f_{\mathrm{b}, \text { max }}\right)$. Since $f \equiv_{\mathrm{bd}} f_{\mathcal{T}, \text { max }} \equiv_{\mathrm{bd}} f_{\mathrm{b}, \text { max }}$, if we let $F=f_{\mathcal{T}, \text { max }}$ we have $f_{\mathrm{b}, \max }=F_{\mathrm{b}, \max }$. For every $x \in \operatorname{domC}\left(F_{\mathrm{b}, \max }\right), x A^{*} \cap \operatorname{domC}(F)$ is finite (by Lemma 3.10); and $F$ can be extended to a right-ideal morphism $F_{0}$ which is defined on $x$. Since $x A^{*} \cap \operatorname{domC}(F)$ is finite, $F_{0} \equiv_{\mathcal{T}} F$. But since $F$ is already $\equiv_{\mathcal{T}}$-maximum, the extension $F_{0}$ of $F$ is $F$ itself. So, $F(x)=F_{0}(x)=f_{\mathrm{b}, \max }(x)$. Since this holds for every $x \in \operatorname{domC}\left(f_{\mathrm{b}, \max }\right)$, it follows that $F=f_{\mathrm{b}, \max }$.

Proposition 3.13 For a right-ideal morphism $f$, the following are equivalent:
(1) $f$ is finitely extendable, to a strictly larger domain;
(2) there exist $x_{0}, y_{0} \in A^{*}$ such that $\left(x_{0} 0, y_{0} 0\right),\left(x_{0} 1, y_{0} 1\right) \in f$, and $\left(x_{0}, y_{0}\right) \notin f$;
(3) $f \neq f_{\mathrm{b}, \text { max }}$.

Proof. The implication $(2) \Rightarrow(1)$ is clear, since $\left(x_{0} 0, y_{0} 0\right),\left(x_{0} 1, y_{0} 1\right) \in f$ implies that $f$ can be extended to $f \cup\left\{\left(x_{0}, y_{0}\right)\right\}$. And (1) implies that $f \varsubsetneqq f \cup\left\{\left(x_{0}, y_{0}\right)\right\} \subseteq f_{\mathrm{b}, \text { max }}$, so (1) implies (3).

Let us prove that that (3) implies (2). By (3) there exists $z \in \operatorname{domC}\left(f_{\mathrm{b}, \max }\right)$ with $z \notin \operatorname{Dom}(f)$. Consider the rooted tree with root $z$ and vertex set and edges set respectively

$$
\begin{aligned}
V & =\left\{z w: w \in A^{*} \text { and } z w \text { is a prefix of a word in } z A^{*} \cap \operatorname{domC}(f)\right\}, \\
E & =\{(v, v a) \in V \times V: a \in A\} .
\end{aligned}
$$

This is a binary tree (every vertex has $\leq 2$ children), and it is saturated (i.e., every vertex has either 2 children or none). The set of leaves is $z A^{*} \cap \operatorname{domC}(f)$, and this set is finite (by Lemma 3.10); hence since the tree is saturated, it is finite. Also, since $z \notin \operatorname{Dom}(f)$ and since the tree is saturated, the tree has at least 3 vertices. Let $d$ be the depth of the tree (number of edges in a longest path from the root). Let $x_{0}$ be any non-leaf vertex at distance $d-1$ from the root; since $x_{0}$ is not a leaf, it has two children, namely $x_{0} 0$ and $x_{0} 1$. Then $x_{0} 0$ and $x_{0} 1$ are at distance $d$ from the root, so they are leaves, i.e., $x_{0} 0, x_{0} 1 \in z A^{*} \cap \operatorname{domC}(f)$. Let $y_{0}=f_{\mathrm{b}, \max }\left(x_{0}\right)$; then for $a \in\{0,1\}$ we have $y_{0} a=f_{\mathrm{b}, \max }\left(x_{0} a\right)$, and the latter is equal to $f\left(x_{0} a\right)$ (since $\left.x_{0} 0, x_{0} 1 \in \operatorname{domC}(f)\right)$. Hence, ( $x_{0}, y_{0}$ ) satisfies (2).

We mention the following, which will however not be used in this paper:
Fact. For every $f \in \mathrm{fP}$, the encoded right-ideal morphism $f^{C}$ is not finitely extendable, and not infinitely extendable. In other words, $f^{C}=\left(f^{C}\right)_{\mathrm{b}, \max }=\left(f^{C}\right)_{\mathrm{e}, \max }$.

Proof. The encoding $f^{C}$ was defined in the Introduction. It follows from that definition that $\operatorname{domC}\left(f^{C}\right) \subseteq\{0,1\}^{*} 11$, so we never have $z 0, z 1 \in \operatorname{domC}\left(f^{C}\right)$ for any $z \in A^{*}$. Hence, by Prop. 3.13, $f^{C}$ is not finitely extendable, so $f^{C}=\left(f^{C}\right)_{\mathrm{b}, \max }$.

Proof that $f^{C}$ is not infinitely extendable: Let $P \subset A^{*}$ be any prefix code such that $\operatorname{domC}\left(f^{C}\right) \subseteq$ $P A^{*}$ and $\operatorname{domC}\left(f^{C}\right) \equiv_{\text {end }} P$; we want to show that $P=\operatorname{domC}\left(f^{C}\right)$, which means that $f^{C}$ cannot be extended to a larger domain. Let us abbreviate $\operatorname{domC}\left(f^{C}\right)$ by $D$.

Since $P \equiv_{\text {end }} D$ we have (by Lemma 2.3): $(\forall p \in P)(\exists x 11 \in D)\left[p \|_{\text {pref }} x 11\right]$; moreover, since $D \subset$ $P A^{*}$, this $p$ is a prefix of $x 11$. Hence, since $D \subset\{00,01\}^{*} 11$, we conclude that $p \in\{00,01\}^{*} \cdot\{\varepsilon, 1,11\}$; i.e., for each $p \in P$ we have three possibilities: $p \in\{00,01\}^{*}, p \in\{00,01\}^{*} 1, p \in\{00,01\}^{*} 11$.

Claim 1. If $p \in\{00,01\}^{*} 11$ then $p \in D$.
Proof. Since $p$ is a prefix of a word $x 11 \in D$, and $p \in\{00,01\}^{*} 11$, we conclude $p=x 11$ (since $\{00,01\}^{*} 11$ is a prefix code). [End, proof of Claim 1.]
Claim 2. $P \cap\{00,01\}^{*} 1=\varnothing$; i.e., $p$ cannot be in $\{00,01\}^{*} 1$.
Proof. If there exists $p \in P \cap\{00,01\}^{*} 1$ and $p$ is a prefix of some $x 11 \in\{00,01\}^{*} 11$, then $p=x 1$. But then $P \not \equiv_{\text {end }} D$, since the right ideal $p 0 A^{*}=x 10 A^{*} \subseteq\{00,01\}^{*} 10 A^{*}$ intersects $P A^{*}$ but not $\{00,01\}^{*} 11 A^{*}$. [End, proof of Claim 2.]
Claim 3. $P \cap\{00,01\}^{*}=\varnothing$; i.e., $p$ cannot be in $\{00,01\}^{*}$.
Proof. Assume there exists $p \in P \cap\{00,01\}^{*}$ such that $p$ is a prefix of some $x 11 \in\{0,1\}^{*} 11$. But then $P \not \equiv \equiv_{\text {end }} D$, since the right ideal $p 10 A^{*}=x 10 A^{*} \subseteq\{00,01\}^{*} 10 A^{*}$ intersects $P A^{*}$ but not $\{00,01\}^{*} 11 A^{*}$. [End, proof of Claim 3.]

We are left with only case 1, i.e., $P \subseteq D$. Hence $P=D$, since $D \equiv_{\text {end }} P$.
Notation: The quotient monoid of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ under the congruence $\equiv_{\mathrm{bd}}$ of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is denoted by $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathrm{bd}}$; this is also the quotient monoid for the action of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ on $A^{\omega}$.

Moreover, $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathrm{bd}}$ will also be denoted by $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$.
Recall that $\mathcal{R} \mathcal{M}^{\text {rec }}$ consists of all right-ideal morphisms that are partial recursive with decidable domain, and that have a total recursive input-output balance.

Proposition 3.14 The monoid $\mathcal{R} \mathcal{M}^{\text {rec }} / \equiv_{\mathrm{bd}}$ is isomorphic to $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathrm{bd}} \quad\left(=\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}\right)$.
Proof. This is proved in the same way as Prop. 2.26.
Lemma 3.15 If $P, Q$ are prefix codes such that $P \equiv_{\mathrm{bd}} Q$, and if $P$ is finite, then $Q$ is finite.
Proof. Let $\beta$ be the length-bounding (total) function associated with $P \equiv_{\text {bd }} Q$. Since $P \equiv_{\text {bd }} Q$, every element $x_{2} \in Q$ is prefix-comparable to some $x_{1} \in P$. Since $P$ is finite, the elements of $P$ can have only finitely many prefixes, hence the set of elements of $Q$ that are prefixes of some element(s) of $P$ is finite. Moreover, each $x_{1} \in P$ can be the prefix of only finitely many $x_{2} \in Q$, since every such $x_{2}$ has length $\leq \beta\left(\left|x_{1}\right|\right)$. Since $P$ is finite, the set of length bounds $\left\{\beta\left(\left|x_{1}\right|\right): x_{1} \in P\right\}$ is finite. Hence $Q$ is finite.

The next theorem refers to the well-known Richard Thompson group $V$ (a.k.a. $G_{2,1}$ ). In order to make the paper self-contained we define $V$ next. First, let

$$
\text { riAut }^{\text {fin }}=\left\{f: f \text { is a right-ideal morphism of } A^{*}\right. \text {, such that }
$$

(1) $f$ is injective,
(2) $\operatorname{domC}(f)$ and $\operatorname{imC}(f)$ are maximal prefix codes,
(3) $\operatorname{domC}(f)$ (and hence $\operatorname{imC}(f)$ ) is finite .

The notation "riAut ${ }^{\text {fin" }}$ stands for right-ideal automorphism with finite domain code. Every element of riAut ${ }^{\text {fin }}$ can be given by a bijection between two finite maximal prefix codes, and it is straightforward
to prove that riAut ${ }^{\text {fin }}$ is a submonoid of $\mathcal{R} \mathcal{M}^{P}$. For every finite maximal prefix code $P$, id $_{P A^{*}}$ is an idempotent of riAut ${ }^{\text {fin }}$, hence riAut ${ }^{\text {fin }}$ is not a group. The Thompson group $V$ is defined by

$$
V=\text { riAut }^{\text {fin }} / \equiv_{\mathrm{bd}} \quad\left(\leq \mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}\right) .
$$

See [8] for details, and a proof that this is a group; riAut ${ }^{\text {fin }}$ is studied in [5]. This group has remarkable properties (e.g., it is finitely presented and simple), and can be defined in several ways. It was introduced by Richard J. Thompson in the 1960s along with two other remarkable groups; see [9] for more background.

## Theorem 3.16.

(1) The monoid $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ is regular, $\mathcal{J}^{0}$-simple, and finitely generated. Every element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is $\equiv_{\mathrm{bd}}{ }^{-}$ equivalent to a regular element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.
(2) For any family $\mathcal{T}$ of total functions as in Def. 3.1, the group of units of $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$ is the Richard Thompson group $V$ (a.k.a. $G_{2,1}$ ). In particular, the group of units of $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ is $V$.
(3) $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$, and in particular $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$, is not congruence-simple.

Proof. (1) Regularity and finite generation are proved in the same way as for $\mathcal{M}_{\text {end }}^{\mathrm{P}}$ (Theorem 2.23, and Lemma (2.22). Since $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is $\mathcal{J}^{0}$-simple, so is its homomorphic image $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathrm{bd}}\left(=\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}\right)$.
(2) By Lemma3.15 (and since $\equiv_{\mathcal{T}}$ implies $\equiv_{\mathrm{bd}}$ ), if $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ has a finite $\operatorname{domC}(f)$ then every element of $[f]_{\mathcal{T}}$ has a finite domain code. Hence when $\operatorname{domC}(f)$ is finite, $[f]_{\mathcal{T}}=[f]_{\text {bd }}$. It now follows immediately from the definition so every element of $V$ belongs to $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$. Thus, $V$ is a subgroup of the group of units.

Conversely, suppose $[F]_{\mathcal{T}} \equiv_{\mathcal{H}}[\mathrm{id}]_{\mathcal{T}}$ in $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$, where $F \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. We note first that if $e \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ satisfies $e \equiv \mathcal{T}$ id, then $e=\operatorname{id}_{P A^{*}}$, for some finite maximal prefix code $P \subset A^{*}$ (finiteness follows from Lemma (3.15). Now $[F]_{\mathcal{T}} \equiv_{\mathcal{H}}[\mathrm{id}]_{\mathcal{T}}$ implies that there is a maximal prefix code $P_{1} \subset A^{*}$ in P such that $\operatorname{id}_{P_{1} A^{*}} \equiv_{\mathcal{T}}$ id, and there exists $g_{2} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $g_{2} \circ F=\operatorname{id}_{P_{1} A^{*}}\left(\right.$ since $\left.[F]_{\mathcal{T}} \equiv_{\mathcal{L}}[\mathrm{id}]_{\mathcal{T}}\right)$. Then, $\operatorname{id}_{P_{1} A^{*}} \equiv \mathcal{T}$ id implies $F \circ \operatorname{id}_{P_{1} A^{*}} \equiv_{\mathcal{T}} F$. Letting $f=F \circ \operatorname{id}_{P_{1} A^{*}}\left(\equiv_{\mathcal{T}} F\right)$, we now have $[f]_{\mathcal{T}} \equiv_{\mathcal{H}}[\mathrm{id}]_{\mathcal{T}}$ and $g_{2} \circ f=\operatorname{id}_{P_{1} A^{*}}$ with $f \circ g_{2} \circ f=f$. So $f$ is regular.

We also have $f \circ g_{1}=\operatorname{id}_{P_{2} A^{*}}$ (since $[f]_{\mathcal{T}} \equiv_{\mathcal{R}}[\mathrm{id}]_{\mathcal{T}}$ ), where $g_{1} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ and $P_{2}$ is a finite maximal prefix code. Since $\operatorname{id}_{P_{2} A^{*}} \equiv_{\mathcal{T}}$ id $\equiv_{\mathcal{T}}$ id $_{P_{2} A^{*}}$, we have $P_{2} \equiv_{\mathcal{T}}\{\varepsilon\} \equiv_{\mathcal{T}} P_{2}$. Since $f$ and id ${ }_{P_{2} A^{*}}$ are regular, Lemma [2.20(1) implies $\operatorname{Im}(f)=P_{2} A^{*}$. Since $f$ and $\operatorname{id}_{P_{1} A^{*}}$ are regular, Lemma 2.20(2) implies that $f$ is injective and $\operatorname{Dom}(f)=P_{1} A^{*}$. Hence, $f$ is a bijection from $P_{1} A^{*}$ onto $P_{2} A^{*}$, with $P_{1}, P_{2}$ finite. Thus, $[f]_{\mathcal{T}}\left(=[F]_{\mathcal{T}}\right)$ belongs to the Thompson group $V$.
(3) Obviously, the congruence $\equiv_{\mathcal{T}}$ is a refinement of $\equiv_{\text {end }}$, so there is a surjective homomorphism $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}} \rightarrow \mathcal{M}_{\text {end }}^{\mathrm{P}}$. By (2) and Prop. 2.25, these two monoids have different groups of units, so they are not isomorphic, hence the above surjective homomorphism is not injective. So, $\equiv_{\mathcal{T}}$ is a strict refinement of $\equiv_{\text {end }}$, so $\mathcal{M}_{\mathcal{T}}^{P}$ is not congruence-simple.

## Proposition 3.17.

(1) In $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ we have: If $\mathbf{1} \equiv_{\mathcal{D}} f$, then $\operatorname{imC}(f)$ is finite.
(2) The monoid $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is not $\mathcal{D}^{0}$-simple.

Proof. (1) If $\mathbf{1} \equiv_{\mathcal{D}} f$ then $f$ is obviously regular (since $\mathbf{1}$ is regular, and the whole $\mathcal{D}$-class is regular if it contains a regular element). By definition of $\equiv_{\mathcal{D}}, \mathbf{1} \equiv_{\mathcal{D}} f$ iff $\mathbf{1} \equiv_{\mathcal{L}} g \equiv_{\mathcal{R}} f$ for some $g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. By Lemma 2.20 this implies that $\operatorname{domC}(g)$ is finite (and, moreover, $g$ is injective, and $\operatorname{domC}(g)$ is a maximal prefix code), and that $\operatorname{Im}(g)=\operatorname{Im}(f)$; hence $\operatorname{imC}(g)=\operatorname{imC}(f)$. Since domC $(g)$ is finite, $\operatorname{imC}(g)$ is finite. Hence $\operatorname{imC}(f)(=\operatorname{imC}(g))$ is finite.
(2) Consider $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ defined (for all $n \geq 0$ ) by $f\left(0^{2 n} 1\right)=0^{2 n+1} 1$ and $f\left(0^{2 n+1} 1\right)=0^{2 n} 1$; so, $\operatorname{domC}(f)=\operatorname{imC}(f)=0^{*} 1$. Then $\operatorname{imC}(f)$ is infinite, so by $(1), f \not \equiv \mathcal{D}$ id in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.

Let $M_{2,1}$ denote the monoid generalization of the Thompson group $V\left(=G_{2,1}\right)$. To define $M_{2,1}$, consider first $\mathcal{R} \mathcal{M}^{\text {fin }}=\left\{f \in \mathcal{R} \mathcal{M}^{\text {P }}: \operatorname{dom} C(f)\right.$ is finite $\}$. Then, $M_{2,1}=\mathcal{R} \mathcal{M}^{\text {fin }} / \equiv_{\text {bd }}$; see [6].

## Proposition 3.18.

(1) In $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$, the $\mathcal{D}$-class of the identity contains $M_{2,1}-\{\mathbf{0}\}$.
(2) In $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$, the $\mathcal{R}$-class of the identity contains some elements that are not in $M_{2,1}$.

Proof. (1) The monoid $M_{2,1}$ is the submonoid $\left\{[f]_{b d} \in \mathcal{M}_{b d}^{P}: \operatorname{dom} C(f)\right.$ is finite $\}$. In Theorem 2.5 of [6] it was proved that $M_{2,1}$ is $\mathcal{D}^{0}$-simple. Therefore, $M_{2,1}-\{\mathbf{0}\}$ is contained in the $\mathcal{D}$-class of $\mathbf{1}$ in $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$.
(2) By Lemma 2.11 in [4, the $\mathcal{R}$-class of $\mathbf{1}$ in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is $\left\{f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}: \varepsilon \in \operatorname{Im}(f)\right\}$. Consider $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ defined by $f\left(0^{n} 1\right)=0^{n}$ for all $n \geq 0$; so, $\operatorname{dom} C(f)=0^{*} 1$ and $\operatorname{imC}(f)=\{\varepsilon\}$. Hence $f$ is in the $\mathcal{R}$-class of 1. But $[f]_{\mathrm{bd}}\left(\in \mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}\right)$ does not belong to $M_{2,1}$, since the infinite prefix code $0^{*} 1$ is not $\bar{E}_{\text {bd }}$-equivalent to a finite prefix code, by Lemma 3.15.

Lemma 3.19 If $R_{2} \subseteq R_{1}$ are right ideals of $A^{*}$ and $R_{2}$ is essential, then $R_{1}$ is also essential. If $R_{2}$ is essential and finitely generated (as a right ideal), then $R_{1}$ is finitely generated.

Proof. Every right ideal $x A^{*}$ intersects $R_{2}$, hence $x A^{*}$ obviously intersects $R_{1}$ (since $R_{2} \subseteq R_{1}$ ). So $R_{1}$ is essential. If $R_{2}$ is a finitely generated essential right ideal then $R_{2}=P_{2} A^{*}$ for a finite maximal prefix code $P_{2}$. It follows that $A^{*}-P_{2} A^{*}$ is a finite set. Moreover, $R_{1}$ is generated by a subset of $P_{2} \cup\left(R_{1}-P_{2} A^{*}\right)$. This is a subset of $P_{2} \cup\left(A^{*}-P_{2} A^{*}\right)$, which is finite.

Lemma 3.20 (See also Lemma 5.9) If $f_{1}, f_{2} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ satisfy $f_{1} \equiv_{\mathrm{bd}} f_{2}$, then $\operatorname{imC}\left(f_{1}\right) \equiv_{\mathrm{bd}} \operatorname{imC}\left(f_{2}\right)$.
Proof. By Prop. 3.2 and Cor. 3.8, $f_{1} \equiv_{\text {bd }} f_{2}$ iff $\operatorname{domC}\left(f_{1}\right) A^{\omega}=\operatorname{domC}\left(f_{2}\right) A^{\omega}$ and $f_{1}, f_{2}$ agree on $\operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right)$. Thus $f_{1} \equiv_{\text {bd }} f_{2}$ implies $\operatorname{imC}\left(f_{1}\right) A^{\omega}=f_{1}\left(\operatorname{domC}\left(f_{1}\right) A^{\omega}\right)=f_{1}\left(\operatorname{domC}\left(f_{1}\right) A^{\omega} \cap\right.$ $\left.\operatorname{domC}\left(f_{2}\right) A^{\omega}\right)=f_{2}\left(\operatorname{domC}\left(f_{1}\right) A^{\omega} \cap \operatorname{domC}\left(f_{2}\right) A^{\omega}\right)=f_{2}\left(\operatorname{domC}\left(f_{2}\right) A^{\omega}\right)=\operatorname{imC}\left(f_{2}\right) A^{\omega}$. So imC $\left(f_{1}\right) A^{\omega}$ $=\operatorname{imC}\left(f_{2}\right) A^{\omega}$, hence (by Prop. 3.2 again), $\operatorname{imC}\left(f_{1}\right) \equiv_{\mathrm{bd}} \operatorname{imC}\left(f_{2}\right)$.

Notation: For any right ideal $R \subseteq A^{*}$, the (unique) prefix code that generates $R$ as a right ideal is denoted by $\operatorname{prefC}(R)$.

## Lemma 3.21.

(1) Let $R \subseteq A^{*}$ be an essential right ideal such that $R \in \mathrm{P}$. If $[\mathbf{1}]_{\mathrm{bd}} \equiv_{\mathcal{D}}\left[\mathbf{1}_{R}\right]_{\mathrm{bd}}$ in $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$, then $R$ is finitely generated (as a right ideal).
(2) The monoid $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ is not $\mathcal{D}^{0}$-simple.

Proof. (1) We have $[\mathbf{1}]_{\mathrm{bd}} \equiv_{\mathcal{D}}[f]_{\mathrm{bd}}$ iff $[\mathbf{1}]_{\mathrm{bd}} \equiv_{\mathcal{L}}[g]_{\mathrm{bd}} \equiv_{\mathcal{R}}[f]_{\mathrm{bd}}$ for some $g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. The relation $[\mathbf{1}]_{\mathrm{bd}} \equiv_{\mathcal{L}}[g]_{\mathrm{bd}}$ is equivalent to $[\mathbf{1}]_{\mathrm{bd}}=[m]_{\mathrm{bd}}[g]_{\mathrm{bd}}$ for some $m \in \mathcal{R} \mathcal{M}^{\mathbf{P}}$, hence $\mathbf{1} \equiv_{\mathrm{bd}} m g$. So (by Lemma 3.15), $\mathbf{1}_{P A^{*}}=m g$ for some finite maximal prefix code $P \subset A^{*}$. It follows from $\mathbf{1}_{P A^{*}}=m g$ that $P A^{*} \subseteq \operatorname{Dom}(g)$. Since $P A^{*}$ is a finitely generated essential right ideal, it follows (by Lemma 3.19) that $\operatorname{Dom}(g)$ is also a finitely generated essential right ideal. In summary, so far we have shown that $[\mathbf{1}]_{\text {bd }} \equiv_{\mathcal{L}}[g]_{\text {bd }} \equiv_{\mathcal{R}}[f]_{\text {bd }}$, where $g$ is such that
$\operatorname{domC}(g)$ is a finite maximal prefix code.
We are interested in the case when $f=\mathbf{1}_{R}$, where $R \subset A^{*}$ is an essential right ideal with $[\mathbf{1}]_{\mathrm{bd}} \equiv_{\mathcal{D}}$ $\left[\mathbf{1}_{R}\right]_{\mathrm{bd}}$. Then $[\mathbf{1}]_{\mathrm{bd}} \equiv_{\mathcal{L}}[g]_{\mathrm{bd}} \equiv_{\mathcal{R}}[f]_{\mathrm{bd}}$, with $g$ as above. The relation $[g]_{\mathrm{bd}} \equiv_{\mathcal{R}}\left[\mathbf{1}_{R}\right]_{\mathrm{bd}}$ implies that $\mathbf{1}_{R} \equiv_{\mathrm{bd}} g h$ for some $h \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. And this implies that $\mathbf{1}_{R_{1}}=g h$ for some right ideal $R_{1}$ such that $\operatorname{prefC}\left(R_{1}\right) \equiv_{\text {bd }} \operatorname{prefC}(R)$; hence, $R_{1}$ is essential (since $R$ is essential). From $\mathbf{1}_{R_{1}}=g h$ it follows that $R_{1} \subseteq \operatorname{Im}(g)$; this implies that $\operatorname{Im}(g)$ is essential (by Lemma 3.19). Moreover, since $\operatorname{Dom}(g)$ is finitely generated (as we saw above), $g(\operatorname{Dom}(g))=\operatorname{Im}(g)$ is finitely generated. $\operatorname{So}, \operatorname{Im}(g)$ is a finitely generated essential right ideal, i.e.,
$\operatorname{imC}(g)$ is a finite maximal prefix code.
The relation $[g]_{\mathrm{bd}} \equiv_{\mathcal{R}}\left[\mathbf{1}_{R}\right]_{\mathrm{bd}}$ also implies that $g \equiv_{\mathrm{bd}} \mathbf{1}_{R} \cdot k^{\prime}$ for some $k^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}} ;$ let $g_{1}^{\prime}=\mathbf{1}_{R} \cdot k^{\prime}$.

Let $g_{1}=g_{1}^{\prime} \cdot \mathbf{1}_{\operatorname{Dom}(g)}$; hence $\mathbf{1}_{\operatorname{Dom}\left(g_{1}\right)}=\mathbf{1}_{\operatorname{Dom}\left(g_{1}^{\prime}\right)} \cdot \mathbf{1}_{\operatorname{Dom}(g)}=\mathbf{1}_{\operatorname{Dom}(g)} \cdot \mathbf{1}_{\operatorname{Dom}\left(g_{1}^{\prime}\right)}$. And let $k=k^{\prime} \cdot \mathbf{1}_{\operatorname{Dom}\left(g_{1}\right)}$. Multiplying $g \equiv_{\text {bd }} g_{1}^{\prime}=\mathbf{1}_{R} \cdot k^{\prime}$ on the right by $\mathbf{1}_{\text {Dom }(g)}$ yields:

$$
g \equiv_{\mathrm{bd}} g_{1}=\mathbf{1}_{R} \cdot k^{\prime} \cdot \mathbf{1}_{\operatorname{Dom}(g)},
$$

and then multiplying this on the right by $\mathbf{1}_{\operatorname{Dom}\left(g_{1}^{\prime}\right)}$ yields

$$
\left(g \equiv_{\mathrm{bd}}\right) g_{1}=\operatorname{id}_{R} \cdot k^{\prime} \cdot \operatorname{id}_{\operatorname{Dom}(g)} \cdot \operatorname{id}_{\operatorname{Dom}\left(g_{1}^{\prime}\right)}=\mathrm{id}_{R} \cdot k^{\prime} \cdot \operatorname{id}_{\operatorname{Dom}\left(g_{1}^{\prime}\right)} \cdot \operatorname{id}_{\operatorname{Dom}(g)},
$$

since $\mathbf{1}_{\operatorname{Dom}\left(g_{1}^{\prime}\right)}$ and $\mathbf{1}_{\operatorname{Dom}(g)}$ commute, and since $\operatorname{Dom}\left(g_{1}\right) \subseteq \operatorname{Dom}\left(g_{1}^{\prime}\right)$. Moreover,

$$
k^{\prime} \cdot \mathbf{1}_{\operatorname{Dom}\left(g_{1}^{\prime}\right)} \cdot \mathbf{1}_{\operatorname{Dom}(g)}=k^{\prime} \cdot \mathbf{1}_{\operatorname{Dom}\left(g_{1}\right)}=k .
$$

Thus we have $g_{1}=\mathbf{1}_{R} \cdot k$; this implies $\operatorname{Im}\left(g_{1}\right) \subseteq R$. Since $g \equiv_{\text {bd }} g_{1}$, and $\operatorname{Im}(g)$ is essential (as we saw above), Lemma 3.20 implies that $\operatorname{Im}\left(g_{1}\right)$ is also essential. And since $\operatorname{Dom}(g)$ is finitely generated (as we saw above), and $g \equiv_{\text {bd }} g_{1}$, it follows that $\operatorname{Dom}\left(g_{1}\right)$ is finitely generated (by Lemma 3.15). Hence, $\operatorname{Im}\left(g_{1}\right)=g_{1}\left(\operatorname{Dom}\left(g_{1}\right)\right)$ is finitely generated. So now we have $\operatorname{Im}\left(g_{1}\right) \subseteq R$ (seen above), where $\operatorname{Im}\left(g_{1}\right)$ is an essential right ideal that is finitely generated. By Lemma 3.19, it follows that $R$ is finitely generated.
(2) Let $R$ be an essential right ideal in P such that prefC $(R)$ is infinite. Such right ideals exist; examples are $0^{*} 1 A^{*}$ and $0^{*} 10^{*} 1 A^{*}$. Then by ( 1 ), $[\mathbf{1}]_{\text {bd }} \not \equiv \mathcal{D}\left[\mathbf{1}_{R}\right]_{\text {bd }}$.
We can now completely characterize the $\mathcal{D}$-relation in $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ :
Theorem 3.22 The monoid $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ has exactly two non-zero $\mathcal{D}$-classes, namely
$D_{1}=\left\{[f]_{\text {bd }}:[f]_{\text {bd }}\right.$ contains $f \neq 0$ such that $\operatorname{imC}(f)$ is finite $\}$ and
$D_{2}=\left\{[f]_{\text {bd }}:[f]_{\text {bd }}\right.$ contains $f$ such that $\operatorname{imC}(f)$ is infinite $\}$.
Proof. The sets $D_{1}$ and $D_{2}$ are disjoint and they form a bipartition of $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}-\left\{[\mathbf{0}]_{\mathrm{bd}}\right\}$. Indeed, if $g \in[f]_{\mathrm{bd}}$ then $\operatorname{imC}(g) \equiv_{\mathrm{bd}} \operatorname{imC}(f)$ (by Lemma [3.20); and finiteness of $\operatorname{imC}(f)$ implies finiteness of $\operatorname{imC}(g)$ (by Prop. (3.15). Thus, if $[f]_{\text {bd }} \in D_{1}$ then all elements of $[f]_{\text {bd }}$ have finite image code, so $[f]_{\text {bd }} \notin D_{2}$. Recall that by definition, $[f]_{\text {bd }}=\left\{\varphi \in \mathcal{R} \mathcal{M}^{P}: \varphi \equiv_{\text {bd }} f\right\}$. Let us also define

$$
[[f]]_{\mathrm{bd}}=\left\{\xi \in \mathcal{R}^{\text {rec }}: \xi \equiv_{\text {bd }} f\right\}
$$

By Prop. 3.14, $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}\left(=\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathrm{bd}}\right)$ is isomorphic to $\mathcal{R} \mathcal{M}^{\text {rec }} / \equiv_{\mathrm{bd}}$. For the remainder of this proof we represent $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ by $\mathcal{R} \mathcal{M}^{\text {rec }} / \equiv_{\mathrm{bd}}$. Let us reformulate the description of the two non-zero $\mathcal{D}$-classes in terms of $\mathcal{R} \mathcal{M}^{\text {rec }}$ :

$$
\begin{aligned}
& \Delta_{1}=\left\{[[f]]_{\mathrm{bd}}:[[f]]_{\mathrm{bd}} \text { contains } f \neq 0 \text { such that } \operatorname{imC}(f) \text { is finite }\right\}, \\
& \Delta_{2}=\left\{[[f]]_{\mathrm{bd}}:[[f]]_{\mathrm{bd}} \text { contains } f \text { such that } \operatorname{imC}(f) \text { is infinite }\right\} .
\end{aligned}
$$

The sets $\Delta_{1}$ and $\Delta_{2}$ are disjoint and form a bipartition of $\mathcal{R} \mathcal{M}^{\text {rec }} / \equiv_{\text {bd }}-\left\{[[\mathbf{0}]]_{\text {bd }}\right\}$; this holds for the same reason as in the case of $D_{1}$ and $D_{2}$ above, since Lemma 3.20 and Prop. 3.15 apply to all right-ideal morphisms.

By Lemma $3.21(2)), \mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$ is not $\mathcal{D}^{0}$-simple, hence $\Delta_{1} \cup \Delta_{2}$ is not one $\mathcal{D}$-class. So to prove the Theorem it suffices to prove that all elements in $\Delta_{1}$ are $\mathcal{D}$-related, and all elements in $\Delta_{2}$ are $\mathcal{D}$-related in $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$.
(1) Let us prove that all elements of $\Delta_{1}$ are $\mathcal{D}$-related.

Every element of $[[i d]]_{\mathrm{bd}}$ is of the form $\operatorname{id}_{P A^{*}}$, where $P$ is a maximal prefix code, and by Lemma 3.15, $P$ is finite. If $g \in \mathcal{R} \mathcal{M}^{\text {rec }}$ is such that $\operatorname{domC}(g)$ is finite, then $g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. By Lemma 2.20, if $g \in \mathcal{R} \mathcal{M}^{\mathbf{P}}$ is injective and $\operatorname{domC}(g)$ is a finite maximal prefix code, then $g \equiv_{\mathcal{L}} \operatorname{id}_{\mathrm{domC}(g) A^{*}}$; hence $[[g]]_{\mathrm{bd}} \equiv_{\mathcal{L}}[[\mathrm{id}]]_{\mathrm{bd}}\left(\right.$ in $\left.\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}\right)$. For any $f \in \mathcal{R} \mathcal{M}^{\text {rec }}$ with finite $\operatorname{imC}(f)$, we want to show that $f \equiv_{\mathcal{R}} g$ in $\mathcal{R} \mathcal{M}^{\text {rec }}$ for some $g$ of the above type. Then we will have $[[f]]_{\mathrm{bd}} \equiv_{\mathcal{R}}[[g]]_{\mathrm{bd}} \equiv_{\mathcal{L}}[[i d]]_{\mathrm{bd}}$ in $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}$; so every $[[f]]_{\mathrm{bd}} \in \Delta_{1}$ will be in the $\mathcal{D}$-class of $[[i d]]_{\text {bd }}$.

Let $\operatorname{imC}(f)=\left\{y_{i}: i=1, \ldots, N\right\}$, where $N=|\operatorname{imC}(f)|$ (finite). Let $X=\left\{x_{i}: i=1, \ldots, N\right\} \subseteq$ $\operatorname{domC}(f)$ be such that $x_{i} \in f^{-1}\left(y_{i}\right)$; i.e., $X$ is a choice set for the restriction of $f^{-1}$ to $\operatorname{imC}(f)$. Since $X \subseteq \operatorname{dom} C(f), X$ is a finite prefix code. Let $P \subset A^{*}$ be any finite maximal prefix code of size $N$, and
let $\left(p_{1}, \ldots, p_{N}\right)$ be any total ordering of $P$. We define an injective right-ideal morphism $\alpha \in \mathcal{R} \mathcal{M}^{P}$ $\left(\subseteq \mathcal{R} \mathcal{M}^{\text {rec }}\right)$ by $\operatorname{domC}(\alpha)=P, \operatorname{imC}(\alpha)=X$, and $\alpha\left(p_{i} w\right)=x_{i} w\left(\right.$ for all $i=1, \ldots, N$, and $\left.w \in A^{*}\right)$.

Let $g=f \circ \alpha$. So, $\operatorname{domC}(g)=P$ (which is a finite maximal prefix code), and $\operatorname{imC}(g)=\operatorname{imC}(f)$. And $g$, restricted to $\operatorname{domC}(g)$, is a bijection from $P$ to $\operatorname{imC}(f)$, so $g$ is injective on $\operatorname{Dom}(g)=P A^{*}$. (In fact, $[g]_{\mathrm{bd}} \in M_{2,1}$, the Thompson-Higman monoid defined in [6]).) Let $\beta=g^{-1} \circ f$; this is well defined since $g$ is injective. Note that $g \circ g^{-1}$ is the restriction of the identity map to $\operatorname{imC}(f) A^{*}=\operatorname{Im}(f)=\operatorname{Im}(g)$; thus, $g \circ \beta=g \circ g^{-1} \circ f=\operatorname{id}_{\operatorname{Im}(f)} \circ f=f$. Thus, $g=f \circ \alpha$ and $g \circ \beta=f$, so $g \equiv_{\mathcal{R}} f$ in $\mathcal{R M}^{\text {rec }}$.
(2) Let us prove that all elements of $\Delta_{2}$ are $\mathcal{D}$-related.

By Prop. 3.14, $\mathcal{M}_{\mathrm{bd}}^{\mathrm{P}}\left(=\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathrm{bd}}\right)$ is isomorphic to $\mathcal{R} \mathcal{M}^{\text {rec }} / \equiv_{\mathrm{bd}}$.
Claim 1: Let $P, Q \subset A^{*}$ be prefix codes that are infinite and decidable. Then there exists a bijection $\alpha \in \mathcal{R} \mathcal{M}^{\text {rec }}$ from $P$ onto $Q$.
Proof: For any infinite set $S \subseteq A^{*}$, the rank function of $S$ is a bijection from $S$ onto $\mathbb{N}$, defined for $x \in S$ by $\operatorname{rank}_{S}(x)=\left|\left\{w \in S: w<_{\ell \ell} x\right\}\right|$. To make $\operatorname{rank}_{S}($.$) a function between words, we represent$ a natural integer $n \in \mathbb{N}$ by $0^{n} 1$, so $\operatorname{Im}\left(\right.$ rank $\left._{S}\right)=0^{*} 1$.

If $S$ is decidable, rank $_{S}$ is partial recursive with decidable domain $S$. And rank ${ }_{S}$ has a computable input-output balance when $S$ is infinite and decidable. Then $\alpha=\operatorname{rank}_{Q}^{-1} \circ \operatorname{rank}_{P}$ is a bijection from $P$ onto $Q$ with the claimed properties. Finally, $\alpha$ can be extended to a bijective right-ideal morphism from $P A^{*}$ onto $Q A^{*}$; thus, $\alpha \in \mathcal{R} \mathcal{M}^{\text {rec }}$. This proves Claim 1.

As a consequence of the proof of Claim 1, $\mathrm{id}_{P A^{*}}=\alpha \circ \mathrm{id}_{Q A^{*}}$ and $\mathrm{id}_{Q A^{*}}=\alpha^{-1} \circ \mathrm{id}_{P A^{*}}$. Hence for all infinite decidable prefix codes $P$ and $Q$ we have: $i d_{P A^{*}} \equiv_{\mathcal{L}}$ id $_{Q A^{*}}$ in $\mathcal{R} \mathcal{M}^{\text {rec }}$.
Claim 2: For any $f \in \mathcal{R} \mathcal{M}^{\text {rec }}, f \equiv_{\mathcal{R}} \operatorname{id}_{\operatorname{Im}(f)}$.
Proof: For $f \in \mathcal{R} \mathcal{M}^{\text {rec }}, \operatorname{Im}(f)$ is a decidable set, because of the computable I/O-balance. Every $f \in \mathcal{R} \mathcal{M}^{\text {rec }}$ has an inverse in $\mathcal{R} \mathcal{M}^{\text {rec }}$, and since $\operatorname{Im}(f)$ is decidable, such an inverse can be restricted to $\operatorname{Im}(f)$. If $f^{\prime}$ is such an inverse with domain $\operatorname{Im}(f)$, we have: $f \circ f^{\prime}=\operatorname{id}_{\operatorname{Im}(f)}$. Moreover, $\operatorname{id}_{\operatorname{Im}(f)} \circ f=f$. This proves Claim 2.

By Claim 2 and the consequence of Claim 1 we now have: If $f, g \in \mathcal{R} \mathcal{M}^{\text {rec }}$ and if $\operatorname{imC}(f), \operatorname{imC}(g)$ are infinite, then $f \equiv_{\mathcal{R}} \operatorname{id}_{\operatorname{Im}(f)} \equiv_{\mathcal{L}} \operatorname{id}_{\operatorname{Im}(g)} \equiv_{\mathcal{R}} g$.

## 4 Polynomial and exponential end-equivalences

The relations $\equiv_{\mathcal{T}}$, in particular $\equiv_{\text {poly }}$ and $\equiv_{\text {E3 }}$, were defined in Def. 3.1 for prefix codes, and in Def. 3.4 for right-ideal morphisms. We call $\equiv_{\text {poly }}$ the polynomial end-equivalence relation, and $\equiv_{\text {E3 }}$ the exponential end-equivalence (or elementary recursive end-equivalence) relation. Throughout this section, $\mathcal{T}$ denotes a family of functions as in Def. 3.1] possibly with additional properties. When $\equiv \mathcal{T}$ is applied between prefix codes, it is an equivalence relation. Transitivity follows from the fact that $\mathcal{T}$ is closed under composition. For two prefix codes $P_{1}, P_{2} \subset A^{*}$ and $\tau \in \mathcal{T}$, we say that lengths in $P_{1}$ and $P_{2}$ are $\tau$-related iff $\left|x_{1}\right| \leq \tau\left(\left|x_{2}\right|\right)$ and $\left|x_{2}\right| \leq \tau\left(\left|x_{1}\right|\right)$ for all $x_{1} \in P_{1}, x_{2} \in P_{2}$ with $x_{1} \|_{\text {pref }} x_{2}$. In particular, when $\mathcal{T}$ is the set of polynomials we say "polynomially related". The latter is the most interesting, due to its connections with NP.

If $L_{1}, L_{2}, M_{1}, M_{2}$ are prefix codes such that $L_{1} \equiv{ }_{\mathcal{T}} L_{2}, M_{1} \subseteq L_{1}, M_{2} \subseteq L_{2}$, and $M_{1} \equiv{ }_{\text {end }} M_{2}$, then $M_{1} \equiv_{\mathcal{T}} M_{2}$. Indeed, the bounding function $\tau \in \mathcal{T}$ that appear in the definition of $L_{1} \equiv_{\mathcal{T}} L_{2}$, also works for $M_{1} \equiv \mathcal{T} M_{2}$.

There exists $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $f(\operatorname{domC}(f)) \not \equiv \mathcal{T} \operatorname{imC}(f)$, and $\equiv \mathcal{T}$ is not reflexive on $f(\operatorname{domC}(f))$. Moreover, $f$ can be chosen so that there exist prefix codes $P_{1}, P_{2} \subset \operatorname{Dom}(f)$ with $P_{1} \equiv_{\mathcal{T}} P_{2}$, such that $f\left(P_{1}\right) \not \equiv \mathcal{T} f\left(P_{2}\right)$. As an example, let $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ be defined by $f\left(0^{n} 1 w\right)=0^{n} w$ for all $n \geq 0$ and $w \in\{0,1\}^{*}$; so $\operatorname{domC}(f)=0^{*} 1$, and $\operatorname{imC}(f)=\{\varepsilon\}$. Then, $f(\operatorname{domC}(f))=0^{*} \not \equiv_{\mathrm{bd}}\{\varepsilon\}=\operatorname{imC}(f)$, and $f(\operatorname{domC}(f))=0^{*} \not \equiv_{\mathrm{bd}} 0^{*}=f(\operatorname{domC}(f))$ (non-reflexive). Note that $\not \equiv_{\mathrm{bd}}$ implies $\not \equiv \mathcal{T}$. To show the possibility of $f\left(P_{1}\right) \not \equiv \mathcal{T} f\left(P_{2}\right)$ when $P_{1} \equiv \mathcal{T} P_{2}$, let $f$ be as in the example above, and let $P_{1}=P_{2}=0^{*} 1$. Then $f\left(P_{1}\right)=f\left(P_{2}\right)=0^{*}$; but $0^{*} \not \equiv_{\text {bd }} 0^{*}$ (non-reflexivity in this case).

There exist $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ and prefix codes $P_{1}, P_{2} \subset \operatorname{Im}(f)$ such that $P_{1} \equiv_{\text {poly }} P_{2}$ but $f^{-1}\left(P_{1}\right) \not \equiv_{\text {end }}$ $f^{-1}\left(P_{2}\right)$. For example, let $\operatorname{domC}(f)=\{00,01,1\}$, and $f(00)=00, f(01)=0, f(1)=\varepsilon$; so $f \in \mathcal{R} \mathcal{M}^{P}$. Then $f^{-1}(\{\varepsilon\})=\{1\}, f^{-1}(\{0\})=\{01,10\}, f^{-1}(\{1\})=\{11\}, f^{-1}(\{01\})=\{101,011\}, f^{-1}(\{00\})=$ $\{100,010,00\}$. Let $P_{1}=\{\varepsilon\}, P_{2}=\{0,1\}$, and $P_{3}=\{00,01,1\}$. Then $P_{1} \equiv_{\text {poly }} P_{2} \equiv_{\text {poly }} P_{3}$, but $f^{-1}\left(P_{1}\right), f^{-1}\left(P_{2}\right)$, and $f^{-1}\left(P_{3}\right)$ are all $\not \equiv_{\text {end }}$, since $f^{-1}\left(P_{1}\right)=\{11\}, f^{-1}\left(P_{2}\right)=\{01,10,11\}$, and $f^{-1}\left(P_{3}\right)=\{11,100,010,00,101,011\}$.

The following is the $\equiv_{\mathcal{T}}$ version of Propositions 2.9 and 3.5.
Proposition 4.1 For any family of functions $\mathcal{T}$, as in Def. 3.1 we have the following.
(1) Let $P_{1}, P_{2} \subset A^{*}$ be prefix codes such that $P_{1} \equiv_{\mathcal{T}} P_{2}$, let $P_{\cap}$ be the prefix code that generates the right ideal $P_{1} A^{*} \cap P_{2} A^{*}$, and let $P_{\cup}$ be the prefix code that generates the right ideal $P_{1} A^{*} \cup P_{2} A^{*}$. Then $P_{1} \equiv_{\mathcal{T}} P_{2} \equiv_{\mathcal{T}} P_{\cap} \equiv_{\mathcal{T}} P_{\cup}$.
(2) Let $f_{1}, f_{2}$ be right-ideal morphisms such that $f_{1} \equiv \mathcal{T} f_{2}$. Then $f_{1} \cap f_{2}$ and $f_{1} \cup f_{2}$ are right-ideal morphisms, and $f_{1} \equiv \mathcal{T} f_{2} \equiv \mathcal{T} f_{1} \cap f_{2} \equiv \mathcal{T} f_{1} \cup f_{2}$.

Proof. (1) By Prop. [2.9, $P_{\cap} \equiv_{\text {end }} P_{\cup} \equiv_{\text {end }} P_{1} \equiv_{\text {end }} P_{2}$.
Length bounds: It is known (and easily proved) that $P_{\cup} \subseteq P_{1} \cup P_{2}$. Hence, if $x \in P_{\cup}, p_{1} \in P_{1}$, and $x \|_{\text {pref }} p_{1}$, then either $x \in P_{1}$ (and then $x=p_{1}$ ), or $x \in P_{2}$ (and then $|x|,\left|p_{1}\right|$ are $\tau$-related since $P_{1} \equiv \mathcal{T} P_{2}$ ). Similarly, if $x \in P_{\cup}, p_{2} \in P_{2}$, and $x \|_{\text {pref }} p_{2}$, then $|x|,\left|p_{2}\right|$ are $\tau$-related.

It is also the case that $P_{\cap} \subseteq P_{1} \cup P_{2}$, and a similar reasoning applies here.
(2) We conclude from (1) that $\operatorname{Dom}\left(f_{i}\right) \equiv \mathcal{T} \operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right) \equiv \mathcal{T} \operatorname{Dom}\left(f_{1}\right) \cup \operatorname{Dom}\left(f_{2}\right)$, for $i=1,2$.

Also, $\operatorname{Dom}\left(f_{1} \cap f_{2}\right)=\operatorname{Dom}\left(f_{1}\right) \cap \operatorname{Dom}\left(f_{2}\right)$. And since $f_{1}=f_{2}$ on $\operatorname{Dom}\left(f_{1} \cap f_{2}\right)$ we have $f_{1}=f_{1} \cap f_{2}$ on $\operatorname{Dom}\left(f_{1} \cap f_{2}\right)$. Hence $f_{1} \equiv \mathcal{T} f_{1} \cap f_{2}$, and similarly for $f_{2}$.

Also, $\operatorname{Dom}\left(f_{1} \cup f_{2}\right)=\operatorname{Dom}\left(f_{1}\right) \cup \operatorname{Dom}\left(f_{2}\right)$. And $f_{1}=f_{1} \cup f_{2}$ on $\operatorname{Dom}\left(f_{1}\right)$, and $f_{2}=f_{1} \cup f_{2}$ on $\operatorname{Dom}\left(f_{2}\right)$, hence $f_{1} \equiv \mathcal{T} f_{1} \cup f_{2} \equiv \mathcal{T} f_{2}$.

Corollary 4.2 Every $\equiv_{\mathcal{T}}$-class in $\mathcal{R} \mathcal{M}^{P}$ is a lattice under $\subseteq, \cup$ and $\cap$.
Theorem 4.3 Let $\mathcal{T}$ be any family of functions as in Def. 3.1. and let $\mathcal{M}$ be any monoid of right-ideal morphisms with $I / O$-balance function in $\mathcal{T}$. Then the relation $\equiv_{\mathcal{T}}$ is a congruence on $\mathcal{M}$.

Proof. Clearly, $\equiv_{\mathcal{T}}$ is an equivalence relation (for transitivity we use the fact that $\mathcal{T}$ has upper bounds for composition, see Def. 3.1(2)). For the multiplicative property, let $f_{1}, f_{2}, g \in \mathcal{M}$, and suppose $f_{1} \equiv \mathcal{T} f_{2}$; we want to prove that $f_{1} g \equiv_{\mathcal{T}} f_{2} g$ and $g f_{1} \equiv \mathcal{T} g f_{2}$. Since $\equiv_{\mathcal{T}}$ implies $\equiv_{\text {bd }}$, the actions of $f_{1}$ and $f_{2}$ on $A^{\omega}$ are the same, hence $f_{1} g \equiv_{\text {bd }} f_{2} g$, and $g f_{1} \equiv_{\text {bd }} g f_{2}$ (using Cor. 3.8). It now suffices to check the $\mathcal{T}$-relation for lengths in the domain codes.

- Proof that $\operatorname{domC}\left(f_{1} g\right) \equiv \mathcal{T} \operatorname{domC}\left(f_{2} g\right)$ :
e want to show that lengths in $\operatorname{domC}\left(f_{1} g\right)$ and $\operatorname{domC}\left(f_{2} g\right)$ are $\mathcal{T}$-related. Let $x_{1} \in \operatorname{domC}\left(f_{1} g\right)$ and $x_{2} \in \operatorname{domC}\left(f_{2} g\right)$ be prefix-comparable. By Prop. 4.1(2) we can assume that $f_{2} \subseteq f_{1}$, hence $\operatorname{Dom}\left(f_{2}\right) \subseteq \operatorname{Dom}\left(f_{1}\right)$; hence, $x_{2} \geq_{\text {pref }} x_{1}$ (i.e., $x_{1}$ is a prefix of $x_{2}$ ).

Since $x_{2} \geq_{\text {pref }} x_{1}$, we have $g\left(x_{2}\right) \geq_{\text {pref }} g\left(x_{1}\right)$. Since $x_{i} \in \operatorname{domC}\left(f_{i} g\right)$, we have $g\left(x_{i}\right) \in \operatorname{Dom}\left(f_{i}\right)$; hence there exists $z_{i} \in \operatorname{domC}\left(f_{i}\right)$ with $g\left(x_{i}\right) \geq_{\text {pref }} z_{i}$ (for $i=1,2$ ). Since $z_{2} \leq_{\text {pref }} g\left(x_{2}\right) \geq_{\text {pref }} g\left(x_{1}\right) \geq_{\text {pref }} z_{1}$, we have $z_{2} \|_{\text {pref }} g\left(x_{1}\right)$ and $z_{2} \|_{\text {pref }} z_{1}$. Since $f_{1} \equiv \mathcal{T} f_{2},\left|z_{1}\right|$ and $\left|z_{2}\right|$ are $\tau_{12}$-related for some $\tau_{12} \in \mathcal{T}$ (depending only on $f_{1}, f_{2}$ ). We can assume $\tau_{12}(n) \geq n$ for all $n \in \mathbb{N}$ and that $\tau_{12}$ is increasing, since $\mathcal{T}$ contains such a function and $\mathcal{T}$ has upper bounds for sum (see Def. 3.1(2)).

Since $f_{2} \subseteq f_{1}, \operatorname{Dom}\left(f_{2}\right) \subseteq \operatorname{Dom}\left(f_{1}\right)$, hence for all $d_{2} \in \operatorname{domC}\left(f_{2}\right)$ and all $d_{1} \in \operatorname{domC}\left(f_{1}\right)$ : if $d_{2} \|_{\text {pref }} d_{1}$ then $d_{2} \geq_{\text {pref }} d_{1}$; therefore, $z_{2} \geq_{\text {pref }} z_{1}$. We now have two cases:
$g\left(x_{2}\right) \geq_{\text {pref }} g\left(x_{1}\right) \geq_{\text {pref }} \quad z_{2} \geq_{\text {pref }} z_{1}$, or $g\left(x_{2}\right) \geq_{\text {pref }} z_{2} \geq_{\text {pref }} g\left(x_{1}\right) \geq_{\text {pref }} z_{1}$.
Case 1: $g\left(x_{1}\right) \geq_{\text {pref }} z_{2}$.

Now $g\left(x_{1}\right) \in \operatorname{Dom}\left(f_{2}\right)\left(\right.$ since $\left.g\left(x_{1}\right) \geq_{\text {pref }} z_{2}\right)$, hence $x_{1} \in g^{-1}\left(\operatorname{Dom}\left(f_{2}\right)\right)$, so $x_{1} \in \operatorname{Dom}\left(f_{2} g\right)$. Since $x_{2} \in \operatorname{domC}\left(f_{2} g\right)$ and $x_{2} \geq_{\text {pref }} x_{1}$ it follows that $x_{2}=x_{1}$.

This implies obviously that $\left|x_{2}\right|=\left|x_{1}\right|$, hence $\left|x_{2}\right|$ and $\left|x_{1}\right|$ are $\tau_{12}$-related (since $\tau_{12}(n) \geq n$ ).
Case 2: $z_{2} \geq_{\text {pref }} g\left(x_{1}\right)$.
Being a right-ideal morphism of $A^{*}, g$ is a one-to-one correspondence between the $\geq_{\text {pref-chains }}$

$$
\begin{aligned}
& x_{2} \geq_{\text {pref }}^{\text {p }} \ldots \geq_{\text {pref }} x_{1} \quad \text { and } \\
& g\left(x_{2}\right) \geq_{\text {pref }} \cdots \geq_{\text {pref }} g\left(x_{1}\right)
\end{aligned}
$$

in $A^{*}$. (It is easy to see that any right-ideal morphism $f$ of $A^{*}$ is injective on any chain $x>_{\text {pref }}$ $x m_{1}>_{\text {pref }} x m_{1} m_{2}>_{\text {pref }} \ldots>_{\text {pref }} x m_{1} m_{2} \ldots m_{k}$, if $x \in \operatorname{Dom}(f)$; this holds even if $f$ is not injective on all of $\operatorname{Dom}(f)$.) Since $g\left(x_{2}\right) \geq_{\text {pref }} z_{2} \geq_{\text {pref }} g\left(x_{1}\right)$, let $t_{2}$ be the (unique) inverse image of $z_{2}$ in the upper chain; so $x_{2} \geq_{\text {pref }} t_{2} \geq_{\text {pref }} x_{1}$, and $g\left(t_{2}\right)=z_{2}\left(\in \operatorname{domC}\left(f_{2}\right)\right)$. Then $t_{2} \in g^{-1}\left(\operatorname{domC}\left(f_{2}\right)\right)$, hence $t_{2} \in \operatorname{Dom}\left(f_{2} g\right)$. Moreover, $t_{2} \leq_{\text {pref }} x_{2} \in \operatorname{domC}\left(f_{2} g\right)$ implies that $t_{2}=x_{2}$ (since $\operatorname{domC}\left(f_{2} g\right)$ is a prefix code, and in a prefix code, prefix comparable elements are equal). Therefore, $g\left(t_{2}\right)=g\left(x_{2}\right)$, hence (since $\left.g\left(t_{2}\right)=z_{2}\right), z_{2}=g\left(x_{2}\right)$. Thus, $g\left(x_{2}\right) \in \operatorname{domC}\left(f_{2}\right)$. Since we also have $z_{1} \in \operatorname{domC}\left(f_{1}\right)$ we conclude that $\left|g\left(x_{2}\right)\right| \leq \tau_{12}\left(\left|z_{1}\right|\right)$ (since lengths in $\operatorname{domC}\left(f_{2}\right)$ and $\operatorname{domC}\left(f_{1}\right)$ are $\tau_{12}$-related). And $\left|z_{1}\right| \leq\left|g\left(x_{1}\right)\right|$ (since $\left.g\left(x_{1}\right) \geq_{\text {pref }} z_{1}\right)$, hence $\left|g\left(x_{2}\right)\right| \leq \tau_{12}\left(\left|g\left(x_{1}\right)\right|\right)$ (since $\tau_{12}$ is increasing). Letting $\tau_{g}$ denote the balance polynomial of $g$, we obtain: $\left|x_{2}\right| \leq \tau_{g}\left(\left|g\left(x_{2}\right)\right|\right) \leq \tau_{g}\left(\tau_{12}\left(\left|g\left(x_{1}\right)\right|\right)\right) \leq \tau_{g} \circ \tau_{12} \circ \tau_{g}\left(\left|x_{1}\right|\right)$.

We also have $\left|x_{2}\right| \geq\left|x_{1}\right| \quad$ (since $\left.x_{2} \geq_{\text {pref }} x_{1}\right)$.
In summary, $\operatorname{domC}\left(f_{1} g\right) \equiv \mathcal{T} \operatorname{domC}\left(f_{2} g\right)$ for any function in $\mathcal{T}$ that bounds $\tau_{g} \circ \tau_{12} \circ \tau_{g}$ from above.

- Proof that $\operatorname{domC}\left(g f_{1}\right) \equiv \mathcal{T} \operatorname{domC}\left(g f_{2}\right)$ :

Let $x_{1} \in \operatorname{domC}\left(g f_{1}\right), x_{2} \in \operatorname{domC}\left(g f_{2}\right)$ be prefix-comparable. We want to show that $\left|x_{1}\right|,\left|x_{2}\right|$ are $\mathcal{T}$-related. As before, we can assume that $f_{2} \subseteq f_{1}$, hence $x_{2} \geq_{\text {pref }} x_{1}$ (i.e., $x_{1}$ is a prefix of $x_{2}$ ).

Since $g f_{1}\left(x_{1}\right)$ is defined, $f_{1}(x)$ is defined for all $x \geq_{\text {pref }} x_{1}$.
Since $f_{2} \subseteq f_{1}$ and since $x_{2} \in \operatorname{Dom}\left(g f_{2}\right) \subseteq \operatorname{Dom}\left(f_{2}\right) \subseteq \operatorname{Dom}\left(f_{1}\right)$, we have $f_{2}\left(x_{2}\right)=f_{1}\left(x_{2}\right)$. Let $z_{2} \in \operatorname{domC}\left(f_{2}\right)$ be such that $x_{2} \geq_{\text {pref }} z_{2}$. Then, $z_{2} \leq_{\text {pref }} x_{2} \geq_{\text {pref }} x_{1}$, hence $z_{2} \|_{\text {pref }} x_{1}$; so we have two cases: $x_{1} \geq_{\text {pref }} z_{2}$, or $x_{2} \geq_{\text {pref }} z_{2} \geq_{\text {pref }} x_{1}$.
Case 1: $x_{1} \geq_{\text {pref }} z_{2}\left(\in \operatorname{domC}\left(f_{2}\right)\right)$ :
Then $x_{1} \in \operatorname{Dom}\left(f_{2}\right)$, thus $f_{2}\left(x_{1}\right)=f_{1}\left(x_{1}\right)$ (since $f_{2} \subseteq f_{1}$, and $f_{2}\left(x_{1}\right)$ is defined). So $g f_{2}\left(x_{1}\right)=$ $g f_{1}\left(x_{1}\right)$, and $g f_{2}\left(x_{1}\right)$ is defined, i.e., $x_{1} \in \operatorname{Dom}\left(g f_{2}\right)$. Since $x_{2} \in \operatorname{domC}\left(f_{2}\right)$ and $x_{2} \|_{\text {pref }} x_{1}$, it follows that $x_{1} \geq_{\text {pref }} x_{2}$. So $x_{1}=x_{2}$ (since we also have $x_{2} \geq_{\text {pref }} x_{1}$ ). So $\left|x_{1}\right|,\left|x_{2}\right|$ are $\mathcal{T}$-related.
Case 2: $x_{2} \geq_{\text {pref }} z_{2} \geq_{\text {pref }} x_{1}$.
Since $z_{2} \in \operatorname{domC}\left(f_{2}\right), f_{2}\left(z_{2}\right)$ is defined, hence $f_{2}\left(z_{2}\right)=f_{1}\left(z_{2}\right)$ (since $f_{2} \subseteq f_{1}$ ). Hence $g f_{2}\left(z_{2}\right)$ is defined (since $g f_{1}(x)$ is defined for all $x \geq_{\text {pref }} x_{1}$, and $f_{2}\left(z_{2}\right)=f_{1}\left(z_{2}\right)$ ), i.e., $z_{2} \in \operatorname{Dom}\left(g f_{2}\right)$. Therefore, $z_{2} \geq_{\text {pref }} x_{2}$, since $z_{2} \|_{\text {pref }} x_{2}$ and $x_{2} \in \operatorname{domC}\left(g f_{2}\right)$. But since we also have $x_{2} \geq_{\text {pref }} z_{2}$, it follows that $z_{2}=x_{2}$.

Let $z_{1} \in \operatorname{domC}\left(f_{1}\right)$ be such that $x_{1} \geq_{\text {pref }} z_{1}$. Then we have the $\geq_{\text {pref }}$-chain $x_{2}=z_{2} \geq_{\text {pref }} x_{1} \geq_{\text {pref }} z_{1}$. Since $\left|z_{2}\right| \leq \tau_{12}\left(\left|z_{1}\right|\right)$ it follows (since $x_{2}=z_{2}$ and $\left.\left|z_{1}\right| \leq\left|x_{1}\right|\right)$ that $\left|x_{2}\right| \leq \tau_{12}\left(\left|x_{1}\right|\right)$. Also, $\left|x_{1}\right| \leq\left|x_{2}\right|$. So, $\left|x_{1}\right|$ and $\left|x_{2}\right|$ are $\tau_{12}$-related.

In summary, $\operatorname{domC}\left(g f_{1}\right) \equiv \mathcal{T} \operatorname{domC}\left(g f_{2}\right)$ for the function $\tau_{12}$.
Notation: Let $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathcal{T}}$ denote the set of $\equiv_{\mathcal{T}}$-congruence classes in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$; we will abbreviate $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv \mathcal{T}$ by $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$. In particular, we will consider $\mathcal{M}_{\text {poly }}^{\mathrm{P}}, \mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$, and $\mathcal{M}_{\text {lin }}^{\mathrm{P}}$.

Similarly, $\mathcal{R} \mathcal{M}^{\mathbf{N P}} / \equiv_{\mathcal{T}}$ denotes the set of $\equiv_{\mathcal{T}}$-congruence classes in $\mathcal{R} \mathcal{M}^{\mathbf{N P}}$. Here, $\mathcal{R} \mathcal{M}^{\mathbf{N P}}$ (also called $\mathcal{R} \mathcal{M}^{\Sigma_{1}}$ ) is the monoid
$\mathcal{R} \mathcal{M}^{\mathrm{NP}}=\left\{f: f\right.$ is a polynomially balanced right-ideal morphism $A^{*} \rightarrow A^{*}$ that is computable by a polynomial-time deterministic Turing machine with an oracle in NP\}.
See [4, section 6 , for more details on the similarly defined $\mathrm{fP}^{\mathrm{NP}}$.
By Theorem 4.3 we have (if poly $\subseteq \mathcal{T}$ ):

Corollary 4.4 Let $\mathcal{T}$ be as in Def. 3.1 with, in addition, poly $\subseteq \mathcal{T}$. Then for every monoid $\mathcal{M}$ of right-ideal morphisms with polynomial I/O-balance, $\mathcal{M} / \equiv_{\mathcal{T}}$ is a monoid. In particular, $\mathcal{R} \mathcal{M}^{\mathbf{P}} / \equiv_{\text {poly }}$ $\left(=\mathcal{M}_{\text {poly }}^{\mathrm{P}}\right)$ and $\mathcal{R} \mathcal{M}^{\mathrm{NP}} / \equiv_{\text {poly }}$ are monoids, and there is a homomorphic embedding

$$
\mathcal{M}_{\text {poly }}^{\mathrm{P}} \hookrightarrow \mathcal{R} \mathcal{M}^{\mathrm{NP}} / \equiv_{\text {poly }} .
$$

Proof. The first statements follow from the fact that $\equiv_{\mathcal{T}}$ is a congruence.
Every $\equiv_{\text {poly }}$-class of $\mathcal{R} \mathcal{M}^{\mathbf{N P}}$ contains at most one $\equiv_{\text {poly }}$-class of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, since $\equiv_{\text {poly }}$ is transitive; hence we have the embedding.

But if $\mathrm{P} \neq \mathrm{NP}$ then $\mathrm{a} \equiv_{\text {poly-class }}$ of $\mathcal{R} \mathcal{M}^{\mathbf{N P}}$ that contains elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ could also contain functions that are not in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$; i.e., a $\equiv_{\text {poly }}$-class of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ could be a strict subset of the corresponding $\equiv_{\text {poly }}$-class if $\mathrm{P} \neq \mathrm{NP}$. So if $\mathrm{P} \neq \mathrm{NP}$, the embedding above is not an inclusion.

Let $\mathcal{T}_{1}, \mathcal{T}_{2}$ be families of functions as in Def. 3.1. If $\mathcal{T}_{1} \subseteq \mathcal{T}_{2}$ then $\equiv \mathcal{T}_{1} \subseteq \equiv \mathcal{T}_{2}$; hence there exists a surjective monoid morphism $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathcal{T}_{1}} \rightarrow \mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\mathcal{T}_{2}}$. In particular we have surjective monoid morphisms

$$
\mathcal{M}_{\text {poly }}^{\mathrm{P}} \rightarrow \mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}} \rightarrow \mathcal{M}_{\mathrm{bd}}^{\mathrm{P}} \rightarrow \mathcal{M}_{\mathrm{end}}^{\mathrm{P}} \rightarrow\{\mathbf{1}\} .
$$

Since $\mathcal{M}_{\text {end }}^{P}$ is congruence-simple, the right-most arrow (onto the one-element monoid) cannot be factored (except by using automorphisms as factors).

Proposition 4.5 Let $\mathcal{T}$ be as in Def. 3.1, with the additional condition that poly $\subseteq \mathcal{T}$. Then every $\equiv \mathcal{T}$-class of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ contains functions whose I/O-balance and time-complexity are linear (bounded from above by the function $n \mapsto 3 n$ ).

Proof. We proceed as in Lemma 2.22(2). For any $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ and any function $\tau \in \mathcal{T}$, we define the right-ideal morphism $F_{f, \tau}$ by

$$
\begin{aligned}
& \operatorname{domC}\left(F_{f, \tau}\right)=\bigcup_{x \in \operatorname{domC}(f)} x\{0,1\}^{|x| \cdot \tau(|x|)}, \quad \text { and } \\
& F_{f, \tau}(x z w)=f(x) z w,
\end{aligned}
$$

for all $x \in \operatorname{domC}(f), z \in\{0,1\}^{|x| \cdot \mathcal{T}(|x|)}$, and $w \in\{0,1\}^{*}$. Then $f \equiv_{\mathcal{T}} F_{f, \tau}$ (since $\{0,1\}^{|x| \cdot \tau(|x|)}$ is a maximal prefix code, see Lemma 2.22(2)).

Let $\tau$ be a polynomial upper bound on the I/O-balance and the time-complexity of $f$. We can choose $\tau$ to be a polynomial of the form $n \mapsto a \cdot\left(n^{d}+1\right)$; then $\tau \in \mathcal{T}$ and $\tau$ is fully time-constructible. Then $F_{f, \tau}$ has linear I/O-balance and time-complexity with coefficient le3, by Lemma [2.22(2).

We saw in [4] that fP and $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ do not contain evaluation maps (contrary to the monoid fR of all partial recursive functions with partial recursive balance function). By definition, a (coded) evaluation map for fR is a partial function eval $\in \mathrm{fR}$ such that for every $f \in \mathrm{fR}$ there exists $w \in\{0,1\}^{*}$ (called a program for $f$ ) such that for all $x \in \operatorname{Dom}(f)$ : eval $(\operatorname{code}(w) 11 x)=f(x)$. We saw that fP and $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ contain partial functions that play the role of evaluation maps in a limited way: For every polynomial $q$ of degree $>1$ there exists eval $_{q}$ that works as an evaluation map for functions whose time-complexity and I/O-balance are less than $q$; for fP, see section 4 of [4], for $\mathcal{R} \mathcal{M}^{P}$, see section 2 of 3].

An interesting property of $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is that it has "evaluation elements" that play the same role as evaluation maps; of course, elements of $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ are not maps but equivalence classes of maps. We will also see that $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$, contrary to $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, is finitely generated.

Definition 4.6 A class $\left[e_{0}\right] \in \mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is called an evaluation element iff there exists $e \in\left[e_{0}\right]$ such that for every $\left[f_{0}\right] \in \mathcal{M}_{\text {poly }}^{\mathrm{P}}$, there exists $u \in\{0,1\}^{*}$ such that $[e(\operatorname{code}(u) 11(\cdot))]=\left[f_{0}(\cdot)\right]$.

Here code $(u) 11(\cdot)$ denotes the function $x \in\{0,1\}^{*} \longmapsto \operatorname{code}(u) 11 x$.

Equivalently,
$\left(\exists e \in\left[e_{0}\right]\right)\left(\forall\left[f_{0}\right] \in \mathcal{M}_{\text {poly }}^{\mathrm{P}}\right)\left(\exists f \in\left[f_{0}\right]\right)\left(\exists u \in\{0,1\}^{*}\right)(\forall x \in \operatorname{Dom}(f)): \quad e(\operatorname{code}(u) 11 x)=f(x)$.
The function code(.) was defined in the Introduction (just before the definition of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ ).
Theorem 4.7 The monoid $\mathcal{M}_{\mathrm{poly}}^{\mathrm{P}}$ has evaluation elements and is finitely generated.
Proof. For any polynomial $q$ of the form $q(n)=a \cdot\left(n^{d}+1\right)$ with $d>1$ and $a \geq 3$, we consider the evaluation function $\operatorname{evalR_{q}^{C}}$ defined by $\operatorname{eval} \mathrm{R}_{q}^{C}(\operatorname{code}(u) 11 x z)=\phi_{u}(x) z$; here $u$ is any program with linear time-complexity and I/O-balance (with coefficient $\leq 3$ ), $x \in \operatorname{domC}\left(\phi_{u}\right)$, and $z \in A^{*}$. By Prop. 4.5. every $\phi_{v} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ is $\equiv_{\text {poly-equivalent to }}$ some $\phi_{u} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that $\phi_{u}$ has time-complexity and I/O-balance less than the function $n \mapsto 3 n$; thus, $\left[\operatorname{evalR}_{q}^{C}\right]$ is an evaluation element.

Defining $\operatorname{evR}_{q}^{C}$ by $\operatorname{evR} R_{q}^{C}(\operatorname{code}(u) 11 x z)=\operatorname{code}(u) 11 \phi_{u}(x) z$, we also have

$$
\phi_{u}=\pi_{|\operatorname{code}(u) 11|}^{\prime} \circ \operatorname{evR}_{q}^{C} \circ \pi_{\operatorname{code}(u) 11} .
$$

So, $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is generated by $\left\{\left[\pi_{0}\right],\left[\pi_{1}\right],\left[\pi_{1}^{\prime}\right]\right.$, $\left.\left[\operatorname{evR}_{q}^{C}\right]\right\}$.
We saw that when poly $\subseteq \mathcal{T}$ then $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$ is a homomorphic image of $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$. Hence we have:
Corollary 4.8 If poly $\subseteq \mathcal{T}$ then $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$ is finitely generated.
We do not know whether $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is regular (and this is equivalent to $\mathrm{P}=\mathrm{NP}$ by Theorem 5.16), but for $\mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$ we can prove:

Proposition 4.9 The monoid $\mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$ is regular.
Proof. Consider $[f] \in \mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$, i.e., an $\equiv_{E 3}$-class in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ for some $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. Suppose $f$ has I/O balance and time-complexity $\leq T$ for some polynomial $T$. To show that $f$ has an inverse with elementary recursive I/O balance and time-complexity, let $y \in \operatorname{Im}(f)$ and consider all words $x$ of length $|x| \leq T(|y|)$; for each such $x$ we test whether $x \in \operatorname{Dom}(f)$, and (if so) we compute $f(x)$, in time $\leq T(|x|)$. On input $y$ we output the first $x$ in length-lexicographic order such that $f(x)=y$. All this takes time $\leq|A|^{\ell} \cdot T(\ell)$, where $\ell$ is the minimum length of $x \in f^{-1}(y)$; so $\ell \leq T(|y|)$ (by I/O-balance). The bound $\tau(|y|)=|A|^{T(|y|)} \cdot T(T(|y|))$ is elementary recursive, and testing whether $y \in \operatorname{Im}(f)$ is also elementary recursive, since $\operatorname{Im}(f) \in \mathrm{NP} \subset \mathrm{E}_{3}$. So $f$ has an inverse $f^{\prime}$ with elementary recursive I/O balance and time-complexity.

Let $\tau(n)$ be a fully time-constructible elementary recursive upper bound on $|A|^{T(n)} \cdot T(T(n))$ and on the time it takes to test whether $y \in \operatorname{Im}(f)$ (when $|y|=n$ ). The function $n \mapsto 2^{n}$ is fully timeconstructible, and if $T(n)=a \cdot\left(n^{d}+1\right)$ then $T(T(n))$ has an upper bound that has that form too. Moreover, the product of fully time-constructible functions is fully time-constructible.

So $f^{\prime}$ has balance and time-complexity bounded by $\tau$. We use Lemma $2.22(2)$ in the same way as in the proof of regularity of $\mathcal{M}_{\mathrm{end}}^{\mathrm{P}}$ (Theorem 2.23); we pad $f^{\prime}$ by taking the restriction $F^{\prime}$ of $f^{\prime}$ to

$$
\operatorname{Dom}\left(F^{\prime}\right)=\bigcup_{y \in \operatorname{domC}\left(f^{\prime}\right)} y A^{|y| \cdot \tau(|y|)} A^{*}
$$

And we restrict $f$ to $F=f \circ F^{\prime} \circ f=\operatorname{id}_{\operatorname{Dom}\left(F^{\prime}\right)} \circ f$. Then $F F^{\prime} F=F$, and $F$ and $F^{\prime}$ have linear time-complexity and balance (by Lemma 2.22(2)), hence $F, F^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. Moreover, $f \equiv_{E 3} F$ and $f^{\prime} \equiv_{E 3} F^{\prime}$, since $A^{|y| \cdot \tau(|y|)}$ is a maximal prefix code and $n \rightarrow n \cdot \tau(n)$ is elementary recursive. So [ $F^{\prime}$ ] is an inverse of $[f](=[F])$ in $\mathcal{M}_{\mathrm{E} 3}^{\mathrm{P}}$.

## 5 Inverses in $\mathcal{M}_{\text {poly }}^{P}$

In this section we study the regular elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ and of $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$, and we eventually show that $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular iff $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is regular.

### 5.1 Properties of inverses in $\mathcal{R} \mathcal{M}^{P}$

Here are a few useful facts about inverses that were not proved in [4], 3].
Lemma 5.1 For any right-ideal morphism $g: \quad g^{-1}(\mathrm{imC}(g)) \subseteq \operatorname{domC}(g)$.
Proof. If $x \in g^{-1}(\operatorname{imC}(g))(\subseteq \operatorname{Dom}(g))$, then $x=p w$ for some $p \in \operatorname{domC}(g)$ and $w \in A^{*}$; hence, $g(x)=g(p) w$, and $g(x) \in \operatorname{imC}(g)$. Since $g(p) w \in \operatorname{imC}(g)$ and $g(p) \in \operatorname{Im}(g)$, we have $g(p) w=g(p)$ (since $\operatorname{imC}(g)$ is a prefix code, and $g(p) w \in \operatorname{imC}(g)$ cannot have a strict prefix in the right ideal generated by $\operatorname{imC}(g))$. So, $w=\varepsilon$, hence $x=p w=p \in \operatorname{domC}(g)$.

Lemma 5.2 For every right-ideal morphism $g$ we have:
(1) $\quad \operatorname{imC}(g) \subseteq g(\operatorname{domC}(g))$.
(2) If $g$ is injective then $\operatorname{imC}(g)=g(\operatorname{domC}(g))$.
(3) If $g^{\prime}$ is an inverse of $g$ and if $\operatorname{Dom}\left(g^{\prime}\right)=\operatorname{Im}(g)$, then the inverse $g^{\prime}$ is injective.

Proof. (1) By applying $g$ to the inclusion in Lemma 5.1 we obtain: $g g^{-1}(\mathrm{imC}(g)) \subseteq g(\operatorname{domC}(g))$; since $g g^{-1}=\mathrm{id}_{\operatorname{Im}(g)}$, the result follows.
(2) When $g$ is injective, let $g(x) \in g(\operatorname{domC}(g))$ with $x \in \operatorname{domC}(g)$. Then $g(x) \in \operatorname{Im}(g)=\operatorname{imC}(g) A^{*}$; so $g(x)=u v$ for some $u \in \operatorname{imC}(g), v \in A^{*}$. Let $z \in \operatorname{Dom}(g)$ be such that $g(z)=u$; then $z=s t$ for some $s \in \operatorname{domC}(g), t \in A^{*}$. Hence, $g(x)=u v=g(z) v=g(z v)=g(s t v)$. Since $g$ is injective, this implies that $x=s t v$, so $s$ and $x$ are prefix-comparable. But then $s=x$, since $x$ and $s$ belong to the prefix code $\operatorname{domC}(g)$. Therefore $t=v=\varepsilon$. It follows that $g(x)=u v=u \in \operatorname{imC}(g)$, so $g(x) \in \operatorname{imC}(g)$. Thus, $g(\operatorname{domC}(g)) \subseteq \operatorname{imC}(g)$ when $g$ is injective.
(3) For all $y_{1}, y_{2} \in \operatorname{Dom}\left(g^{\prime}\right)=\operatorname{Im}(g), g^{\prime}\left(y_{i}\right) \in g^{-1}\left(y_{i}\right)(i=1,2)$. If $y_{1} \neq y_{2}$ then $g^{-1}\left(y_{1}\right)$ is disjoint from $g^{-1}\left(y_{2}\right)$, so $g^{\prime}\left(y_{1}\right) \neq g^{\prime}\left(y_{2}\right)$.
Proposition 5.3 If $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ (or $\in \mathrm{fP}$ ) is regular then $f$ has an injective inverse $f^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ (respectively $\in \mathrm{fP}$ ) with the additional property that $\operatorname{Dom}\left(f^{\prime}\right)=\operatorname{Im}(f)$.
Proof. Let $F^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ (or $\in \mathrm{fP}$ ) be an inverse of $f$, so $\operatorname{Im}(f) \subseteq \operatorname{Dom}\left(F^{\prime}\right)$. Since $f$ is regular we know (by Prop. 1.9 in [4]) that $\operatorname{Im}(f)$ is in P . Hence the restriction $f^{\prime}=\left.F^{\prime}\right|_{\operatorname{Im}(f)}$ belongs to $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ (respectively to fP ). Moreover, the restriction of an inverse of $f$ to $\operatorname{Im}(f)$ is always an injective inverse of $f$ (by Lemma 5.2(3)).
As a consequence of Prop. 5.3 we have:
Corollary 5.4 Every regular $\mathcal{D}$-class of fP and of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ contains injective partial functions.
Question: Do non-regular $\mathcal{D}$-classes also contain injective partial functions (that are thus not regular)? This is equivalent to the existence of injective one-way functions.

## Lemma 5.5.

(1) For every right-ideal morphism $f: A^{*} \rightarrow A^{*}$ and every prefix code $P \subset A^{*}$ we have: $f^{-1}(P)$ is a prefix code, and $f^{-1}(P) A^{*} \subseteq f^{-1}\left(P A^{*}\right)$.
(2) There exists $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ and a prefix code $P$ such that $f^{-1}(P) A^{*} \neq f^{-1}\left(P A^{*}\right)$.
(3) There exists $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ and a prefix code $P$ such that $f(P)$ is not a prefix code.

Proof. (1) If $x_{1}$ is a prefix of $x_{2}=x_{1} u$, with $x_{1}, x_{2} \in f^{-1}(P)$, then $f\left(x_{1}\right)$ and $f\left(x_{2}\right)=f\left(x_{1}\right) u$ both belong to the prefix code $P$, hence $f\left(x_{1}\right)=f\left(x_{1}\right) u$, hence $u=\varepsilon$. Now $x_{2}=x_{1} u$ implies $x_{2}=x_{1}$. So, in $f^{-1}(P)$, prefix-related words are equal, hence $f^{-1}(P)$ is a prefix code.

Obviously, $f^{-1}(P) \subseteq f^{-1}\left(P A^{*}\right)$. Moreover, $P A^{*}$ is a right ideal, hence $f^{-1}\left(P A^{*}\right)$ is a right ideal. Therefore, $f^{-1}(P) A^{*} \subseteq f^{-1}\left(P A^{*}\right) A^{*}=f^{-1}\left(P A^{*}\right)$.
(2) Example: Let $f\left(0^{n} 1\right)=0^{n}$ for all $n \geq 0$, with $\operatorname{domC}(f)=0^{*} 1$, and $\operatorname{imC}(f)=\{\varepsilon\}$. Let $P=\{\varepsilon\}$. Then $f^{-1}(P) A^{*}=f^{-1}(\{\varepsilon\})\{0,1\}^{*}=1\{0,1\}^{*}$, and $f^{-1}\left(P A^{*}\right)=f^{-1}\left(\{0,1\}^{*}\right)=0^{*} 1\{0,1\}^{*}$.
(3) Example: For $f$ as in (2), let $P=0^{*} 1$. We obtain $f(P)=0^{*}$, which is not a prefix code.

Definition 5.6 (normal morphism). A right-ideal morphism $f$ is called normal iff $f(\operatorname{domC}(f))=$ $\operatorname{imC}(f)$.

Thus, $f$ is normal iff its restriction to domC $(f)$ maps into (hence onto) $\operatorname{imC}(f)$; in other words, $f$ is entirely defined by the way it relates $\operatorname{domC}(f)$ to $\operatorname{imC}(f)$. On the other hand, a non-normal right-ideal morphism $g$ will map domC $(g)$ to a larger set than $\operatorname{imC}(g)$, i.e., $\operatorname{imC}(g) \varsubsetneqq g(\operatorname{domC}(g))$.
Examples of normal and non-normal right-ideal morphisms:
Every injective right-ideal morphism is normal (by Lemma 5.2).
The encodings of the elements of fP are normal; we saw near the beginning of the Introduction that for all $f \in \mathrm{fP}, f^{C}$ is normal.

The following is a non-normal regular element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ : Let $\operatorname{dom} \mathrm{C}(g)=01^{*}$; and let $g\left(0^{n} 1 w\right)=$ $0^{n} w$ for all $n \geq 0$ and all $w \in\{0,1\}^{*}$. Then $\operatorname{imC}(g)=\{\varepsilon\} \neq 0^{*}=g(\operatorname{domC}(g)$. So in this example, $\operatorname{imC}(g)$ and $g(\operatorname{domC}(g)$ are extremely different.

Lemma 5.7 A right-ideal morphism $f$ is normal iff $\operatorname{domC}(f)=f^{-1}(\operatorname{imC}(f))$.
Proof. The right-to-left implication is trivial since $f f^{-1}=\mathbf{1}_{\operatorname{lm}(f)}$.
Conversely, let us assume normality, i.e., $f(\operatorname{domC}(f))=\operatorname{imC}(f)$. Then $f^{-1}(\operatorname{imC}(f)) \subseteq \operatorname{domC}(f)$ by Lemma 5.1. To prove that $\operatorname{domC}(f) \subseteq f^{-1}(\operatorname{imC}(f))$, let $x \in \operatorname{domC}(f)$. Then $f(x) \in f(\operatorname{domC}(f))=$ $\operatorname{imC}(f)$; the latter equality holds by the assumption of normality. So, $f(x) \in \operatorname{imC}(f)$, hence $x \in$ $f^{-1}(\operatorname{imC}(f))$. Thus, $\operatorname{domC}(f) \subseteq f^{-1}(\operatorname{imC}(f))$.
Proposition 5.8 (1) The set $\left\{f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}: f\right.$ is normal $\}$ is not closed under composition, i.e., it is not a submonoid. In fact, there exist regular normal elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ whose composite is regular but not normal.
(2) There exist a regular normal $g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ and a prefix code $P \subset \operatorname{Dom}(g)$ with $P \equiv_{\text {poly }} \operatorname{domC}(g)$, such that $g(P)$ is not a prefix code.

Proof. (1) This is shown by the following example. Let $f, g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ be defined by $\operatorname{domC}(f)=\{0,1\}$ and $f(0)=0, f(1)=10 ; \operatorname{domC}(g)=\{0,1\}$ and $g(0)=g(1)=0$. Then $f(\operatorname{domC}(f))=\operatorname{imC}(f)=$ $\{0,10\}$, and $g(\operatorname{domC}(g))=\operatorname{imC}(g)=\{0\}$; so, $f$ and $g$ are normal. Now, $\operatorname{domC}(g f)=\{0,1\}$ and $g f(0)=0, g f(1)=00$. So, $g f(\operatorname{domC}(g f))=\{0,00\}$, which is not a prefix code; and $\operatorname{imC}(g f)=\{0\}$. Thus, $g f$ is not normal.
(2) Take $g$ as above, and $P=\{0,10,11\}$. Then $g(P)=\{0,00,01\}$, which is not a prefix code.

## Miscellaneous

The remaining Definition and Facts of this subsection will not be used in the rest of the paper.
Definition 5.M1 For any right-ideal morphism $f: A^{*} \rightarrow A^{*}$, the normalization $f_{N}$ of $f$ is the restriction of $f$ to $f^{-1}(\mathrm{imC}(f)) A^{*}$. So, $\operatorname{domC}\left(f_{N}\right)=f^{-1}(\operatorname{imC}(f))(\subseteq \operatorname{domC}(f))$.
Then $f_{N}$ is normal: $f_{N}\left(\operatorname{domC}\left(f_{N}\right)=f\left(\operatorname{domC}\left(f_{N}\right)\right)=f\left(f^{-1}(\operatorname{imC}(f))=\operatorname{imC}(f)\right.\right.$, and this is equal to $\operatorname{imC}\left(f_{N}\right)$, by Prop. 5.M2 (next). Moreover, $f^{-1}(\mathrm{imC}(f))$ is a prefix code (by Lemma 5.5)(1)).

Note that $f_{N} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ iff $f^{-1}(\operatorname{imC}(f)) \in \mathrm{P}$. Indeed, if $f_{N} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ then $\operatorname{domC}\left(f_{N}\right) \in \mathrm{P}$; and if $\operatorname{domC}\left(f_{N}\right) \in \mathrm{P}$ then the restriction of $f\left(\in \mathcal{R} \mathcal{M}^{\mathrm{P}}\right)$ is in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$.

We conjecture that $f_{N}$ is not always in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ when $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$; this conjecture is motivated by Prop. 5.M3 below.
Proposition 5.M2 For any right-ideal morphism $f$ and its normalization $f_{N}$, we have: $\operatorname{lm}\left(f_{N}\right)=$ $\operatorname{Im}(f)$, and $\operatorname{imC}\left(f_{N}\right)=\operatorname{imC}(f)$.
Proof. Obviously, $\operatorname{Im}\left(f_{N}\right) \subseteq \operatorname{Im}(f)$. Conversely, let $y \in \operatorname{Im}(f)$, so $y=q w$ for some $q \in \operatorname{imC}(f)$ and $w \in A^{*}$. Then $q=f(p)$ for some $p \in \operatorname{domC}(f) \cap f^{-1}(\operatorname{imC}(f))=\operatorname{domC}\left(f_{N}\right)$, hence $y=f(p) w=$ $f(p w) \in \operatorname{Im}\left(f_{N}\right)$.

We know that for all $f \in \mathrm{fP}, \operatorname{Dom}(f)$ is in P and $\operatorname{Im}(f)$ is in NP (Prop. 1.9 in [4). What can be said about the complexity of $\operatorname{imC}(f)$ ? The complexity class $\mathrm{DP}\left(\subseteq \Delta_{2}^{\mathrm{P}}=\mathrm{P}^{N P}\right)$ is defined by

$$
\mathrm{DP}=\left\{L_{1}-L_{0}: L_{1}, L_{0} \in \mathrm{NP}\right\} .
$$

Obviously, $\mathrm{NP} \cup$ coNP $\subseteq$ DP. There exist DP-complete problems, e.g., the following: critical3SAT $=$ $\{\beta: \beta$ is a boolean formula in 3 cnf that is not satisfiable, but for every clause $c$ in $\beta$, the removal of $c$ results in a boolean formula $\beta-\{c\}$ that is satisfiable $\}$. See e.g. [14].
Proposition 5.M3 For all $f \in \mathrm{fP}, \operatorname{imC}(f)$ is in DP , and when $f$ is normal then $\operatorname{imC}(f)$ is in NP . Proof. We have $x \in \operatorname{imC}(f)$ iff the following hold: (1) $x \in \operatorname{Im}(f)$, and (2) for every strict prefix $p$ of $x: p \notin \operatorname{Im}(f)$. The second condition is equivalent to $\operatorname{not}(\exists p)\left[p<_{\text {pref }} x\right.$ and $\left.p \in \operatorname{Im}(f)\right]$. Hence, $\operatorname{imC}(f) \in \mathrm{DP}$.

However since $\operatorname{domC}(f)$ is in $\mathrm{P}, f(\operatorname{domC}(f))$ is in NP; so when $f$ is normal then $\operatorname{imC}(f)(=$ $f(\operatorname{domC}(f)))$ is in NP.

Normalization works well with inverses:
Proposition 5.M4 For any right-ideal morphism $f$ and its normalization $f_{N}$ we have:
(1) If $f_{1}^{\prime}$ is an inverse of $f_{N}$ then $f_{1}^{\prime}$ is also an inverse of $f$.
(2) If $f^{\prime}$ is any inverse of $f$ then the restriction $F^{\prime}$ of $f^{\prime}$ to $\operatorname{Im}(f)$ is an injective (hence normal) inverse of $f$. Moreover, if $f, f^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ then $F^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$.
Proof. (1) Let us show that $f_{1}^{\prime}$ is an inverse of $f$. Since $\operatorname{Im}(f)=\operatorname{Im}\left(f_{N}\right)$, for all $x \in \operatorname{Dom}(f)$ there exists $x_{1} \in \operatorname{Dom}\left(f_{N}\right)$ such that $f(x)=f_{N}\left(x_{1}\right)$. Now, $f(x)=f_{N}\left(x_{1}\right)=f_{N} f_{1}^{\prime} f_{N}\left(x_{1}\right)=f_{N} f_{1}^{\prime} f(x) \subseteq$ $f f_{1}^{\prime} f(x)$; the latter holds since $f_{N}$ is a restriction of $f$. But since $f f_{1}^{\prime} f$ is a function, and $f_{N} f_{1}^{\prime} f(x)$ is defined, we have $f_{N} f_{1}^{\prime} f(x)=f f_{1}^{\prime} f(x)$. Thus, $f(x)=f f_{1}^{\prime} f(x)$.
(2) The only part of the domain of an inverse of $f$ that matters (in the relation $f f^{\prime} f=f$ ) is $\operatorname{Im}(f)$ (which is always a subset of $\operatorname{Dom}\left(f^{\prime}\right)$ ). So the restriction $F^{\prime}$ of $f^{\prime}$ to $\operatorname{Im}(f)$ is an inverse of $f$. For any inverse $f^{\prime}$ of $f$ we have: $f^{\prime}(\mathrm{imC}(f)) \subseteq f^{-1}(\mathrm{imC}(f))$, which is a prefix code by Lemma $5.5(1)$; so, $F^{\prime}$ is normal. For any inverse, the restriction to $\operatorname{Im}(f)$ is injective, since for all $y_{1} \neq y_{2}$ in $\operatorname{Im}(f), f^{-1}\left(y_{1}\right)$ and $f^{-1}\left(y_{2}\right)$ are disjoint.

If $f, f^{\prime} \in \mathcal{R} \mathcal{M}_{2}^{P}$ then $f$ is regular, so by Prop. 1.9 in [4], $\operatorname{Im}(f)$ is in P . Hence the restriction of $f^{\prime}$ to $\operatorname{Im}(f)$ is in $\mathcal{R M}_{2}^{\mathrm{P}}$.

We know (Prop. 6.1 in 4]) that every element of fP , and in particular, every element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, has an inverse in $\mathrm{fP}^{N P}$. We show next that every element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ has an inverse in $\mathcal{R} \mathcal{M}^{\mathrm{NP}}$. We first extend Prop. 2.6 of $\left[4\right.$ to $f^{N P}$ and to $\mathcal{R} \mathcal{M}^{\mathrm{NP}}$.
Lemma 5.M5 If an element $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ has an inverse in fP (or in $\mathrm{fP}^{\mathrm{NP}}$ ), then $f$ also has an inverse in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ (respectively in $\mathcal{R} \mathcal{M}^{\mathbf{N P}}$ ).

Moreover, this inverse in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ (resp. $\mathrm{fP}^{\mathrm{NP}}$ ) can be chosen to be injective (and hence normal).
Proof. If $f$ has an inverse in fP then the result was proved in Prop. 2.6 of 44 .
Let $f_{0}^{\prime} \in \mathrm{fP}^{\mathrm{NP}}$ be an inverse of $f$; we want to construct an inverse $f^{\prime}$ of $f$ that belongs to $\mathcal{R} \mathcal{M}^{\mathrm{NP}}$. We know (Prop. 1.9 of [4]) that $\operatorname{Im}(f)$ is in NP. Hence we can restrict $f_{0}^{\prime}$ to $\operatorname{Im}(f)$, i.e., $\operatorname{Dom}\left(f_{0}^{\prime}\right)=\operatorname{Im}(f)$. We proceed to define $f^{\prime}(y)$ for $y \in \operatorname{Im}(f)$.

First, we compute the shortest prefix $p$ of $y$ that satisfies $p \in \operatorname{Dom}\left(f_{0}^{\prime}\right)=\operatorname{Im}(f)$. Since $\operatorname{Im}(f) \in$ NP, this can be done in polynomial time with calls to an NP oracle. Now, $y=p z$ for some string $z$.

Second, we define $f^{\prime}(y)=f_{0}^{\prime}(p) z$, where $p$ and $z$ are as above. Thus, $f^{\prime}$ is a right-ideal morphism.
Let us verify that $f^{\prime}$ has the claimed properties. Clearly, $f^{\prime}$ is computable in polynomial time with calls to an NP oracle, and is polynomially balanced (the latter following from the fact that $f^{\prime}$ is an inverse of $f$, which we prove next); thus, $f^{\prime}$ is a right-ideal morphism in $\mathrm{fP}^{\mathrm{NP}}$, so $f^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{NP}}$. To prove that $f^{\prime}$ is an inverse of $f$, let $x \in \operatorname{Dom}(f)$. Then $f\left(f^{\prime}(f(x))\right)=f\left(f^{\prime}(p z)\right)$, where $y=f(x)=p z$, and $p$ is the shortest prefix of $y$ such that $p \in \operatorname{Im}(f)$. Then, $f^{\prime}(p z)=f_{0}^{\prime}(p) z$, by the definition of $f^{\prime}$.

Then, since $f$ is a right-ideal morphism, $f\left(f_{0}^{\prime}(p) z\right)=f\left(f_{0}^{\prime}(p)\right) z=p z$ (the latter since $f_{0}^{\prime}$ is an inverse of $f$, and since $p \in \operatorname{Im}(f))$. Hence, $\left.f f^{\prime}\right|_{\operatorname{Im}(f)}=\mathbf{1}_{\operatorname{Im}(f)}$. Thus, $f^{\prime}$ is an inverse of $f$.

Note that since $\operatorname{Dom}\left(f^{\prime}\right)=\operatorname{Im}(f)$, the inverse $f^{\prime}$ described above is injective. Indeed, if $\operatorname{Dom}\left(f^{\prime}\right)=$ $\operatorname{Im}(f)$ then $f^{\prime}=\left.f^{\prime}\right|_{\operatorname{Im}(f)}$, so $f f^{\prime}=\mathbf{1}_{\operatorname{Dom}\left(f^{\prime}\right)}$, which implies that $f^{\prime}$ is injective (hence normal by Lemma 5.21).

Proposition 5.M6 Every element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ has an inverse in $\mathcal{R} \mathcal{M}^{\mathrm{NP}}$, and this inverse can be chosen to be injective (and hence normal).
Proof. By Prop. 6.1 in [4], every element of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ has an inverse in $\mathrm{fP}^{\mathrm{NP}}$. The result then follows from Lemma 5.M5.

Proposition 5.M7 Let $f_{0} \in \mathcal{R M}^{\mathrm{P}}$, and let $f$ be any right-ideal morphism such that $f \equiv_{\text {poly }} f_{0}$. Then $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ iff $\operatorname{Dom}(f) \in \mathrm{P}$.

Hence, if $f_{0} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$, and $\operatorname{Dom}(f) \in \mathrm{P}$, and $f \notin \mathcal{R} \mathcal{M}^{\mathrm{P}}$, then $f \not \equiv_{\text {poly }} f_{0}$.
Proof. We know that for all $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}, \operatorname{Dom}(f) \in \mathrm{P}$. For the converse, if $x \in \operatorname{Dom}(f)$ (which can be checked in polynomial time), then either $x \in \operatorname{Dom}\left(f_{0}\right)$ or $x$ is a prefix of a word $x u \in \operatorname{domC}\left(f_{0}\right)$. If $x \in \operatorname{Dom}\left(f_{0}\right)$ we can immediately compute $f_{0}(x)(=f(x))$, using the polynomial-time algorithm of $f_{0}$.

If $x u \in \operatorname{domC}\left(f_{0}\right)$ for some $u \in A^{*}$, we can compute $f_{0}(x u)$ in polynomial time (as a function of $|x u|)$. Here, $u$ is the shortest word such that $x u \in \operatorname{domC}\left(f_{0}\right)$. So, $|u|$ is polynomially bounded in terms of $|x|$ (because domC $(f) \equiv_{\text {poly }}$ domC $\left(f_{0}\right)$ ). Therefore, the computation of $f_{0}(x u)$ takes polynomial time as a function of $|x|$.

Also, $f_{0}(x u)=f(x u)=f(x) u$; so we obtain $f(x)$ by removing the suffix $u$ from $f_{0}(x u)$; we know $u$, since it is the shortest word such that $x u \in \operatorname{dom} \mathrm{C}\left(f_{0}\right)$ (and $\operatorname{domC}\left(f_{0}\right) \in \mathrm{P}$ when $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ ).

## Proposition 5.M8

(1) There exist prefix codes $P_{1}, P_{0} \subset A^{*}$ such that $P_{1} \equiv_{\text {poly }} P_{0}$, and $P_{0} \in \mathrm{P}$, but $P_{1} \notin \mathrm{P}$. The prefix code $P_{1}$ can be chosen to have any complexity above polynomial, or to be undecidable; if $\mathrm{P} \neq \mathrm{NP}$ then $P_{1}$ can be chosen in DP.
(2) There exist right ideal morphisms $f_{1}, f_{0}$ such that $f_{1} \equiv_{\text {poly }} f_{0}$, and $f_{0} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$, but $f_{1} \notin \mathcal{R} \mathcal{M}^{\mathrm{P}}$. If $\mathrm{P} \neq \mathrm{NP}$ then $f_{1}$ can be chosen in $\mathcal{R} \mathcal{M}^{\mathbf{N P}}$.
Proof. (1) We construct a family of examples. Let $L \subset A^{*}$ be any set that is not in P. Let

$$
\begin{aligned}
& P_{0}=\{00,01\}^{*} 11, \quad \text { and } \\
& P_{1}=\{\operatorname{code}(x) 11: x \in L\} \cup\{\operatorname{code}(x) 110: x \notin L\} \cup\{\operatorname{code}(x) 111: x \notin L\} .
\end{aligned}
$$

Then $P_{1}, P_{0}$ are prefix codes, $P_{1} \equiv_{\text {poly }} P_{0}$, and $P_{0} \in \mathrm{P}$. But $P_{1} \notin \mathrm{P}$ since $L$ is polynomial-time reducible to $P_{1}$.
(2) Let $f_{0}, f_{1}$ be the identity map restricted to $P_{0} A^{*}$, respectively $P_{1} A^{*}$ (with $P_{0}, P_{1}$ as above). Then $f_{1} \equiv_{\text {poly }} f_{0}$, and $f_{0} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$; but $f_{1} \notin \mathcal{R} \mathcal{M}^{\mathrm{P}}$ since $\operatorname{dom} \mathrm{C}\left(f_{1}\right)=P_{1} \notin \mathrm{P}$. If $\mathrm{P} \neq \mathrm{NP}$ then $L$ can be chosen in NP -P , and then $f_{1} \notin \mathcal{R} \mathcal{M}^{\mathrm{P}}$.

Question: Assuming $\mathrm{P} \neq \mathrm{NP}$, is there $F \in \mathcal{M}_{\text {poly }}^{\mathrm{P}}$ such that for all $f \in F: \operatorname{Dom}(f) \notin \mathrm{P}$ ?

## $5.2 \mathcal{M}_{\text {poly }}^{\mathrm{P}}$ vs. $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, regarding regularity

It is obvious that if $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular then $\mathcal{M}_{\text {poly }}^{\mathrm{P}}\left(=\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\text {poly }}\right)$ is regular, being a homomorphic image of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. The converse is also true, but the proof is not obvious, mainly because of the existence on non-normal functions in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. Many of the results of this sub-section hold for $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$ (where $\mathcal{T}$ is any family of functions as in Def. (3.1).

Lemma 5.9 If $f_{0}, f$ are right-ideal morphisms with $f_{0} \equiv_{\text {end }} f$ and $f_{0} \subseteq f$, then $\operatorname{imC}\left(f_{0}\right) \equiv_{\text {end }} \operatorname{imC}(f)$. If poly $\subseteq \mathcal{T}$ and $f_{0}, f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ satisfy $f_{0} \equiv_{\mathcal{T}} f$ and $f_{0} \subseteq f$, then $\operatorname{imC}\left(f_{0}\right) \equiv_{\mathcal{T}} \operatorname{imC}(f)$.

Compare with Lemma 3.20 .
Proof. Since $f_{0} \subseteq f$ we also have $\operatorname{Im}\left(f_{0}\right) \subseteq \operatorname{Im}(f)$. Suppose a right ideal $R$ intersects $\operatorname{Im}(f)$; so there exists $x \in \operatorname{Dom}(f)$ such that $f(x) \in R \cap \operatorname{Im}(f)$. Hence, $x \in f^{-1}(R \cap \operatorname{Im}(f))=f^{-1}(R) \cap \operatorname{Dom}(f)$; so the right ideal $f^{-1}(R)$ intersects $\operatorname{Dom}(f)$, hence by end-equivalence, $f^{-1}(R)$ also intersects $\operatorname{Dom}\left(f_{0}\right)$. So there exists $x_{0} \in f^{-1}(R) \cap \operatorname{Dom}\left(f_{0}\right)$, and this implies that $f\left(x_{0}\right) \in f f^{-1}(R) \cap f\left(\operatorname{Dom}\left(f_{0}\right)\right)=R \cap \operatorname{Im}\left(f_{0}\right)$. So, $R$ intersects $\operatorname{Im}\left(f_{0}\right)$.

For the second statement, let $y_{0} \in \operatorname{imC}\left(f_{0}\right)$ and $y \in \operatorname{imC}(f)$ be such that $y \leq_{\text {pref }} y_{0}=y w$ (for some $\left.w \in A^{*}\right)$. We want to show that $\left|y_{0}\right|$ and $|y|$ are related by some function in $\mathcal{T}$ that depends only on $f$ and $f_{0}$. Since $f_{0}^{-1}\left(\operatorname{imC}\left(f_{0}\right)\right) \subseteq \operatorname{domC}\left(f_{0}\right)$ and $f^{-1}(\operatorname{imC}(f)) \subseteq \operatorname{domC}(f)$ (by Lemma 5.11), there exists $x \in \operatorname{domC}(f)$ such that $y=f(x)$, and hence $y_{0}=f(x) w=f(x w)$; and $x w \in f_{0}^{-1}\left(\operatorname{imC}\left(f_{0}\right)\right) \subseteq$ $\operatorname{domC}\left(f_{0}\right)$. So, $x \in \operatorname{domC}(f)$ and $x w \in \operatorname{domC}\left(f_{0}\right)$, and $x w \geq_{\text {pref }} x$, hence $|x|$ and $|x w|$ are length-related by a function in $\mathcal{T}$ (because $f_{0} \equiv \mathcal{T} f$ ). Moreover, $|f(x)|$ and $|x|$ are polynomially related (because of the I/O-balance of $f$ ), and $\left|f_{0}(x w)\right|$ and $|x w|$ are polynomially related (because of the I/O-balance of $f_{0}$ ). Thus, $|y|$ and $\left|y_{0}\right|$ are length-related by a function in $\mathcal{T}$.

Lemma 5.10 Let $h, g$ be any right-ideal morphisms such that $h g h \equiv_{\mathrm{bd}} h$. Then $h g h \subseteq h$, and $h g h g h g h=h g h$.

Proof. For all functions we have $\operatorname{Dom}(h g h) \subseteq \operatorname{Dom}(h)$, so since $h g h \equiv_{\text {bd }} h$, we have $h g h \subseteq h$. Hence, for all $x \in \operatorname{Dom}(h g h)$ we have $h g h(x)=h(x)$, and $h g$ is defined on $h(x)$. Since $h g h(x)=h(x)$ and $h g$ is defined on $h(x), h g$ is defined on $h g h(x)$, and we have $\operatorname{hghgh}(x)=h g h(x)=h(x)$ for all $x \in \operatorname{Dom}(h g h)$. By the same argument, $h g$ is defined on $\operatorname{hghgh}(x)$, on $h g h(x)$, and on $h(x)$, and we have: $\operatorname{hghghg}(x)=\operatorname{hghg}(x)=h g h(x)=h(x)$. In particular: $h g h g h g h(x)=h g h(x)$ for all $x \in \operatorname{Dom}(h g h)$.

Proposition 5.11 Let $F, G$ be any $\equiv_{\mathcal{T}}$-equivalence classes in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. Then we have:
(Inverses) $F G F=F$ iff there exist $f \in F$ and $g \in G$ such that $f g f=f$.
(Mutual inverses) $F G F=F$ and $G F G=G$ iff there exist $f \in F$ and $g \in G$ such that $f g f=f$ and $g f g=g$.

Proof. For the first statement: If $f \in F$ and $g \in G$ satisfy $f g f=f$ then $F G F=F$ since $\equiv_{\mathcal{T}}$ is a congruence, and $F=[f], G=[g]$. Conversely, if $F G F=F$ then for any $h \in F, g \in G$, we have $h g h \equiv \mathcal{T} h$. Hence, for any $g \in G$, letting $f=h g h \in F G F=F$ we have $f g f=f$ (by Lemma (5.10).

For the second statement: The right-to-left implication is obvious since $\equiv \mathcal{T}$ is a congruence. Conversely, $F G F=F$ implies $f g_{1} f=f$ for some $f \in F$ and $g_{1} \in G$ (by the "Inverses" statement of the Proposition, that we just proved). Let $g=g_{1} f g_{1} \in G F G=G$. Then $f g f=f g_{1} f g_{1} f=f g_{1} f=f$, and $g f g=g_{1} f g_{1} f g_{1} f g_{1}=g_{1} f g_{1} f g_{1}=g_{1} f g_{1}=g$.

Lemma 5.12 Let $P_{0}, P_{1} \subset A^{*}$ be prefix codes with $P_{0} A^{*} \subseteq P_{1} A^{*}$ and $P_{0} \equiv_{\mathcal{T}} P_{1}$; let $\tau \in \mathcal{T}$ be the function used for $P_{0} \equiv_{\mathcal{T}} P_{1}$. Then for every $y_{1} \in P_{1}$ and every $t \in A^{*}$ with $|t| \geq \tau\left(\left|y_{1}\right|\right): y_{1} t \in P_{0} A^{*}$.

Proof. By definition, $\equiv_{\mathcal{T}}$ implies $\equiv_{\text {bd }}$, so $P_{0} A^{\omega}=P_{1} A^{\omega}$ (by Prop. 3.2). Hence for all $y_{1} \in P_{1}, t \in A^{*}$, and $w \in A^{\omega}$ : the end $y_{1} t w$ intersects $P_{0}$, i.e., some prefix of $y_{1} t w$ is in $P_{0}$. If $|t| \geq \tau\left(\left|y_{1}\right|\right)$ then this prefix is a prefix of $y_{1} t$ (by the definition of $\equiv \mathcal{T}$ and the choice of $\tau$ ). Hence, $y_{1} t \in P_{0} A^{*}$.

Terminology: A normal inverse of a right-ideal morphism $f$ is any normal right-ideal morphism $f^{\prime}$ (i.e., $f^{\prime}\left(\operatorname{domC}\left(f^{\prime}\right)\right)=\operatorname{imC}\left(f^{\prime}\right)$, by Def. 5.6) such that $f^{\prime}$ is an inverse of $f$.

Lemma 5.13 Let $g, f$ be right-ideal morphisms such that $g \subseteq f$, and let $f^{\prime}$ be any inverse of $f$ such that $f^{\prime}(\operatorname{Im}(g)) \subseteq \operatorname{Dom}(g)$. Then $f^{\prime}$ is also an inverse of $g$.

Proof. For $x \in \operatorname{Dom}(g), f^{\prime} g(x)$ is defined, since $g \subseteq f$ and since $f(x)(=g(x))$ is defined, and $f^{\prime}$ is defined on $f(x)$. And $g f^{\prime} g(x)$ is defined since $f^{\prime}(\operatorname{lm}(g)) \subseteq \operatorname{Dom}(g)$. Hence, $g f^{\prime} g(x)=f f^{\prime} f(x)=$ $f(x)=g(x)$, since $g \subseteq f$.

Lemma 5.14 and Theorem 5.16 below are only proved for $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$; it is not clear for what other $\mathcal{M}_{\mathcal{T}}^{\mathrm{P}}$ they hold.

For the next Lemma, recall that if $g \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular then $g$ has a normal inverse $g^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$; in fact, we can choose $g^{\prime}$ to be injective such that $\operatorname{Dom}\left(g^{\prime}\right)=\operatorname{Im}(g)$ (see Prop. 5.3, Lemma 5.2, and Def. 5.6).

Main Lemma 5.14 (inverse of a $\equiv_{\text {poly }}$-equivalent extension). Suppose $f, f_{0} \in \mathcal{R M}^{\mathrm{P}}$ are such that $f_{0} \subseteq f$ and $f_{0} \equiv_{\text {poly }} f$. Suppose also that $f$ is normal. Then we have:
(1) If $f_{0}$ is regular then $f$ is regular.
(2) For every injective inverse $f_{0}^{\prime} \in \mathcal{R} \mathcal{M}^{\mathbf{P}}$ of $f_{0}$ such that $\operatorname{Dom}\left(f_{0}^{\prime}\right)=\operatorname{Im}\left(f_{0}\right)$, there exists an injective inverse $f_{1}^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ of $f$ such that $\operatorname{dom} \mathrm{C}\left(f_{0}^{\prime}\right) \equiv_{\text {poly }} \operatorname{dom} \mathrm{C}\left(f_{1}^{\prime}\right)$.
(3) Moreover, $f_{1}^{\prime}$ is also an inverse of $f_{0}$. But $f_{1}^{\prime}$ cannot always be chosen to be an extension of the given $f_{0}^{\prime}$.

Proof. (1) follows from (2), since every regular element $f_{0}$ of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ has an injective inverse $f_{0}^{\prime}$ satisfying $\operatorname{Dom}\left(f_{0}^{\prime}\right)=\operatorname{Im}\left(f_{0}\right)$ (by Prop. 5.3 and Lemma 5.2).
(2) Let $f_{0}^{\prime}$ be an inverse of $f_{0}$ as assumed, hence $f_{0}^{\prime}\left(\operatorname{imC}\left(f_{0}\right)\right) \subseteq f_{0}^{-1}\left(\operatorname{imC}\left(f_{0}\right)\right) \subseteq \operatorname{domC}\left(f_{0}\right)$ (the latter " $\subseteq$ " holds by Lemma 5.1).
Claim: If $f$ is normal then for all $y \in \operatorname{imC}(f)$ and all $t \in A^{*}: f^{-1}(y t)=f^{-1}(y) t$.
Proof of Claim: $[\subseteq]: x \in f^{-1}(y t)$ iff $f(x)=y t$. Since $x \in \operatorname{Dom}(f)$ we have $x=p w$ for $p \in \operatorname{domC}(f)$, $x w \in A^{*}$, so $f(x)=y t=f(p) w$; hence $f(p)$ and $y$ are prefix-comparable. By normality, $f(p) \in$ $\operatorname{imC}(f)$; hence $f(p)=y$ (since $y \in \operatorname{imC}(f)$ by assumption, and $\operatorname{imC}(f)$ is a prefix code). Thus, $f(x)=y t=f(p) w=y w$, so $w=t$ (since $y=f(p))$. Hence $x=p w=p t \in f^{-1}(y) t$.
[〕] (this holds also when $f$ is not normal): $x \in f^{-1}(y) t$ implies $f(x) \in f\left(f^{-1}(y) t\right)$. Since $f^{-1}(y) \subseteq$ $f^{-1}(\operatorname{imC}(f)) \subseteq \operatorname{domC}(f)$ (the latter " $\subseteq$ " holds by Lemma 5.1), we have $f\left(f^{-1}(y) t\right)=f\left(f^{-1}(y)\right) t=$ $\{y t\}$. Hence, $x \in f^{-1}(y t)$. [End, Proof of Claim]

Now let $y \in \operatorname{imC}(f)$, and let $t \in A^{*}$ be any string such that $y t \in \operatorname{imC}\left(f_{0}\right)$. Since $f_{0} \subseteq f$ and $f_{0} \equiv_{\text {poly }} f$, we have $\operatorname{imC}\left(f_{0}\right) \equiv_{\text {poly }} \operatorname{imC}(f)$ (by Lemma 5.9); hence, $|t| \leq q(|y|)$ for some polynomial $q$. And by Lemma 5.12 (with $\mathcal{T}=$ poly), we can pick $t$ to be $t=0^{q(|y|)}$; then $t$ can be computed from $y$ in polynomial time. Since $f_{0}^{\prime}(y t) \in f_{0}^{-1}(y t) \subseteq f^{-1}(y t)$, we have by the Claim: $f_{0}^{\prime}(y t) \in f^{-1}(y) t$. Since $f_{1}^{\prime}(y)$ should belong to $f^{-1}(y)$, we define:
$f_{1}^{\prime}(y)$ is is prefix of $f_{0}^{\prime}(y t)$ obtained by removing the suffix $t$.
Then, indeed, $f_{1}^{\prime}(y) \in f^{-1}(y)$. In general, for all $y \in \operatorname{imC}(f)$ and all $z \in A^{*}$, we define $f_{1}^{\prime}(y z)=f_{1}^{\prime}(y) z$. Then $f_{1}^{\prime}(y z) \in f^{-1}(y z)$, hence $f_{1}^{\prime}$ is an inverse of $f$. By construction, $\operatorname{dom} C\left(f_{1}^{\prime}\right)=\operatorname{imC}(f)$, hence $f_{1}^{\prime}$ is injective (by Lemma $5.2(3))$. And $f_{1}^{\prime}(y)$ is polynomial-time computable, since $t=0^{q(|y|)}$ and since $f_{0}^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$. Finally, $f_{1}^{\prime}$ is polynomially balanced, since $f_{0}^{\prime}$ is polynomially balanced and $|t| \leq q(|y|)$.

By Lemma [5.9, $\operatorname{imC}\left(f_{0}\right) \equiv_{\text {poly }} \operatorname{imC}(f)$. Hence, $\operatorname{domC}\left(f_{0}^{\prime}\right)=\operatorname{imC}\left(f_{0}\right) \equiv_{\text {poly }} \operatorname{imC}(f)=\operatorname{domC}\left(f_{1}^{\prime}\right)$, so $\operatorname{domC}\left(f_{0}^{\prime}\right) \equiv_{\text {poly }} \operatorname{domC}\left(f_{1}^{\prime}\right)$.
(3) By Lemma 5.13, $f_{1}^{\prime}$ is an inverse of $f_{0}$. In order to apply Lemma 5.13 we need to check that $f_{1}^{\prime}\left(\operatorname{Im}\left(f_{0}\right)\right) \subseteq \operatorname{Dom}\left(f_{0}\right)$. For all $y z \in \operatorname{imC}\left(f_{0}\right)$ (with $y \in \operatorname{imC}(f)$ ) we have $f_{1}^{\prime}(y z) \in f^{-1}(y z)$; and $f^{-1}(y z)=f_{0}^{-1}(y z)$ when $y z \in \operatorname{imC}\left(f_{0}\right)$. Moreover, $f_{0}^{-1}(y z) \subseteq \operatorname{Dom}\left(f_{0}\right)$. Hence, $f_{1}^{\prime}\left(\operatorname{imC}\left(f_{0}\right)\right) \subseteq$ $\operatorname{Dom}\left(f_{0}\right)$, and thus $f_{1}^{\prime}\left(\operatorname{Im}\left(f_{0}\right)\right) \subseteq \operatorname{Dom}\left(f_{0}\right)$.

By Lemma 5.15 below, the inverse $f_{0}^{\prime}$ of $f_{0}$ is not necessarily a restriction of an inverse of $f$. So $f_{0}^{\prime}$ is not always extendable to an inverse of $f$.

Lemma 5.15 There exist $f, g \in \mathcal{R M}^{\mathbf{P}}$ such that $g \subseteq f, g \equiv_{\text {poly }} f$, and $g$ is regular, but such that not every inverse $g^{\prime} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ (not even every injective inverse) of $g$ is extendable to an inverse of $f$.

Proof. This is illustrated by the following example:

$$
\begin{aligned}
& f(0)=f(1)=1, \text { with } \operatorname{domC}(f)=\{0,1\}, \quad \operatorname{imC}(f)=\{1\} ; \text { and } \\
& g(00)=g(10)=10, g(01)=g(11)=11, \text { with } \operatorname{domC}(g)=\{00,10,01,11\}, \quad \operatorname{imC}(g)=\{10,11\} .
\end{aligned}
$$

Then every inverse $f^{\prime}$ of $f$ satisfies either $f^{\prime}(1)=0$ or $f^{\prime}(1)=1$. In particular, $f$ has two injective inverses with domain code $\{1\}(=\operatorname{imC}(f))$, namely $f_{0}^{\prime}$ and $f_{1}^{\prime}$, given by $f_{0}^{\prime}(1)=0$ and $f_{1}^{\prime}(1)=1$.

And $g$ has four injective inverses with domain code $\{10,11\}(=\mathrm{imC}(g))$. Two of them, namely $g_{0}^{\prime}$ and $g_{0}^{\prime}$, are restrictions of $f_{0}^{\prime}$, respectively $f_{1}^{\prime}$, defined by $g_{0}^{\prime}(10)=00, g_{0}^{\prime}(11)=01$, and $g_{1}^{\prime}(10)=$ $10, g_{1}^{\prime}(11)=11$. The two other injective inverses of $g$ with domain code $\{10,11\}$ are $g_{2}^{\prime}$ and $g_{3}^{\prime}$, defined by $g_{2}^{\prime}(10)=00, g_{2}^{\prime}(11)=10$, and $g_{3}^{\prime}(10)=10, g_{3}^{\prime}(11)=01$. These are not restrictions of inverses of $f$, since every inverse $f^{\prime}$ of $f$ satisfies either $f^{\prime}(1)=0$ or $f^{\prime}(1)=1$, hence $f^{\prime}(10)=10, f^{\prime}(11)=11$, or $f^{\prime}(10)=00, f^{\prime}(11)=01$; in either case, $g_{2}^{\prime}$, $g_{3}^{\prime}$ are not restrictions of $f^{\prime}$.

Theorem 5.16 The monoid $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is regular iff $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular.
Proof. Obviously, if $\mathcal{R} \mathcal{M}^{P}$ is regular then its homomorphic image $\mathcal{M}_{\text {poly }}^{P}$ is regular, since $\equiv_{\text {poly }}$ is a congruence.

For the converse we will show that if $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is regular then fP is regular; the latter implies that $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular (by Prop. 2.6 of [4]). For any $f \in \mathrm{fP}$, let $f^{C} \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ be the encoding of $f$, as defined near the beginning of the Introduction. Then $f^{C}$ is normal (see the Examples after Def. 5.6). Let $F=\left[f^{C}\right] \in \mathcal{M}_{\text {poly }}^{\mathrm{P}}$ be the $\equiv_{\text {poly }}$-class of $f^{C}$ in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, and let $F^{\prime} \in \mathcal{M}_{\text {poly }}^{\mathrm{P}}$ be an inverse of $F$. A consequence of $F F^{\prime} F=F$ in $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is that for all $h \in F$ and all $g \in F^{\prime}: h g h \equiv_{\text {poly }} h$. Then by Lemma 5.10, $h g h \in F$ and $h g h$ is regular with inverse $g \in F^{\prime}$. Also, $h g h \subseteq h$. Let $h=f^{C}$, which is normal. Then the Lemma 5.14(1) applies since $h g h \subseteq h, h g h \equiv_{\text {poly }} h$ (with $h=f^{C}$ ), $h$ is normal, and $h g h$ is regular. Hence Lemma 5.14(1) implies that $h=f^{C}$ is regular in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. Hence by Prop. 3.4(2) in [4, $f$ is regular in fP .

Comments: The proof of Theorem 5.16 also shows the following fact: If all normal elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ are regular then $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular. Thus the set of all normal elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ plays a crucial role. It remains an open question whether we have the following element-wise properties: Let $F \in \mathcal{M}_{\text {poly }}^{\mathrm{P}}$ (hence $F \subset \mathcal{R} \mathcal{M}^{\mathbf{P}}$ ); if $F$ is regular in $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$, does that imply that every $f \in F$ is regular in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ ? Equivalently, let $f_{0}, f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ be such that $f_{0} \subseteq f, f_{0} \equiv_{\text {poly }} f$, and $f_{0}$ is regular; does that imply that $f$ is regular? Lemma [.14(1) yields this statement when $f$ is normal.

## 6 A non-regular monoid that maps onto $\mathcal{M}_{\text {poly }}^{P}$

We show that there is a non-regular submonoid of $\mathcal{R} \mathcal{M}^{P}$ that maps homomorphically onto $\mathcal{M}_{\text {poly }}^{P}$. The fact that some non-regular monoid maps onto $\mathcal{M}_{\text {poly }}^{\mathrm{p}}$ is trivial, by itself, because we could use a (finitely generated) free monoid for this. However, there is a non-regular submonoid $\mathcal{R} \mathcal{M}^{n+o(n)}$ of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ such that the following monoid homomorphisms (where $\nearrow$ is injective) form a commutative diagram:


The construction of $\mathcal{R} \mathcal{M}^{n+o(n)}$ is intuitive, but we need some definitions.

We will use the classical Landau symbol $o$. For two total functions $t_{1}, t_{2}: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ we say that " $t_{1}$ is $o\left(t_{2}\right)$ " iff there exists a total function $\epsilon: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ such that $\lim _{n \rightarrow \infty} \epsilon(n)=0$, and for all $n \in \mathbb{N}$, $t_{1}(n) \leq \epsilon(n) \cdot t_{2}(n)$. In particular, a total function $t: \mathbb{N} \rightarrow \mathbb{N}$ is said to be $n+o(n)$ iff there exists a total function $\epsilon: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ such that $\lim _{n \rightarrow \infty} \epsilon(n)=0$, and for all $n \in \mathbb{N}$ : $t(n) \leq n+\epsilon(n) \cdot n$. Since $n+\epsilon(n) \cdot n=(1+\epsilon(n)) \cdot n$, we can also write $(1+o(1)) \cdot n$ for $n+o(n)$. (By the definition of the Landau symbol, a function $t: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ is $o(1)$ iff $\lim _{n \rightarrow \infty} t(n)=0$.)

Clearly, the set of total functions $\mathbb{N} \rightarrow \mathbb{N}$ that are $n+o(n)$ is closed under composition.
An $\mathcal{R} \mathcal{M}^{\mathrm{P}}$-machine is a multi-tape Turing machine $M$ with a read-only input-tape that contains the input, and with a write-only output-tape, such that the input-tape head and the output-tape head never move left. The machine has an accept state; when $M$ halts, the content of the output-tape is a valid output iff $M$ is in the accept state (when $M$ halts in a non-accept state, the output is undefined). A convention of this sort is necessary, otherwise there is always an output (possibly the empty string). Let $f_{M}$ denote the input-output function of $M$. We assume that for every $x \in \operatorname{Dom}\left(f_{M}\right)$ and every word $z \in A^{*}$, the computation of $M$ on input $x z$ has the following property: the input-tape head does not start reading $z$ until $f_{M}(x)$ has been written on the output tape. (To "read" a letter $\ell$ means to make a transition whose input-tape letter is this letter $\ell$.) This is not the complete definition of an $\mathcal{R} \mathcal{M}^{\mathrm{P}}$-machine, but that is all we need here; the details are given at the beginning of Section 2 in (3). We define the following submonoid of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ :

$$
\begin{aligned}
& \mathcal{R} \mathcal{M}^{n+o(n)}=\left\{f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}: f \text { can be computed by an } \mathcal{R} \mathcal{M}^{\mathrm{P}}\right. \text {-machine whose input-output } \\
& \text { balance and time-complexity are } n+o(n)\} \text {. }
\end{aligned}
$$

It should be pointed out that the bound $|x|+o(|x|)$ is only assumed when $f(x)$ is defined; for $x \notin$ $\operatorname{Dom}(f)$, we do not assume any time-bound. Of course, there exists also a machine that runs in polynomial time for all inputs, but then it is not guaranteed that the running time is $|x|+o(|x|)$ for accepted inputs. An $\mathcal{R} \mathcal{M}^{\mathrm{P}}$-machine whose time and balance on accepted inputs are $n+o(n)$ is called an $\mathcal{R M}^{n+o(n)}$-machine.

Note that $\mathcal{R} \mathcal{M}^{n+o(n)}$ is a strict subset of $\mathcal{R} \mathcal{M}^{\text {lin }}$, that consists of the elements of $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ that have linear upper-bounds on their balance and their time-complexity (where by "linear" we mean any function of the form $n \mapsto a n+b$ for some natural integers $a, b)$. Indeed, if $t($.$) is n+o(n)$ then $t(n) \leq 2 n+c$ for some constant $c$; the strictness of the inclusion comes from the fact that $\mathcal{R} \mathcal{M}^{\text {lin }}$ contains, for example, functions whose output-length is twice the input-length, and the function $n \mapsto 2 n$ is not $n+o(n)$.

## Lemma 6.1 $\mathcal{R M}^{n+o(n)}$ is a monoid.

Proof. Let $f_{1}, f_{2} \in \mathcal{R} \mathcal{M}^{n+o(n)}$ and let $M_{1}, M_{2}$ be $\mathcal{R} \mathcal{M}^{n+o(n)}$-machines that compute $f_{1}$, respectively $f_{2}$. Since the set of functions that are $n+o(n)$ is closed under composition, the I/O-balance of $f_{2} \circ f_{1}$ is $n+o(n)$.

To compute $f_{2} \circ f_{1}(x)$ in time $n+o(n)$ (where $n=|x|$ ), we combine $M_{1}$ and $M_{2}$ into an $\mathcal{R} \mathcal{M}^{n+o(n)}$ _ machine $M$, as follows. The output-tape of $M_{1}$ and the input-tape of $M_{2}$ are combined into one work-tape of $M$; we call this work-tape the intermediate tape. On input $x$, the machine $M$ starts simulating $M_{1}$ and starts writing $f_{1}(x)$ on the intermediate tape; as soon as there is something on this intermediate tape, $M$ starts the simulation of $M_{2}$ on $f_{1}(x)$. The writing of $f_{1}(x)$ by $M_{1}$ takes at most $o(n)$ more steps than it takes to read $x$; the computation of $f_{1}(x)$, except for this $o(n)$-step delay, is done in parallel (simultaneously) with the reading of $x$. Similarly, when $M_{2}$ reads $f_{1}(x)$ as an input, it computes $f_{2}\left(f_{1}(x)\right)$ at the same time as it reads $f_{1}(x)$, except for a $o\left(\left|f_{1}(x)\right|\right)$-step delay; but $o\left(\left|f_{1}(x)\right|\right)$ means $\leq \epsilon_{1}(|x|) \cdot\left|f_{1}(x)\right| \leq \epsilon_{1}(|x|) \cdot\left(|x|+\epsilon_{2}(|x|) \cdot|x|\right)$, and this is $\leq \epsilon(|x|) \cdot|x|$ (for some functions $\epsilon$ with limit 0$)$; hence, $o\left(\left|f_{1}(x)\right|\right)$ is $o(|x|)$. So when $x \in \operatorname{Dom}\left(f_{2} \circ f_{1}\right)$ the total time taken by $M$ (i.e., $M_{1}$ and $M_{2}$ working together, mostly in parallel) is $|x|+o(|x|)$.

Proposition 6.2 The monoid $\mathcal{R} \mathcal{M}^{n+o(n)}$ is non-regular. In fact, there exists a real-time function in $\mathcal{R} \mathcal{M}^{n+o(n)}$ that has no inverse in $\mathcal{R M}^{n+o(n)}$.

Proof. We use the encoding function code: $\{0,1, \#\}^{*} \longmapsto\{00,01,11\}^{*}$, replacing 0 by 00,1 by 01 , and \# by 11 (as discussed at the beginning of the Introduction). For a string $x, x^{\text {rev }}$ denotes the string in reverse (i.e., backwards) order.

Consider the right-ideal morphism defined for all $x, w \in\{0,1\}^{*}$ by

$$
s: \operatorname{code}(x) 110^{|x|} w \longmapsto 0^{2|x|} 11 x^{\mathrm{rev}} w,
$$

where $\operatorname{domC}(s)=\bigcup_{k \geq 0}\{00,01\}^{k} 110^{k}$. Thus $s$ is injective and length-preserving, and belongs to $\mathcal{R} \mathcal{M}^{\mathrm{P}}$. Moreover, $s$ belongs to $\mathcal{R} \mathcal{M}^{n+o(n)}$ since an $\mathcal{R} \mathcal{M}^{n+o(n)}$-machine can compute $s\left(\operatorname{code}(x) 110^{|x|} w\right)$ in time $\leq n+o(n)$, where $n=2|x|+2+|x|+|w|$, as follows: The machine reads code $(x)$ in time $2|x|$, while writing $x$ on a work-tape and while writing $0^{2|x|}$ on the output-tape. When 11 is encountered in the input, the head of the work-tape is at the right end of $x$. The machine copies 11 to the output-tape, then reads $0^{|x|}$ in the input, while copying the work-tape from right to left to the output-tape; thus, $x^{\text {rev }}$ is written. When the work-tape head reaches the left end of the work-tape, the input-tape head reaches $w$ on the input-tape while copying it to the output-tape. Note that the above machine is a real-time Turing machine, with running time $\leq n+c$ for some constant $c \geq 0$.

We show next that $s$ does not have an inverse in $\mathcal{R} \mathcal{M}^{n+o(n)}$; hence $\mathcal{R} \mathcal{M}^{n+o(n)}$ is not regular. For every inverse $s^{\prime}$ of $s$ we have $s^{\prime}\left(0^{2|x|} 11 x^{\text {rev }}\right)=\operatorname{code}(x) 110^{|x|}$. It is easy to see that although $s^{\prime}$ can be chosen so as to belong to $\mathcal{R} \mathcal{M}^{\mathrm{P}}, s^{\prime}$ cannot be evaluated in time $\leq n+o(n)$; here, $n=$ $\mid$ code $(x)|+2+|x|=2| x\left|+2+|x|\right.$. Indeed, an $\mathcal{R} \mathcal{M}^{n+o(n)}$-machine reads the input $0^{2|x|} 11 x^{\text {rev }}$ only once, from left to right. While $0^{2|x|}$ is being read, $x^{\text {rev }}$ has not yet been seen, so no letter-pair of $\operatorname{code}(x)$ can be written on the output-tape; indeed, the machine is deterministic, so anything written on the output-tape up to this moment would be false for some input $x$. At the moment $x^{\text {rev }}$ starts being read, the machine has made $2|x|+2$ steps, and no output has been written yet (except perhaps one 0 ). To write down the output (which has length $n$ ) will take at least $n$ steps from here onward. So the total time will be $\geq n+2|x|+2 \geq n+n / 2$. But $n+n / 2$ does not have $n+o(n)$ as an upper-bound.

Proposition 6.3 The monoid $\mathcal{R} \mathcal{M}^{n+o(n)} / \equiv_{\text {poly }}$ is isomorphic to $\mathcal{R} \mathcal{M}^{\mathrm{P}} / \equiv_{\text {poly }}\left(=\mathcal{M}_{\text {poly }}^{P}\right)$.
Proof. We show that the embedding $[g]_{\text {poly }} \in \mathcal{R} \mathcal{M}^{n+o(n)} / \equiv_{\text {poly }} \longmapsto[g]_{\text {poly }} \in \mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is surjective. More precisely, for every $f \in \mathcal{R} \mathcal{M}^{\mathrm{P}}$ with time-complexity and balance $\leq p($.$) (a polynomial), we$ construct a function $F \in \mathcal{R} \mathcal{M}^{n+o(n)}$ such that $F \equiv_{\text {poly }} f$. This is done by a padding argument similar to the one in Lemma 2.22; $F$ is the restriction of $f$ to $\bigcup_{x \in \operatorname{domC(f)}} x A^{q(|x|)} A^{*}$, where $q($.$) is a fully$ time-constructible function that satisfies $(n \cdot p(n))^{2}<q(n)$ for all $n \in \mathbb{N}$. The function $q($.$) exists by$ Lemma 2.22(1); in fact, since $p($.$) is a polynomial, q($.$) can be chosen to be a fully time-constructible$ polynomial of the form $n \mapsto a(n+1)^{d}$. Since every $A^{q(|x|)}$ is a maximal prefix code and since $q$ is polynomial bound, $F \equiv_{\text {poly }} f$.

We want to show that $F \in \mathcal{R} \mathcal{M}^{n+o(n)}$. We construct a Turing machine $M_{F}$ that on input xvw computes $f(x) v w$ in time $n+o(n)$, where $x \in \operatorname{domC}(f), v \in A^{q(|x|)}, w \in A^{*}$, and $n=|x v w|$. On input $z \notin \operatorname{Dom}(f), M_{F}$ should reject, but we do not care how much time $M_{F}$ takes in that case. On input $x v w, M_{F}$ works as follows. First it finds $x$ as the first prefix of the input that belongs to $\operatorname{Dom}(f)$, and writes $f(x)$ on the output-tape; this takes time $\leq 2|x| \cdot p(|x|)$ (see the proof of Lemma 2.22(2)). If no prefix of the input is in $\operatorname{Dom}(f), M_{F}$ rejects. If $x \in \operatorname{domC}(f)$ is found, since $\operatorname{domC}(f)$ is a prefix code, the end of $x$, and the beginning of $v$, are uniquely determined within the input $x v w$. Next, $M_{F}$ finds $v$, consisting of the next $q(|x|)$ letters, and concatenates this to the right of $f(x)$ on the output-tape; since $q$ is fully time-constructible, this can be done in time $q(|x|)$ exactly. If the remainder $v w$ of the input has length $<q(|x|), M_{F}$ rejects. If $v$ is found, the remainder $w$ of the input is copied to the output-tape.

So the total time of the computation (if there is an output) is $\leq 2|x| \cdot p(|x|)+q(|x|)+|w|$. Since $n=|x v w|,|v|=q(|x|)$, and since $2|x| \cdot p(|x|)=\sqrt{q(|x|)}$, the total time is $\leq \sqrt{n}+n$, which is $\leq n+o(n)$.

As a consequence of Prop. 6.3 and earlier results we have:
Corollary 6.4 $\mathrm{P} \neq \mathrm{NP}$ iff there exists a function in $\mathcal{R M}^{n+o(n)}$ that has no inverse in $\mathcal{R M}^{\mathrm{P}}$.
Proof. If some $F \in \mathcal{R} \mathcal{M}^{n+o(n)}\left(\subset \mathcal{R} \mathcal{M}^{\mathrm{P}}\right)$ has no inverse in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$, then $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is not regular, hence $\mathrm{P} \neq \mathrm{NP}$ (by results from [4), as we saw in the Introduction).

Conversely, suppose every $F \in \mathcal{R} \mathcal{M}^{n+o(n)}$ has some inverse $F^{\prime} \in \mathcal{R} \mathcal{M}^{\text {P }}$. By Prop. 6.3, every element of $\mathcal{M}_{\text {poly }}^{P}$ is an $\equiv_{\text {poly }}$-class of the form $[F]$ for some $F \in \mathcal{R} \mathcal{M}^{n+o(n)}$. Since $F F^{\prime} F=F$, we have $[F]\left[F^{\prime}\right][F]=[F]$ (since $\equiv_{\text {poly }}$ is a congruence by Theorem 4.3). Hence $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is regular. By Theorem 5.16 this implies that $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is regular.

Remark. By Corollary 6.4 if $\mathrm{P} \neq \mathrm{NP}$ then this is "witnessed" by an element of $\mathcal{R} \mathcal{M}^{n+o(n)}$. Although $\mathcal{R} \mathcal{M}^{n+o(n)}$ is not regular by itself, its non-regularity in $\mathcal{R} \mathcal{M}^{\mathrm{P}}$ is not obvious (and equivalent to $\mathrm{P} \neq \mathrm{NP}$ ).

It is not especially surprising that $\mathcal{R} \mathcal{M}^{n+o(n)}$ is non-regular; ultimately, this is due to the limitations of tapes as storage devices. By itself, it is not too surprising either that $\mathcal{R} \mathcal{M}^{n+o(n)}$ is $\equiv_{\text {poly }}$-equivalent to $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$. The $\equiv_{\text {poly }}$-equivalence of $\mathcal{R} \mathcal{M}^{n+o(n)}$ and $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is proved by pushing the familiar padding argument a little further. The combination of the two facts is interesting, however, because $\equiv_{\text {poly }}{ }^{-}$ equivalence means that $\mathcal{R} \mathcal{M}^{n+o(n)}$ and $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ are very close to each other; yet, $\mathcal{R} \mathcal{M}^{n+o(n)}$ is nonregular, while the non-regularity of $\mathcal{M}_{\text {poly }}^{\mathrm{P}}$ is equivalent to $\mathrm{P} \neq \mathrm{NP}$.

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