# COMMUTATOR SUBGROUPS OF VIRTUAL AND WELDED BRAID GROUPS 

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#### Abstract

Let $V B_{n}$, resp. $W B_{n}$ denote the virtual, resp. welded, braid group on $n$ strands. We study their commutator subgroups $V B_{n}^{\prime}=\left[V B_{n}, V B_{n}\right]$ and, $W B_{n}^{\prime}=\left[W B_{n}, W B_{n}\right]$ respectively. We obtain a set of generators and defining relations for these commutator subgroups. In particular, we prove that $V B_{n}^{\prime}$ is finitely generated if and only if $n \geq 4$, and $W B_{n}^{\prime}$ is finitely generated for $n \geq 3$. Also we prove that $V B_{3}^{\prime} / V B_{3}^{\prime \prime}=\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}^{\infty}, V B_{4}^{\prime} / V B_{4}^{\prime \prime}=\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3}$, $W B_{3}^{\prime} / W B_{3}^{\prime \prime}=\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}, W B_{4}^{\prime} / W B_{4}^{\prime \prime}=\mathbb{Z}_{3}$, and for $n \geq 5$ the commutator subgroups $V B_{n}^{\prime}$ and $W B_{n}^{\prime}$ are perfect, i. e. the commutator subgroup is equal to the second commutator subgroup.


## 1. Introduction

Virtual braid groups $V B_{n}$ on $n$ strands are certain extensions of the classical braid groups. It was introduced by L. Kauffman [10] (see also [14). Virtual braids play the same role in the virtual knot theory that classical braids played in the classical knot theory. In particular, like closures of classical braids represent classical knots and links, the closure of virtual braids represent the virtual knots and links (see 9, [10]). On connections of virtual braids with the virtual knot theory, see [3, 4]. For a structure of the virtual braid groups, see [2].

The welded braid group $W B_{n}$ is a quotient of $V B_{n}$. This group is called the group of conjugating automorphisms [13, 1], the braid-permutation group [7] and so on. For several notions of this group and their equivalence, see [5].

The commutator subgroup $B_{n}^{\prime}$ of the classical braid group $B_{n}$ is studied in the paper [8] (see also [12]). The following facts follow from these papers:

- $B_{n}^{\prime}$ is finitely presented for all $n \geq 2$;
- $B_{3}^{\prime}$ is a free group of rank two;
- $B_{4}^{\prime}$ is a semi-direct product of two free groups of rank two;
- for $n>4$ the second commutator subgroup $B_{n}^{\prime \prime}$ of $B_{n}$ coincides with the first commutator subgroup $B_{n}^{\prime}$, i.e. $B_{n}^{\prime}$ is perfect.

[^0]In the present paper we investigate the commutator subgroups $V B_{n}^{\prime}$ and $W B_{n}^{\prime}$. Our main result is the following.

Theorem 1.1. The commutator subgroup $V B_{3}^{\prime}$ is infinitely generated. For $n \geq 4$ the commutator subgroup $V B_{n}^{\prime}$ can be generated by $2 n-3$ elements.

To prove Theorem 1.1]we obtain a presentation of $V B_{n}^{\prime}$ using the classical method of Reidemeister-Schreier, and then remove certain generators and relations using Tietze transformations. As a consequence of Theorem 1.1, we further have the following corollaries.

Corollary 1.2. (1) The quotient $V B_{3}^{\prime} / V B_{3}^{\prime \prime}$ is isomorphic to the direct product $\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}^{\infty}$, where $\mathbb{Z}^{\infty}$ is the direct product of counting number of $\mathbb{Z}$.
(2) The quotient $V B_{4}^{\prime} / V B_{4}^{\prime \prime}$ is isomorphic to the direct product $\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3}$.
(3) For $n \geq 5, V B_{n}^{\prime}$ is perfect, that is $V B_{n}^{\prime}=V B_{n}^{\prime \prime}$.

Recently the commutator subgroup $W B_{n}^{\prime}$ of the welded braid group has been investigated by Zaremsky in [15], who proved that $W B_{n}^{\prime}$ is finitely presented if and only if $n \geq 4$. Zaremsky proved this result using discrete Morse theory, without constructing explicit finite presentation. Dey and Gongopadhyay 6 also proved that $W B_{n}^{\prime}$ is finitely generated for all $n \geq 3$. In the present paper we have found a better bound on the number of generators than in [6]. We prove the following result.
Theorem 1.3. (1) The commutator subgroup $W B_{n}^{\prime}$ can be generated by $n$ elements for all $n \geq 4$, and $W B_{3}^{\prime}$ can be generated by 4 elements.
(2) The quotient $W B_{3}^{\prime} / W B_{3}^{\prime \prime}$ is isomorphic to the direct product $\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}$.
(3) The quotient $W B_{4}^{\prime} / W B_{4}^{\prime \prime}$ is isomorphic to $\mathbb{Z}_{3}$.
(4) For $n \geq 5, W B_{n}^{\prime}$ is perfect.

The presentation of $W B_{n}^{\prime}$ obtained in this paper is slightly different from the one obtained in [6. We obtain this presentation using the presentation of $V B_{n}$, while computing that we use successive conjugation rule in the rewritting process, see Lemma 3.1. This simple conjugation tirck has given an alternative presentation of $W B_{n}^{\prime}$ where the elimination of generators become simpler, and consequently we get a better bound on the number of generators.

We now briefly describe the structure of the paper. We recall the necessary preliminaries in Section 2. Using Reidemeister-Schreier method, we first obtain a general presentation of $V B_{n}^{\prime}$, see Theorem 3.9 in Section 3. We prove Theorem 1.1 in Section 4. Because of the differences of the nature of the proofs for $n \geq 4$ and $n=3$, As the cases $n=3$ and $n \geq 4$ are different, accordingly, the proof of Theorem 1.1 is divided over two subsections. In Section 4.1, first, we apply Tietze transformations to remove certain generators from the presentation in Theorem 3.9. This gives a finite generating set for $V B_{n}^{\prime}$ for $n \geq 4$. In Section 4.2, we show that $V B_{3}^{\prime}$ is infinitely generated. Combining these results, Theorem 1.1 is obtained. We prove Theorem 1.3 in Section 5 .

Finally, we note the following problems that remain to be answered.

Problem 1. Is it true that the commutator subgroup $V B_{n}^{\prime}$ is not finitely presented for $n \geq 4$.

We expect the answer to be yes, but it is not clear how.
Problem 2. Construct explicit finite presentation of $W B_{n}^{\prime}$ for $n \geq 3$.
Problem 3. Let $G$ is a group from the set $\left\{V B_{3}, V B_{4}, W B_{3}, W B_{4}\right\}$. Find the quotients $G^{(i)} / G^{(i+1)}, i=2,3, \ldots$, where $G^{(k)}$ is the $k$-th commutator subgroup:

$$
G^{(1)}=G^{\prime}, G^{(k+1)}=\left[G^{(k)}, G^{(k)}\right], k=1,2, \ldots
$$

## 2. Preliminaries

2.1. Group of Virtual Braids. The virtual braid group of $n$ strands $V B_{n}$ is generated by the classical braid group $B_{n}=\left\langle\sigma_{1}, \ldots, \sigma_{n-1}\right\rangle$ and the symmetric group $S_{n}=\left\langle\rho_{1}, \ldots, \rho_{n-1}\right\rangle$. The generators $\sigma_{i}, i=1, \ldots, n-1$ satisfy the relations

$$
\begin{array}{ccc}
\sigma_{i} \sigma_{j}=\sigma_{j} \sigma_{i} & \text { for } & |i-j| \geq 2 \\
\sigma_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \sigma_{i+1} & \text { for } & i=1, \ldots, n-2
\end{array}
$$

The generators $\rho_{i}, i=1, \ldots, n-1$ satisfy the relations of symmetric group $S_{n}$ :

$$
\begin{array}{ccc}
\rho_{i}^{2}=1 & \text { for } & i=1,2, \ldots, n-1 \\
\rho_{i} \rho_{j}=\rho_{j} \rho_{i} & \text { for } & |i-j| \geq 2 \\
\rho_{i} \rho_{i+1} \rho_{i}=\rho_{i+1} \rho_{i} \rho_{i+1} & \text { for } & i=1,2 \ldots, n-2
\end{array}
$$

Other defining relations of $V B_{n}$ are mixed and have the form

$$
\begin{array}{cl}
\sigma_{i} \rho_{j}=\rho_{j} \sigma_{i} & \text { for } \\
\rho_{i} \rho_{i+1} \sigma_{i}=\sigma_{i+1} \rho_{i} \rho_{i+1} & \text { for } \quad i=1,2, \ldots, n-2
\end{array}
$$

2.2. Reidemeister-Schreier Algorithm. Given a presentation of a group $G$, this algorithm allows one to find a presentation of a subgroup $H \subset G$. To obtain the presentation of $H$, it is necessary to find a Schreier's set of right coset of the group $G$ over the subgroup $H$. We give a formal description of this process, for more details see [11].

Let $a_{1}, \ldots, a_{n}$ be the generators of the group $G$ and $R_{1}, \ldots, R_{m}$ be the set of defining relations for the given set of generators. System of words $N=\left\{K_{\alpha}, \alpha \in A\right\}$ on generators $a_{1}, \ldots, a_{n}$ defines a Schreier's system for the subgroup $H \subset G$ relative to the system of generators $a_{1}, \ldots, a_{n}$ if the next conditions are satisfied:

1) in every right coset of the group $G$ over $H$ there is only one word from the system $N$;
2) if the word $K_{\alpha}=a_{i_{1}}^{\varepsilon_{1}} \ldots a_{i_{p-1}}^{\varepsilon_{p-1}} a_{i_{p}}^{\varepsilon_{p}},\left(\varepsilon_{j}= \pm 1\right)$ lies in $N$, then the word $a_{i_{1}}^{\varepsilon_{1}} \ldots a_{i_{p-1}}^{\varepsilon_{p-1}}$ also lies in $N$.

Suppose that some Schreier's system $N$ is chosen for the subgroup $H \subset G$ relative to the system generators $a_{1}, \ldots, a_{n}$ of $G$. For every word $Q$ on $a_{1}, \ldots, a_{n}$, we denote by $\bar{Q}$ the only word from $N$ which lies in the same right coset of $G$ over the subgroup $H$. Denote

$$
S_{K_{\alpha}, a_{\nu}}=K_{\alpha} a_{\nu} \cdot\left(\overline{K_{\alpha} a_{\nu}}\right)^{-1}, \quad \alpha \in A, \nu=1, \ldots, n
$$

Theorem of Reidemeister-Schreier states that the elements $S_{K_{\alpha}, a_{\nu}}$ generate subgroup $H$ and the set of defining relations for this set of generators is divided in two parts. First part consists of trivial relations $S_{K_{\alpha}, a_{\nu}}=1$, where the pair $K_{\alpha}, a_{\nu}$ is such that the word $K_{\alpha} a_{\nu} \cdot\left(\overline{K_{\alpha} a_{\nu}}\right)^{-1}$ is freely equivalent to the word 1 . Second part consists of all relations of the form $\tau\left(K_{\alpha} R_{\mu} K_{\alpha}^{-1}\right)$, where $\alpha \in A, \mu=1, \ldots, m$, and $\tau$ is Reidemeister's transformation, which maps every nonempty word $a_{i_{1}}^{\varepsilon_{1}} \ldots a_{i_{p}}^{\varepsilon_{p}},\left(\varepsilon_{j}=\right.$ $\pm 1$ ) from symbols $a_{1}, \ldots, a_{n}$ to the word from symbols $S_{K_{\alpha}, a_{\nu}}$ by the rule:

$$
\tau\left(a_{i_{1}}^{\varepsilon_{1}} \ldots a_{i_{p}}^{\varepsilon_{p}}\right)=S_{K_{i_{1}}, a_{i_{1}}}^{\varepsilon_{1}} \ldots S_{K_{i_{p}}, a_{i_{p}}}^{\varepsilon_{p}}
$$

where $K_{i_{j}}=\overline{a_{i_{1}}^{\varepsilon_{1}} \ldots a_{i_{j-1}}^{\varepsilon_{j-1}}}$, if $\varepsilon_{j}=1$, and $K_{i_{j}}=\overline{a_{i_{1}}^{\varepsilon_{1}} \ldots a_{i_{j}}^{\varepsilon_{j}}}$, if $\varepsilon_{j}=-1$.

## 3. Commutator subgroup $V B_{n}^{\prime}$

3.1. Generating set of $V B_{n}^{\prime}$. From the above relations it follows that the quotient $V B_{n} / V B_{n}^{\prime}$ is isomorphic to the direct product $\mathbb{Z} \times \mathbb{Z}_{2}$. One can define the map $\varphi$ from the following short exact sequence:

$$
1 \rightarrow V B_{n}^{\prime} \rightarrow V B_{n} \xrightarrow{\varphi} \mathbb{Z} \times \mathbb{Z}_{2} \rightarrow 1
$$

where, for $i=1, \ldots, n-1, \varphi\left(\sigma_{i}\right)$ is the generator of $\mathbb{Z}$ and $\phi\left(\rho_{i}\right)$ is the generator of $\mathbb{Z}_{2}$ respectively when viewing it as $V B_{n} / V B_{n}^{\prime}$. The map $\varphi$ does have a section in the above short exact sequence for $n \geq 3$, and $\operatorname{ker} \varphi=V B_{n}^{\prime}$.

As a Schreier set of coset representatives of $V B_{n}$ by $V B_{n}^{\prime}$ take the words

$$
\Lambda=\left\{\sigma_{1}^{i} \rho_{1}^{\varepsilon} \mid i \in \mathbb{Z}, \varepsilon=0,1\right\}
$$

The commutator subgroup $V B_{n}^{\prime}$ is generated by the words

$$
S_{\lambda, a}=\lambda a(\overline{\lambda a})^{-1}, \quad \lambda \in \Lambda, \quad a \in\left\{\sigma_{1}, \ldots, \sigma_{n-1}, \rho_{1}, \ldots, \rho_{n-1}\right\} .
$$

Here $\bar{w}$ is a coset representative in $\Lambda$ of $w V B_{n}^{\prime}$. Find the elements $S_{\lambda, a}$. For this put $\lambda=\sigma_{1}^{i} \rho_{1}^{\varepsilon}$ and considering different $a$ we will get the following cases:

1) If $a=\sigma_{1}$, then

$$
S_{\lambda, \sigma_{1}}=\sigma_{1}^{i} \rho_{1}^{\varepsilon} \sigma_{1}\left(\sigma_{1}^{i+1} \rho_{1}^{\varepsilon}\right)^{-1}
$$

For $\varepsilon=0$ we have $S_{\lambda, \sigma_{1}}=1$ and for $\varepsilon=1$ we have $S_{\lambda, \sigma_{1}}=\sigma_{1}^{i}\left(\rho_{1} \sigma_{1} \rho_{1} \sigma_{1}^{-1}\right) \sigma_{1}^{-i}$, which we will denote by $a_{i}$.
2) If $a=\sigma_{2}$, then

$$
S_{\lambda, \sigma_{2}}=\sigma_{1}^{i}\left(\rho_{1}^{\varepsilon} \sigma_{2} \rho_{1}^{\varepsilon} \sigma_{1}^{-1}\right) \sigma_{1}^{-i}
$$

and we will denote this element by $b_{i, \varepsilon}$.
3) If $a=\sigma_{l}, l>2$, then

$$
S_{\lambda, \sigma_{l}}=\sigma_{l} \sigma_{1}^{-1}
$$

and we will denote this element by $c_{l}$.
4) If $a=\rho_{1}$, then

$$
S_{\lambda, \rho_{1}}=1
$$

5) If $a=\rho_{2}$, then

$$
S_{\lambda, \rho_{2}}=\sigma_{1}^{i}\left(\rho_{1}^{\varepsilon} \rho_{2} \rho_{1}^{\varepsilon+1}\right) \sigma_{1}^{-i},
$$

and we will denote this element by $f_{i, \varepsilon}$.
6) If $a=\rho_{l}, l>2$, then

$$
S_{\lambda, \rho_{l}}=\sigma_{1}^{i}\left(\rho_{l} \rho_{1}\right) \sigma_{1}^{-i}
$$

and we will denote this element by $g_{i, l}$.
To find defining relations of $V B_{n}^{\prime}$ we will use the following conjugation rules by elements $\rho_{1}$ and $\sigma_{1}^{-m}$.
Lemma 3.1. The following formulas hold
(1) $a_{i}^{\sigma_{1}^{-m}}=a_{i+m}, \quad b_{i, \varepsilon}^{\sigma_{1}^{-m}}=b_{i+m, \varepsilon}, \quad c_{l}^{\sigma_{1}^{-m}}=c_{l}, \quad f_{i, \varepsilon}^{\sigma_{1}^{-m}}=f_{i+m, \varepsilon}, \quad g_{i, \varepsilon}^{\sigma_{1}^{-m}}=$ $g_{i+m, \varepsilon} ;$
(2) $a_{0}^{\rho_{1}}=a_{0}^{-1}, \quad b_{0,0}^{\rho_{1}}=b_{0,1} a_{0}^{-1}, \quad b_{0,1}^{\rho_{1}}=b_{0,0} a_{0}^{-1}, \quad b_{1,0}^{\rho_{1}}=a_{0} b_{1,1} a_{1}^{-1} a_{0}^{-1}, \quad b_{2,0}^{\rho_{1}}=$ $a_{0} a_{1}\left(b_{2,1} a_{2}^{-1}\right) a_{1}^{-1} a_{0}^{-1} ;$
(3) $c_{l}^{\rho_{1}}=c_{l} a_{0}^{-1}, \quad f_{0,0}^{\rho_{1}}=f_{0,1}, \quad f_{0,1}^{\rho_{1}}=f_{0,0}, \quad f_{1,0}^{\rho_{1}}=a_{0} f_{1,1} a_{0}^{-1}, \quad f_{1,1}^{\rho_{1}}=a_{0} f_{1,0} a_{0}^{-1}$;
(4) $g_{0, i}^{\rho_{1}}=g_{0, i}, \quad g_{1, i}^{\rho_{1}}=a_{0} g_{1, i} a_{0}^{-1}, \quad i>2$.

Proof. (1) follow from the definition.
For proving (2) note that:

$$
\begin{gathered}
\rho_{1} a_{0} \rho_{1}=\rho_{1} \rho_{1} \sigma_{1} \rho_{1} \sigma_{1}^{-1} \rho_{1}=S_{1, \sigma_{1}} S_{\sigma_{1}, \rho_{1}} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}}=a_{0}^{-1}, \\
\rho_{1} b_{0,0} \rho_{1}=\rho_{1} \sigma_{2} \sigma_{1}^{-1} \rho_{1}=S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{2}} S_{\rho_{1,, \sigma_{1}}^{-1}}^{-1} S_{\rho_{1}, \rho_{1}}=b_{0,1} a_{0}^{-1}, \\
b_{0,1}^{\rho_{1}}=\rho_{1} \rho_{1} \sigma_{2} \rho_{1} \sigma_{1}^{-1} \rho_{1}=S_{1, \sigma_{2}} S_{\sigma_{1}, \rho_{1}}^{-1} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}}=b_{0,0} a_{0}^{-1}, \\
\rho_{1} b_{1,0} \rho_{1}=\rho_{1} \sigma_{1} \sigma_{2} \sigma_{1}^{-1} \sigma_{1}^{-1} \rho_{1}=S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{1}} S_{\sigma_{1} \rho_{1}, \sigma_{2}}^{-1} S_{\sigma_{1} \rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \sigma_{1}}=a_{0} b_{1,1} a_{1}^{-1} a_{0}^{-1} . \\
\rho_{1} b_{2,0} \rho_{1}=\rho_{1} \sigma_{1}^{2} \sigma_{2}\left(\sigma_{1}^{-1}\right)^{3} \rho_{1}= \\
=S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{1}} S_{\sigma_{1} \rho_{1}, \sigma_{1}} S_{\sigma_{1}^{2} \rho_{1}, \sigma_{2}} S_{\sigma_{1}^{2} \rho_{1}, \sigma_{1}}^{-1} S_{\sigma_{1} \rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}}=a_{0} a_{1}\left(b_{2,1} a_{2}^{-1}\right) a_{1}^{-1} a_{0}^{-1} .
\end{gathered}
$$

For (3):

$$
\begin{gathered}
\rho_{1} c_{l} \rho_{1}=\rho_{1} \sigma_{l} \sigma_{1}^{-1} \rho_{1}=S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{l}} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}}=c_{l} a_{0}^{-1}, \\
f_{0,0}^{\rho_{1}}=\rho_{1} \rho_{2} \rho_{1} \rho_{1}=\rho_{1} \rho_{2}=f_{0,1}, \\
f_{0,1}^{\rho_{1}}=\rho_{1} \rho_{1} \rho_{2} \rho_{1}=\rho_{2} \rho_{1}=f_{0,0}, \\
f_{1,0}^{\rho_{1}}=\rho_{1} \sigma_{1} \rho_{2} \rho_{1} \sigma_{1}^{-1} \rho_{1}=S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{1}} S_{\sigma_{1} \rho_{1}, \rho_{2}} S_{\sigma_{1}, \rho_{1}} S_{\rho_{1, \sigma_{1}}^{-1}}^{-1} S_{\rho_{1}, \rho_{1}}=a_{0} f_{1,1} a_{0}^{-1} . \\
f_{1,1}^{\rho_{1}}=\rho_{1} \sigma_{1} \rho_{1} \rho_{2} \sigma_{1}^{-1} \rho_{1}=S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{1}} S_{\sigma_{1} \rho_{1}, \rho_{1}} S_{\sigma_{1}, \rho_{2}} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}}=a_{0} f_{1,0} a_{0}^{-1} .
\end{gathered}
$$

$$
\begin{equation*}
g_{0, i}^{\rho_{1}}=\rho_{1} \rho_{i} \rho_{1} \rho_{1}=\rho_{1} \rho_{i}=g_{0, i}, \tag{4}
\end{equation*}
$$

$$
g_{1, i}^{\rho_{1}}=\rho_{1} \sigma_{1} \rho_{i} \rho_{1} \sigma_{1}^{-1} \rho_{1}=S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{1}} S_{\sigma_{1} \rho_{1}, \rho_{i}} S_{\sigma_{1}, \rho_{1}} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}}=a_{0} g_{1, i} a_{0}^{-1}, \quad i>2
$$

This proves the lemma.
3.2. Defining Relations in $V B_{n}^{\prime}$. In this subsection we will consider the defining relations of $V B_{n}$, rewrite them in the generators of $V B_{n}^{\prime}$, and conjugating by elements $\lambda \in \Lambda$, we get the defining relations of $V B_{n}^{\prime}$.
3.2.1. Defining relation of $V B_{n}^{\prime}$ that follow from the relation $\sigma_{i} \sigma_{j}=\sigma_{j} \sigma_{i}$. Rewrite this relation in the form

$$
r_{1}=\sigma_{i} \sigma_{j} \sigma_{i}^{-1} \sigma_{j}^{-1}, 1 \leq i<j \leq n-1, i+1<j
$$

Using the rewritting process we get:

$$
\begin{aligned}
& \text { for } i=1: r_{1}=\sigma_{1} \sigma_{j} \sigma_{1}^{-1} \sigma_{j}^{-1}=S_{1, \sigma_{1}} S_{\sigma_{1}, \sigma_{j}} S_{\sigma_{1}, \sigma_{1}}^{-1} S_{\sigma_{1}, \sigma_{j}}^{-1}=c_{j} c_{j}^{-1}=1 \\
& \text { for } i=2: r_{1}=\sigma_{2} \sigma_{j} \sigma_{2}^{-1} \sigma_{j}^{-1}=S_{1, \sigma_{2}} S_{\sigma_{1}, \sigma_{j}} S_{\sigma_{1}, \sigma_{2}}^{-} S_{\sigma_{1}, \sigma_{j}}^{-1}=b_{0,0} c_{j} b_{1,0}^{-1} c_{j}^{-1} \\
& \text { for } i>2: r_{1}=\sigma_{i} \sigma_{j} \sigma_{i}^{-1} \sigma_{j}^{-1}=S_{1, \sigma_{i}} S_{\sigma_{1}, \sigma_{j}} S_{\sigma_{1}, \sigma_{i}}^{-1} S_{\sigma_{1}, \sigma_{j}}^{-1}=c_{i} c_{j} c_{i}^{-1} c_{j}^{-1}
\end{aligned}
$$

Lemma 3.2. The following four types of relations in $V B_{n}^{\prime}$ follow from the relation $r_{1}$ of $V B_{n}$ :

$$
\begin{gathered}
b_{m, 0} c_{j} b_{m+1,0}^{-1} c_{j}^{-1}=1, \quad j \geq 4, \\
c_{i} c_{j} c_{i}^{-1} c_{j}^{-1}=1, \quad i \geq 3, \quad j>i+1 \\
b_{m, 1} a_{m}^{-1} c_{j} a_{m+1} b_{m+1,1}^{-1} c_{j}^{-1}=1, \quad j \geq 4 \\
c_{i} a_{m}^{-1} c_{j} c_{i}^{-1} a_{m} c_{j}^{-1}=1, \quad i \geq 3, \quad j>i+1 .
\end{gathered}
$$

Proof. Conjugating of $r_{1}$ by $\rho_{1}$ and using Lemma 3.1 we get

$$
\begin{gathered}
\rho_{1}\left(b_{0,0} c_{j} b_{1,0}^{-1} c_{j}^{-1}\right) \rho_{1}=b_{0,1} a_{0}^{-1} c_{j} a_{1} b_{1,1}^{-1} c_{j}^{-1} \\
\rho_{1}\left(c_{i} c_{j} c_{i}^{-1} c_{j}^{-1}\right) \rho_{1}=c_{i} a_{0}^{-1} c_{j} c_{i}^{-1} a_{0}^{-1} c_{j}^{-1}
\end{gathered}
$$

Conjugating $r_{1}$ and $\rho_{1} r_{1} \rho_{1}$ by $\sigma_{1}^{-m}$, we get the need relations.
3.2.2. Defining relations of $V B_{n}^{\prime}$ that follow from the relation $\sigma_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \sigma_{i+1}$. Rewrite this relation in the form

$$
r_{2}=\sigma_{i} \sigma_{i+1} \sigma_{i} \sigma_{i+1}^{-1} \sigma_{i}^{-1} \sigma_{i+1}^{-1}
$$

Then
for $i=1$ : $r_{2}=\sigma_{1} \sigma_{2} \sigma_{1} \sigma_{2}^{-1} \sigma_{1}^{-1} \sigma_{2}^{-1}=S_{1, \sigma_{1}} S_{\sigma_{1}, \sigma_{2}} S_{\sigma_{1}^{2}, \sigma_{1}} S_{\sigma_{1}^{2}, \sigma_{1}}^{-1} S_{\sigma_{1}, \sigma_{1}}^{-1} S_{1, \sigma_{2}}^{-1}=b_{1,0} b_{2,0}^{-1} b_{0,0}^{-1}$.
for $i=2: \quad r_{2}=\sigma_{2} \sigma_{3} \sigma_{2} \sigma_{3}^{-1} \sigma_{2}^{-1} \sigma_{3}^{-1}=S_{1, \sigma_{2}} S_{\sigma_{1}, \sigma_{3}} S_{\sigma_{1}^{2}, \sigma_{2}} S_{\sigma_{1}^{2}, \sigma_{3}}^{-1} S_{\sigma_{1}, \sigma_{2}}^{-1} S_{1, \sigma_{3}}^{-1}=b_{0,0} c_{3} b_{2,0} c_{3}^{-1} b_{1,0}^{-1} c_{3}^{-1}$.
for $i>2: \quad r_{2}=\sigma_{i} \sigma_{i+1} \sigma_{i} \sigma_{i+1}^{-1} \sigma_{i}^{-1} \sigma_{i+1}^{-1}=S_{1, \sigma_{i}} S_{\sigma_{1}, \sigma_{i+1}} S_{\sigma_{1}^{2}, \sigma_{i}} S_{\sigma_{1}^{2}, \sigma_{i+1}}^{-1} S_{\sigma_{1}, \sigma_{i}}^{-1} S_{1, \sigma_{i+1}}^{-1}=$ $c_{i} c_{i+1} c_{i} c_{i+1}^{-1} c_{i}^{-1} c_{i+1}^{-1}$.
Lemma 3.3. The following six types of relations in $V B_{n}^{\prime}$ follow from the relation $r_{2}$ of $V B_{n}$ :

$$
\begin{gathered}
b_{m+1,0} b_{m+2,0}^{-1} b_{m, 0}^{-1}=1, \\
b_{m, 0} c_{3} b_{m+2,0} c_{3}^{-1} b_{m+1,0}^{-1} c_{3}^{-1}=1, \\
c_{i} c_{i+1} c_{i} c_{i+1}^{-1} c_{i}^{-1} c_{i+1}^{-1}=1, \quad i \geq 3, \\
a_{m} b_{m+1,1} a_{m+2} b_{m+2,1}^{-1} a_{m+1}^{-1} b_{m, 1}^{-1}=1, \\
b_{m, 1} a_{m}^{-1} c_{3} a_{m+1} b_{m+2,1} a_{m+2}^{-1} a_{m+1}^{-1} c_{3}^{-1} a_{m} a_{m+1} b_{m+1,1}^{-1} c_{3}^{-1}=1, \\
c_{i} a_{m}^{-1} c_{i+1} a_{m}^{-1} c_{i} c_{i+1}^{-1} a_{m} c_{i}^{-1} a_{m} c_{i+1}^{-1}=1, \quad i \geq 3 .
\end{gathered}
$$

Proof. Conjugating $r_{2}$ by $\rho_{1}$ and using Lemma 3.1, we get

$$
\begin{gathered}
\left(b_{1,0} b_{2,0}^{-1} b_{0,0}^{-1}\right)^{\rho_{1}}=a_{0} b_{1,1} a_{2} b_{2,1}^{-1} a_{1}^{-1} b_{0,1}^{-1}, \\
\left(b_{0,0} c_{3} b_{2,0} c_{3}^{-1} b_{1,0}^{-1} c_{3}^{-1}\right)^{\rho_{1}}=b_{0,1} a_{0}^{-1} c_{3} a_{1} b_{2,1} a_{2}^{-1} a_{1}^{-1} c_{3}^{-1} a_{0} a_{1} b_{1,1}^{-1} c_{3}^{-1}, \\
\left(c_{i} c_{i+1} c_{i} c_{i+1}^{-1} c_{i}^{-1} c_{i+1}^{-1}\right)^{\rho_{1}}=c_{i} a_{0}^{-1} c_{i+1} a_{0}^{-1} c_{i} c_{i+1}^{-1} a_{0} c_{i}^{-1} a_{0} c_{i+1}^{-1} .
\end{gathered}
$$

Conjugating $r_{2}$ and $\rho_{1} r_{2} \rho_{1}$ by $\sigma_{1}^{-m}$, we get the need relations.
3.2.3. Defining relation that follow from the relation $\rho_{i}^{2}=1$. Rewrite this relation in the form

$$
r_{3}=\rho_{i}^{2} .
$$

Then
for $i=1: r_{3}=\rho_{1} \rho_{1}=S_{1, \rho_{1}} S_{\rho_{1}, \rho_{1}}=1$.
for $i=2: r_{3}=\rho_{3} \rho_{3}=S_{1, \rho_{2}} S_{\rho_{1}, \rho_{2}}=f_{0,0} f_{0,1}$.
for $i>2: r_{3}=\rho_{i} \rho_{i}=S_{1, \rho_{i}} S_{\rho_{1}, \rho_{i}}=g_{0, i}^{2}$.
The following lemma holds
Lemma 3.4. From $r_{3}$ follow two types of relations in $V B_{n}^{\prime}$ :

$$
f_{m, 0} f_{m, 1}=g_{m, i}^{2}=1, \quad i>2
$$

Proof. Conjugating $r_{3}$ by $\rho_{1}$ and using Lemma 3.1 4, we get

$$
\begin{aligned}
& \left(f_{0,0} f_{0,1}\right)^{\rho_{1}}=f_{0,1} f_{0,0} \\
& \left(g_{0, i}^{2}\right)^{\rho_{1}}=g_{0, i}^{2}, \quad i>2
\end{aligned}
$$

Conjugating $r_{3}$ and $\rho_{1} r_{3} \rho_{1}$ by $\sigma_{1}^{-m}$, we get the need relations.
3.2.4. Defining relations of $V B_{n}^{\prime}$ that follow from the relation $\rho_{i} \rho_{j}=\rho_{j} \rho_{i}$. Rewrite this relation in the form

$$
r_{4}=\rho_{i} \rho_{j} \rho_{i} \rho_{j}, 1 \leq i<j \leq n-1, i+1<j
$$

Then
for $i=1: r_{4}=\rho_{1} \rho_{j} \rho_{1} \rho_{j}=S_{1, \rho_{1}} S_{\rho_{1}, \rho_{j}} S_{1, \rho_{1}} S_{\rho_{1}, \rho_{j}}=g_{0, j}^{2}, j>2$.
We have got this relation when we considered the relation $r_{3}$.
for $i=2: r_{4}=\rho_{2} \rho_{k} \rho_{2} \rho_{k}=S_{1, \rho_{2}} S_{\rho_{1}, \rho_{k}} S_{1, \rho_{2}} S_{\rho_{1}, \rho_{k}}=\left(f_{0,0} g_{0, k}\right)^{2}, k>3$.
for $i>2: r_{4}=\rho_{i} \rho_{j} \rho_{i} \rho_{j}=S_{1, \rho_{i}} S_{\rho_{1}, \rho_{j}} S_{1, \rho_{i}} S_{\rho_{1}, \rho_{j}}=\left(g_{0, i} g_{0, j}\right)^{2}, 3 \leq i<j \leq n-1$, $i+1<j$.

The following lemma holds
Lemma 3.5. From the relation $r_{4}$ of $V B_{n}$, the following three types of relations of $V B_{n}^{\prime}$ follow:

$$
\begin{gathered}
\left(f_{m, 0} g_{m, k}\right)^{2}=\left(f_{m, 1} g_{m, k}\right)^{2}=1, \quad k>3 \\
\left(g_{m, i} g_{m, j}\right)^{2}=1, \quad 3 \leq i<j \leq n-1, \quad i+1<j
\end{gathered}
$$

Proof. Conjugated $r_{4}$ by $\rho_{1}$ and using Lemma 3.1 4), we get

$$
\begin{aligned}
\rho_{1}\left(f_{0,0} g_{0, k}\right)^{2} \rho_{1}=\left(f_{0,1} g_{0, k}\right)^{2}, \quad k>3, \\
\rho_{1}\left(g_{0, i} g_{0, j}\right)^{2} \rho_{1}=\left(g_{0, i} g_{0, j}\right)^{2}, \quad 3 \leq i<j \leq n-1, \quad i+1<j
\end{aligned}
$$

Conjugating $r_{4}$ and $\rho_{1} r_{4} \rho_{1}$ by $\sigma_{1}^{-m}$, we get the need relations.
3.2.5. Defining relations of $V B_{n}^{\prime}$ that follow from the relation $\rho_{i} \rho_{i+1} \rho_{i}=\rho_{i+1} \rho_{i} \rho_{i+1}$. Rewrite this relation in the form

$$
r_{5}=\rho_{i} \rho_{i+1} \rho_{i} \rho_{i+1} \rho_{i} \rho_{i+1} .
$$

Then
for $i=1: r_{5}=\rho_{1} \rho_{2} \rho_{1} \rho_{2} \rho_{1} \rho_{2}=S_{1, \rho_{1}} S_{\rho_{1}, \rho_{2}} S_{1, \rho_{1}} S_{\rho_{1}, \rho_{2}} S_{1, \rho_{1}} S_{\rho_{1}, \rho_{2}}=f_{0,1}^{3}$.
for $i=2: r_{5}=\rho_{2} \rho_{3} \rho_{2} \rho_{3} \rho_{2} \rho_{3}=S_{1, \rho_{2}} S_{\rho_{1}, \rho_{3}} S_{1, \rho_{2}} S_{\rho_{1}, \rho_{3}} S_{1, \rho_{2}} S_{\rho_{1}, \rho_{3}}=\left(f_{0,0} g_{0,3}\right)^{3}$.
for $i>2: r_{5}=\rho_{i} \rho_{i+1} \rho_{i} \rho_{i+1} \rho_{i} \rho_{i+1}=S_{1, \rho_{i}} S_{\rho_{1}, \rho_{i+1}} S_{1, \rho_{i}} S_{\rho_{1}, \rho_{i+1}} S_{1, \rho_{i}} S_{\rho_{1}, \rho_{i+1}}=\left(g_{0, i} g_{0, i+1}\right)^{3}$. The following lemma holds

Lemma 3.6. From the relation $r_{5}$ of $V B_{n}$, we have the following five types of relations of $V B_{n}^{\prime}$ :

$$
\begin{gathered}
f_{m, 1}^{3}=1 \\
\left(f_{m, 0} g_{m, 3}\right)^{3}=1 \\
\left(g_{m, i} g_{m, i+1}\right)^{3}=1 \\
f_{m, 0}^{3}=1 \\
\left(f_{m, 1} g_{m, 3}\right)^{3}=1
\end{gathered}
$$

Proof. Conjugating $r_{5}$ by $\rho_{1}$ and using Lemma 3.14), we get

$$
\begin{gathered}
\left(f_{0,1}^{3}\right)^{\rho_{1}}=f_{0,0}^{3} \\
\left(\left(f_{0,0} g_{0,3}\right)^{3}\right)^{\rho_{1}}=\left(f_{0,1} g_{0,3}\right)^{3}, \\
\left(\left(g_{0, i} g_{0, i+1}\right)^{3}\right)^{\rho_{1}}=\left(g_{0, i} g_{0, i+1}\right)^{3}, \quad i>2
\end{gathered}
$$

Conjugating $r_{5}$ and $\rho_{1} r_{5} \rho_{1}$ by $\sigma_{1}^{-m}$, we get the need relations.
3.2.6. Defining relations of $V B_{n}^{\prime}$ that follow from the relation $\sigma_{i} \rho_{j}=\rho_{j} \sigma_{i}$. Rewrite this relation in the form

$$
r_{6}=\sigma_{i} \rho_{j} \sigma_{i}^{-1} \rho_{j},|i-j|>1
$$

In dependence of $i$ and $j$ we will consider the next cases
a) $r_{6}=\sigma_{1} \rho_{i} \sigma_{1}^{-1} \rho_{i}=S_{1, \sigma_{1}} S_{\sigma_{1}, \rho_{i}} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{i}}=g_{1, i} a_{0}^{-1} g_{0, i}$, for $i>2$.
b) $r_{6}=\sigma_{2} \rho_{j} \sigma_{2}^{-1} \rho_{j}=S_{1, \sigma_{2}} S_{\sigma_{1}, \rho_{j}} S_{\rho_{1}, \sigma_{2}}^{-1} S_{\rho_{1}, \rho_{j}}=b_{0,0} g_{1, j} b_{0,1}^{-1} g_{0, j}$, for $j>3$.
c) $r_{6}=\sigma_{k} \rho_{l} \sigma_{k}^{-1} \rho_{l}=S_{1, \sigma_{k}} S_{\sigma_{1}, \rho_{l}} S_{\rho_{1}, \sigma_{k}}^{-1} S_{\rho_{1}, \rho_{l}}=c_{k} g_{1, l} c_{k}^{-1} g_{0, l}, k, l \geq 3,|l-k|>1$.
d) $r_{6}=\sigma_{i} \rho_{1} \sigma_{i}^{-1} \rho_{1}=S_{1, \sigma_{i}} S_{\sigma_{1}, \rho_{1}} S_{\rho_{1}, \sigma_{i}}^{-1} S_{\rho_{1}, \rho_{1}}=c_{i} c_{i}^{-1}=1$, for $i>2$.
e) $r_{6}=\sigma_{j} \rho_{2} \sigma_{j}^{-1} \rho_{2}=S_{1, \sigma_{j}} S_{\sigma_{1}, \rho_{2}} S_{\rho_{1}, \sigma_{j}}^{-1} S_{\rho_{1}, \rho_{2}}=c_{j} f_{1,0} c_{j}^{-1} f_{0,1}$, for $j>3$.

Now, the following lemma holds.

Lemma 3.7. From the relation $r_{6}$ of $V B_{n}$, the following seven types of relations of $V B_{n}^{\prime}$ follow:

$$
\begin{gathered}
g_{m+1, i} a_{m}^{-1} g_{m, i}=1, \quad i \geq 3 \\
a_{m} g_{m+1, i} g_{m, i}=1, \quad i \geq 3 \\
b_{m, 0} g_{m+1, j} b_{m, 1}^{-1} g_{m, j}=1, \quad j \geq 4, \\
b_{m, 1} g_{m+1, j} b_{m, 0}^{-1} g_{m, j}=1, \quad j \geq 4, \\
c_{k} g_{m+1, l} c_{k}^{-1} g_{m, l}=1, \quad k, l \geq 3, \quad|l-k|>1 \\
c_{j} f_{m+1,0} c_{j}^{-1} f_{m, 1}=1, \quad j \geq 4, \\
c_{j} f_{m+1,1} c_{j}^{-1} f_{m, 0}=1, \quad j \geq 4
\end{gathered}
$$

Proof. Conjugating $r_{6}$ by $\rho_{1}$ and using Lemma 3.1, we get

$$
\begin{gathered}
\left(g_{1, i} a_{0}^{-1} g_{0, i}\right)^{\rho_{1}}=a_{0} g_{1, i} g_{0, i}, \quad i \geq 3 \\
\left(b_{0,0} g_{1, j} b_{0,1}^{-1} g_{0, j}\right)^{\rho_{1}}=b_{0,1} g_{1, j} b_{0,0}^{-1} g_{0, j}, \quad j \geq 4 \\
\left(c_{k} g_{1, l} c_{k}^{-1} g_{0, l}\right)^{\rho_{1}}=c_{k} g_{1, l} c_{k}^{-1} g_{0, l}, \quad k, l \geq 3, \quad|l-k|>1 \\
\left(c_{j} f_{1,0} c_{j}^{-1} f_{0,1}\right)^{\rho_{1}}=c_{j} f_{1,0} c_{j}^{-1} f_{0,1}, \quad j \geq 4
\end{gathered}
$$

Conjugating $r_{6}$ and $\rho_{1} r_{6} \rho_{1}$ by $\sigma_{1}^{-m}$, we get the need relations.
3.2.7. Defining relations of $V B_{n}^{\prime}$ that follow from the relation $\rho_{i} \rho_{i+1} \sigma_{i}=\sigma_{i+1} \rho_{i} \rho_{i+1}$. Rewrite this relation in the form

$$
r_{7}=\rho_{i} \rho_{i+1} \sigma_{i} \rho_{i+1} \rho_{i} \sigma_{i+1}^{-1}
$$

Then
for $i=1$ : $r_{7}=\rho_{1} \rho_{2} \sigma_{1} \rho_{2} \rho_{1} \sigma_{2}^{-1}=S_{1, \rho_{1}} S_{\rho_{1}, \rho_{2}} S_{1, \sigma_{1}} S_{\sigma_{1}, \rho_{2}} S_{\sigma_{1} \rho_{1}, \rho_{1}} S_{1, \sigma_{2}}^{-1}=f_{0,1} f_{1,0} b_{0,0}^{-1}$.
for $i=2: r_{7}=\rho_{2} \rho_{3} \sigma_{2} \rho_{3} \rho_{2} \sigma_{3}^{-1}=S_{1, \rho_{2}} S_{\rho_{1}, \rho_{3}} S_{1, \sigma_{2}} S_{\sigma_{1}, \rho_{3}} S_{\sigma_{1} \rho_{1}, \rho_{2}} S_{1, \sigma_{3}}^{-1}=f_{0,0} g_{0,3} b_{0,0} g_{1,3} f_{1,1} c_{3}^{-1}$.
for $i>2: r_{7}=\rho_{i} \rho_{i+1} \sigma_{i} \rho_{i+1} \rho_{i} \sigma_{i+1}^{-1}=S_{1, \rho_{i}} S_{\rho_{1}, \rho_{i+1}} S_{1, \sigma_{i}} S_{\sigma_{1}, \rho_{i+1}} S_{\sigma_{1} \rho_{1}, \rho_{i}} S_{1, \sigma_{i+1}}^{-1}=$ $g_{0, i} g_{0, i+1} c_{i} g_{1, i+1} g_{1, i} c_{i+1}^{-1}$.

Now the following defining relations of $V B_{n}^{\prime}$ follow from the relation $r_{7}$.
Lemma 3.8. The following five types of relations in $V B_{n}^{\prime}$ follow from the relation $r_{7}$ of $V B_{n}$ :

$$
\begin{gathered}
f_{m, 1} f_{m+1,0} b_{m, 0}^{-1}=1 \\
f_{m, 0} a_{m} f_{m+1,1} b_{m, 1}^{-1}=1 \\
f_{m, 0} g_{m, 3} b_{m, 0} g_{m+1,3} f_{m+1,1} c_{3}^{-1}=1 \\
f_{m, 1} g_{m, 3} b_{m, 1} g_{m+1,3} f_{m+1,0} c_{3}^{-1}=1 \\
g_{m, i} g_{m, i+1} c_{i} g_{m+1, i+1} g_{m+1, i} c_{i+1}^{-1}=1, \quad i>2
\end{gathered}
$$

Proof. Conjugating $r_{7}$ by $\rho_{1}$ and using Lemma 3.1 2), 4) 5) and 6), we get

$$
\begin{gathered}
\left(f_{0,1} f_{1,0} b_{0,0}^{-1}\right)^{\rho_{1}}=f_{0,0} a_{0} f_{1,1} b_{0,1}^{-1}, \\
\left(f_{0,0} g_{0,3} b_{0,0} g_{1,3} f_{1,1} c_{3}^{-1}\right)^{\rho_{1}}=f_{0,1} g_{0,3} b_{0,1} g_{1,3} f_{1,0} c_{3}^{-1} \\
\left(g_{0, i} g_{0, i+1} c_{i} g_{1, i+1} g_{1, i} c_{i+1}^{-1}\right)^{\rho_{1}}=g_{0, i} g_{0, i+1} c_{i} g_{1, i+1} g_{1, i} c_{i+1}^{-1}, \quad i>2 .
\end{gathered}
$$

Conjugating $r_{7}$ and $\rho_{1} r_{7} \rho_{1}$ by $\sigma_{1}^{-m}$, we get the need relations.
Using the relations

$$
f_{m, 0} f_{m, 1}=1, \text { (see Lemma 3.4), }
$$

we can remove elements $f_{m, 1}, m \in \mathbb{Z}$, from the generating set and keep only elements

$$
f_{m, 0}=f_{m}, \quad m \in \mathbb{Z}
$$

The following result gives a presentation of $V B_{n}^{\prime}$.
Theorem 3.9. The commutator subgroup $V B_{n}^{\prime}$ is generated by elements

$$
a_{m}, \quad b_{m, \varepsilon}, \quad c_{l}, \quad f_{m}, \quad g_{m, l},
$$

where $m \in \mathbb{Z}, \varepsilon=0,1,2<l<n$ and is defined by the relations

$$
\begin{gathered}
b_{m, 0} c_{j} b_{m+1,0}^{-1} c_{j}^{-1}=1, \quad j \geq 4, \\
c_{i} c_{j} c_{i}^{-1} c_{j}^{-1}=1, \quad i \geq 3, \quad j>i+1, \\
b_{m, 1} a_{m}^{-1} c_{j} a_{m+1} b_{m+1,1}^{-1} c_{j}^{-1}=1, \quad j \geq 4, \\
c_{i} a_{m}^{-1} c_{j} c_{i}^{-1} a_{m} c_{j}^{-1}=1, \quad i \geq 3, \quad j>i+1 . \\
b_{m+1,0} b_{m+2,0}^{-1} b_{m, 0}^{-1}=1, \\
b_{m, 0} c_{3} b_{m+2,0} c_{3}^{-1} b_{m+1,0}^{-1} c_{3}^{-1}=1, \\
c_{i} c_{i+1} c_{i} c_{i+1}^{-1} c_{i}^{-1} c_{i+1}^{-1}=1, \quad i \geq 3, \\
a_{m} b_{m+1,1} a_{m+2} b_{m+2,1}^{-1} a_{m+1}^{-1} b_{m, 1}^{-1}=1, \\
b_{m, 1} a_{m}^{-1} c_{3} a_{m+1} b_{m+2,1} a_{m+2}^{-1} a_{m+1}^{-1} c_{3}^{-1} a_{m} a_{m+1}^{-1} b_{m+1,1} c_{3}^{-1}=1, \\
c_{i} a_{m}^{-1} c_{i+1} a_{m}^{-1} c_{i} c_{i+1}^{-1} a_{m} c_{i}^{-1} a_{m} c_{i+1}^{-1}=1, \quad i \geq 3 . \\
g_{m, i}^{2}=1, \quad i>2 . \\
\left(f_{m} g_{m, k}\right)^{2}=1, \quad k>3, \\
\left(g_{m, i} g_{m, j}\right)^{2}=1, \quad i<i \leq j \leq n-1, \quad i+1<j \\
f_{m}^{3}=1, \\
\left(f_{m} g_{m, 3}^{3}=1,\right. \\
\left(g_{m, i} g_{m, i+1}\right)^{3}=1, \quad i>2,
\end{gathered}
$$

$$
\begin{gathered}
g_{m+1, i} a_{m}^{-1} g_{m, i}=1, \quad i \geq 3 \\
b_{m, 1} g_{m+1, j} b_{m, 0}^{-1} g_{m, j}=1, \quad j \geq 4 \\
c_{k} g_{m+1, l} c_{k}^{-1} g_{m, l}=1, \quad k, l \geq 3, \quad|l-k|>1 \\
c_{j} f_{m+1} c_{j}^{-1} f_{m}^{-1}=1, \quad j \geq 4 \\
f_{m}^{-1} f_{m+1} b_{m, 0}^{-1}=1 \\
f_{m} a_{m} f_{m+1}^{-1} b_{m, 1}^{-1}=1 \\
f_{m} g_{m, 3} b_{m, 0} g_{m+1,3} f_{m+1}^{-1} c_{3}^{-1}=1 \\
f_{m}^{-1} g_{m, 3} b_{m, 1} g_{m+1,3} f_{m+1} c_{3}^{-1}=1 \\
g_{m, i} g_{m, i+1} c_{i} g_{m+1, i+1} g_{m+1, i} c_{i+1}^{-1}=1, \quad i>2
\end{gathered}
$$

Proof. The theorem is obtained by combining the set of relations we have obtained in Lemma 3.2-Lemma 3.8.

## 4. Proof of Theorem 1.1

4.1. Finite generation of $V B_{n}^{\prime}, n \geq 4$. For the next calculations we remove the generators $b_{m, 1}$ and $b_{m, 0}$ from the presentation in Theorem 3.9.

Using the relations

$$
b_{m, 1}=f_{m} a_{m} f_{m+1}^{-1}
$$

we can remove generators $b_{m, 1}$ from the set of generators.
We have

$$
\begin{gathered}
b_{m, 0} c_{j} b_{m+1,0}^{-1} c_{j}^{-1}=1, \quad j>4 \\
c_{i} c_{j} c_{i}^{-1} c_{j}^{-1}=1, \quad i \geq 3, \quad j \geq i+1 \\
f_{m} a_{m} f_{m+1}^{-1} a_{m}^{-1} c_{j} a_{m+1} f_{m+2} a_{m+1}^{-1} f_{m+1}^{-1} c_{j}^{-1}=1, \quad j \geq 4 \\
c_{i} a_{m}^{-1} c_{j} c_{i}^{-1} a_{m} c_{j}^{-1}=1, \quad i \geq 3, \quad j>i+1 \\
b_{m+1,0} b_{m+2,0}^{-1} b_{m, 0}^{-1}=1 \\
b_{m, 0} c_{3} b_{m+2,0} c_{3}^{-1} b_{m+1,0}^{-1} c_{3}^{-1}=1, \\
c_{i} c_{i+1} c_{i} c_{i+1}^{-1} c_{i}^{-1} c_{i+1}^{-1}=1, \quad i \geq 3, \\
f_{m} a_{m} f_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1}=c_{3} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1}, \\
c_{i} a_{m}^{-1} c_{i+1} a_{m}^{-1} c_{i} c_{i+1}^{-1} a_{m} c_{i}^{-1} a_{m} c_{i+1}^{-1}=1, \quad i \geq 3 \\
g_{m, i}^{2}=1, \quad i>2
\end{gathered}
$$

$$
\begin{gathered}
\left(f_{m} g_{m, k}\right)^{2}=1, \quad k>3 \\
\left(g_{m, i} g_{m, j}\right)^{2}=1, \quad 3 \leq i<j \leq n-1, \quad i+1<j \\
f_{m}^{3}=1 \\
\left(f_{m} g_{m, 3}\right)^{3}=1, \\
\left(g_{m, i} g_{m, i+1}\right)^{3}=1, \quad i>2 \\
g_{m+1, i} a_{m}^{-1} g_{m, i}=1, \quad i \geq 3 \\
f_{m} a_{m} f_{m+1}^{-1} g_{m+1, j} b_{m, 0}^{-1} g_{m, j}=1, \quad j \geq 4 \\
c_{k} g_{m+1, l} c_{k}^{-1} g_{m, l}=1, \quad k, l \geq 3, \quad|l-k|>1 \\
c_{j} f_{m+1} c_{j}^{-1} f_{m}^{-1}=1, \quad j \geq 4, \\
f_{m}^{-1} f_{m+1} b_{m, 0}^{-1}=1, \\
f_{m} g_{m, 3} b_{m, 0} g_{m+1,3} f_{m+1}^{-1} c_{3}^{-1}=1, \\
f_{m}^{-1} g_{m, 3} f_{m} a_{m} f_{m+1}^{-1} g_{m+1,3} f_{m+1} c_{3}^{-1}=1 \\
g_{m, i} g_{m, i+1} c_{i} g_{m+1, i+1} g_{m+1, i} c_{i+1}^{-1}=1, \quad i>2
\end{gathered}
$$

Using the relations

$$
b_{m, 0}=f_{m}^{-1} f_{m+1}
$$

we can remove the generators $b_{m, 0}$ from the generating set.
After removing $b_{m, 0}$ and $b_{m, 1}$ we have following set of defining relations of $V B_{n}^{\prime}$ :

$$
\begin{gathered}
f_{m}^{-1} f_{m+1} c_{j}=c_{j} f_{m+1}^{-1} f_{m+2}, \quad j \geq 4 \\
c_{i} c_{j} c_{i}^{-1} c_{j}^{-1}=1, \quad i \geq 3, \quad j>i+1, \\
f_{m} a_{m} f_{m+1}^{-1} a_{m}^{-1} c_{j} a_{m+1} f_{m+2} a_{m+1}^{-1} f_{m+1}^{-1} c_{j}^{-1}=1, \quad j \geq 4 \\
c_{i} a_{m}^{-1} c_{j} c_{i}^{-1} a_{m} c_{j}^{-1}=1, \quad i \geq 3, \quad j>i+1 \\
f_{m} f_{m+1}^{-1} f_{m+2}=f_{m+1} f_{m+2}^{-1} f_{m+3} \\
f_{m}^{-1} f_{m+1} c_{3} f_{m+2}^{-1} f_{m+3}=c_{3} f_{m+1}^{-1} f_{m+2} c_{3} \\
c_{i} c_{i+1} c_{i} c_{i+1}^{-1} c_{i}^{-1} c_{i+1}^{-1}=1, \quad i \geq 3, \\
f_{m} a_{m} f_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1}=c_{3} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1}, \\
c_{i} a_{m}^{-1} c_{i+1} a_{m}^{-1} c_{i} c_{i+1}^{-1} a_{m} c_{i}^{-1} a_{m} c_{i+1}^{-1}=1, \quad i \geq 3 \\
a_{m, i}^{2}=1, \quad i>2
\end{gathered}
$$

$$
\begin{gathered}
\left(g_{m, i} g_{m, j}\right)^{2}=1, \quad 3 \leq i<j \leq n-1, \quad i+1<j \\
f_{m}^{3}=1 \\
\left(f_{m} g_{m, 3}\right)^{3}=1, \\
\left(g_{m, i} g_{m, i+1}\right)^{3}=1, \quad i>2 \\
g_{m+1, i} a_{m}^{-1} g_{m, i}=1, \quad i \geq 3, \\
g_{m, j} f_{m} a_{m} f_{m+1}^{-1} g_{m+1, j}=f_{m}^{-1} f_{m+1}, \quad j \geq 4 \\
c_{k} g_{m+1, l}=g_{m, l} c_{k}, \quad k, l \geq 3, \quad|l-k|>1 \\
c_{j} f_{m+1}=f_{m} c_{j}, \quad j \geq 4, \\
f_{m} g_{m, 3} f_{m}^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1}=c_{3}^{-1} \\
f_{m}^{-1} g_{m, 3} f_{m} a_{m} f_{m+1}^{-1} g_{m+1,3} f_{m+1}=c_{3} \\
g_{m, i} g_{m, i+1} c_{i}=c_{i+1} g_{m+1, i} g_{m+1, i+1}, \quad i>2
\end{gathered}
$$

We will use this set of relations to prove that $V B_{n}^{\prime}$ is finitely generated for all $n \geq 4$.

Lemma 4.1. The commutator subgroup $V B_{n}^{\prime}$ is finitely generated for all $n \geq 4$. In particular, $V B_{4}^{\prime}$ is generated by 5 elements: $c_{3}, f_{0}, f_{1}, f_{2}, g_{0,3}$, and $V B_{n}^{\prime}, n \geq 5$, is generated by $2 n-3$ elements: $c_{3}, \ldots, c_{n-1}, f_{0}, f_{1}, f_{2}, g_{0,3}, \ldots, g_{0, n-1}$.

Proof. 1) Using the relations

$$
g_{m, i} g_{m+1, i}=a_{m}, \quad i \geq 3,
$$

we will remove the generators $a_{m}, m \in \mathbb{Z}$, and express them by $g_{m, i}, m \in \mathbb{Z}, i \geq 3$.
2) Using the relations

$$
f_{m} f_{m+1}^{-1} f_{m+2}=f_{m+1} f_{m+2}^{-1} f_{m+3}
$$

we can remove the generators $f_{m}$, for $m \in \mathbb{Z}$, and keep only $f_{0}, f_{1}, f_{2}$.
3) Using the relations

$$
f_{m} g_{m, 3} f_{m}^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1}=c_{3}^{-1}
$$

we can remove the generators $g_{m, 3}$, for $m \in \mathbb{Z}$, and keep only $g_{0,3}$.
If $n=4$, then we have only generators $g_{0,3}, f_{0}, f_{1}, f_{2}, c_{3}$. Hence, $V B_{4}^{\prime}$ is finitely generated.

If $n>4$, then
4) Using the relations

$$
g_{m, i} g_{m, i+1} c_{i}=c_{i+1} g_{m+1, i} g_{m+1, i+1}, \quad i>2
$$

we can remove the generators $g_{m, i}$, for $m \in \mathbb{Z}, i>3$, and keep only $g_{0, i}$.
4.2. Infinite generation of $V B_{3}^{\prime}$. Consider the case $n=3$. From Theorem 3.9 follows that $V B_{3}^{\prime}$ is generated by elements

$$
a_{m}, \quad b_{m, \varepsilon}, \quad f_{m}, \quad m \in \mathbb{Z}
$$

and is defined by the relations

$$
\begin{gather*}
a_{m} b_{m+1,1} a_{m+2} b_{m+2,1}^{-1} a_{m+1}^{-1} b_{m, 1}^{-1}=1  \tag{4.2}\\
f_{m}^{3}=1  \tag{4.3}\\
f_{m}^{-1} f_{m+1} f_{m, 0}^{-1}=1  \tag{4.4}\\
f_{m} a_{m} f_{m+1}^{-1} b_{m, 1}^{-1}=1 \tag{4.5}
\end{gather*}
$$

Now we apply Tietze transformations to the presentation of $V B_{3}^{\prime}$. Using relations (4.5) we can remove the generator $b_{m, 1}=f_{m} a_{m} f_{m+1}^{-1}$. Then the modified set of defining relations take the form:

$$
\begin{gather*}
b_{m+1,0} b_{m+2,0}^{-1} b_{m, 0}^{-1}=1  \tag{4.6}\\
a_{m} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} f_{m+3} a_{m+2}^{-1} f_{m+2}^{-1} a_{m+1}^{-1} f_{m+1} a_{m}^{-1} f_{m}^{-1}=1  \tag{4.7}\\
f_{m}^{3}=1  \tag{4.8}\\
f_{m}^{-1} f_{m+1} b_{m, 0}^{-1}=1 \tag{4.9}
\end{gather*}
$$

Using relations (4.9) we can remove the generator $b_{m, 0}=f_{m}^{-1} f_{m+1}$. Then $V B_{3}^{\prime}$ is generated by elements

$$
a_{m}, \quad f_{m}, \quad m \in \mathbb{Z}
$$

and is defined by relation:

$$
\begin{gather*}
f_{m+1}^{-1} f_{m+2} f_{m+3}^{-1} f_{m+2} f_{m+1}^{-1} f_{m}=1  \tag{4.10}\\
a_{m} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} f_{m+3} a_{m+2}^{-1} f_{m+2}^{-1} a_{m+1}^{-1} f_{m+1} a_{m}^{-1} f_{m}^{-1}=1  \tag{4.11}\\
f_{m}^{3}=1 \tag{4.12}
\end{gather*}
$$

So, we have the following lemma.
Lemma 4.2. The group $V B_{3}^{\prime}$ has a presentation with $\left\{a_{m}, f_{m}, m \in \mathbb{Z}\right\}$ as the generating set, and the relations (4.10)- (4.12) as the defining relations.
Lemma 4.3. $V B_{3}^{\prime}$ is not finitely generated.
Proof. If we put $f_{m}=1$, for all $m \in \mathbb{Z}$, then all the relations (4.10)- (4.12) will vanish, i. e. the subgroup $\left\langle a_{m} \mid m \in \mathbb{Z}\right\rangle$ is infinitely generated free group with the set of free generators $a_{m}, m \in \mathbb{Z}$ and we have an epimorphism

$$
V B_{3}^{\prime} \longrightarrow F_{\infty}=\left\langle a_{m}, m \in \mathbb{Z}\right\rangle
$$

with kernel $\left\langle f_{m}, m \in \mathbb{Z}\right\rangle^{V B_{3}^{\prime}}$.
4.3. Proof of Theorem 1.1. Note that $V B_{2}=F_{2} \rtimes S_{2}$ and hence $V B_{2}^{\prime}$ is infinitely generated. Then the Theorem 1.1 follows by combining Lemma 4.1 and Lemma 4.3 .
4.4. Proof of Corollary 1.2. In the quotient $V B_{3}^{\prime} / V B_{3}^{\prime \prime}$ relations have the form

$$
\begin{gathered}
f_{m} f_{m+1}=f_{m+2} f_{m+3} \\
f_{m}^{3}=1
\end{gathered}
$$

In the generators $f_{0}, f_{1}, f_{2}, a_{m}, m \in \mathbb{Z}$, we have relations

$$
f_{0}^{3}=f_{1}^{3}=f_{2}^{3}=1
$$

Hence, $V B_{3}^{\prime} / V B_{3}^{\prime \prime}$ is isomorphic to the direct sum

$$
\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}^{\infty}
$$

(2) Consider the case $n=4$. Then $V B_{4}^{\prime}$ is generated by elements

$$
a_{m}, \quad c_{3}, \quad f_{m}, \quad g_{m, 3}
$$

where $m \in \mathbb{Z}$, and the defining relations have the form

$$
\begin{gathered}
f_{m} f_{m+1}^{-1} f_{m+2}=f_{m+1} f_{m+2}^{-1} f_{m+3}, \\
f_{m}^{-1} f_{m+1} c_{3} f_{m+2}^{-1} f_{m+3}=c_{3} f_{m+1}^{-1} f_{m+2} c_{3} \\
a_{m} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2}=f_{m} a_{m} f_{m+1}^{-1} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} \\
f_{m} a_{m} f_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1}=c_{3} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1}, \\
g_{m, 3}^{2}=1, \\
f_{m}^{3}=1, \\
\left(f_{m} g_{m, 3}\right)^{3}=1, \\
g_{m+1,3} a_{m}^{-1} g_{m, 3}=1, \\
f_{m} g_{m, 3} f_{m}^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1}=c_{3}^{-1} \\
f_{m}^{-1} g_{m, 3} f_{m} a_{m} f_{m+1}^{-1} g_{m+1,3} f_{m+1}=c_{3}
\end{gathered}
$$

Consider these relations in the quotient $V B_{4}^{\prime} / V B_{4}^{\prime \prime}$ and for the images of the generators

$$
c_{3}, \quad a_{m}, \quad f_{m}, \quad g_{m, 3}, \quad m \in \mathbb{Z},
$$

we will use the same symbols.
From relation $g_{m, 3}^{2}=f_{m}^{3}=\left(f_{m} g_{m, 3}\right)^{3}=1$ we get $g_{m, 3}=1$.
Then from the relation $g_{m+1,3} a_{m}^{-1} g_{m, 3}=1$ follows that $a_{m}=1$.
The other relations have the form

$$
c_{3}=f_{m}^{3}=1, \quad f_{m} f_{m+1}=f_{m+2} f_{m+3}
$$

Hence, we can keep only generators $f_{0}, f_{1}, f_{2}$ and defining relations $f_{0}^{3}=f_{1}^{3}=f_{2}^{3}=1$.
Hence the quotient $V B_{4}^{\prime} / V B_{4}^{\prime \prime}$ is isomorphic to the direct product $\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3}$ of three cyclic groups $\mathbb{Z}_{3}$ of order 3 .

Consider the case $n>4$. We will consider relations of $V B_{n}^{\prime}$ in the quotient $V B_{n}^{\prime} / V B_{n}^{\prime \prime}$ and will denote the images of the generators

$$
a_{m}, \quad b_{m, \varepsilon}, \quad c_{l}, \quad f_{m}, \quad g_{m, l},
$$

where $m \in \mathbb{Z}, \varepsilon=0,1,2<l<n$ by the same symbols.
As in the case $n=4$ we get $g_{m, 3}=a_{m}=1$.
Then from the relations

$$
g_{m, i}^{2}=1, \quad\left(g_{m, i} g_{m, i+1}\right)^{3}=1, \quad i>2,
$$

follows that $g_{m, i}=1, i>2$.
From the relations

$$
f_{m}^{3}=1, \quad\left(f_{m} g_{m, k}\right)^{2}=1, \quad k>3,
$$

follows that $f_{m}=1$.
Remaining relations have the form

$$
c_{i}=1, \quad i \geq 3
$$

This completes the proof.

## 5. Commutator subgroup of the welded braid group

The welded braid group $W B_{n}, n \geq 2$, is the quotient of $V B_{n}$ by the relations

$$
\rho_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \rho_{i+1}, i=1,2, \ldots, n-2
$$

In this section we will find a presentation of $W B_{n}^{\prime}$. We will use the same set of generators that we used for $V B_{n}$ and $V B_{n}^{\prime}$. Hence to find defining relations for $W B_{n}^{\prime}$ we need to add relations that follow from the relation

$$
r_{8}=\rho_{i} \sigma_{i+1} \sigma_{i} \rho_{i+1} \sigma_{i}^{-1} \sigma_{i+1}^{-1}
$$

Depending on $i$ we will consider 3 cases:
if $i=1$, then

$$
r_{8}=\rho_{1} \sigma_{2} \sigma_{1} \rho_{2} \sigma_{1}^{-1} \sigma_{2}^{-1}=S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{2}} S_{\sigma_{1} \rho_{1}, \sigma_{1}} S_{\sigma_{1}^{2} \rho_{1}, \rho_{2}} S_{\sigma_{1}, \sigma_{1}}^{-1} S_{1, \sigma_{2}}^{-1}=b_{0,1} a_{1} f_{2,1} b_{0,0}^{-1}
$$

if $i=2$, then

$$
r_{8}=\rho_{2} \sigma_{3} \sigma_{2} \rho_{3} \sigma_{2}^{-1} \sigma_{3}^{-1}=S_{1, \rho_{2}} S_{\rho_{1}, \sigma_{3}} S_{\sigma_{1} \rho_{1}, \sigma_{2}} S_{\sigma_{1}^{2} \rho_{1}, \rho_{3}} S_{\sigma_{1}, \sigma_{2}}^{-1} S_{1, \sigma_{3}}^{-1}=f_{0,0} c_{3} b_{1,1} g_{2,3} b_{1,0}^{-1} c_{3}^{-1}
$$

if $i>2$, then

$$
\begin{aligned}
r_{8} & =\rho_{i} \sigma_{i+1} \sigma_{i} \rho_{i+1} \sigma_{i}^{-1} \sigma_{i+1}^{-1} \\
& =S_{1, \rho_{i}} S_{\rho_{1}, \sigma_{i+1}} S_{\sigma_{1} \rho_{1}, \sigma_{i}} S_{\sigma_{1}^{2} \rho_{1}, \rho_{i+1}} S_{\sigma_{1}, \sigma_{i}}^{-1} S_{1, \sigma_{i+1}}^{-1} \\
& =g_{0, i} c_{i+1} c_{i} g_{2, i+1} c_{i}^{-1} c_{i+1}^{-1} .
\end{aligned}
$$

We will use the following conjugation rules
Lemma 5.1. In $W B_{n}$ the following conjugation rules hold:
(1) $a_{1}^{\rho_{1}}=a_{0} a_{1}^{-1} a_{0}^{-1}$,
(2) $f_{2,1}^{\rho_{1}}=a_{0} a_{1} f_{2,0} a_{1}^{-1} a_{0}^{-1}$,
(3) $g_{2, i}^{\rho_{1}}=a_{0} a_{1} g_{2, i} a_{1}^{-1} a_{0}^{-1}$ for $i>2$.

Proof. (1) Note that

$$
\begin{aligned}
\rho_{1} a_{1} \rho_{1} & =\rho_{1} \sigma_{1} \rho_{1} \sigma_{1} \rho_{1} \sigma_{1}^{-1} \sigma_{1}^{-1} \rho_{1} \\
& =S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{1}} S_{\sigma_{1} \rho_{1}, \rho_{1}} S_{\sigma_{1}, \sigma_{1}} S_{\sigma_{1}^{2}, \rho_{1}} S_{\sigma_{1} \rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}} \\
& =a_{0} a_{1}^{-1} a_{0}^{-1} ;
\end{aligned}
$$

(2) Next we have,

$$
\begin{aligned}
\rho_{1} f_{2,1} \rho_{1} & =\rho_{1} \sigma_{1} \sigma_{1} \rho_{1} \rho_{2} \sigma_{1}^{-1} \sigma_{1}^{-1} \rho_{1} \\
& =S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{1}} S_{\sigma_{1} \rho_{1}, \sigma_{1}} S_{\sigma_{1}^{2} \rho_{1}, \rho_{1}} S_{\sigma_{1}^{2}, \rho_{2}} S_{\sigma_{1} \rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}} \\
& =a_{0} a_{1} f_{2,0} a_{1}^{-1} a_{0}^{-1}
\end{aligned}
$$

(3) Finally,

$$
\begin{aligned}
\rho_{1} g_{2, i} \rho_{1} & =\rho_{1} \sigma_{1} \sigma_{1} \rho_{i} \rho_{1} \sigma_{1}^{-1} \sigma_{1}^{-1} \rho_{1} \\
& =S_{1, \rho_{1}} S_{\rho_{1}, \sigma_{1}} S_{\sigma_{1} \rho_{1}, \sigma_{1}} S_{\sigma_{1}^{2} \rho_{1}, \rho_{i}} S_{\sigma_{1}^{2}, \rho_{1}} S_{\sigma_{1} \rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \sigma_{1}}^{-1} S_{\rho_{1}, \rho_{1}} \\
& =a_{0} a_{1} g_{2, i} a_{1}^{-1} a_{0}^{-1} .
\end{aligned}
$$

This proves the lemma.
Lemma 5.2. From the relation $r_{8}$ of $W B_{n}$, the following six types of relations of $W B_{n}^{\prime}$ follow:

$$
\begin{aligned}
& b_{m, 1} a_{m+1} f_{m+2,1} b_{m, 0}^{-1}=1, \\
& f_{m, 0} c_{3} b_{m+1,1} g_{m+2,3} b_{m+1,0} c_{3}^{-1}=1, \\
& g_{m, i} c_{i+1} c_{i} g_{m+2, i+1} c_{i}^{-1} c_{i+1}^{-1}=1, \\
& b_{m, 0} f_{m+2,0} a_{m+1}^{-1} b_{m, 1}^{-1}=1, \\
& f_{m, 1} c_{3} f_{m+1,0} a_{m+1} g_{m+2,3} b_{m+1,1}^{-1} c_{3}^{-1}=1, \\
& g_{m, i} c_{i+1} a_{m}^{-1} c_{i} a_{m+1} g_{m+2, i+1} a_{m+1}^{-1} c_{i}^{-1} a_{m} c_{i+1}^{-1}=1 .
\end{aligned}
$$

Proof. Conjugating relations $r_{8}$ by $\rho_{1}$ and using Lemma 5.1, we get 3 relations:

$$
\begin{aligned}
\left(b_{0,1} a_{1} f_{2,1} b_{0,0}^{-1}\right)^{\rho_{1}} & =b_{0,0} f_{2,0} a_{1}^{-1} b_{0,1}^{-1}, \\
\left(f_{0,0} c_{3} b_{1,1} g_{2,3} b_{1,0}^{-1} c_{3}^{-1}\right)^{\rho_{1}} & =f_{0,1} c_{3} f_{1,0} a_{1} g_{2,3} b_{1,1}^{-1} c_{3}^{-1},
\end{aligned}
$$

$$
\left(g_{0, i} c_{i+1} c_{i} g_{2, i+1} c_{i}^{-1} c_{i+1}^{-1}\right)^{\rho_{1}}=g_{0, i} c_{i+1} a_{0}^{-1} c_{i} a_{1} g_{2, i+1} a_{1}^{-1} c_{i}^{-1} a_{0} c_{i+1}^{-1} .
$$

Conjugating relations $r_{8}$ and $\rho_{1} r_{8} \rho_{1}$ by $\sigma_{1}^{-m}$, we get the six relations from the lemma.

Thus we have the following.
Corollary 5.3. The commutator subgroup $W B_{n}^{\prime}$ is generated by elements

$$
a_{m}, \quad b_{m, \varepsilon}, \quad c_{l}, \quad f_{m}, \quad g_{m, l}
$$

where $m \in \mathbb{Z}, \varepsilon=0,1,2<l<n$ and is defined by the relations in Theorem 3.9 and Lemma 5.2.
5.1. Presentation of $W B_{3}^{\prime}$. We have found a presentation of $V B_{3}^{\prime}$. To get a presentation of $W B_{3}^{\prime}$ we need to add two series of relations:

$$
\begin{align*}
& b_{m, 1} a_{m+1} f_{m+2,1} b_{m, 0}^{-1}=1  \tag{5.1}\\
& b_{m, 0} f_{m+2,0} a_{m+1}^{-1} b_{m, 1}^{-1}=1 \tag{5.2}
\end{align*}
$$

that follow from Lemma 5.2 .
As in the case of $V B_{3}^{\prime}$ we can remove the generator $f_{m, 1}$, using the relation $f_{m, 0} f_{m, 1}=1$. Then the relations (5.1)-(5.2) have the form

$$
\begin{align*}
& b_{m, 1} a_{m+1} f_{m+2}^{-1} b_{m, 0}^{-1}=1,  \tag{5.3}\\
& b_{m, 0} f_{m+2} a_{m+1}^{-1} b_{m, 1}^{-1}=1, \tag{5.4}
\end{align*}
$$

where we denote $f_{m}=f_{m, 0}$.
Using the relations

$$
b_{m, 1}=f_{m} a_{m} f_{m+1}^{-1},
$$

that hold in $V B_{3}^{\prime}$, we can remove $b_{m, 1}$. Then the relations (5.3)-(5.4) have the form

$$
\begin{gather*}
f_{m} a_{m} f_{m+1}^{-1} a_{m+1} f_{m+2}^{-1} b_{m, 0}^{-1}=1  \tag{5.5}\\
b_{m, 0} f_{m+2} a_{m+1}^{-1} f_{m+1} a_{m}^{-1} f_{m}^{-1}=1 \tag{5.6}
\end{gather*}
$$

We see that the second relation is inverse to the first one. Hence, we can remove the second relation.

Next, using the relations $f_{m}^{-1} f_{m+1} b_{m, 0}^{-1}=1$, which hold in $V B_{3}^{\prime}$, we can remove the generator $b_{m, 0}$. Then (5.5) has the form

$$
\begin{equation*}
f_{m} a_{m} f_{m+1}^{-1} a_{m+1} f_{m+2}^{-1} f_{m+1}^{-1} f_{m}=1 \tag{5.7}
\end{equation*}
$$

Using the presentation of $V B_{3}^{\prime}$ we get
Proposition 5.4. The group $W B_{3}^{\prime}$ is generated by elements

$$
a_{m}, \quad f_{m}, \quad m \in \mathbb{Z}
$$

and is defined by relation:

$$
\begin{equation*}
f_{m+1}^{-1} f_{m+2} f_{m+3}^{-1} f_{m+2} f_{m+1}^{-1} f_{m}=1 \tag{5.8}
\end{equation*}
$$

$$
\begin{gather*}
a_{m} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} f_{m+3} a_{m+2}^{-1} f_{m+2}^{-1} a_{m+1}^{-1} f_{m+1} a_{m}^{-1} f_{m}^{-1}=1,  \tag{5.9}\\
f_{m}^{3}=1,  \tag{5.10}\\
a_{m} f_{m+1}^{-1} a_{m+1} f_{m+2}^{-1} f_{m+1}^{-1} f_{m}^{-1}=1 . \tag{5.11}
\end{gather*}
$$

As consequence we get
Corollary 5.5. $W B_{3}^{\prime}$ is generated by $a_{0}, f_{0}, f_{1}, f_{2}$.
Proof. From the set of relations (5.8) we can express the generators $f_{k}$, where $k>2$ or $k<0$, as words in the generators $f_{0}, f_{1}, f_{2}$ and analogously, from the set of relations (5.11) we can express the generators $a_{l}$, where $l \neq 0$, as words in the generators $a_{0}, f_{0}, f_{1}, f_{2}$.

Corollary 5.6. $W B_{3}^{\prime} / W B_{3}^{\prime \prime}$ is isomorphic to the direct sum

$$
\mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}
$$

Proof. In the quotient $W B_{3}^{\prime} / W B_{3}^{\prime \prime}$ the relations have the form

$$
\begin{gathered}
f_{m} f_{m+1}=f_{m+2} f_{m+3} \\
f_{m}^{3}=1 \\
a_{m} a_{m+1}=f_{m} f_{m+1}^{-1} f_{m+2}
\end{gathered}
$$

In the generators $a_{0}, f_{0}, f_{1}, f_{2}$ we have relations

$$
f_{0}^{3}=f_{1}^{3}=f_{2}^{3}=1
$$

This completes the proof.
5.2. The commutator subgroup $W B_{4}^{\prime}$. In $W B_{4}^{\prime}$ we have relations of $V B_{4}^{\prime}$ and the following relations:

$$
\begin{gathered}
b_{m, 1} a_{m+1} f_{m+2,1} b_{m, 0}^{-1}=1, \\
b_{m, 0} f_{m+2,0} a_{m+1}^{-1} b_{m, 1}^{-1}=1 \\
f_{m, 0} c_{3} b_{m+1,1} g_{m+2,3} b_{m+1,0}^{-1} c_{3}^{-1}=1, \\
f_{m, 1} c_{3} f_{m+1,0} a_{m+1} g_{m+2,3} b_{m+1,1}^{-1} c_{3}^{-1}=1 .
\end{gathered}
$$

Excluding the generators

$$
b_{m, 0}=f_{m}^{-1} f_{m+1}, \quad b_{m, 1}=f_{m} a_{m} f_{m+1}^{-1}, \quad f_{m, 1}=f_{m, 0}^{-1}=f_{m}^{-1}
$$

from these relations. We get relations

$$
\begin{gathered}
f_{m} a_{m} f_{m+1}^{-1} a_{m+1} f_{m+2}^{-1} f_{m+1}^{-1} f_{m}=1, \\
f_{m}^{-1} f_{m+1} f_{m+2} a_{m+1}^{-1} f_{m+1} a_{m}^{-1} f_{m}^{-1}=1, \\
f_{m} c_{3} f_{m+1} a_{m+1} f_{m+2}^{-1} g_{m+2,3} f_{m+2}^{-1} f_{m+1} c_{3}^{-1}=1, \\
f_{m}^{-1} c_{3} f_{m+1} a_{m+1} g_{m+2,3} f_{m+2} a_{m+1}^{-1} f_{m+1}^{-1} c_{3}^{-1}=1
\end{gathered}
$$

The second relation is inverse of the first relation. Hence, we can keep only the first relation. Rewrite it in the form

$$
a_{m} f_{m+1}^{-1} a_{m+1}=f_{m} f_{m+1} f_{m+2}
$$

Rewrite the third and the forth relations in the form

$$
\begin{aligned}
& c_{3} f_{m+1} a_{m+1} f_{m+2}^{-1} g_{m+2,3} f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m}=1 \\
& c_{3} f_{m+1} a_{m+1} g_{m+2,3} f_{m+2} a_{m+1}^{-1} f_{m+1}^{-1} c_{3}^{-1} f_{m}^{-1}=1
\end{aligned}
$$

From these relations:

$$
f_{m+2}^{-1} g_{m+2,3} f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m}=g_{m+2,3} f_{m+2} a_{m+1}^{-1} f_{m+1}^{-1} c_{3}^{-1} f_{m}^{-1}
$$

Since in $V B_{4}^{\prime}$ holds

$$
\left(g_{m+2,3} f_{m+2}\right)^{3}=1, \quad g_{m+2,3}^{2}=1
$$

then

$$
\left(g_{m+2,3} f_{m+2}\right)^{-2} f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m}=a_{m+1}^{-1} f_{m+1}^{-1} c_{3}^{-1} f_{m}^{-1}
$$

and

$$
g_{m+2,3} f_{m+2} f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m}=a_{m+1}^{-1} f_{m+1}^{-1} c_{3}^{-1} f_{m}^{-1}
$$

Therefore,

$$
g_{m+2,3}=a_{m+1}^{-1} f_{m+1}^{-1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1}
$$

Including this expression of $g_{m+2,3}$ in the forth relation:

$$
f_{m}^{-1} c_{3} f_{m+1} a_{m+1} a_{m+1}^{-1} f_{m+1}^{-1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} f_{m+2} a_{m+1}^{-1} f_{m+1}^{-1} c_{3}^{-1}=1
$$

We get after cancelation

$$
a_{m+1}=f_{m+1} f_{m+2}
$$

Including this expression of $a_{m+1}$ in the expression for $g_{m+2,3}$, we get

$$
g_{m+2,3}=f_{m+2}^{-1} f_{m+1}^{-2} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1}
$$

or

$$
g_{m+2,3}=f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} .
$$

Next, the relation $a_{m} f_{m+1}^{-1} a_{m+1}=f_{m} f_{m+1} f_{m+2}$ after substitution $a_{m}=f_{m} f_{m+1}$, $a_{m+1}=f_{m+1} f_{m+2}$ becomes an identity.

Hence, the new relations in $W B_{4}^{\prime}$ are equal to relations

$$
\begin{aligned}
g_{m+2,3}= & f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} \\
& a_{m}=f_{m} f_{m+1}
\end{aligned}
$$

The full set of relations in $W B_{4}^{\prime}$ has the form:

$$
\begin{gathered}
f_{m} f_{m+1}^{-1} f_{m+2}=f_{m+1} f_{m+2}^{-1} f_{m+3}, \\
f_{m}^{-1} f_{m+1} c_{3} f_{m+2}^{-1} f_{m+3}=c_{3} f_{m+1}^{-1} f_{m+2} c_{3}, \\
a_{m} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2}=f_{m} a_{m} f_{m+1}^{-1} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} \\
f_{m} a_{m} f_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1}=c_{3} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1}, \\
g_{m, 3}^{2}=1, \\
f_{m}^{3}=1,
\end{gathered}
$$

$$
\begin{gathered}
\left(f_{m} g_{m, 3}\right)^{3}=1 \\
g_{m+1,3} a_{m}^{-1} g_{m, 3}=1 \\
f_{m} g_{m, 3} f_{m}^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1}=c_{3}^{-1} \\
f_{m}^{-1} g_{m, 3} f_{m} a_{m} f_{m+1}^{-1} g_{m+1,3} f_{m+1}=c_{3} \\
g_{m+2,3}=f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} \\
a_{m}=f_{m} f_{m+1}
\end{gathered}
$$

Transform these relations, excluding $a_{m}$ and $g_{m, 3}$.

1) The relation $a_{m} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2}=f_{m} a_{m} f_{m+1}^{-1} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1}$, after substitution

$$
a_{m}=f_{m} f_{m+1}, \quad a_{m+1}=f_{m+1} f_{m+2}, \quad a_{m+2}=f_{m+2} f_{m+3}
$$

has the form

$$
\begin{gathered}
f_{m} f_{m+1} f_{m+1} f_{m+1} f_{m+2} f_{m+2}^{-1} f_{m+2} f_{m+3}= \\
=f_{m} f_{m} f_{m+1} f_{m+1}^{-1} f_{m+1} f_{m+2} f_{m+2} f_{m+2} f_{m+3} f_{m+3}^{-1}
\end{gathered}
$$

Using the relation $f_{m}^{3}=1$, we get

$$
f_{m} f_{m+1}=f_{m+2} f_{m+3}
$$

2) The relation

$$
f_{m} a_{m} f_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1}=c_{3} f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_{m}^{-1} c_{3} a_{m+1}
$$

after substitution

$$
a_{m}=f_{m} f_{m+1}, \quad a_{m+1}=f_{m+1} f_{m+2}, \quad a_{m+2}=f_{m+2} f_{m+3}
$$

has the form

$$
\begin{aligned}
& f_{m} f_{m} f_{m+1} f_{m+1}^{-1} f_{m+1}^{-1} f_{m}^{-1} c_{3} f_{m+1} f_{m+2} f_{m+2} f_{m+2} f_{m+3} f_{m+3}^{-1} f_{m+3}^{-1} f_{m+2}^{-1}= \\
& \quad=c_{3} f_{m+1} f_{m+1} f_{m+2} f_{m+2}^{-1} f_{m+2}^{-1} f_{m+1}^{-1} f_{m+1}^{-1} f_{m}^{-1} c_{3} f_{m+1} f_{m+2}
\end{aligned}
$$

or, after cancelation and using the relation $f_{m}^{3}=1$ we get

$$
f_{m}^{-1} f_{m+1}^{-1} f_{m}^{-1} c_{3} f_{m+1} f_{m+3}^{-1} f_{m+2}=c_{3} f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} f_{m}^{-1} c_{3} f_{m+1}
$$

3) The relation $g_{m+2,3}^{2}=1$ after substitution

$$
g_{m+2,3}=f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1}
$$

has the form

$$
f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1}=1
$$

4) The relation $\left(f_{m} g_{m, 3}\right)^{3}=1$ after substitution

$$
g_{m, 3}=f_{m}^{-1} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1},
$$

has the form

$$
\left(f_{m} f_{m}^{-1} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1}\right)^{3}=1
$$

and is identity since $f_{m}^{3}=1$.
5) The relation $g_{m+1,3} a_{m}^{-1} g_{m, 3}=1$ after substitution

$$
g_{m+1,3}=f_{m+1}^{-1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1}, \quad g_{m, 3}=f_{m}^{-1} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1}, \quad a_{m}=f_{m} f_{m+1}
$$

has the form

$$
f_{m+1}^{-1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1} f_{m+1}^{-1} f_{m}^{-1} f_{m}^{-1} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1}=1
$$

Using the relation $f_{m}^{3}=1$ and changing the index $m$ on $m+1$, we get

$$
f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1}=1
$$

6) The relation $f_{m} g_{m, 3} f_{m}^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1}=c_{3}^{-1}$ after substitution

$$
g_{m+1,3}=f_{m+1}^{-1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1}, \quad g_{m, 3}=f_{m}^{-1} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1}
$$

has the form

$$
f_{m} f_{m}^{-1} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1} f_{m}^{-1} f_{m+1} f_{m+1}^{-1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1} f_{m+1}^{-1}=c_{3}^{-1}
$$

or after cancelation

$$
f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1} f_{m+1}^{-1}=c_{3}^{-1}
$$

7) The relation $f_{m}^{-1} g_{m, 3} f_{m} a_{m} f_{m+1}^{-1} g_{m+1,3} f_{m+1}=c_{3}$ after substitution

$$
g_{m+1,3}=f_{m+1}^{-1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1}, \quad g_{m, 3}=f_{m}^{-1} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1}, \quad a_{m}=f_{m} f_{m+1}
$$

has the form

$$
f_{m}^{-1} f_{m}^{-1} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1} f_{m} f_{m} f_{m+1} f_{m+1}^{-1} f_{m+1}^{-1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1} f_{m+1}=c_{3}
$$

or, after cancelation and using the relation $f_{m}^{3}=1$ we get

$$
f_{m} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1} f_{m}^{-1} f_{m+1}^{-1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1} f_{m+1}=c_{3}
$$

Hence, we have proven
Theorem 5.7. The group $W B_{4}^{\prime}$ is generated by $c_{3}, f_{m}, m \in \mathbb{Z}$, and is defined by the relations

$$
\begin{gathered}
f_{m} f_{m+1}^{-1} f_{m+2}=f_{m+1} f_{m+2}^{-1} f_{m+3}, \\
f_{m}^{-1} f_{m+1} c_{3} f_{m+2}^{-1} f_{m+3}=c_{3} f_{m+1}^{-1} f_{m+2} c_{3}, \\
f_{m} f_{m+1}=f_{m+2} f_{m+3} \\
f_{m}^{-1} f_{m+1}^{-1} f_{m}^{-1} c_{3} f_{m+1} f_{m+3}^{-1} f_{m+2}=c_{3} f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} f_{m}^{-1} c_{3} f_{m+1} \\
f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1}=1, \\
f_{m}^{3}=1, \\
f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1}=1, \\
f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1} f_{m+1}^{-1}=c_{3}^{-1}, \\
f_{m} f_{m-1} c_{3}^{-1} f_{m-2} c_{3} f_{m-1}^{-1} f_{m}^{-1} f_{m+1}^{-1} f_{m} c_{3}^{-1} f_{m-1} c_{3} f_{m}^{-1} f_{m+1}=c_{3}
\end{gathered}
$$

Corollary 5.8. The group $W B_{4}^{\prime}$ is generated by $c_{3}, f_{0}, f_{1}, f_{2}$.
Indeed, using the relations

$$
f_{m} f_{m+1}^{-1} f_{m+2}=f_{m+1} f_{m+2}^{-1} f_{m+3}
$$

we can save from the generators $f_{m}, m \in \mathbb{Z}$, only the relations $f_{0}, f_{1}, f_{2}$.
Corollary 5.9. $W B_{4}^{\prime} / W B_{4}^{\prime \prime} \cong \mathbb{Z}_{3}$.

Indeed, considering relations of $W B_{4}^{\prime}$ by modulo $W B_{4}^{\prime \prime}$ we see that $f_{m} f_{m+1}=1$, $f_{m}^{3}=1$ and $c_{3}=1$.

### 5.3. The commutator subgroup $W B_{n}^{\prime}$ for $n \geq 5$.

Theorem 5.10. The group $W B_{n}^{\prime}, n \geq 5$, is generated by $n$ elements $f_{0}, f_{1}, f_{2}, c_{3}$, $\ldots, c_{n-1}$.

Proof. As we proved before, $V B_{n}^{\prime}, n \geq 5$, is generated by elements $c_{3}, \ldots, c_{n-1}, f_{0}$, $f_{1}, f_{2}, g_{0,3}, \ldots, g_{0, n-1}$.

The group $W B_{n}^{\prime}, n \geq 5$, is defined by relations of $V B_{n}^{\prime}$ and the relations:

$$
\begin{gathered}
b_{m, 1} a_{m+1} f_{m+2,1} b_{m, 0}^{-1}=1, \\
b_{m, 0} f_{m+2,0} a_{m+1}^{-1} b_{m, 1}^{-1}=1, \\
f_{m, 0} c_{3} b_{m+1,1} g_{m+2,3} b_{m+1,0}^{-1} c_{3}^{-1}=1, \\
f_{m, 1} c_{3} f_{m+1,0} a_{m+1} g_{m+2,3} b_{m+1,1}^{-1} c_{3}^{-1}=1 . \\
g_{m, i} c_{i+1} c_{i} g_{m+2, i+1} c_{i}^{-1} c_{i+1}^{-1}=1, \\
g_{m, i} c_{i+1} a_{m}^{-1} c_{i} a_{m+1} g_{m+2, i+1} a_{m+1}^{-1} c_{i}^{-1} a_{m} c_{i+1}^{-1}=1 .
\end{gathered}
$$

Similar to the group $W B_{4}^{\prime}$, the firs for relations are equivalent to the relations

$$
g_{m+2,3}=f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1}, \quad a_{m}=f_{m} f_{m+1}
$$

Hence, the additional relations of $W B_{n}^{\prime}, \geq 5$ have the form

$$
\begin{gathered}
g_{m+2,3}=f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1} \\
a_{m}=f_{m} f_{m+1} \\
g_{m, i} c_{i+1} c_{i} g_{m+2, i+1} c_{i}^{-1} c_{i+1}^{-1}=1 \\
g_{m, i} c_{i+1} f_{m+1}^{-1} f_{m}^{-1} c_{i} f_{m+1} f_{m+2} g_{m+2, i+1} f_{m+2}^{-1} f_{m+1}^{-1} c_{i}^{-1} f_{m} f_{m+1} c_{i+1}^{-1}=1
\end{gathered}
$$

Using the relations $g_{m, i} c_{i+1} c_{i} g_{m+2, i+1} c_{i}^{-1} c_{i+1}^{-1}=1$, we can express the generators $g_{m, i}$, $i \geq 4$, as words in the generators $c_{3}, \ldots, c_{n-1}, f_{m}, g_{m, 3}, m \in \mathbb{Z}$. Also, as in the case of the group $W B_{4}^{\prime}$, we can express the generators $f_{m}, g_{m, 3}, m \in \mathbb{Z}$, as words in the generators $c_{3}, f_{0}, f_{1}, f_{2}$.

### 5.4. Proof of Theorem 1.3,

Proof. Parts (1) of Theorem 1.3 follows by combining Corollary 5.5, Corollary 5.8 and Theorem 5.7. Part (2) and (3) follow from Corollary 5.6, and Corollary 5.9.

For $n \geq 5$, note that $W B_{n}^{\prime}$ is perfect as a quotient of the perfect group $V B_{n}^{\prime}$. This proves (4).

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