ON THE LOCAL K-ELASTICITIES OF PUISEUX MONOIDS

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ABSTRACT. If M is an atomic monoid and x is a nonzero non-unit element of M, then the set of lengths L(x) of x is the set of all possible lengths of factorizations of x, where the length of a factorization is the number of irreducible factors (counting repetitions). In a recent paper, F. Gotti and C. O'Neil studied the sets of elasticities $\mathcal{R}(P) := \{\sup L(x)/\inf L(x) : x \in P\}$ of Puiseux monoids P. Here we take this study a step further and explore the local k-elasticities of the same class of monoids. We find conditions under which Puiseux monoids have all their local elasticities finite as well as conditions under which they have infinite local k-elasticities for sufficiently large k. Finally, we focus our study of the k-elasticities on the class of primary Puiseux monoids, proving that they have finite local k-elasticities if either they are boundedly generated and do not have any stable atoms or if they do not contain 0 as a limit point.

1. Introduction

Rings of integers of algebraic number fields are not necessarily factorial. We can use the class group of a ring of integers R to measure to which extent elements in R fail to have a unique factorization. In [2], L. Carlitz characterized the half-factorial rings of integers in terms of their class groups. A friendly survey illustrating this characterization when $R = \mathbb{Z}[\sqrt{-5}]$ is provided in [3]. After the publication of Carlitz's result, many authors attempted to characterize the class group of a general ring of integers in terms of further arithmetical properties describing the non-uniqueness of factorizations in such a ring (Rush [15] was the first to give a complete characterization).

More generally, the algebraic invariants of several non-factorial Noetherian domains can be used to understand how far are such domains from being factorial. Because many of the factorization-related questions on integral domains are independent of the ring additive structure, in the last few decades the study of the phenomenon of non-unique factorization has been extended to the setting of atomic monoids. The monograph [6] by A. Geroldinger and F. Halter-Koch has significantly influenced the shape of the modern non-unique factorization theory, which considers not only integral domains but also atomic monoids.

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The purpose of modern non-unique factorization theory is to measure how far an atomic monoid is from being factorial (or half-factorial). To carry out this measurement, we focus attention on several factorization invariants, which include the system of sets of lengths, the union of sets of lengths (first introduced in [14]), and the elasticity. Roughly speaking, the elasticity of an atomic monoid M is given by

$$\rho(M) = \sup\{\rho(x) : x \in M\},\$$

where $\rho(x)$ is the quotient of the maximum possible length of factorizations of x by the minimum possible length of factorizations of x. Our aim in this paper is to study an \mathbb{N} -parametrized local version of the elasticity of Puiseux monoids, which are, up to isomorphism, the additive submonoids of $(\mathbb{Q}, +)$ that are not groups.

This paper is organized as follows. In Section 2, we review notation and introduce most of the concepts we shall be using later. In Section 3, we begin our exploration of the local elasticities of atomic Puiseux monoids. In particular, we find conditions under which Puiseux monoids have all their local elasticities finite as well as conditions under which they have infinite local k-elasticities for sufficiently large k. Lastly, in Section 4, we target the class of primary Puiseux monoids, proving that they have finite local elasticities if they do not contain stable atoms or if they do not contain 0 as a limit point.

2. Definitions & Notations

Throughout this paper, we let \mathbb{N} denote the set of positive integers, and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each subset A of \mathbb{Q} , we let A^{\bullet} denote $A \setminus \{0\}$. Also, for all $q \in \mathbb{Q}$ such that q > 0, we let $\mathsf{n}(q)$ and $\mathsf{d}(q)$ denote the unique pair of relatively prime positive integers satisfying that $q = \mathsf{n}(q)/\mathsf{d}(q)$. In this case, we call $\mathsf{n}(q)$ and $\mathsf{d}(q)$ the numerator and denominator of q, respectively. Moreover, for a subset A of positive rationals, we set

$$n(A) := \{ n(a) : a \in A \} \text{ and } d(A) := \{ d(a) : a \in A \}.$$

Although most of the definitions given in this section make sense in a much broader context, for the sake of simplicity we will present them in a particular setting which is enough for the treatment of Puiseux monoids. Every monoid here is tacitly assumed to be commutative, cancellative, and reduced (i.e., only the identity is a unit). As we shall be working in a commutative environment, unless otherwise specified we will use additive notation. Let M be a monoid.

Definition 2.1. An element $a \in M \setminus \{0\}$ is called an *atom* (i.e., *irreducible*) provided that for all $x, y \in M$ the fact that a = x + y implies that either x = 0 or y = 0. Let $\mathcal{A}(M)$ denote the set of all atoms of M.

If A is a subset of M, then the minimal submonoid of M containing A is denoted by $\langle A \rangle$. If $M = \langle A \rangle$, then we say that M is generated by A or that A is a generating set of M. The monoid M is called *finitely generated* provided that it contains a finite generating set.

Definition 2.2. If $M = \langle \mathcal{A}(M) \rangle$, then we call the monoid M atomic.

Clearly, every generating set of an atomic monoid M contains $\mathcal{A}(M)$. In addition, it is not hard to prove that M is atomic if and only if it contains exactly one minimal generating set, namely $\mathcal{A}(M)$; see, for instance, [6, Proposition 1.1.7].

Definition 2.3. A *Puiseux monoid* is an additive submonoid of $(\mathbb{Q}, +)$ consisting of nonnegative rationals.

As we mentioned in the introduction, each additive submonoid of $(\mathbb{Q}, +)$ that is not a group is isomorphic to a Puiseux monoid [8, Theorem 2.9]. Puiseux monoids have a fascinating atomic structure. Some of them contain no atoms at all, as it is the case of $\langle 1/2^n : n \in \mathbb{N} \rangle$, while there are others whose sets of atoms are dense in the nonnegative real line [11, Theorem 3.5]. The atomicity of members of the family of Puiseux monoids has only been recently studied (see [9] and [10]).

There are three classes of Puiseux monoids we shall be studying, namely the classes of bounded, strongly bounded, and primary Puiseux monoids. Let P be a Puiseux monoid. We say that P is bounded (respectively, strongly bounded) if it can be generated by a set A of rational numbers such that A is bounded (respectively, $\mathbf{n}(A)$ is bounded). In addition, P is primary if it can be generated by a subset of positive rationals whose denominators are pairwise distinct prime numbers.

Bounded and strongly bounded Puiseux monoids are not necessarily atomic; see, for example, $\langle 1/2^n : n \in \mathbb{N} \rangle$. However, it is not hard to verify that primary monoids are always atomic. Indeed, it was proved in [10] that every submonoid of a primary Puiseux monoid is atomic. The class of atomic Puiseux monoids is plentiful as the following theorem indicates.

Theorem 2.4. [10, Theorem 3.10] Let P be a Puiseux monoid. If 0 is not a limit point of P, then P is atomic.

Given a set S, it is not hard to verify that the formal sums of elements of S (up to permutation) is a monoid, which is called the *free commutative monoid* on S. For an atomic monoid M, we let $\mathsf{Z}(M)$ denote the free commutative monoid on $\mathcal{A}(M)$. The elements of $\mathsf{Z}(M)$ have the form $a_1+\cdots+a_n$ for some $a_1,\ldots,a_n\in\mathcal{A}(M)$ and are called *factorizations*. It follows immediately that the function $\phi\colon\mathsf{Z}(M)\to M$ defined by $\phi(a_1+\cdots+a_n)=a_1+\cdots+a_n$ is a monoid homomorphism.

Definition 2.5. The homomorphism ϕ given above is called the *factorization homomorphism* of M.

If $x \in M$, then the set of factorizations of x, denoted by $\mathsf{Z}_M(x)$, is defined to be the preimage of x by ϕ , i.e.,

$$\mathsf{Z}_M(x) = \phi^{-1}(x) \subseteq \mathsf{Z}(M).$$

It follows that M is atomic if and only if $\mathsf{Z}_M(x)$ is nonempty for all $x \in M$. If $z = a_1 + \cdots + a_n$ for some a_1, \ldots, a_n , then n is called the *length* of z and is denoted by |z|. For $x \in M$, the set of lengths of x is the set

$$L_M(x) := \{ |z| : z \in Z_M(x) \}.$$

We write Z(x) and L(x) for the respective sets $Z_M(x)$ and $L_M(x)$ when there is no risk of ambiguity. In addition, the collection of sets

$$\mathcal{L}(M) := \{ \mathsf{L}(x) : x \in M \}$$

is called the *system of sets of lengths* of M. Systems of sets of lengths of many families of atomic monoids have been the focus of a great deal of research during the last few decades (see, for example, [1, 7, 12]).

We proceed to introduce unions of sets of lengths and local elasticities. Similar to the system of sets of lengths, the elasticity is another arithmetical invariant used to measure up to what extent factorizations in monoids (or domains) fail to be unique. The concept of elasticity was introduced by R. Valenza [13] in the context of algebraic number theory. The *elasticity* $\rho(M)$ of an atomic monoid M is given by

$$\rho(M) = \sup\{\rho(x) : x \in M\}, \text{ where } \rho(x) = \frac{\sup \mathsf{L}(x)}{\min \mathsf{L}(x)}.$$

For $n \in \mathbb{N}_0$, we define $\mathsf{L}^{-1}(n) := \{x \in M : n \in \mathsf{L}(x)\}$. Now, the union of sets of lengths of M containing n is defined to be

$$\mathcal{U}_n(M) = \{|z| : z \in \mathsf{Z}(x) \text{ for some } x \in \mathsf{L}^{-1}(n)\}.$$

Definition 2.6. The *n*-th local elasticity of M is defined by

$$\rho_n(M) = \sup \mathcal{U}_n(M).$$

A numerical semigroup is a cofinite additive submonoid of $(\mathbb{N}_0, +)$. It is well known that every numerical semigroup is finitely generated and, therefore, atomic [6, Proposition 2.7.8(4)]. See [5] for an introduction to numerical semigroups. For a numerical semigroup N with minimal generating set A, it was proved in [4, Section 2] that the elasticity of N is given by max $A/\min A$. On the other hand, it is not hard to verify that $\mathcal{U}_n(N)$ is bounded and, therefore, every local elasticity of N is finite. In the next two sections, we will generalize this fact in two different ways to Puiseux monoids.

3. The General Case

We begin this section proposing a sufficient condition under which most of the local elasticities of an atomic Puiseux monoid have infinite cardinality. On the other hand, we describe a subclass of Puiseux monoids (containing isomorphic copies of each numerical semigroup) whose local k-elasticities are finite.

If P is a Puiseux monoid, then we say that $a_0 \in \mathcal{A}(P)$ is *stable* provided that the set $\{a \in \mathcal{A}(P) : \mathsf{n}(a) = \mathsf{n}(a_0)\}$ is infinite.

Proposition 3.1. Let P be an atomic Puiseux monoid. If P contains a stable atom, then $\rho_k(P)$ is infinite for all sufficiently large k.

Proof. Suppose that for some $m \in \mathbb{N}$ the set $A := \{a \in \mathcal{A}(P) : \mathsf{n}(a) = m\}$ contains infinitely many elements. Let $\{a_n\}$ be an enumeration of the elements of A. Because the elements of A have the same numerator, namely m, we can assume that the sequence $\{a_n\}$ is decreasing. Setting $d = \mathsf{d}(a_1)$, we can easily see that $da_1 = m = \mathsf{d}(a_j)a_j$ for each $j \in \mathbb{N}$. Therefore $\mathsf{d}(a_j) \in \mathcal{U}_d(P)$ for each $j \in \mathbb{N}$. As $\mathsf{d}(A)$ is an infinite set so is $\mathcal{U}_d(P)$. The fact that $|\mathcal{U}_d(P)| = \infty$ immediately implies that $|\mathcal{U}_k(P)| = \infty$ for all $k \geq d$. Hence $\rho_k(P) = \sup \mathcal{U}_k(P) = \infty$ for every $k \geq d$.

Recall that a Puiseux monoid P is strongly bounded if it can be generated by a set of rationals A whose numerator set $\mathbf{n}(A)$ is bounded. As a direct consequence of Proposition 3.1 we obtain the following result.

Corollary 3.2. If P is a non-finitely generated strongly bounded atomic Puiseux monoid, then $\rho_k(P)$ is infinite for all k sufficiently large.

In contrast to the previous proposition, the next result gives a condition under which Puiseux monoids have finite k-elasticity for each $k \in \mathbb{N}$.

Proposition 3.3. Let P be a Puiseux monoid that does not contain 0 as a limit point. If P is bounded, then $\rho_k(P) < \infty$ for every $k \in \mathbb{N}$.

Proof. Because 0 is not a limit point of P, it follows by Theorem 2.4 that P is atomic As P is a bounded Puiseux monoid, $\mathcal{A}(P)$ is a bounded set of rational numbers. Take $q, Q \in \mathbb{Q}$ such that 0 < q < a < Q for all $a \in \mathcal{A}(P)$. Now fix $k \in \mathbb{N}$, and suppose that $\ell \in \mathcal{U}_k(P)$. Then there exists $x \in \mathsf{L}^{-1}(k)$ such that $\ell \in \mathsf{L}(x)$. Because x has a factorization of length k, it follows that x < kQ. Taking $a_1, \ldots, a_\ell \in \mathcal{A}(P)$ such that $x = a_1 + \cdots + a_\ell$, we find that

$$q\ell < a_1 + \dots + a_\ell = x < kQ$$
.

Therefore $\ell < kQ/q$. Because neither q nor Q depends on the choice of x, one obtains that $\mathcal{U}_k(P)$ is bounded from above by kQ/q. Hence $\rho_k(P) = \sup \mathcal{U}_k(P)$ is finite, and the proof follows.

With the following two examples, we shall verify that the conditions of containing a stable atom and not having 0 as a limit point are not superfluous in Proposition 3.1 and Proposition 3.3, respectively.

Example 3.4. Let $\{p_n\}$ be a strictly increasing enumeration of the prime numbers, and consider the following Puiseux monoid:

$$P = \langle A \rangle$$
, where $A = \left\{ \frac{p_n - 1}{p_n} : n \in \mathbb{N} \right\}$.

As the denominators of elements in A are pairwise distinct primes, it immediately follows that $\mathcal{A}(P) = A$. Therefore P is atomic. Clearly, P does not contain stable atoms. Because A is bounded so is P (as a Puiseux monoid). On the other hand, 0 is not a limit point of P. Thus, it follows by Proposition 3.3 that $\rho_k(P)$ is finite for every $k \in \mathbb{N}$. Notice also that

- (1) if $q \in P$ has at least two factorizations with no atoms in common, then $q \in \mathbb{N}$;
- (2) by Proposition 3.3, we have both a lower and an upper bound for any $q \in L^{-1}(k)$.

Using the previous two observations, we have created an R-script that generates the sets U_k for $k \in \{1, ..., 15\}$. Each U_k appears as the k-th column in Table 1.

U ₁	U ₂	U ₃	U ₄	U ₅	U ₆	U ₇	U ₈	U ₉	U ₁₀	U ₁₁	U ₁₂	U ₁₃	U ₁₄	U ₁₅
1	2	3	3	4	5	5	5	6	7	7	7	8	9	10
		4	4	5	6	6	6	7	8	8	8	9	10	11
			5	6	7	7	7	8	9	9	9	10	11	12
				7	8	8	8	9	10	10	10	11	12	13
				8	9	9	9	10	11	11	11	12	13	14
						10	10	11	12	12	12	13	14	15
						11	11	12	13	13	13	14	15	16
						12	12	13	14	14	14	15	16	17
							13	14	15	15	15	16	17	18
									16	16	16	17	18	19
										17	17	18	19	20
										18	18	19	20	21
										19	19	20	21	22
										20	20	21	22	23
											21	22	23	24
												23	24	25
												24	25	26

Table 1. U_k for $k \in \{1, ..., 15\}$.

Example 3.5. Let $\{p_n\}$ be an enumeration of the prime numbers, and consider the Puiseux monoid $P = \langle 1/p_n : n \in \mathbb{N} \rangle$. It is not difficult to argue that P is atomic with $\mathcal{A}(P) = \{1/p_n : n \in \mathbb{N}\}$. As $\mathcal{A}(P)$ is a bounded subset of positive rationals, the Puiseux monoid P is bounded. Notice, however, that 0 is a limit point of P. By Proposition 3.1, it follows that the local elasticities $\rho_k(P)$ are infinite for all k sufficiently large.

The condition of boundedness on Proposition 3.3 is also required, as shown by the following proposition.

Proposition 3.6. There exist infinitely many non-isomorphic Puiseux monoids without 0 as a limit point that have no finite local elasticities.

Proof. Let $\mathcal{P} = \{S_n : n \in \mathbb{N}\}$ be a family of disjoint infinite sets of odd prime numbers. For each set S_n , we will construct an atomic Puiseux monoid M_n . Then we will show that $M_i \cong M_j$ implies i = j.

Fix $j \in \mathbb{N}$ and take $p \in S_j$. To construct the Puiseux monoid M_j , let us inductively create a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of positive rationals with $A_1 \subsetneq A_2 \subsetneq \cdots$ such that, for each $k \in \mathbb{N}$, the following three conditions hold:

- (1) $d(A_k)$ consists of odd prime numbers;
- (2) $d(\max A_k) = \max d(A_k);$
- (3) A_k minimally generates the Puiseux monoid $P_k = \langle A_k \rangle$.

Take $A_1 = \{1/p\}$, with p an odd prime number, and assume we have already constructed the sets A_1, \ldots, A_n for some $n \in \mathbb{N}$ satisfying our three conditions. To construct A_{n+1} , we take $a = \max A_n$ and let

$$b_1 = \frac{\mathsf{n}(a)\lfloor q/2 \rfloor}{q} \text{ and } b_2 = \frac{\mathsf{n}(a)(q - \lfloor q/2 \rfloor)}{q},$$

where q is an odd prime in S_j satisfying $q > \max d(A_n)$ and $q \nmid n(a)$. Using the fact that $q \geq 5$ and $d(a) \geq 3$, one obtains that

$$b_2 > b_1 = \frac{\lfloor q/2 \rfloor}{q} \mathsf{n}(a) > \frac{1}{3} \mathsf{n}(a) \ge a.$$

Now set $A_{n+1} = A_n \cup \{b_1, b_2\}$. Notice that $b_1 + b_2 = \mathsf{n}(a)$. Clearly, $A_n \subsetneq A_{n+1}$, and condition (1) is an immediate consequence of our inductive construction. In addition,

$$d(\max A_{n+1}) = d(b_2) = q = \max d(A_{n+1}),$$

which is condition (2). Therefore it suffices to verify that A_{n+1} minimally generates $P_{n+1} = \langle A_{n+1} \rangle$. Because both b_1 and b_2 are greater than every element in A_n , we only need to check that $b_1 \notin P_n$ and $b_2 \notin \langle A_n \cup \{b_1\} \rangle$. Let d be the product of all the elements in $d(A_n)$. Assuming that $b_1 = a_1 + \cdots + a_r$ for some $a_1, \ldots, a_r \in A_n$, and multiplying both sides of the same equality by qd, we would obtain that $q \mid \mathsf{n}(b_1)$,

which contradicts that $q \nmid \mathsf{n}(a)$. Hence $b_1 \notin P_n$. Similarly, one finds that $b_2 \notin P_n$. Suppose, again by contradiction, that $b_2 \in \langle A_n \cup \{b_1\} \rangle$. Then there exist $a'_1, \ldots, a'_s \in A_n$ and $m \in \mathbb{N}$ such that $b_2 = mb_1 + a'_1 + \cdots + a'_s$. Notice that $2b_1 = \mathsf{n}(a)(q-1)/q > b_2$, which implies that $m \leq 1$. As $b_2 \notin P_n$, it follows that m = 1. Then we can write

(3.1)
$$\frac{\mathsf{n}(a)}{q} = b_2 - b_1 = \sum_{i=1}^s a_i'.$$

Once again, we can multiply the extreme parts of the equality (3.1) by $q d(\{a'_1, \ldots, a'_s\})$, to obtain that $q \mid \mathbf{n}(a)$, a contradiction. As a result, condition (3) follows.

Now set $M_j := \bigcup_{n \in \mathbb{N}} P_n$. As $P_1 \subsetneq P_2 \subsetneq \ldots$, the set M_j is, indeed, a Puiseux monoid. We can easily see that M_j is generated by the set $A := \bigcup_{n \in \mathbb{N}} A_n$. Let us verify now that $\mathcal{A}(M_j) = A$. It is clear that $\mathcal{A}(M_j) \subseteq A$. To check the reverse inclusion, suppose that $a \in A$ is the sum of atoms $a_1, \ldots, a_r \in \mathcal{A}(M_j)$. Take $t \in \mathbb{N}$ such that $a, a_1, \ldots, a_r \in A_t$. Because A_t minimally generates P_t it follows that r = 1 and $a = a_1$ and, therefore, that $a \in \mathcal{A}(M_j)$. Hence $\mathcal{A}(M_j) = A$, which implies that M_j is an atomic monoid.

To disregard 0 as a limit point of M_j , it is enough to observe that $\min \mathcal{A}(M_j) = 1/p$. We need to show then that $\rho_k(M_j) = \infty$ for $k \geq 2$. Set $a_n = \max A_n$. When constructing the sequence $\{A_n\}$, we observed that $\mathsf{n}(a_n) = b_{n_1} + b_{n_2}$, where $\{b_{n_1}, b_{n_2}\} = A_{n+1} \setminus A_n$. Because $\mathsf{n}(a_n) \in M_j$ and

$$b_{n_1} + b_{n_2} = \mathsf{n}(a_n) = \mathsf{d}(a_n)a_n,$$

one has that the factorizations $z = b_{n_1} + b_{n_2}$ and $z' = \mathsf{d}(a_n)a_n$ are both in $\mathsf{Z}(\mathsf{n}(a_n))$. Since |z| = 2 and $|z'| = \mathsf{d}(a_n)$ it follows that $\mathsf{d}(a_n) \in \mathcal{U}_2(M_j)$. By condition (2) above, $\mathsf{d}(a_n) = \mathsf{d}(\max A_n) = \max \mathsf{d}(A_n)$. This implies that the set $\{\mathsf{d}(a_n) : n \in \mathbb{N}\}$ contains infinitely many elements. As $\{\mathsf{d}(a_n) : n \in \mathbb{N}\} \subseteq \mathcal{U}_2(M_j)$, we obtain that $\rho_2(M_j) = \infty$. Hence $\rho_k(M_j) = \infty$ for all $k \geq 2$.

We have just constructed an infinite family $\mathcal{F} := \{M_n : n \in \mathbb{N}\}$ of atomic Puiseux monoids with infinite k-elasticities. Let us show now that the monoids in \mathcal{F} are pairwise non-isomorphic. To do this we use the fact that the only homomorphisms between Puiseux monoids are given by rational multiplication [11, Lemma 3.3]. Take $i, j \in \mathbb{N}$ such that $M_i \cong M_j$. Then there exists $r \in \mathbb{Q}$ such that $M_i = rM_j$. Let $m \in M_j$ such that d(m) = p and $p \nmid \mathbf{n}(r)$ for some prime p in S_j . Since the element $rm \in M_i$ and $p \mid \mathbf{d}(rm)$, we must have that the prime p belongs to S_i . Because the sets in \mathcal{P} are pairwise disjoint, we conclude that i = j. This completes the proof. \square

Proposition 3.1 (respectively, Proposition 3.3) establishes sufficient conditions under which a Puiseux monoid has most of its local elasticities infinite (respectively, finite). In addition, we have verified that such conditions are not necessary. For the sake of completeness, we now exhibit a Puiseux monoid that does not satisfy the

conditions of either of the propositions above and has no finite k-elasticity for any $k \geq 2$.

Example 3.7. Consider the Puiseux monoid

$$P = \left\langle \left(\frac{2}{3}\right)^n : n \in \mathbb{N} \right\rangle.$$

It was proved in [10, Theorem 6.2] that P is atomic and $\mathcal{A}(P) = \{(2/3)^n : n \in \mathbb{N}\}$. In addition, it is clear that P is bounded, has 0 as a limit point, and does not contain any stable atoms. So neither Proposition 3.1 nor Proposition 3.3 applies to P. Now we argue that $\rho_k(P) = \infty$ for each $k \in \mathbb{N}$ such that $k \geq 2$.

Take $k \geq 2$ and set $x = k^{\frac{2}{3}} \in P$. Notice that, by definition, $x \in \mathsf{L}^{-1}(k)$. We can conveniently rewrite x as

$$x = ((k-2)+2)\frac{2}{3} = (k-2)\frac{2}{3} + 3 \cdot (\frac{2}{3})^2$$

which reveals that $z = (k-2)\frac{2}{3} + 3(\frac{2}{3})^2$ is a factorization of x with |z| = k+1. Taking k' = 3 to play the role of k and repeating this process as many times as needed, one can obtain factorizations of x of lengths as large as one desires. The fact that k was chosen arbitrarily implies now that $\rho_k(P) = \infty$ for each $k \ge 2$.

4. The Primary Case

Recall that a Puiseux monoid is said to be primary if it can be generated by a subset of rational numbers whose denominators are pairwise distinct primes. In Proposition 3.3, we established a sufficient condition on Puiseux monoids to ensure that all their local k-elasticities are finite. Here we restrict our study to the case of primary Puiseux monoids, providing two more sufficient conditions to guarantee the finiteness of all the local k-elasticities.

Theorem 4.1. For a primary Puiseux monoid P, the following two conditions hold.

- (1) If 0 is not a limit point of P, then $\rho_k(P) < \infty$ for every $k \in \mathbb{N}$.
- (2) If P is bounded and has no stable atoms, then $\rho_k(P) < \infty$ for every $k \in \mathbb{N}$.

Proof. Because every finitely generated Puiseux monoid is isomorphic to a numerical semigroup, and numerical semigroups have finite k-elasticities, we can assume, without loss of generality, that P is not finitely generated.

To prove condition (1), suppose, by way of contradiction, that $\rho_k(P) = \infty$ for some $k \in \mathbb{N}$. Because 0 is not a limit point of P there exists $q \in \mathbb{Q}$ such that 0 < q < a for each $a \in \mathcal{A}(P)$. Let

$$\ell = \min\{n \in \mathbb{N} : |\mathcal{U}_n(P)| = \infty\}.$$

Clearly, $\ell \geq 2$. Let $m = \max \mathcal{U}_{\ell-1}(P)$. Now take $N \in \mathbb{N}$ sufficiently large such that, for each $a \in \mathcal{A}(P)$, a > N implies that $\mathsf{d}(a) > \ell$. As $\mathcal{U}_{\ell}(P)$ contains infinitely many elements, there exists $k \in \mathcal{U}_{\ell}(P)$ such that

$$k > \max\left\{\frac{\ell}{q}N, \ m+1\right\}.$$

In particular, k-1 is a strict upper bound for $\mathcal{U}_{\ell-1}(P)$. As $k \in \mathcal{U}_{\ell}(P)$, we can choose an element $x \in P$ such that $\{k,\ell\} \subseteq \mathsf{L}(x)$. Take $A = \{a_1,\ldots,a_k\} \subsetneq \mathcal{A}(P)$ and $B = \{b_1,\ldots,b_\ell\} \subsetneq \mathcal{A}(P)$ with

$$(4.1) a_1 + \dots + a_k = x = b_1 + \dots + b_{\ell}.$$

Observe that the sets A and B must be disjoint, for if $a \in A \cap B$, canceling a in (4.1) would yield that $\{\ell - 1, k - 1\} \subseteq \mathsf{L}(x - a)$, which contradicts that k - 1 is a strict upper bound for $\mathcal{U}_{\ell-1}(P)$. Because $k > (\ell/q)N$, it follows that

$$x > kq > \ell N$$
.

Therefore $b := \max\{b_1, \ldots, b_\ell\} > N$, which implies that $p = \mathsf{d}(b) > \ell$. Since $a_i \neq b$ for each $i = 1, \ldots, k$, it follows that $p \notin \mathsf{d}(\{a_1, \ldots, a_k\})$. We can assume, without loss of generality, that there exists $j \in \{1, \ldots, \ell\}$ such that $b_i \neq b$ for every $i \leq j$ and $b_{j+1} = \cdots = b_\ell = b$. This allows us to rewrite (4.1) as

(4.2)
$$(\ell - j)b = \sum_{i=1}^{k} a_i - \sum_{i=1}^{j} b_i.$$

After multiplying 4.2 by p times the product d of all the denominators of the atoms $\{a_1, \ldots, a_k, b_1, \ldots, b_j\}$, we find that p divides $d(\ell-j)b$. As $\gcd(p,d) = 1$ and $\ell-j < p$, it follows that p divides $\mathbf{n}(b)$, which is a contradiction. Hence we conclude that $\rho_k(P) < \infty$ for every $k \in \mathbb{N}$.

Now we argue the second condition. Let $\{a_n\}$ be an enumeration of the elements of $\mathcal{A}(P)$ such that $\{d(a_n)\}$ is an increasing sequence. Set $p_n = d(a_n)$. Since P has no stable atoms, $\lim \mathsf{n}(a_n) = \infty$. Let B be an upper bound for $\mathcal{A}(P)$.

Suppose, by way of contradiction, that $\rho_n(P) = \infty$ for some $n \in \mathbb{N}$. Let k be the smallest natural number such that $|\mathcal{U}_k(P)| = \infty$. Now take $\ell \in \mathcal{U}_k(P)$ large enough such that $\ell - 1 > \max \mathcal{U}_{k-1}(P)$ and for each $a \in \mathcal{A}(P)$ satisfying $a \leq Bk/\ell$ we have that $\mathsf{n}(a) > Bk$. Take $x \in \mathsf{L}^{-1}(k)$ such that $a_1 + \cdots + a_k = x = b_1 + \cdots + b_\ell$ for some $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in \mathcal{A}(P)$. Now set $b = \min\{b_1, \ldots, b_\ell\}$. Then

$$b \le \frac{b_1 + \dots + b_\ell}{\ell} = \frac{a_1 + \dots + a_k}{\ell} \le \frac{Bk}{\ell}.$$

Therefore $\mathsf{n}(b) > Bk$. We claim that $\mathsf{d}(b) \notin \mathsf{d}(\{a_1, \ldots, a_k\})$. Suppose by contradiction that this is not the case. Then $b = a_i$ for some $i \in \{1, \ldots, k\}$. This implies that $\{k-1, \ell-1\} \subseteq \mathsf{L}(x-b)$, contradicting that $\ell-1 > \max \mathcal{U}_{k-1}(P)$. Hence

 $d(b) \notin d(\{a_1, \ldots, a_k\})$. Now assume, without loss of generality, that there exists $j \in \{1, \ldots, \ell\}$ such that $b_i \neq b$ for each $i \leq j$ and $b_{j+1} = \cdots = b_{\ell} = b$. Write

(4.3)
$$(\ell - j)b = \sum_{i=1}^{k} a_i - \sum_{i=1}^{j} b_i.$$

From (4.3) we obtain that p_{ℓ} divides $\ell - j$. As a consequence,

$$Bk \ge \sum_{i=1}^{k} a_i \ge \frac{\ell - j}{p_\ell} \mathsf{n}(b) \ge \mathsf{n}(b) > Bk,$$

which is a contradiction. Hence $\rho_k(P) < \infty$ for every $k \in \mathbb{N}$.

The sufficient conditions in part (1) of Theorem 4.1(1) and the condition of boundedness in part (2) of Theorem 4.1 are not necessary, as the following example illustrates.

Example 4.2.

(1) Consider the primary Puiseux monoid

$$P = \left\langle \frac{n}{p_n} : n \in \mathbb{N} \right\rangle,$$

where $\{p_n\}$ is the increasing sequence of all prime numbers. Since $\mathcal{A}(P) = \{n/p_n : n \in \mathbb{N}\}$, it follows that P does not contain any stable atom. It is well known that the sequence $\{n/p_n\}$ converges to 0, which implies that P is bounded. Hence part (2) of Theorem 4.1 ensures that $\rho_k(P) < \infty$ for all $k \in \mathbb{N}$. Thus, the reverse implication of part (1) in Theorem 4.1 does not hold.

(2) Consider now the Puiseux monoid

$$P = \left\langle \frac{p_n^2 - 1}{p_n} : n \in \mathbb{N} \right\rangle,$$

where $\{p_n\}$ is any enumeration of the prime numbers. Since 0 is not a limit point of P, we can apply part (1) of Theorem 4.1 to conclude that $\rho_k(P) < \infty$ for all $k \in \mathbb{N}$. Notice, however, that P is not bounded. Therefore, the boundedness in part (2) of Theorem 4.1 is not a necessary condition.

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