COMMUTATOR SUBGROUPS OF VIRTUAL AND WELDED **BRAID GROUPS**

VALERIY G. BARDAKOV, KRISHNENDU GONGOPADHYAY, AND MIKHAIL V. NESHCHADIM

ABSTRACT. Let VB_n , resp. WB_n denote the virtual, resp. welded, braid group on n strands. We study their commutator subgroups $VB'_n = [VB_n, VB_n]$ and, $WB'_n = [WB_n, WB_n]$ respectively. We obtain a set of generators and defining relations for these commutator subgroups. In particular, we prove that VB'_n is finitely generated if and only if $n \ge 4$, and WB'_n is finitely generated for $n \ge 3$. Also we prove that $VB'_3/VB''_3 = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^\infty$, $VB'_4/VB''_4 = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $WB'_3/WB''_3 = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}, WB'_4/WB''_4 = \mathbb{Z}_3$, and for $n \geq 5$ the commutator subgroups VB'_n and WB'_n are perfect, i. e. the commutator subgroup is equal to the second commutator subgroup.

1. INTRODUCTION

Virtual braid groups VB_n on n strands are certain extensions of the classical braid groups. It was introduced by L. Kauffman [10] (see also [14]). Virtual braids play the same role in the virtual knot theory that classical braids played in the classical knot theory. In particular, like closures of classical braids represent classical knots and links, the closure of virtual braids represent the virtual knots and links (see [9], [10]). On connections of virtual braids with the virtual knot theory, see [3, 4]. For a structure of the virtual braid groups, see [2].

The welded braid group WB_n is a quotient of VB_n . This group is called the group of conjugating automorphisms [13, 1], the braid-permutation group [7] and so on. For several notions of this group and their equivalence, see [5].

The commutator subgroup B'_n of the classical braid group B_n is studied in the paper [8] (see also [12]). The following facts follow from these papers:

 $-B'_n$ is finitely presented for all $n \ge 2$;

 $-B'_3$ is a free group of rank two;

 $-B'_4$ is a semi-direct product of two free groups of rank two; - for n > 4 the second commutator subgroup B''_n of B_n coincides with the first commutator subgroup B'_n , i.e. B'_n is perfect.

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In the present paper we investigate the commutator subgroups VB'_n and WB'_n . Our main result is the following.

Theorem 1.1. The commutator subgroup VB'_3 is infinitely generated. For $n \ge 4$ the commutator subgroup VB'_n can be generated by 2n - 3 elements.

To prove Theorem 1.1 we obtain a presentation of VB'_n using the classical method of Reidemeister-Schreier, and then remove certain generators and relations using Tietze transformations. As a consequence of Theorem 1.1, we further have the following corollaries.

Corollary 1.2. (1) The quotient VB'_3/VB''_3 is isomorphic to the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^{\infty}$, where \mathbb{Z}^{∞} is the direct product of counting number of \mathbb{Z} .

- (2) The quotient VB'_4/VB''_4 is isomorphic to the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.
- (3) For $n \ge 5$, VB'_n is perfect, that is $VB'_n = VB''_n$.

Recently the commutator subgroup WB'_n of the welded braid group has been investigated by Zaremsky in [15], who proved that WB'_n is finitely presented if and only if $n \ge 4$. Zaremsky proved this result using discrete Morse theory, without constructing explicit finite presentation. Dey and Gongopadhyay [6] also proved that WB'_n is finitely generated for all $n \ge 3$. In the present paper we have found a better bound on the number of generators than in [6]. We prove the following result.

Theorem 1.3. (1) The commutator subgroup WB'_n can be generated by n elements for all $n \ge 4$, and WB'_3 can be generated by 4 elements.

- (2) The quotient WB'_3/WB''_3 is isomorphic to the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$.
- (3) The quotient WB'_4/WB''_4 is isomorphic to \mathbb{Z}_3 .
- (4) For $n \ge 5$, WB'_n is perfect.

The presentation of WB'_n obtained in this paper is slightly different from the one obtained in [6]. We obtain this presentation using the presentation of VB_n , while computing that we use successive conjugation rule in the rewritting process, see Lemma 3.1. This simple conjugation tirck has given an alternative presentation of WB'_n where the elimination of generators become simpler, and consequently we get a better bound on the number of generators.

We now briefly describe the structure of the paper. We recall the necessary preliminaries in Section 2. Using Reidemeister-Schreier method, we first obtain a general presentation of VB'_n , see Theorem 3.9 in Section 3. We prove Theorem 1.1 in Section 4. Because of the differences of the nature of the proofs for $n \ge 4$ and n = 3, As the cases n = 3 and $n \ge 4$ are different, accordingly, the proof of Theorem 1.1 is divided over two subsections. In Section 4.1, first, we apply Tietze transformations to remove certain generators from the presentation in Theorem 3.9. This gives a finite generating set for VB'_n for $n \ge 4$. In Section 4.2, we show that VB'_3 is infinitely generated. Combining these results, Theorem 1.1 is obtained. We prove Theorem 1.3 in Section 5.

Finally, we note the following problems that remain to be answered.

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Problem 1. Is it true that the commutator subgroup VB'_n is not finitely presented for $n \ge 4$.

We expect the answer to be yes, but it is not clear how.

Problem 2. Construct explicit finite presentation of WB'_n for $n \ge 3$.

Problem 3. Let G is a group from the set $\{VB_3, VB_4, WB_3, WB_4\}$. Find the quotients $G^{(i)}/G^{(i+1)}$, i = 2, 3, ..., where $G^{(k)}$ is the k-th commutator subgroup:

 $G^{(1)} = G', \ G^{(k+1)} = [G^{(k)}, G^{(k)}], \ k = 1, 2, \dots$

2. Preliminaries

2.1. Group of Virtual Braids. The virtual braid group of n strands VB_n is generated by the classical braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and the symmetric group $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$. The generators $\sigma_i, i = 1, \ldots, n-1$ satisfy the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| \ge 2,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad i = 1, \dots, n-2$$

The generators ρ_i , i = 1, ..., n - 1 satisfy the relations of symmetric group S_n :

$ ho_{i}^{2} = 1$	for	$i=1,2,\ldots,n-1,$
$\rho_i \rho_j = \rho_j \rho_i$	for	$ i-j \ge 2,$
$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$	for	$i=1,2\ldots,n-2.$

Other defining relations of VB_n are mixed and have the form

$$\sigma_i \rho_j = \rho_j \sigma_i \quad \text{for} \quad |i - j| \ge 2,$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} \quad \text{for} \quad i = 1, 2, \dots, n-2$$

2.2. Reidemeister-Schreier Algorithm. Given a presentation of a group G, this algorithm allows one to find a presentation of a subgroup $H \subset G$. To obtain the presentation of H, it is necessary to find a Schreier's set of right coset of the group G over the subgroup H. We give a formal description of this process, for more details see [11].

Let a_1, \ldots, a_n be the generators of the group G and R_1, \ldots, R_m be the set of defining relations for the given set of generators. System of words $N = \{K_\alpha, \alpha \in A\}$ on generators a_1, \ldots, a_n defines a Schreier's system for the subgroup $H \subset G$ relative to the system of generators a_1, \ldots, a_n if the next conditions are satisfied:

1) in every right coset of the group G over H there is only one word from the system N;

2) if the word $K_{\alpha} = a_{i_1}^{\varepsilon_1} \dots a_{i_{p-1}}^{\varepsilon_{p-1}} a_{i_p}^{\varepsilon_p}$, $(\varepsilon_j = \pm 1)$ lies in N, then the word $a_{i_1}^{\varepsilon_1} \dots a_{i_{p-1}}^{\varepsilon_{p-1}}$ also lies in N.

Suppose that some Schreier's system N is chosen for the subgroup $H \subset G$ relative to the system generators a_1, \ldots, a_n of G. For every word Q on a_1, \ldots, a_n , we denote by \overline{Q} the only word from N which lies in the same right coset of G over the subgroup H. Denote

$$S_{K_{\alpha},a_{\nu}} = K_{\alpha}a_{\nu} \cdot (\overline{K_{\alpha}a_{\nu}})^{-1}, \quad \alpha \in A, \ \nu = 1, \dots, n.$$

Theorem of Reidemeister-Schreier states that the elements $S_{K_{\alpha},a_{\nu}}$ generate subgroup H and the set of defining relations for this set of generators is divided in two parts. First part consists of trivial relations $S_{K_{\alpha},a_{\nu}} = 1$, where the pair K_{α}, a_{ν} is such that the word $K_{\alpha}a_{\nu} \cdot (\overline{K_{\alpha}}a_{\nu})^{-1}$ is freely equivalent to the word 1. Second part consists of all relations of the form $\tau(K_{\alpha}R_{\mu}K_{\alpha}^{-1})$, where $\alpha \in A$, $\mu = 1, \ldots, m$, and τ is Reidemeister's transformation, which maps every nonempty word $a_{i_1}^{\varepsilon_1} \ldots a_{i_p}^{\varepsilon_p}$, $(\varepsilon_j = \pm 1)$ from symbols a_1, \ldots, a_n to the word from symbols $S_{K_{\alpha},a_{\nu}}$ by the rule:

$$\tau(a_{i_1}^{\varepsilon_1} \dots a_{i_p}^{\varepsilon_p}) = S_{K_{i_1}, a_{i_1}}^{\varepsilon_1} \dots S_{K_{i_p}, a_{i_p}}^{\varepsilon_p},$$

where $K_{i_j} = \overline{a_{i_1}^{\varepsilon_1} \dots a_{i_{j-1}}^{\varepsilon_{j-1}}}$, if $\varepsilon_j = 1$, and $K_{i_j} = \overline{a_{i_1}^{\varepsilon_1} \dots a_{i_j}^{\varepsilon_j}}$, if $\varepsilon_j = -1$.

3. Commutator subgroup VB'_n

3.1. Generating set of VB'_n . From the above relations it follows that the quotient VB_n/VB'_n is isomorphic to the direct product $\mathbb{Z} \times \mathbb{Z}_2$. One can define the map φ from the following short exact sequence:

$$1 \to VB'_n \to VB_n \xrightarrow{\varphi} \mathbb{Z} \times \mathbb{Z}_2 \to 1$$

where, for i = 1, ..., n - 1, $\varphi(\sigma_i)$ is the generator of \mathbb{Z} and $\phi(\rho_i)$ is the generator of \mathbb{Z}_2 respectively when viewing it as VB_n/VB'_n . The map φ does have a section in the above short exact sequence for $n \ge 3$, and ker $\varphi = VB'_n$.

As a Schreier set of coset representatives of VB_n by VB'_n take the words

$$\Lambda = \left\{ \, \sigma_1^i \rho_1^{\varepsilon} \, | \, i \in \mathbb{Z}, \, \, \varepsilon = 0, 1 \, \right\}$$

The commutator subgroup VB'_n is generated by the words

$$S_{\lambda,a} = \lambda a(\overline{\lambda a})^{-1}, \quad \lambda \in \Lambda, \quad a \in \{\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1}\}$$

Here \overline{w} is a coset representative in Λ of wVB'_n . Find the elements $S_{\lambda,a}$. For this put $\lambda = \sigma_1^i \rho_1^{\varepsilon}$ and considering different a we will get the following cases:

1) If $a = \sigma_1$, then

$$S_{\lambda,\sigma_1} = \sigma_1^i \rho_1^\varepsilon \sigma_1 (\sigma_1^{i+1} \rho_1^\varepsilon)^{-1}.$$

For $\varepsilon = 0$ we have $S_{\lambda,\sigma_1} = 1$ and for $\varepsilon = 1$ we have $S_{\lambda,\sigma_1} = \sigma_1^i (\rho_1 \sigma_1 \rho_1 \sigma_1^{-1}) \sigma_1^{-i}$, which we will denote by a_i .

2) If $a = \sigma_2$, then

$$S_{\lambda,\sigma_2} = \sigma_1^i (\rho_1^\varepsilon \sigma_2 \rho_1^\varepsilon \sigma_1^{-1}) \sigma_1^{-i},$$

and we will denote this element by $b_{i,\varepsilon}$.

3) If $a = \sigma_l, l > 2$, then

$$S_{\lambda,\sigma_l} = \sigma_l \sigma_1^{-1},$$

and we will denote this element by c_l .

4) If $a = \rho_1$, then

 $S_{\lambda,\rho_1} = 1.$

5) If $a = \rho_2$, then

$$S_{\lambda,\rho_2} = \sigma_1^i (\rho_1^{\varepsilon} \rho_2 \rho_1^{\varepsilon+1}) \sigma_1^{-i},$$

and we will denote this element by $f_{i,\varepsilon}$.

6) If $a = \rho_l, l > 2$, then

$$S_{\lambda,\rho_l} = \sigma_1^i(\rho_l\rho_1)\sigma_1^{-i},$$

and we will denote this element by $g_{i,l}$.

To find defining relations of VB'_n we will use the following conjugation rules by elements ρ_1 and σ_1^{-m} .

Lemma 3.1. The following formulas hold

(1) $a_i^{\sigma_1^{-m}} = a_{i+m}, \quad b_{i,\varepsilon}^{\sigma_1^{-m}} = b_{i+m,\varepsilon}, \quad c_l^{\sigma_1^{-m}} = c_l, \quad f_{i,\varepsilon}^{\sigma_1^{-m}} = f_{i+m,\varepsilon}, \quad g_{i,\varepsilon}^{\sigma_1^{-m}} = g_{i+m,\varepsilon},$ (2) $a_0^{\rho_1} = a_0^{-1}, \quad b_{0,0}^{\rho_1} = b_{0,1}a_0^{-1}, \quad b_{0,1}^{\rho_1} = b_{0,0}a_0^{-1}, \quad b_{1,0}^{\rho_1} = a_0b_{1,1}a_1^{-1}a_0^{-1}, \quad b_{2,0}^{\rho_1} = a_0b_{1,1}a_1^{-1}a_0^{$

(3)
$$c_l^{\rho_1} = c_l a_0^{-1}, \quad f_{0,1}^{\rho_1} = f_{0,1}, \quad f_{0,1}^{\rho_1} = f_{0,0}, \quad f_{1,0}^{\rho_1} = a_0 f_{1,1} a_0^{-1}, \quad f_{1,1}^{\rho_1} = a_0 f_{1,0} a_0^{-1};$$

(4) $g_{0,i}^{\rho_1} = g_{0,i}, \quad g_{1,i}^{\rho_1} = a_0 g_{1,i} a_0^{-1}, \quad i > 2.$

Proof. (1) follow from the definition.

For proving (2) note that:

$$\rho_{1}a_{0}\rho_{1} = \rho_{1}\rho_{1}\sigma_{1}\rho_{1}\sigma_{1}^{-1}\rho_{1} = S_{1,\sigma_{1}}S_{\sigma_{1},\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\rho_{1},\rho_{1}} = a_{0}^{-1},$$

$$\rho_{1}b_{0,0}\rho_{1} = \rho_{1}\sigma_{2}\sigma_{1}^{-1}\rho_{1} = S_{1,\rho_{1}}S_{\rho_{1},\sigma_{2}}S_{\rho_{1},\sigma_{1}}^{-1}S_{\rho_{1},\rho_{1}} = b_{0,1}a_{0}^{-1},$$

$$b_{0,1}^{\rho_{1}} = \rho_{1}\rho_{1}\sigma_{2}\rho_{1}\sigma_{1}^{-1}\rho_{1} = S_{1,\sigma_{2}}S_{\sigma_{1},\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\rho_{1},\rho_{1}} = b_{0,0}a_{0}^{-1},$$

$$\rho_{1}b_{1,0}\rho_{1} = \rho_{1}\sigma_{1}\sigma_{2}\sigma_{1}^{-1}\sigma_{1}^{-1}\rho_{1} = S_{1,\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\sigma_{1}\rho_{1},\sigma_{2}}S_{\sigma_{1}\rho_{1},\sigma_{1}}S_{\rho_{1},\sigma_{1}}^{-1} = a_{0}b_{1,1}a_{1}^{-1}a_{0}^{-1}.$$

$$\rho_{1}b_{2,0}\rho_{1} = \rho_{1}\sigma_{1}^{2}\sigma_{2}(\sigma_{1}^{-1})^{3}\rho_{1} =$$

$$= S_{1,\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\sigma_{1}\rho_{1},\sigma_{2}}S_{\sigma_{1}\rho_{1},\sigma_{1}}S_{\sigma_{1}\rho_{1},\sigma_{1}}S_{\rho_{1},\rho_{1}}^{-1} = a_{0}a_{1}(b_{2,1}a_{2}^{-1})a_{1}^{-1}a_{0}^{-1}.$$
For (3):

$$\rho_{1}c_{l}\rho_{1} = \rho_{1}\sigma_{l}\sigma_{1}^{-1}\rho_{1} = S_{1,\rho_{1}}S_{\rho_{1},\sigma_{l}}S_{\rho_{1},\sigma_{1}}S_{\rho_{1},\rho_{1}} = c_{l}a_{0}^{-1},$$

$$f_{0,0}^{\rho_{1}} = \rho_{1}\rho_{2}\rho_{1}\rho_{1} = \rho_{1}\rho_{2} = f_{0,1},$$

$$f_{0,1}^{\rho_{1}} = \rho_{1}\rho_{1}\rho_{2}\rho_{1} = \rho_{2}\rho_{1} = f_{0,0},$$

$$f_{1,0}^{\rho_{1}} = \rho_{1}\sigma_{1}\rho_{2}\rho_{1}\sigma_{1}^{-1}\rho_{1} = S_{1,\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\sigma_{1}\rho_{1},\rho_{2}}S_{\sigma_{1},\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\rho_{1},\rho_{1}} = a_{0}f_{1,1}a_{0}^{-1}.$$

$$f_{1,1}^{\rho_{1}} = \rho_{1}\sigma_{1}\rho_{1}\rho_{2}\sigma_{1}^{-1}\rho_{1} = S_{1,\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\sigma_{1}\rho_{1},\rho_{2}}S_{\rho_{1},\sigma_{1}}S_{\rho_{1},\rho_{1}} = a_{0}f_{1,0}a_{0}^{-1}.$$

$$(4):$$

 $g_{0,i}^{\rho_1} = \rho_1 \rho_i \rho_1 \rho_1 = \rho_1 \rho_i = g_{0,i},$ $g_{1,i}^{\rho_1} = \rho_1 \sigma_1 \rho_i \rho_1 \sigma_1^{-1} \rho_1 = S_{1,\rho_1} S_{\rho_1,\sigma_1} S_{\sigma_1,\rho_1,\rho_i} S_{\sigma_1,\rho_1} S_{\rho_1,\sigma_1} S_{\rho_1,\rho_1} = a_0 g_{1,i} a_0^{-1}, \quad i > 2.$ This proves the lemma. \Box

3.2. Defining Relations in VB'_n . In this subsection we will consider the defining relations of VB_n , rewrite them in the generators of VB'_n , and conjugating by elements $\lambda \in \Lambda$, we get the defining relations of VB'_n .

3.2.1. Defining relation of VB'_n that follow from the relation $\sigma_i\sigma_j = \sigma_j\sigma_i$. Rewrite this relation in the form

$$r_1 = \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}, \ 1 \le i < j \le n - 1, \ i + 1 < j.$$

Using the rewritting process we get:

for
$$i = 1$$
: $r_1 = \sigma_1 \sigma_j \sigma_1^{-1} \sigma_j^{-1} = S_{1,\sigma_1} S_{\sigma_1,\sigma_j} S_{\sigma_1,\sigma_1}^{-1} S_{\sigma_1,\sigma_j}^{-1} = c_j c_j^{-1} = 1$;
for $i = 2$: $r_1 = \sigma_2 \sigma_j \sigma_2^{-1} \sigma_j^{-1} = S_{1,\sigma_2} S_{\sigma_1,\sigma_j} S_{\sigma_1,\sigma_2}^{-1} S_{\sigma_1,\sigma_j}^{-1} = b_{0,0} c_j b_{1,0}^{-1} c_j^{-1}$;
for $i > 2$: $r_1 = \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = S_{1,\sigma_i} S_{\sigma_1,\sigma_j} S_{\sigma_1,\sigma_i}^{-1} S_{\sigma_1,\sigma_j}^{-1} = c_i c_j c_i^{-1} c_j^{-1}$.

Lemma 3.2. The following four types of relations in VB'_n follow from the relation r_1 of VB_n : 1 1

$$b_{m,0}c_{j}b_{m+1,0}^{-1}c_{j}^{-1} = 1, \quad j \ge 4,$$

$$c_{i}c_{j}c_{i}^{-1}c_{j}^{-1} = 1, \quad i \ge 3, \quad j > i+1$$

$$b_{m,1}a_{m}^{-1}c_{j}a_{m+1}b_{m+1,1}^{-1}c_{j}^{-1} = 1, \quad j \ge 4,$$

$$c_{i}a_{m}^{-1}c_{j}c_{i}^{-1}a_{m}c_{j}^{-1} = 1, \quad i \ge 3, \quad j > i+1.$$

Proof. Conjugating of r_1 by ρ_1 and using Lemma 3.1 we get

$$\rho_1(b_{0,0}c_jb_{1,0}^{-1}c_j^{-1})\rho_1 = b_{0,1}a_0^{-1}c_ja_1b_{1,1}^{-1}c_j^{-1},$$

$$\rho_1(c_ic_jc_i^{-1}c_j^{-1})\rho_1 = c_ia_0^{-1}c_jc_i^{-1}a_0^{-1}c_j^{-1}.$$

$$r_1\rho_1 \text{ by } \sigma_1^{-m}, \text{ we get the need relations.} \square$$

Conjugating r_1 and $\rho_1 r_1 \rho_1$ by σ_1^{-m} , we get the need relations.

3.2.2. Defining relations of VB'_n that follow from the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. Rewrite this relation in the form

$$r_2 = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}.$$

Then

for
$$i = 1$$
: $r_2 = \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} = S_{1,\sigma_1} S_{\sigma_1,\sigma_2} S_{\sigma_1^2,\sigma_1} S_{\sigma_1^2,\sigma_1}^{-1} S_{\sigma_1,\sigma_1}^{-1} S_{1,\sigma_2}^{-1} = b_{1,0} b_{2,0}^{-1} b_{0,0}^{-1}.$
for $i = 2$: $r_2 = \sigma_2 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} = S_{1,\sigma_2} S_{\sigma_1,\sigma_3} S_{\sigma_1^2,\sigma_2} S_{\sigma_1^2,\sigma_3}^{-1} S_{\sigma_1,\sigma_2}^{-1} S_{1,\sigma_3}^{-1} = b_{0,0} c_3 b_{2,0} c_3^{-1} b_{1,0}^{-1} c_3^{-1}.$
for $i > 2$: $r_2 = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = S_{1,\sigma_i} S_{\sigma_1,\sigma_{i+1}} S_{\sigma_1^2,\sigma_i} S_{\sigma_1^2,\sigma_{i+1}}^{-1} S_{\sigma_1,\sigma_i}^{-1} S_{1,\sigma_i}^{-1} S_{1,\sigma_{i+1}}^{-1} = c_i c_{i+1} c_i c_{i+1}^{-1} c_i^{-1} c_{i+1}^{-1}.$

Lemma 3.3. The following six types of relations in VB'_n follow from the relation r_2 of VB_n : 1 _1

$$b_{m+1,0}b_{m+2,0}^{-1}b_{m,0}^{-1} = 1,$$

$$b_{m,0}c_{3}b_{m+2,0}c_{3}^{-1}b_{m+1,0}^{-1}c_{3}^{-1} = 1,$$

$$c_{i}c_{i+1}c_{i}c_{i+1}^{-1}c_{i}^{-1}c_{i+1}^{-1} = 1, \quad i \ge 3,$$

$$a_{m}b_{m+1,1}a_{m+2}b_{m+2,1}^{-1}a_{m+1}^{-1}b_{m,1}^{-1} = 1,$$

$$b_{m,1}a_{m}^{-1}c_{3}a_{m+1}b_{m+2,1}a_{m+2}^{-1}a_{m+1}^{-1}c_{3}^{-1}a_{m}a_{m+1}b_{m+1,1}^{-1}c_{3}^{-1} = 1,$$

$$c_{i}a_{m}^{-1}c_{i+1}a_{m}^{-1}c_{i}c_{i+1}^{-1}a_{m}c_{i}^{-1}a_{m}c_{i+1}^{-1} = 1, \quad i \ge 3.$$

Proof. Conjugating r_2 by ρ_1 and using Lemma 3.1, we get

$$(b_{1,0}b_{2,0}^{-1}b_{0,0}^{-1})^{\rho_1} = a_0b_{1,1}a_2b_{2,1}^{-1}a_1^{-1}b_{0,1}^{-1},$$

$$(b_{0,0}c_3b_{2,0}c_3^{-1}b_{1,0}^{-1}c_3^{-1})^{\rho_1} = b_{0,1}a_0^{-1}c_3a_1b_{2,1}a_2^{-1}a_1^{-1}c_3^{-1}a_0a_1b_{1,1}^{-1}c_3^{-1},$$

$$(c_ic_{i+1}c_ic_{i+1}^{-1}c_i^{-1}c_{i+1}^{-1})^{\rho_1} = c_ia_0^{-1}c_{i+1}a_0^{-1}c_ic_{i+1}^{-1}a_0c_i^{-1}a_0c_{i+1}^{-1}.$$

Conjugating r_2 and $\rho_1 r_2 \rho_1$ by σ_1^{-m} , we get the need relations.

3.2.3. Defining relation that follow from the relation $\rho_i^2 = 1$. Rewrite this relation in the form

 $r_3 = \rho_i^2$.

Then

for i = 1: $r_3 = \rho_1 \rho_1 = S_{1,\rho_1} S_{\rho_1,\rho_1} = 1$. for i = 2: $r_3 = \rho_3 \rho_3 = S_{1,\rho_2} S_{\rho_1,\rho_2} = f_{0,0} f_{0,1}$. for i > 2: $r_3 = \rho_i \rho_i = S_{1,\rho_i} S_{\rho_1,\rho_i} = g_{0,i}^2$. The following lemma holds

Lemma 3.4. From r_3 follow two types of relations in VB'_n :

$$f_{m,0}f_{m,1} = g_{m,i}^2 = 1, \quad i > 2.$$

Proof. Conjugating r_3 by ρ_1 and using Lemma 3.1 4, we get

$$(f_{0,0}f_{0,1})^{\rho_1} = f_{0,1}f_{0,0},$$

$$(g_{0,i}^2)^{\rho_1} = g_{0,i}^2, \quad i > 2,$$

Conjugating r_3 and $\rho_1 r_3 \rho_1$ by σ_1^{-m} , we get the need relations.

3.2.4. Defining relations of VB'_n that follow from the relation $\rho_i\rho_j = \rho_j\rho_i$. Rewrite this relation in the form

$$r_4 = \rho_i \rho_j \rho_i \rho_j, \ 1 \le i < j \le n - 1, \ i + 1 < j.$$

Then

for i = 1: $r_4 = \rho_1 \rho_j \rho_1 \rho_j = S_{1,\rho_1} S_{\rho_1,\rho_j} S_{1,\rho_1} S_{\rho_1,\rho_j} = g_{0,j}^2, j > 2$. We have got this relation when we considered the relation r_3 .

for
$$i = 2$$
: $r_4 = \rho_2 \rho_k \rho_2 \rho_k = S_{1,\rho_2} S_{\rho_1,\rho_k} S_{1,\rho_2} S_{\rho_1,\rho_k} = (f_{0,0}g_{0,k})^2, k > 3.$

for i > 2: $r_4 = \rho_i \rho_j \rho_i \rho_j = S_{1,\rho_i} S_{\rho_1,\rho_j} S_{1,\rho_i} S_{\rho_1,\rho_j} = (g_{0,i}g_{0,j})^2, \ 3 \le i < j \le n-1, i+1 < j.$

The following lemma holds

Lemma 3.5. From the relation r_4 of VB_n , the following three types of relations of VB'_n follow:

$$(f_{m,0}g_{m,k})^2 = (f_{m,1}g_{m,k})^2 = 1, \quad k > 3,$$

$$(g_{m,i}g_{m,j})^2 = 1, \quad 3 \le i < j \le n-1, \quad i+1 < j.$$

Proof. Conjugated r_4 by ρ_1 and using Lemma 3.1 4), we get

$$\rho_1(f_{0,0}g_{0,k})^2\rho_1 = (f_{0,1}g_{0,k})^2, \quad k > 3,$$

$$\rho_1(g_{0,i}g_{0,j})^2 \rho_1 = (g_{0,i}g_{0,j})^2, \quad 3 \le i < j \le n-1, \quad i+1 < j.$$

Conjugating r_4 and $\rho_1 r_4 \rho_1$ by σ_1^{-m} , we get the need relations.

3.2.5. Defining relations of VB'_n that follow from the relation $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$. Rewrite this relation in the form

$$r_5 = \rho_i \rho_{i+1} \rho_i \rho_{i+1} \rho_i \rho_{i+1}.$$

Then

for
$$i = 1$$
: $r_5 = \rho_1 \rho_2 \rho_1 \rho_2 \rho_1 \rho_2 = S_{1,\rho_1} S_{\rho_1,\rho_2} S_{1,\rho_1} S_{\rho_1,\rho_2} S_{1,\rho_1} S_{\rho_1,\rho_2} = f_{0,1}^3$.
for $i = 2$: $r_5 = \rho_2 \rho_3 \rho_2 \rho_3 \rho_2 \rho_3 = S_{1,\rho_2} S_{\rho_1,\rho_3} S_{1,\rho_2} S_{\rho_1,\rho_3} S_{1,\rho_2} S_{\rho_1,\rho_3} = (f_{0,0}g_{0,3})^3$.
for $i > 2$: $r_5 = \rho_i \rho_{i+1} \rho_i \rho_{i+1} \rho_i \rho_{i+1} = S_{1,\rho_i} S_{\rho_1,\rho_{i+1}} S_{1,\rho_i} S_{\rho_1,\rho_{i+1}} S_{1,\rho_i} S_{\rho_1,\rho_{i+1}} = (g_{0,i}g_{0,i+1})^3$
The following lemma holds

Lemma 3.6. From the relation r_5 of VB_n , we have the following five types of relations of VB'_n :

$$f_{m,1}^{3} = 1,$$

$$(f_{m,0}g_{m,3})^{3} = 1,$$

$$(g_{m,i}g_{m,i+1})^{3} = 1,$$

$$f_{m,0}^{3} = 1,$$

$$(f_{m,1}g_{m,3})^{3} = 1.$$

Proof. Conjugating r_5 by ρ_1 and using Lemma 3.1 4), we get

$$(f_{0,1}^3)^{\rho_1} = f_{0,0}^3,$$

$$((f_{0,0}g_{0,3})^3)^{\rho_1} = (f_{0,1}g_{0,3})^3,$$

$$((g_{0,i}g_{0,i+1})^3)^{\rho_1} = (g_{0,i}g_{0,i+1})^3, \quad i > 2.$$

Conjugating r_5 and $\rho_1 r_5 \rho_1$ by σ_1^{-m} , we get the need relations.

3.2.6. Defining relations of VB'_n that follow from the relation $\sigma_i \rho_j = \rho_j \sigma_i$. Rewrite this relation in the form

$$r_6 = \sigma_i \rho_j \sigma_i^{-1} \rho_j, \ |i - j| > 1.$$

In dependence of i and j we will consider the next cases

a)
$$r_6 = \sigma_1 \rho_i \sigma_1^{-1} \rho_i = S_{1,\sigma_1} S_{\sigma_1,\rho_i} S_{\rho_1,\sigma_1}^{-1} S_{\rho_1,\rho_i} = g_{1,i} a_0^{-1} g_{0,i}$$
, for $i > 2$.
b) $r_6 = \sigma_2 \rho_j \sigma_2^{-1} \rho_j = S_{1,\sigma_2} S_{\sigma_1,\rho_j} S_{\rho_1,\sigma_2}^{-1} S_{\rho_1,\rho_j} = b_{0,0} g_{1,j} b_{0,1}^{-1} g_{0,j}$, for $j > 3$.
c) $r_6 = \sigma_k \rho_l \sigma_k^{-1} \rho_l = S_{1,\sigma_k} S_{\sigma_1,\rho_l} S_{\rho_1,\sigma_k}^{-1} S_{\rho_1,\rho_l} = c_k g_{1,l} c_k^{-1} g_{0,l}$, $k, l \ge 3$, $|l-k| > 1$.
d) $r_6 = \sigma_i \rho_1 \sigma_i^{-1} \rho_1 = S_{1,\sigma_i} S_{\sigma_1,\rho_1} S_{\rho_1,\sigma_i}^{-1} S_{\rho_1,\rho_1} = c_i c_i^{-1} = 1$, for $i > 2$.
e) $r_6 = \sigma_j \rho_2 \sigma_j^{-1} \rho_2 = S_{1,\sigma_j} S_{\sigma_1,\rho_2} S_{\rho_1,\sigma_j}^{-1} S_{\rho_1,\rho_2} = c_j f_{1,0} c_j^{-1} f_{0,1}$, for $j > 3$.
Now, the following lemma holds.

Lemma 3.7. From the relation r_6 of VB_n , the following seven types of relations of VB'_n follow:

$$g_{m+1,i}a_m^{-1}g_{m,i} = 1, \quad i \ge 3,$$

$$a_m g_{m+1,i}g_{m,i} = 1, \quad i \ge 3,$$

$$b_{m,0}g_{m+1,j}b_{m,1}^{-1}g_{m,j} = 1, \quad j \ge 4,$$

$$b_{m,1}g_{m+1,j}b_{m,0}^{-1}g_{m,j} = 1, \quad j \ge 4,$$

$$c_k g_{m+1,l}c_k^{-1}g_{m,l} = 1, \quad k,l \ge 3, \quad |l-k| > 1,$$

$$c_j f_{m+1,0}c_j^{-1}f_{m,1} = 1, \quad j \ge 4,$$

$$c_j f_{m+1,1}c_j^{-1}f_{m,0} = 1, \quad j \ge 4.$$

Proof. Conjugating r_6 by ρ_1 and using Lemma 3.1, we get

$$(g_{1,i}a_0^{-1}g_{0,i})^{\rho_1} = a_0g_{1,i}g_{0,i}, \quad i \ge 3,$$

$$(b_{0,0}g_{1,j}b_{0,1}^{-1}g_{0,j})^{\rho_1} = b_{0,1}g_{1,j}b_{0,0}^{-1}g_{0,j}, \quad j \ge 4,$$

$$(c_kg_{1,l}c_k^{-1}g_{0,l})^{\rho_1} = c_kg_{1,l}c_k^{-1}g_{0,l}, \quad k,l \ge 3, \quad |l-k| > 1,$$

$$(c_jf_{1,0}c_j^{-1}f_{0,1})^{\rho_1} = c_jf_{1,0}c_j^{-1}f_{0,1}, \quad j \ge 4.$$

Conjugating r_6 and $\rho_1 r_6 \rho_1$ by σ_1^{-m} , we get the need relations.

3.2.7. Defining relations of VB'_n that follow from the relation $\rho_i\rho_{i+1}\sigma_i = \sigma_{i+1}\rho_i\rho_{i+1}$. Rewrite this relation in the form

$$r_7 = \rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i \sigma_{i+1}^{-1}.$$

Then

for
$$i = 1$$
: $r_7 = \rho_1 \rho_2 \sigma_1 \rho_2 \rho_1 \sigma_2^{-1} = S_{1,\rho_1} S_{\rho_1,\rho_2} S_{1,\sigma_1} S_{\sigma_1,\rho_2} S_{\sigma_1\rho_1,\rho_1} S_{1,\sigma_2}^{-1} = f_{0,1} f_{1,0} b_{0,0}^{-1}$.
for $i = 2$: $r_7 = \rho_2 \rho_3 \sigma_2 \rho_3 \rho_2 \sigma_3^{-1} = S_{1,\rho_2} S_{\rho_1,\rho_3} S_{1,\sigma_2} S_{\sigma_1,\rho_3} S_{\sigma_1\rho_1,\rho_2} S_{1,\sigma_3}^{-1} = f_{0,0} g_{0,3} b_{0,0} g_{1,3} f_{1,1} c_3^{-1}$.
for $i > 2$: $r_7 = \rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i \sigma_{i+1}^{-1} = S_{1,\rho_i} S_{\rho_1,\rho_{i+1}} S_{1,\sigma_i} S_{\sigma_1,\rho_{i+1}} S_{\sigma_1\rho_1,\rho_i} S_{1,\sigma_{i+1}}^{-1} = g_{0,i} g_{0,i+1} c_{i} g_{1,i+1} g_{1,i} c_{i+1}^{-1}$.

Now the following defining relations of VB'_n follow from the relation r_7 .

Lemma 3.8. The following five types of relations in VB'_n follow from the relation r_7 of VB_n :

$$\begin{split} f_{m,1}f_{m+1,0}b_{m,0}^{-1} &= 1, \\ f_{m,0}a_mf_{m+1,1}b_{m,1}^{-1} &= 1, \\ f_{m,0}g_{m,3}b_{m,0}g_{m+1,3}f_{m+1,1}c_3^{-1} &= 1, \\ f_{m,1}g_{m,3}b_{m,1}g_{m+1,3}f_{m+1,0}c_3^{-1} &= 1, \\ g_{m,i}g_{m,i+1}c_{i}g_{m+1,i+1}g_{m+1,i}c_{i+1}^{-1} &= 1, \quad i > 2. \end{split}$$

Proof. Conjugating r_7 by ρ_1 and using Lemma 3.1 2), 4) 5) and 6), we get

$$(f_{0,1}f_{1,0}b_{0,0}^{-1})^{\rho_1} = f_{0,0}a_0f_{1,1}b_{0,1}^{-1},$$

$$(f_{0,0}g_{0,3}b_{0,0}g_{1,3}f_{1,1}c_3^{-1})^{\rho_1} = f_{0,1}g_{0,3}b_{0,1}g_{1,3}f_{1,0}c_3^{-1},$$

$$(g_{0,i}g_{0,i+1}c_ig_{1,i+1}g_{1,i}c_{i+1}^{-1})^{\rho_1} = g_{0,i}g_{0,i+1}c_ig_{1,i+1}g_{1,i}c_{i+1}^{-1}, \quad i > 2.$$

Conjugating r_7 and $\rho_1 r_7 \rho_1$ by σ_1^{-m} , we get the need relations.

Using the relations

 $f_{m,0}f_{m,1} = 1$, (see Lemma 3.4),

we can remove elements $f_{m,1}, m \in \mathbb{Z}$, from the generating set and keep only elements

$$f_{m,0} = f_m, \quad m \in \mathbb{Z}.$$

The following result gives a presentation of VB'_n .

Theorem 3.9. The commutator subgroup VB'_n is generated by elements

$$a_m, b_{m,\varepsilon}, c_l, f_m, g_{m,l},$$

where $m \in \mathbb{Z}$, $\varepsilon = 0, 1, 2 < l < n$ and is defined by the relations

$$b_{m,0}c_jb_{m+1,0}^{-1}c_j^{-1} = 1, \quad j \ge 4,$$

$$c_ic_jc_i^{-1}c_j^{-1} = 1, \quad i \ge 3, \quad j > i+1,$$

$$b_{m,1}a_m^{-1}c_ja_{m+1}b_{m+1,1}^{-1}c_j^{-1} = 1, \quad j \ge 4,$$

$$c_ia_m^{-1}c_jc_i^{-1}a_mc_j^{-1} = 1, \quad i \ge 3, \quad j > i+1$$

$$\begin{split} b_{m+1,0}b_{m+2,0}^{-1}b_{m,0}^{-1} &= 1, \\ b_{m,0}c_{3}b_{m+2,0}c_{3}^{-1}b_{m+1,0}^{-1}c_{3}^{-1} &= 1, \\ c_{i}c_{i+1}c_{i}c_{i+1}^{-1}c_{i}^{-1}c_{i+1}^{-1} &= 1, \quad i \geq 3, \\ a_{m}b_{m+1,1}a_{m+2}b_{m+2,1}^{-1}a_{m+1}^{-1}b_{m,1}^{-1} &= 1, \\ b_{m,1}a_{m}^{-1}c_{3}a_{m+1}b_{m+2,1}a_{m+2}^{-1}a_{m+1}^{-1}c_{3}^{-1}a_{m}a_{m+1}b_{m+1,1}^{-1}c_{3}^{-1} &= 1, \\ c_{i}a_{m}^{-1}c_{i+1}a_{m}^{-1}c_{i}c_{i+1}^{-1}a_{m}c_{i}^{-1}a_{m}c_{i+1}^{-1} &= 1, \quad i \geq 3. \end{split}$$

$$g_{m,i}^2 = 1, \quad i > 2$$

$$(f_m g_{m,k})^2 = 1, \quad k > 3,$$

 $(g_{m,i} g_{m,j})^2 = 1, \quad 3 \le i < j \le n - 1, \quad i + 1 < j.$

$$f_m^3 = 1,$$

 $(f_m g_{m,3})^3 = 1,$
 $(g_{m,i}g_{m,i+1})^3 = 1, \quad i > 2,$

$$\begin{split} g_{m+1,i}a_m^{-1}g_{m,i} &= 1, \quad i \geq 3, \\ b_{m,1}g_{m+1,j}b_{m,0}^{-1}g_{m,j} &= 1, \quad j \geq 4, \\ c_kg_{m+1,l}c_k^{-1}g_{m,l} &= 1, \quad k,l \geq 3, \quad |l-k| > 1, \\ c_jf_{m+1}c_j^{-1}f_m^{-1} &= 1, \quad j \geq 4, \\ & f_m^{-1}f_{m+1}b_{m,0}^{-1} &= 1, \\ f_ma_mf_{m+1}^{-1}b_{m,1}^{-1} &= 1, \\ f_mg_{m,3}b_{m,0}g_{m+1,3}f_{m+1}^{-1}c_3^{-1} &= 1, \\ f_m^{-1}g_{m,3}b_{m,1}g_{m+1,3}f_{m+1}c_3^{-1} &= 1, \\ g_{m,i}g_{m,i+1}c_ig_{m+1,i+1}g_{m+1,i}c_{i+1}^{-1} &= 1, \quad i > 2. \end{split}$$

Proof. The theorem is obtained by combining the set of relations we have obtained in Lemma 3.2–Lemma 3.8. $\hfill \Box$

4. Proof of Theorem 1.1

4.1. Finite generation of VB'_n , $n \ge 4$. For the next calculations we remove the generators $b_{m,1}$ and $b_{m,0}$ from the presentation in Theorem 3.9.

Using the relations

$$b_{m,1} = f_m a_m f_{m+1}^{-1},$$

we can remove generators $b_{m,1}$ from the set of generators.

We have

$$b_{m,0}c_jb_{m+1,0}^{-1}c_j^{-1} = 1, \quad j > 4,$$

$$c_ic_jc_i^{-1}c_j^{-1} = 1, \quad i \ge 3, \quad j \ge i+1,$$

$$f_m a_m f_{m+1}^{-1}a_m^{-1}c_ja_{m+1}f_{m+2}a_{m+1}^{-1}f_{m+1}^{-1}c_j^{-1} = 1, \quad j \ge 4,$$

$$c_i a_m^{-1}c_jc_i^{-1}a_m c_j^{-1} = 1, \quad i \ge 3, \quad j > i+1.$$

$$\begin{split} b_{m+1,0}b_{m+2,0}^{-1}b_{m,0}^{-1} &= 1, \\ b_{m,0}c_{3}b_{m+2,0}c_{3}^{-1}b_{m+1,0}^{-1}c_{3}^{-1} &= 1, \\ c_{i}c_{i+1}c_{i}c_{i+1}^{-1}c_{i}^{-1}c_{i+1}^{-1} &= 1, \quad i \geq 3, \\ a_{m}f_{m+1}a_{m+1}f_{m+2}^{-1}a_{m+2} &= f_{m}a_{m}f_{m+1}^{-1}a_{m+1}f_{m+2}a_{m+2}f_{m+3}^{-1}, \\ f_{m}a_{m}f_{m+1}^{-1}a_{m}^{-1}c_{3}a_{m+1}f_{m+2}a_{m+2}f_{m+3}^{-1}a_{m+2}^{-1} &= c_{3}f_{m+1}a_{m+1}f_{m+2}^{-1}a_{m}^{-1}c_{3}a_{m+1}, \\ c_{i}a_{m}^{-1}c_{i+1}a_{m}^{-1}c_{i}c_{i+1}^{-1}a_{m}c_{i}^{-1}a_{m}c_{i+1}^{-1} &= 1, \quad i \geq 3. \end{split}$$

$$g_{m,i}^2 = 1, \quad i > 2.$$

$$(f_m g_{m,k})^2 = 1, \quad k > 3,$$

$$(g_{m,i}g_{m,j})^2 = 1, \quad 3 \le i < j \le n-1, \quad i+1 < j.$$

$$f_m^3 = 1,$$

$$(f_m g_{m,3})^3 = 1,$$

$$(g_{m,i}g_{m,i+1})^3 = 1, \quad i > 2,$$

$$g_{m+1,i}a_m^{-1}g_{m,i} = 1, \quad i \ge 3,$$

$$f_m a_m f_{m+1}^{-1}g_{m+1,j}b_{m,0}^{-1}g_{m,j} = 1, \quad j \ge 4,$$

$$c_k g_{m+1,l}c_k^{-1}g_{m,l} = 1, \quad k, l \ge 3, \quad |l-k| > 1,$$

$$c_j f_{m+1}c_j^{-1}f_m^{-1} = 1, \quad j \ge 4,$$

$$\begin{aligned} f_m^{-1} f_{m+1} b_{m,0}^{-1} &= 1, \\ f_m g_{m,3} b_{m,0} g_{m+1,3} f_{m+1}^{-1} c_3^{-1} &= 1, \\ f_m^{-1} g_{m,3} f_m a_m f_{m+1}^{-1} g_{m+1,3} f_{m+1} c_3^{-1} &= 1, \\ g_{m,i} g_{m,i+1} c_i g_{m+1,i+1} g_{m+1,i} c_{i+1}^{-1} &= 1, \quad i > 2. \end{aligned}$$

Using the relations

$$b_{m,0} = f_m^{-1} f_{m+1},$$

we can remove the generators $b_{m,0}$ from the generating set.

After removing $b_{m,0}$ and $b_{m,1}$ we have following set of defining relations of VB'_n :

$$\begin{split} f_m^{-1} f_{m+1} c_j &= c_j f_{m+1}^{-1} f_{m+2}, \quad j \geq 4, \\ c_i c_j c_i^{-1} c_j^{-1} &= 1, \quad i \geq 3, \quad j > i+1, \\ f_m a_m f_{m+1}^{-1} a_m^{-1} c_j a_{m+1} f_{m+2} a_{m+1}^{-1} f_{m+1}^{-1} c_j^{-1} &= 1, \quad j \geq 4, \\ c_i a_m^{-1} c_j c_i^{-1} a_m c_j^{-1} &= 1, \quad i \geq 3, \quad j > i+1. \\ & f_m f_{m+1}^{-1} f_{m+2} &= f_{m+1} f_{m+2}^{-1} f_{m+3}, \\ f_m^{-1} f_{m+1} c_3 f_{m+2}^{-1} f_{m+3} &= c_3 f_{m+1}^{-1} f_{m+2} c_3, \\ c_i c_{i+1} c_i c_{i+1}^{-1} c_i^{-1} c_{i+1}^{-1} &= 1, \quad i \geq 3, \\ a_m f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} &= f_m a_m f_{m+1}^{-1} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1}, \\ f_m a_m f_{m+1}^{-1} a_m^{-1} c_3 a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1} &= c_3 f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_m^{-1} c_3 a_{m+1}, \\ c_i a_m^{-1} c_{i+1} a_m^{-1} c_i c_{i+1}^{-1} a_m c_i^{-1} a_m c_{i+1}^{-1} &= 1, \quad i \geq 3. \\ g_{m,i}^2 &= 1, \quad i > 2. \end{split}$$

 $(g_{m,i}g_{m,j})^2 = 1, \quad 3 \le i < j \le n-1, \quad i+1 < j.$

$$f_m^3 = 1,$$

 $(f_m g_{m,3})^3 = 1,$
 $(g_{m,i}g_{m,i+1})^3 = 1, \quad i > 2,$

$$g_{m+1,i}a_m^{-1}g_{m,i} = 1, \quad i \ge 3,$$

$$g_{m,j}f_m a_m f_{m+1}^{-1}g_{m+1,j} = f_m^{-1}f_{m+1}, \quad j \ge 4,$$

$$c_k g_{m+1,l} = g_{m,l}c_k, \quad k, l \ge 3, \quad |l-k| > 1,$$

$$c_j f_{m+1} = f_m c_j, \quad j \ge 4,$$

$$f_m g_{m,3} f_m^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1} = c_3^{-1},$$

$$f_m^{-1} g_{m,3} f_m a_m f_{m+1}^{-1} g_{m+1,3} f_{m+1} = c_3,$$

$$g_{m,i} g_{m,i+1} c_i = c_{i+1} g_{m+1,i} g_{m+1,i+1}, \quad i > 2.$$

We will use this set of relations to prove that VB'_n is finitely generated for all $n \ge 4$.

Lemma 4.1. The commutator subgroup VB'_n is finitely generated for all $n \ge 4$. In particular, VB'_4 is generated by 5 elements: c_3 , f_0 , f_1 , f_2 , $g_{0,3}$, and VB'_n , $n \ge 5$, is generated by 2n-3 elements: c_3, \ldots, c_{n-1} , f_0 , f_1 , f_2 , $g_{0,3}, \ldots, g_{0,n-1}$.

Proof. 1) Using the relations

$$g_{m,i}g_{m+1,i} = a_m, \quad i \ge 3,$$

we will remove the generators a_m , $m \in \mathbb{Z}$, and express them by $g_{m,i}$, $m \in \mathbb{Z}$, $i \ge 3$. 2) Using the relations

$$f_m f_{m+1}^{-1} f_{m+2} = f_{m+1} f_{m+2}^{-1} f_{m+3}$$

we can remove the generators f_m , for $m \in \mathbb{Z}$, and keep only f_0, f_1, f_2 .

3) Using the relations

$$f_m g_{m,3} f_m^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1} = c_3^{-1},$$

we can remove the generators $g_{m,3}$, for $m \in \mathbb{Z}$, and keep only $g_{0,3}$.

If n = 4, then we have only generators $g_{0,3}$, f_0 , f_1 , f_2 , c_3 . Hence, VB'_4 is finitely generated.

If n > 4, then

4) Using the relations

$$g_{m,i}g_{m,i+1}c_i = c_{i+1}g_{m+1,i}g_{m+1,i+1}, \quad i > 2$$

we can remove the generators $g_{m,i}$, for $m \in \mathbb{Z}$, i > 3, and keep only $g_{0,i}$.

4.2. Infinite generation of VB'_3 . Consider the case n = 3. From Theorem 3.9 follows that VB'_3 is generated by elements

$$a_m, \quad b_{m,\varepsilon}, \quad f_m, \quad m \in \mathbb{Z},$$

and is defined by the relations

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(4.1)
$$b_{m+1,0}b_{m+2,0}^{-1}b_{m,0}^{-1} = 1,$$

(4.2)
$$a_m b_{m+1,1} a_{m+2} b_{m+2,1}^{-1} a_{m+1}^{-1} b_{m,1}^{-1} = 1,$$

(4.3)
$$f_m^3 = 1,$$

(4.4)
$$f_m^{-1} f_{m+1} f_{m,0}^{-1} = 1,$$

(4.5)
$$f_m a_m f_{m+1}^{-1} b_{m,1}^{-1} = 1.$$

Now we apply Tietze transformations to the presentation of VB'_3 . Using relations (4.5) we can remove the generator $b_{m,1} = f_m a_m f_{m+1}^{-1}$. Then the modified set of defining relations take the form:

(4.6)
$$b_{m+1,0}b_{m+2,0}^{-1}b_{m,0}^{-1} = 1,$$

(4.7)
$$a_m f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} f_{m+3} a_{m+2}^{-1} f_{m+2}^{-1} a_{m+1}^{-1} f_{m+1} a_m^{-1} f_m^{-1} = 1,$$

(4.8)
$$f_m^3 = 1,$$

(4.9)
$$f_m^{-1} f_{m+1} b_{m,0}^{-1} = 1,$$

Using relations (4.9) we can remove the generator $b_{m,0} = f_m^{-1} f_{m+1}$. Then VB'_3 is generated by elements

$$a_m, f_m, m \in \mathbb{Z},$$

and is defined by relation:

(4.10)
$$f_{m+1}^{-1}f_{m+2}f_{m+3}^{-1}f_{m+2}f_{m+1}^{-1}f_m = 1,$$

(4.11)
$$a_m f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} f_{m+3} a_{m+2}^{-1} f_{m+2}^{-1} a_{m+1}^{-1} f_{m+1} a_m^{-1} f_m^{-1} = 1,$$

(4.12)
$$f_m^3 = 1,$$

So, we have the following lemma.

with kernel

Lemma 4.2. The group VB'_3 has a presentation with $\{a_m, f_m, m \in \mathbb{Z}\}$ as the generating set, and the relations (4.10)– (4.12) as the defining relations.

Lemma 4.3. VB'_3 is not finitely generated.

Proof. If we put $f_m = 1$, for all $m \in \mathbb{Z}$, then all the relations (4.10)– (4.12) will vanish, i. e. the subgroup $\langle a_m | m \in \mathbb{Z} \rangle$ is infinitely generated free group with the set of free generators $a_m, m \in \mathbb{Z}$ and we have an epimorphism

$$VB'_{3} \longrightarrow F_{\infty} = \langle a_{m}, \ m \in \mathbb{Z} \rangle$$
$$\langle f_{m}, \ m \in \mathbb{Z} \rangle^{VB'_{3}}.$$

4.3. **Proof of Theorem 1.1.** Note that $VB_2 = F_2 \rtimes S_2$ and hence VB'_2 is infinitely generated. Then the Theorem 1.1 follows by combining Lemma 4.1 and Lemma 4.3.

4.4. Proof of Corollary 1.2. In the quotient VB'_3/VB''_3 relations have the form

$$f_m f_{m+1} = f_{m+2} f_{m+3},$$

$$f_m^3 = 1.$$
 In the generators $f_0, f_1, f_2, a_m, m \in \mathbb{Z}$, we have relations

$$f_0^3 = f_1^3 = f_2^3 = 1.$$

Hence, VB'_3/VB''_3 is isomorphic to the direct sum

$$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^{\infty}.$$

(2) Consider the case n = 4. Then VB'_4 is generated by elements

$$a_m, \quad c_3, \quad f_m, \quad g_{m,3},$$

where $m \in \mathbb{Z}$, and the defining relations have the form

$$\begin{split} f_m f_{m+1}^{-1} f_{m+2} &= f_{m+1} f_{m+2}^{-1} f_{m+3}, \\ f_m^{-1} f_{m+1} c_3 f_{m+2}^{-1} f_{m+3} &= c_3 f_{m+1}^{-1} f_{m+2} c_3, \\ a_m f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} &= f_m a_m f_{m+1}^{-1} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1}, \\ f_m a_m f_{m+1}^{-1} a_m^{-1} c_3 a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1} &= c_3 f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_m^{-1} c_3 a_{m+1}, \\ g_{m,3}^2 &= 1, \\ f_m^3 &= 1, \\ (f_m g_{m,3})^3 &= 1, \\ g_{m+1,3} a_m^{-1} g_{m,3} &= 1, \\ f_m g_{m,3} f_m^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1} &= c_3^{-1}, \\ f_m^{-1} g_{m,3} f_m a_m f_{m+1}^{-1} g_{m+1,3} f_{m+1} &= c_3, \end{split}$$

Consider these relations in the quotient VB'_4/VB''_4 and for the images of the generators

$$c_3, a_m, f_m, g_{m,3}, m \in \mathbb{Z}$$

we will use the same symbols.

From relation $g_{m,3}^2 = f_m^3 = (f_m g_{m,3})^3 = 1$ we get $g_{m,3} = 1$.

Then from the relation $g_{m+1,3}a_m^{-1}g_{m,3} = 1$ follows that $a_m = 1$.

The other relations have the form

$$c_3 = f_m^3 = 1, \quad f_m f_{m+1} = f_{m+2} f_{m+3}$$

Hence, we can keep only generators f_0 , f_1 , f_2 and defining relations $f_0^3 = f_1^3 = f_2^3 = 1$.

Hence the quotient VB'_4/VB''_4 is isomorphic to the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ of three cyclic groups \mathbb{Z}_3 of order 3.

Consider the case n > 4. We will consider relations of VB'_n in the quotient VB'_n/VB''_n and will denote the images of the generators

$$a_m, b_{m,\varepsilon}, c_l, f_m, g_{m,l},$$

where $m \in \mathbb{Z}$, $\varepsilon = 0, 1, 2 < l < n$ by the same symbols.

As in the case n = 4 we get $g_{m,3} = a_m = 1$.

Then from the relations

$$g_{m,i}^2 = 1, \quad (g_{m,i}g_{m,i+1})^3 = 1, \quad i > 2,$$

follows that $g_{m,i} = 1, i > 2$.

From the relations

$$f_m^3 = 1, \quad (f_m g_{m,k})^2 = 1, \quad k > 3,$$

follows that $f_m = 1$.

Remaining relations have the form

$$c_i = 1, \quad i \ge 3.$$

This completes the proof.

5. Commutator subgroup of the welded braid group

The welded braid group WB_n , $n \ge 2$, is the quotient of VB_n by the relations

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \ i = 1, 2, \dots, n-2.$$

In this section we will find a presentation of WB'_n . We will use the same set of generators that we used for VB_n and VB'_n . Hence to find defining relations for WB'_n we need to add relations that follow from the relation

$$r_8 = \rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}.$$

Depending on i we will consider 3 cases:

if
$$i = 1$$
, then
 $r_8 = \rho_1 \sigma_2 \sigma_1 \rho_2 \sigma_1^{-1} \sigma_2^{-1} = S_{1,\rho_1} S_{\rho_1,\sigma_2} S_{\sigma_1 \rho_1,\sigma_1} S_{\sigma_1^2 \rho_1,\rho_2} S_{\sigma_1,\sigma_1}^{-1} S_{1,\sigma_2}^{-1} = b_{0,1} a_1 f_{2,1} b_{0,0}^{-1};$
if $i = 2$, then
 $r_8 = \rho_2 \sigma_3 \sigma_2 \rho_3 \sigma_2^{-1} \sigma_3^{-1} = S_{1,\rho_2} S_{\rho_1,\sigma_3} S_{\sigma_1 \rho_1,\sigma_2} S_{\sigma_1^2 \rho_1,\rho_3} S_{\sigma_1,\sigma_2}^{-1} S_{1,\sigma_3}^{-1} = f_{0,0} c_3 b_{1,1} g_{2,3} b_{1,0}^{-1} c_3^{-1};$

if i > 2, then

$$r_{8} = \rho_{i}\sigma_{i+1}\sigma_{i}\rho_{i+1}\sigma_{i}^{-1}\sigma_{i+1}^{-1}$$

= $S_{1,\rho_{i}}S_{\rho_{1},\sigma_{i+1}}S_{\sigma_{1}\rho_{1},\sigma_{i}}S_{\sigma_{1}^{2}\rho_{1},\rho_{i+1}}S_{\sigma_{1},\sigma_{i}}^{-1}S_{1,\sigma_{i+1}}^{-1}$
= $g_{0,i}c_{i+1}c_{i}g_{2,i+1}c_{i}^{-1}c_{i+1}^{-1}$.

We will use the following conjugation rules

Lemma 5.1. In WB_n the following conjugation rules hold:

(1)
$$a_1^{\rho_1} = a_0 a_1^{-1} a_0^{-1},$$

(2) $f_{2,1}^{\rho_1} = a_0 a_1 f_{2,0} a_1^{-1} a_0^{-1},$
(3) $g_{2,i}^{\rho_1} = a_0 a_1 g_{2,i} a_1^{-1} a_0^{-1}$ for $i > 2.$

Proof. (1) Note that

$$\rho_{1}a_{1}\rho_{1} = \rho_{1}\sigma_{1}\rho_{1}\sigma_{1}\rho_{1}\sigma_{1}^{-1}\sigma_{1}^{-1}\rho_{1}
= S_{1,\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\sigma_{1}\rho_{1},\rho_{1}}S_{\sigma_{1},\sigma_{1}}S_{\sigma_{1}^{2},\rho_{1}}S_{\sigma_{1}\rho_{1},\sigma_{1}}^{-1}S_{\rho_{1},\sigma_{1}}S_{\rho_{1},\rho_{1}}
= a_{0}a_{1}^{-1}a_{0}^{-1};$$

(2) Next we have,

$$\rho_{1}f_{2,1}\rho_{1} = \rho_{1}\sigma_{1}\sigma_{1}\rho_{1}\rho_{2}\sigma_{1}^{-1}\sigma_{1}^{-1}\rho_{1}
= S_{1,\rho_{1}}S_{\rho_{1},\sigma_{1}}S_{\sigma_{1}\rho_{1},\sigma_{1}}S_{\sigma_{1}^{2}\rho_{1},\rho_{1}}S_{\sigma_{1}^{2}\rho_{2}}S_{\sigma_{1}\rho_{1},\sigma_{1}}^{-1}S_{\rho_{1},\sigma_{1}}S_{\rho_{1},\rho_{1}}
= a_{0}a_{1}f_{2,0}a_{1}^{-1}a_{0}^{-1};$$

(3) Finally,

$$\rho_1 g_{2,i} \rho_1 = \rho_1 \sigma_1 \sigma_1 \rho_i \rho_1 \sigma_1^{-1} \sigma_1^{-1} \rho_1
= S_{1,\rho_1} S_{\rho_1,\sigma_1} S_{\sigma_1 \rho_1,\sigma_1} S_{\sigma_1^2 \rho_1,\rho_i} S_{\sigma_1^2,\rho_1} S_{\sigma_1 \rho_1,\sigma_1}^{-1} S_{\rho_1,\sigma_1}^{-1} S_{\rho_1,\rho_1}
= a_0 a_1 g_{2,i} a_1^{-1} a_0^{-1}.$$

This proves the lemma.

Lemma 5.2. From the relation r_8 of WB_n , the following six types of relations of WB'_n follow:

$$\begin{split} b_{m,1}a_{m+1}f_{m+2,1}b_{m,0}^{-1} &= 1, \\ f_{m,0}c_{3}b_{m+1,1}g_{m+2,3}b_{m+1,0}^{-1}c_{3}^{-1} &= 1, \\ g_{m,i}c_{i+1}c_{i}g_{m+2,i+1}c_{i}^{-1}c_{i+1}^{-1} &= 1, \\ b_{m,0}f_{m+2,0}a_{m+1}^{-1}b_{m,1}^{-1} &= 1, \\ f_{m,1}c_{3}f_{m+1,0}a_{m+1}g_{m+2,3}b_{m+1,1}^{-1}c_{3}^{-1} &= 1, \\ g_{m,i}c_{i+1}a_{m}^{-1}c_{i}a_{m+1}g_{m+2,i+1}a_{m+1}^{-1}c_{i}^{-1}a_{m}c_{i+1}^{-1} &= 1. \end{split}$$

Proof. Conjugating relations r_8 by ρ_1 and using Lemma 5.1, we get 3 relations:

$$(b_{0,1}a_1f_{2,1}b_{0,0}^{-1})^{\rho_1} = b_{0,0}f_{2,0}a_1^{-1}b_{0,1}^{-1},$$

$$(f_{0,0}c_3b_{1,1}g_{2,3}b_{1,0}^{-1}c_3^{-1})^{\rho_1} = f_{0,1}c_3f_{1,0}a_1g_{2,3}b_{1,1}^{-1}c_3^{-1},$$

 $(g_{0,i}c_{i+1}c_ig_{2,i+1}c_i^{-1}c_{i+1}^{-1})^{\rho_1} = g_{0,i}c_{i+1}a_0^{-1}c_ia_1g_{2,i+1}a_1^{-1}c_i^{-1}a_0c_{i+1}^{-1}.$

Conjugating relations r_8 and $\rho_1 r_8 \rho_1$ by σ_1^{-m} , we get the six relations from the lemma.

Thus we have the following.

Corollary 5.3. The commutator subgroup WB'_n is generated by elements

 $a_m, \quad b_{m,\varepsilon}, \quad c_l, \quad f_m, \quad g_{m,l},$

where $m \in \mathbb{Z}$, $\varepsilon = 0, 1, 2 < l < n$ and is defined by the relations in Theorem 3.9 and Lemma 5.2.

5.1. **Presentation of** WB'_3 . We have found a presentation of VB'_3 . To get a presentation of WB'_3 we need to add two series of relations:

(5.1)
$$b_{m,1}a_{m+1}f_{m+2,1}b_{m,0}^{-1} = 1$$

(5.2)
$$b_{m,0}f_{m+2,0}a_{m+1}^{-1}b_{m,1}^{-1} = 1,$$

that follow from Lemma 5.2.

As in the case of VB'_3 we can remove the generator $f_{m,1}$, using the relation $f_{m,0}f_{m,1} = 1$. Then the relations (5.1)–(5.2) have the form

(5.3)
$$b_{m,1}a_{m+1}f_{m+2}^{-1}b_{m,0}^{-1} = 1,$$

(5.4)
$$b_{m,0}f_{m+2}a_{m+1}^{-1}b_{m,1}^{-1} = 1,$$

where we denote $f_m = f_{m,0}$.

Using the relations

$$b_{m,1} = f_m a_m f_{m+1}^{-1},$$

that hold in VB'_3 , we can remove $b_{m,1}$. Then the relations (5.3)–(5.4) have the form

(5.5)
$$f_m a_m f_{m+1}^{-1} a_{m+1} f_{m+2}^{-1} b_{m,0}^{-1} = 1$$

(5.6)
$$b_{m,0}f_{m+2}a_{m+1}^{-1}f_{m+1}a_m^{-1}f_m^{-1} = 1.$$

We see that the second relation is inverse to the first one. Hence, we can remove the second relation.

Next, using the relations $f_m^{-1} f_{m+1} b_{m,0}^{-1} = 1$, which hold in VB'_3 , we can remove the generator $b_{m,0}$. Then (5.5) has the form

(5.7)
$$f_m a_m f_{m+1}^{-1} a_{m+1} f_{m+2}^{-1} f_{m+1}^{-1} f_m = 1.$$

Using the presentation of VB'_3 we get

Proposition 5.4. The group WB'_3 is generated by elements

$$a_m, f_m, m \in \mathbb{Z}$$

and is defined by relation:

(5.8)
$$f_{m+1}^{-1}f_{m+2}f_{m+3}^{-1}f_{m+2}f_{m+1}^{-1}f_m = 1,$$

(5.9)
$$a_m f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} f_{m+3} a_{m+2}^{-1} f_{m+2}^{-1} a_{m+1}^{-1} f_{m+1} a_m^{-1} f_m^{-1} = 1,$$

(5.10)
$$f_m^3 = 1,$$

(5.11)
$$a_m f_{m+1}^{-1} a_{m+1} f_{m+2}^{-1} f_{m+1}^{-1} f_m^{-1} = 1$$

As consequence we get

Corollary 5.5. WB'_3 is generated by a_0, f_0, f_1, f_2 .

Proof. From the set of relations (5.8) we can express the generators f_k , where k > 2 or k < 0, as words in the generators f_0, f_1, f_2 and analogously, from the set of relations (5.11) we can express the generators a_l , where $l \neq 0$, as words in the generators a_0, f_0, f_1, f_2 .

Corollary 5.6. WB'_3/WB''_3 is isomorphic to the direct sum

$$\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}.$$

Proof. In the quotient WB'_3/WB''_3 the relations have the form

$$f_m f_{m+1} = f_{m+2} f_{m+3},$$

$$f_m^3 = 1,$$

$$a_m a_{m+1} = f_m f_{m+1}^{-1} f_{m+2}.$$

In the generators a_0, f_0, f_1, f_2 we have relations

$$f_0^3 = f_1^3 = f_2^3 = 1$$

This completes the proof.

5.2. The commutator subgroup WB'_4 . In WB'_4 we have relations of VB'_4 and the following relations:

$$b_{m,1}a_{m+1}f_{m+2,1}b_{m,0}^{-1} = 1,$$

$$b_{m,0}f_{m+2,0}a_{m+1}^{-1}b_{m,1}^{-1} = 1,$$

$$f_{m,0}c_{3}b_{m+1,1}g_{m+2,3}b_{m+1,0}^{-1}c_{3}^{-1} = 1,$$

$$f_{m,1}c_{3}f_{m+1,0}a_{m+1}g_{m+2,3}b_{m+1,1}^{-1}c_{3}^{-1} = 1$$

Excluding the generators

 $b_{m,0} = f_m^{-1} f_{m+1}, \quad b_{m,1} = f_m a_m f_{m+1}^{-1}, \quad f_{m,1} = f_{m,0}^{-1} = f_m^{-1}$

from these relations. We get relations

$$f_m a_m f_{m+1}^{-1} a_{m+1} f_{m+2}^{-1} f_{m+1}^{-1} f_m = 1,$$

$$f_m^{-1} f_{m+1} f_{m+2} a_{m+1}^{-1} f_{m+1} a_m^{-1} f_m^{-1} = 1,$$

$$f_m c_3 f_{m+1} a_{m+1} f_{m+2}^{-1} g_{m+2,3} f_{m+2}^{-1} f_{m+1} c_3^{-1} = 1,$$

$$f_m^{-1} c_3 f_{m+1} a_{m+1} g_{m+2,3} f_{m+2} a_{m+1}^{-1} f_{m+1}^{-1} c_3^{-1} = 1.$$

The second relation is inverse of the first relation. Hence, we can keep only the first relation. Rewrite it in the form

$$a_m f_{m+1}^{-1} a_{m+1} = f_m f_{m+1} f_{m+2}$$

Rewrite the third and the forth relations in the form

$$c_{3}f_{m+1}a_{m+1}f_{m+2}^{-1}g_{m+2,3}f_{m+2}f_{m+1}c_{3}^{-1}f_{m} = 1,$$

$$c_{3}f_{m+1}a_{m+1}g_{m+2,3}f_{m+2}a_{m+1}^{-1}f_{m+1}c_{3}^{-1}f_{m}^{-1} = 1.$$

From these relations:

$$f_{m+2}^{-1}g_{m+2,3}f_{m+2}^{-1}f_{m+1}c_3^{-1}f_m = g_{m+2,3}f_{m+2}a_{m+1}^{-1}f_{m+1}^{-1}c_3^{-1}f_m^{-1}.$$

Since in VB'_4 holds

$$(g_{m+2,3}f_{m+2})^3 = 1, \quad g_{m+2,3}^2 = 1,$$

then

$$(g_{m+2,3}f_{m+2})^{-2}f_{m+2}^{-1}f_{m+1}c_3^{-1}f_m = a_{m+1}^{-1}f_{m+1}^{-1}c_3^{-1}f_m^{-1}$$

and

$$g_{m+2,3}f_{m+2}f_{m+2}^{-1}f_{m+1}c_3^{-1}f_m = a_{m+1}^{-1}f_{m+1}^{-1}c_3^{-1}f_m^{-1}.$$

Therefore,

$$g_{m+2,3} = a_{m+1}^{-1} f_{m+1}^{-1} c_3^{-1} f_m c_3 f_{m+1}^{-1}.$$

Including this expression of $g_{m+2,3}$ in the forth relation:

$$f_m^{-1}c_3f_{m+1}a_{m+1}a_{m+1}^{-1}f_{m+1}^{-1}c_3^{-1}f_mc_3f_{m+1}^{-1}f_{m+2}a_{m+1}^{-1}f_{m+1}^{-1}c_3^{-1} = 1$$

We get after cancelation

$$a_{m+1} = f_{m+1} f_{m+2}$$

Including this expression of a_{m+1} in the expression for $g_{m+2,3}$, we get

$$g_{m+2,3} = f_{m+2}^{-1} f_{m+1}^{-2} c_3^{-1} f_m c_3 f_{m+1}^{-1}$$

or

$$g_{m+2,3} = f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1}.$$

Next, the relation $a_m f_{m+1}^{-1} a_{m+1} = f_m f_{m+1} f_{m+2}$ after substitution $a_m = f_m f_{m+1}$, $a_{m+1} = f_{m+1} f_{m+2}$ becomes an identity.

Hence, the new relations in WB'_4 are equal to relations

$$g_{m+2,3} = f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1},$$
$$a_m = f_m f_{m+1}.$$

The full set of relations in WB'_4 has the form:

$$\begin{split} f_m f_{m+1}^{-1} f_{m+2} &= f_{m+1} f_{m+2}^{-1} f_{m+3}, \\ f_m^{-1} f_{m+1} c_3 f_{m+2}^{-1} f_{m+3} &= c_3 f_{m+1}^{-1} f_{m+2} c_3, \\ a_m f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} &= f_m a_m f_{m+1}^{-1} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1}, \\ f_m a_m f_{m+1}^{-1} a_m^{-1} c_3 a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1} &= c_3 f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_m^{-1} c_3 a_{m+1}, \\ g_{m,3}^2 &= 1, \\ f_m^3 &= 1, \end{split}$$

$$(f_m g_{m,3})^3 = 1,$$

$$g_{m+1,3} a_m^{-1} g_{m,3} = 1,$$

$$f_m g_{m,3} f_m^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1} = c_3^{-1},$$

$$f_m^{-1} g_{m,3} f_m a_m f_{m+1}^{-1} g_{m+1,3} f_{m+1} = c_3,$$

$$g_{m+2,3} = f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1},$$

$$a_m = f_m f_{m+1}.$$

Transform these relations, excluding a_m and $g_{m,3}$. 1) The relation $a_m f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} = f_m a_m f_{m+1}^{-1} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1}$, after substitution

$$a_m = f_m f_{m+1}, \quad a_{m+1} = f_{m+1} f_{m+2}, \quad a_{m+2} = f_{m+2} f_{m+3}$$

has the form

$$f_m f_{m+1} f_{m+1} f_{m+1} f_{m+2} f_{m+2}^{-1} f_{m+2} f_{m+3} =$$

= $f_m f_m f_{m+1} f_{m+1}^{-1} f_{m+1} f_{m+2} f_{m+2} f_{m+2} f_{m+3} f_{m+3}^{-1}.$

Using the relation $f_m^3 = 1$, we get

$$f_m f_{m+1} = f_{m+2} f_{m+3}.$$

2) The relation

 $f_m a_m f_{m+1}^{-1} a_m^{-1} c_3 a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1} a_{m+2}^{-1} = c_3 f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+1}^{-1} a_m^{-1} c_3 a_{m+1},$ after substitution

$$a_m = f_m f_{m+1}, \quad a_{m+1} = f_{m+1} f_{m+2}, \quad a_{m+2} = f_{m+2} f_{m+3}$$

has the form

$$f_m f_m f_{m+1} f_{m+1}^{-1} f_{m+1}^{-1} f_m^{-1} c_3 f_{m+1} f_{m+2} f_{m+2} f_{m+2} f_{m+3} f_{m+3}^{-1} f_{m+3}^{-1} f_{m+2}^{-1} = c_3 f_{m+1} f_{m+1} f_{m+2} f_{m+2}^{-1} f_{m+2}^{-1} f_{m+2}^{-1} f_{m+1}^{-1} f_m^{-1} c_3 f_{m+1} f_{m+2},$$

or, after cancelation and using the relation $f_m^3 = 1$ we get

$$f_m^{-1} f_{m+1}^{-1} f_m^{-1} c_3 f_{m+1} f_{m+3}^{-1} f_{m+2} = c_3 f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} f_m^{-1} c_3 f_{m+1}$$

3) The relation $g_{m+2,3}^2 = 1$ after substitution

$$g_{m+2,3} = f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1},$$

has the form

$$f_{m+2}^{-1}f_{m+1}c_3^{-1}f_mc_3f_{m+1}^{-1}f_{m+2}^{-1}f_{m+1}c_3^{-1}f_mc_3f_{m+1}^{-1} = 1.$$

4) The relation $(f_m g_{m,3})^3 = 1$ after substitution

$$g_{m,3} = f_m^{-1} f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1},$$

has the form

$$(f_m f_m^{-1} f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1})^3 = 1$$

and is identity since $f_m^3 = 1$. 5) The relation $g_{m+1,3}a_m^{-1}g_{m,3} = 1$ after substitution

$$g_{m+1,3} = f_{m+1}^{-1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1}, \quad g_{m,3} = f_m^{-1} f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1}, \quad a_m = f_m f_{m+1} f_m f_m f_m^{-1} f_$$

has the form

$$f_{m+1}^{-1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1} f_{m+1}^{-1} f_m^{-1} f_m^{-1} f_m^{-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1} = 1.$$

Using the relation $f_m^3 = 1$ and changing the index m on m + 1, we get

$$f_{m+2}^{-1}f_{m+1}c_3^{-1}f_mc_3f_{m+1}^{-1}f_{m+2}^{-1}f_{m+1}f_mc_3^{-1}f_{m-1}c_3f_m^{-1} = 1$$

6) The relation $f_m g_{m,3} f_m^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1} = c_3^{-1}$ after substitution

$$g_{m+1,3} = f_{m+1}^{-1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1}, \quad g_{m,3} = f_m^{-1} f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1},$$

form

has the form

$$f_m f_m^{-1} f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1} f_m^{-1} f_{m+1} f_{m+1}^{-1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1} f_{m+1}^{-1} = c_3^{-1}$$

or after cancelation

$$f_{m-1}c_3^{-1}f_{m-2}c_3f_{m-1}^{-1}c_3^{-1}f_{m-1}c_3f_m^{-1}f_{m+1}^{-1} = c_3^{-1}.$$

7) The relation $f_m^{-1}g_{m,3}f_m a_m f_{m+1}^{-1}g_{m+1,3}f_{m+1} = c_3$ after substitution

 $g_{m+1,3} = f_{m+1}^{-1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1}, \quad g_{m,3} = f_m^{-1} f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1}, \quad a_m = f_m f_{m+1}$ has the form

$$f_m^{-1} f_m^{-1} f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1} f_m f_m f_{m+1} f_{m+1}^{-1} f_{m+1}^{-1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1} f_{m+1} = c_3$$

or, after cancelation and using the relation $f_m^3 = 1$ we get

$$f_m f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1} f_m^{-1} f_{m+1}^{-1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1} f_{m+1} = c_3.$$

Hence, we have proven

Theorem 5.7. The group WB'_4 is generated by c_3 , f_m , $m \in \mathbb{Z}$, and is defined by the relations

$$\begin{split} f_m f_{m+1}^{-1} f_{m+2} &= f_{m+1} f_{m+2}^{-1} f_{m+3}, \\ f_m^{-1} f_{m+1} c_3 f_{m+2}^{-1} f_{m+3} &= c_3 f_{m+1}^{-1} f_{m+2} c_3, \\ f_m f_{m+1} &= f_{m+2} f_{m+3}, \\ f_m^{-1} f_{m+1}^{-1} f_m^{-1} c_3 f_{m+1} f_{m+3}^{-1} f_{m+2} &= c_3 f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} f_m^{-1} c_3 f_{m+1}, \\ f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1} &= 1, \\ f_m^3 &= 1, \\ f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1} f_{m+2}^{-1} f_{m+1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1} &= 1, \\ f_{m-1} c_3^{-1} f_m c_3 f_{m-1}^{-1} c_3^{-1} f_{m-1} c_3 f_m^{-1} f_{m+1}^{-1} &= c_3^{-1}, \\ f_m f_{m-1} c_3^{-1} f_{m-2} c_3 f_{m-1}^{-1} f_m^{-1} f_{m+1}^{-1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1} f_{m+1} &= c_3. \end{split}$$

Corollary 5.8. The group WB'_4 is generated by c_3 , f_0 , f_1 , f_2 .

Indeed, using the relations

$$f_m f_{m+1}^{-1} f_{m+2} = f_{m+1} f_{m+2}^{-1} f_{m+3}$$

we can save from the generators f_m , $m \in \mathbb{Z}$, only the relations f_0 , f_1 , f_2 . Corollary 5.9. $WB'_4/WB''_4 \cong \mathbb{Z}_3$. Indeed, considering relations of WB'_4 by modulo WB''_4 we see that $f_m f_{m+1} = 1$, $f_m^3 = 1$ and $c_3 = 1$.

5.3. The commutator subgroup WB'_n for $n \ge 5$.

Theorem 5.10. The group WB'_n , $n \ge 5$, is generated by n elements f_0 , f_1 , f_2 , c_3 , ..., c_{n-1} .

Proof. As we proved before, VB'_n , $n \ge 5$, is generated by elements $c_3, \ldots, c_{n-1}, f_0, f_1, f_2, g_{0,3}, \ldots, g_{0,n-1}$.

The group WB'_n , $n \ge 5$, is defined by relations of VB'_n and the relations:

$$b_{m,1}a_{m+1}f_{m+2,1}b_{m,0}^{-1} = 1,$$

$$b_{m,0}f_{m+2,0}a_{m+1}^{-1}b_{m,1}^{-1} = 1,$$

$$f_{m,0}c_{3}b_{m+1,1}g_{m+2,3}b_{m+1,0}^{-1}c_{3}^{-1} = 1,$$

$$f_{m,1}c_{3}f_{m+1,0}a_{m+1}g_{m+2,3}b_{m+1,1}^{-1}c_{3}^{-1} = 1,$$

$$g_{m,i}c_{i+1}c_{i}g_{m+2,i+1}c_{i}^{-1}c_{i+1}^{-1} = 1,$$

$$g_{m,i}c_{i+1}a_{m}^{-1}c_{i}a_{m+1}g_{m+2,i+1}a_{m+1}^{-1}c_{i}^{-1}a_{m}c_{i+1}^{-1} = 1,$$

Similar to the group WB'_4 , the firs for relations are equivalent to the relations

$$g_{m+2,3} = f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1}, \quad a_m = f_m f_{m+1}.$$

Hence, the additional relations of WB'_n , ≥ 5 have the form

$$g_{m+2,3} = f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1},$$

$$a_m = f_m f_{m+1},$$

$$g_{m,i} c_{i+1} c_i g_{m+2,i+1} c_i^{-1} c_{i+1}^{-1} = 1,$$

$$g_{m,i} c_{i+1} f_m^{-1} c_i f_{m+1} f_{m+2} g_{m+2,i+1} f_m^{-1} f_m^{-1} f_m f_{m+1} c_{i+1}^{-1} = 1.$$

Using the relations $g_{m,i}c_{i+1}c_ig_{m+2,i+1}c_i^{-1}c_{i+1}^{-1} = 1$, we can express the generators $g_{m,i}$, $i \ge 4$, as words in the generators $c_3, \ldots, c_{n-1}, f_m, g_{m,3}, m \in \mathbb{Z}$. Also, as in the case of the group WB'_4 , we can express the generators $f_m, g_{m,3}, m \in \mathbb{Z}$, as words in the generators c_3, f_0, f_1, f_2 .

5.4. Proof of Theorem 1.3.

Proof. Parts (1) of Theorem 1.3 follows by combining Corollary 5.5, Corollary 5.8 and Theorem 5.7. Part (2) and (3) follow from Corollary 5.6, and Corollary 5.9.

For $n \ge 5$, note that WB'_n is perfect as a quotient of the perfect group VB'_n . This proves (4).

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SOBOLEV INSTITUTE OF MATHEMATICS AND NOVOSIBIRSK STATE UNIVERSITY, NOVOSIBIRSK 630090, RUSSIA.

DEPARTMENT OF AOI, NOVOSIBIRSK STATE AGRARIAN UNIVERSITY, DOBROLYUBOVA STREET, 160, NOVOSIBIRSK, 630039, RUSSIA.

E-mail address: bardakov@math.nsc.ru

Indian Institute of Science Education and Research (IISER) Mohali, Sector 81, S. A. S. Nagar, P. O. Manauli, Punjab 140306, India.

E-mail address: krishnendu@iisermohali.ac.in

SOBOLEV INSTITUTE OF MATHEMATICS AND NOVOSIBIRSK STATE UNIVERSITY, NOVOSIBIRSK 630090, RUSSIA.

E-mail address: neshch@math.nsc.ru