Low-complexity computations for nilpotent subgroup problems

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Abstract

We solve the following algorithmic problems using TC^0 circuits, or in logspace and quasilinear time, uniformly in the class of nilpotent groups with bounded nilpotency class and rank: subgroup conjugacy, computing the normalizer and isolator of a subgroup, coset intersection, and computing the torsion subgroup. Additionally, if any input words are provided in compressed form as straight-line programs or in Mal'cev coordinates the algorithms run in quartic time.

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1 Introduction

This is the third paper in a series on complexity of algorithmic problems in finitely generated nilpotent groups. In the first paper [MMNV15], we showed

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that the basic algorithmic problems (normal forms, conjugacy of elements, subgroup membership, centralizers, presentation of subgroups, etc.) can be solved by algorithms running in logarithmic space and quasilinear time. Further, if the problems are considered in 'compressed' form with each input word provided as a straight-line program, we showed that the problems are solvable in polynomial time. The second paper [MW17] pushed the complexity of these problems lower, showing that they may be solved by TC^0 circuits. Here we expand the list of algorithmic problems for nilpotent groups which may be solved in these low complexity conditions to include several fundamental problems concerning subgroups.

Note that in group theory algorithmic problems for subgroups of groups are usually much harder then the basic algorithmic problems mentioned above. Nevertheless, we present here algorithms for deciding the conjugacy of two subgroups of a finitely generated nilpotent group G, finding the normalizer and the isolator of a given subgroup of G, finding the torsion subgroup T(G) of G, and finding the intersection of two cosets of subgroups of G, all of which may be implemented by TC^0 circuits, or run in logarithmic space and quasilinear time on a (multi-tape) Turing machine. Furthermore, the compressed versions of these problems are solvable in polynomial (specifically, quartic) time. All of the algorithms work uniformly over finitely generated nilpotent groups (i.e. the group may be included in the algorithm's input), however the complexity bounds depend on the nilpotency class and the rank (number of generators) of the presentation. When both are bounded, we solve all the problems uniformly in TC⁰ or logspace and quasilinear time.

Algorithmic problems in nilpotent groups have been studied for a long time. On the one hand, it was shown that many of them are decidable and many sophisticated decision algorithms were designed (see, for example, the pioneering paper [KRR⁺69] by Kargapolov et al. published in 1969 and the books [Sim94] and [HEO05] for more recent techniques); on the other hand, there are some which have been known to be undecidable for some time (for instance, decidability of equations [Rom77]). Recent work by a variety of authors has introduced a host of decidable/undecidable problems. New undecidable problems, including the knapsack problem, commutator and rectract problems are described in [Loh15], [KLZ15], [MT16], and [Rom16], while positive decidability results for direct product decompositions and equations in the Heisenberg group are described in [BMO16] and [DLS15]. Decidability and undecidability results for equations over random nilpotent groups are also given in [GMO16a] and [GMO16b].

However, it seems that this paper together with [MMNV15] and [MW17] present the first thorough attempt to study the complexity of the problems, beyond the decidable/undecidable dichotomy. In fact, it seems this is currently the only known large class of non-abelian groups where the major algorithmic problems are shown to have low space and time complexity. Another large class of such groups is, perhaps, the class of finitely generated free groups given by the standard presentations. Even there, if the free groups are given by arbitrary finite presentations the complexity of the algorithmic problems is still mostly

unknown.

We have not yet mentioned one of the fundamental algorithmic problems in nilpotent groups: the isomorphism problem. It is decidable due to the famous result of Grunewald and Segal [GS80]. Nevertheless, not much is known about its complexity.

Problem 1. Is the isomorphism problem in finitely generated nilpotent groups decidable in polynomial time? Exponential time?

2 Background

This section describes, summarizing from [MMNV15] and [MW17], how we will represent finitely generated nilpotent groups (§2.1) and their subgroups (§2.2), and gives black-box descriptions of several algorithms that we will be using as subroutines (§2.4). We also give a brief introduction to the TC⁰ circuit model of computation, logspace computations, and the use of compressed words in algorithmic problems over groups (§2.3).

2.1 Nilpotent presentations

Let G be a finitely generated nilpotent group of nilpotency class c. Then G has lower central series

$$G = \Gamma_1 \triangleright \Gamma_2 \triangleright \ldots \triangleright \Gamma_c \triangleright \Gamma_{c+1} = 1$$

with $\Gamma_{i+1} = [G, \Gamma_i]$ for i > 1. From this series we derive a presentation for G, as follows.

Each Γ_i/Γ_{i+1} is a finitely generated abelian group. We select and fix a finite generating set $a_{s_{i-1}+1}\Gamma_{i+1}, \ldots, a_{s_i}\Gamma_{i+1}$ for Γ_i/Γ_{i+1} and put

$$A = \{a_1, a_2, \ldots, a_m\}.$$

For each j = 1, ..., m, if $s_{i-1} + 1 \le j \le s_i$, we denote by e_j the order of $a_j \Gamma_{i+1}$ in Γ_i / Γ_{i+1} , using $e_j = \infty$ when the order is infinite. Denote

$$\mathcal{T} = \{i \mid e_i < \infty\}.$$

Provided that each generating set above is chosen to correspond to a primary or invariant factor decomposition of Γ_i/Γ_{i+1} , every element $g \in G$ may be written uniquely in *Mal'cev normal form* as

$$g = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m} \tag{1}$$

where $\alpha_i \in \mathbb{Z}$ and if $i \in \mathcal{T}$ then $0 \leq \alpha_i < e_i$. The set A is called a *Mal'cev basis* of G and the integers $(\alpha_1, \ldots, \alpha_m)$ are the *Mal'cev coordinates* of g.

For each i = 1, ..., m, denote $G_i = \langle a_i, ..., a_m \rangle$. An essential fact, which follows from the definition of the lower central series, is that for any i < j,

$$[a_i, a_j] \in G_\ell$$

for some $\ell > j$. From this it follows that relations of the form

$$a_j a_i = a_i a_j a_\ell^{\alpha_{ij\ell}} a_{\ell+1}^{\alpha_{ij(\ell+1)}} \cdots a_m^{\alpha_{ijm}} \tag{2}$$

$$a_{j}^{-1}a_{i} = a_{i}a_{j}^{-1}a_{\ell}^{\beta_{ij\ell}}a_{\ell+1}^{\beta_{ij(\ell+1)}}\cdots a_{m}^{\beta_{ijm}},$$
(3)

with $\ell > j$, hold in G. In addition, for each $i \in \mathcal{T}$ there is a relation of the form

$$a_i^{e_i} = a_\ell^{\mu_{i\ell}} a_{\ell+1}^{\mu_{i(\ell+1)}} \cdots a_m^{\mu_{im}} \tag{4}$$

where $\ell > j$. The set $\{a_1, \ldots, a_m\}$, viewed as an abstract set of symbols, together with relators (2)–(4) then form a presentation for *G* called a *nilpotent presentation*. In fact, any presentation of this form defines a nilpotent group. Such a presentation is called *consistent* if the order of each $a_i \mod \langle a_{i+1}, \ldots, a_m \rangle$ is precisely e_i . Note that $e_i = 1$ is permitted in a nilpotent presentation.

For low-complexity algorithms, an essential property of nilpotent presentations is the following (see [MMNV15] Thm. 2.3 and Lem. 2.5): if w is any word over A^{\pm} , then the length of the Mal'cev normal form (1) of the element gcorresponding to w in G is bounded by a polynomial function of the length of w, with the degree of the polynomial depending on the nilpotency class c and number of generators r of G. This fact plays a crucial role in solving efficiently the fundamental algorithmic problems in finitely generated nilpotent groups.

2.2 Subgroups

All of our results concern subgroups of finitely generated nilpotent groups. For every subgroup $H \leq G$ (all of which are, necessarily, finitely generated), one may define a unique generating set (h_1, \ldots, h_s) called the *full-form sequence* for H. The precise definition was given in [Sim94] (and is reviewed in [MMNV15]), but we mention here only three facts about (h_1, \ldots, h_s) that we will need.

First, let B be the matrix in which row i is the row vector consisting of the Mal'cev coordinates of h_i . Then B is in row echelon form and does not contain zero rows. We denote by π_i the pivot column of row i of B. Since this column corresponds to generator a_{π_i} , the Mal'cev normal form of h_i begins with a_{π_i} , so $h_i \in G_{\pi_i} = \langle a_{\pi_i}, \ldots, a_m \rangle$.

Second, the number of generators s is bounded by the length m of the Mal'cev basis. Third, every element $h \in H$ can be uniquely presented in the form

$$h = h_1^{\beta_1} \cdots h_s^{\beta_s},$$

where $\beta_i \in \mathbb{Z}$ and $0 \leq \beta_i < e_{\pi_i}$ if $\pi_i \in \mathcal{T}$. Hence

$$H = \{ h_1^{\beta_1} \cdots h_s^{\beta_s} \mid \beta_i \in \mathbb{Z} \text{ and } 0 \le \beta_i < e_{\pi_i} \text{ if } \pi_i \in \mathcal{T} \}.$$

2.3 Logspace, TC^0 , and compressed words

Let \mathcal{A} be a finite language. We are interested in both decision and search problems, and we may regard each such problem as a function $f : \mathcal{A}^* \to \mathcal{A}^*$.

The set \mathcal{A} consists of a set of symbols, say x_1, \ldots, x_n , which denote group generators, and a few extra symbols used to separate different parts of the input (commas to separate relators etc.). We will be computing f(x) using *logarithmic space* or using TC^0 circuits. We recall both of these notions below.

Logspace. A *c*-logspace transducer, where c > 0 is a constant, is a multi-tape Turing machine consisting of the following tapes: an 'input' tape which is readonly, a constant number of read-write 'work' tapes, and a write-only 'output' tape. For any input of length L, which is provided on the input tape, the amount of space the transducer is allowed to use on each work tape is $c \log(L)$. The output of the machine is the content of the output tape. A function f is said to be logspace computable, or more casually the associated problem is solvable in logarithmic space, if there exists a constant c and a *c*-logspace transducer that produces f(x) on the output tape for any input x appearing on the input tape.

Though computation on a *c*-logspace transducer puts a bound only on space resources, a polynomial time bound of $O(L^c)$ is forced by the fact that the machine may not enter the same configuration twice (otherwise it will loop infinitely) and the number of configurations is bounded by a polynomial function of the input length. The degree *c* may be very high, and for this reason it is also desirable to show directly that our algorithms run in low-degree polynomial time, in particular quasilinear time (i.e. $O(L \log^k(L))$ for some constant *k*).

Most of our algorithms invoke other logspace algorithms as subroutines, and as such we need to compute compositions $f \circ g$ of logspace computable functions. A standard argument shows that $f \circ g$ is again logspace computable, but in computing $(f \circ g)(x)$ in this way, each symbol of g(x) is recomputed each time it is needed in computation of f, which may give a significant increase in time complexity. However, if the output g(x) is always of size $O(\log(L))$, one may simply compute g(x) first, store the output on the work tape, and then proceed to compute f(g(x)). This is the case in all of our algorithms, so in this case the time complexity of g is simply added to the overall time complexity.

TC⁰ circuits. A TC^0 circuit with n inputs is a boolean circuit of constant depth using NOT gates and unbounded fan-in AND, OR, and MAJORITY gates, such that the total number of gates is bounded by a polynomial function of n. A MAJORITY gate outputs 1 when more than half of its inputs are 1. A function f(x) is TC^0 -computable (more casually, 'an algorithm is in TC^0 ') if for each n there is a TC^0 circuit F_n with n inputs which produces f(x) on every input x of length n. Essential for our purposes is the fact that the composition of two TC^0 -computable functions is again TC^0 -computable.

Since this definition of being computable only asserts that such a family $\{F_n\}_{n=1}^{\infty}$ of circuits exists, one normally imposes in addition a *uniformity* condition stating that each F_n is constructible in some sense. We will only be concerned here with standard notion of DLOGTIME uniformity, which asserts that there is a random-access Turing machine which decides in logarithmic time

whether in circuit F_n the output of gate number *i* is connected to the input of gate *j*, and determines the types of gates *i* and *j*. We refer the reader to [Vol99] for further details on TC⁰.

To put our results in context, we remind the reader of the following inclusions of complexity classes:

$$TC^0 \subseteq LOGSPACE \subseteq P \subseteq NP.$$

It is not known whether any of these inclusions is strict. Though every TC⁰computable function is also logspace-computable and polynomial-time computable, our algorithm descriptions also give direct proofs of membership in these classes.

Compressed words. We are also interested in algorithms that run efficiently when the input is given in compressed format. The use of Mal'cev coordinates provides a natural compression scheme for elements of G: each $g \in G$ may be encoded by a tuple of integers (its Mal'cev coordinates) written in binary. Notice that if the size m of the Mal'cev basis is bounded, a normal form of length n may be encoded by $O(\log n)$ bits. Since every finitely generated nilpotent group has a Mal'cev basis, it is natural to consider algorithmic problems in which input words represented in this compact way. Of course, such 'compressed problems' are, in terms of computational complexity, more difficult than their uncompressed siblings.

Since we will consider uniform algorithms, in which a finitely generated nilpotent group G is given by an *arbitrary* presentation as part of the input, we also consider two other compression schemes which do not depend on a the specification of a Mal'cev basis. First, we may simply allow exponents to be encoded in binary. In this scheme, a word is encoded as a product of tuples (g, m), representing g^m , where g is a group generator or, recursively, a word of this form, and m is a binary integer. For example, $(x^4y^2)^8x^{-6}$ is encoded as $(((x, 0100)(y, 0010)), 1000)(x^{-1}, 0110)$.

Second, we consider straight-line programs, that is, context-free grammars that generate exactly one string. Formally, a straight-line program or compressed word over an alphabet \mathcal{A} consists of a set $\{A_1, A_2, \ldots, A_n\}$ called the nonterminal symbols and for each non-terminal symbol A_i a production rule either of the form $A_i \to A_j A_k$ with j, k < i, or of the form $A_i \to a$ where $a \in \mathcal{A} \cup \{\epsilon\}$ with ϵ denoting the empty word. The non-terminal A_n is termed the root, and one 'expands' the compressed word by starting with the one-character word A_n and successively replacing any non-terminal with the right side of its production rule until only symbols from \mathcal{A} remain. The number n of non-terminal symbols is the size of the program. Compression arises since a program of size n may expand to a word of length 2^{n-1} . We refer the reader to the survey article [Loh12] and the monograph [Loh14] for further information on compressed words, or to the introduction of [MMNV15] for some brief remarks.

2.4 Fundamental algorithms for nilpotent groups

Throughout this paper, we make extensive use of algorithms described in [MMNV15] and [MW17]. We give below a summary of some of the most heavily-used ones, and we will use the names listed here, in **bold text**, to refer to their use.

- Full-form Sequence: Given $H \leq G$, compute the full-form generating sequence for H.
- Membership: Given $g \in G$ and $H \leq G$, determine if $g \in H$ and if so, compute the unique expression $g = h_1^{\alpha_1} \cdots h_s^{\alpha_s}$ where (h_1, \ldots, h_s) is the full-form sequence for H.
- Subgroup Presentation: Given $H \leq G$, compute a consistent nilpotent presentation for H.
- Conjugacy: Given $g, h \in G$, produce $x \in G$ such that $g^x = h$ or determine that no such x exists.
- Centralizer: Given $g \in G$, compute a generating set for the centralizer of g in G.
- Kernel: Given $K \leq G$ and $\phi: K \to G_1$, produce a generating set for the kernel of ϕ .
- **Preimage**: Given $K \leq G$, $\phi : K \to G_1$, and $h \in G_1$ guaranteed to be in $\phi(K)$, produce $k \in K$ such that $\phi(k) = h$.

We will need some further details regarding the input/output of these algorithms as well as their complexity.

Input. In each algorithm, we fix in advance two integers c and r. The ambient nilpotent groups G and G_1 are part of the input (thus the algorithms are 'uniform') but must be of nilpotency class at most c and be presented using at most r generators for the complexity bounds given below to be valid. Group elements are given as words over the generating set(s), subgroups are specified by finite generating sets, and ϕ is given by listing the elements $\phi(k)$ for each given generator k of K. The length L of the input is the sum of the lengths of all relators in G and G_1 plus the lengths of all input words.

Output. Each output word is given as a word over the original generating set except possibly in **Full-form sequence** and **Membership**. In these cases, the algorithm converts to a nilpotent presentation of G, if one is not already provided, and provides the output words in the new generators (the isomorphism may also be provided, see Lemma 3 below). In **Centralizer** and **Kernel**, if the original presentation of G is already a nilpotent presentation, one may assume that the subgroup generating set in the output is the full-form sequence.

In every case, the total length of each output word is bounded by a polynomial function of L and the number of output words is bounded by a constant. Optionally, the output words may be given by their Mal'cev coordinates.

Complexity. Each algorithm may be implemented on a logspace transducer, and if so runs in time quasilinear in L. The proofs are given in [MMNV15]. Alternatively, each problem may be solved using TC^0 circuits, as proved in [MW17].

Compressed inputs. Each algorithm may also be run 'with compressed inputs'. In this case, any input word (including group relators) may be provided by (binary) Mal'cev coordinates, words with binary exponents, or straight-line programs, as described in §2.3. We will measure the size of the input in terms of the number n of input words and the maximum size M of any single input word (in number of bits or number of non-terminal symbols). The space complexity of each algorithm is then O(M) (it does not depend on n) and the time complexity is $O(nM^3)$. All output is provided in the corresponding compressed format. Although each input word, in its expanded form, may have length $O(2^M)$, the polynomial bound for the length of output words implies that each output word, in expanded form, has length $O(2^{dM})$ where d is the degree of the aformentioned polynomial bound. Since d is constant, the compressed size of each output remains $O(\log(2^{dM})) = O(M)$.

Remark 2. We place no restriction on the number n of input words. In all of the algorithms, any variable-sized set of input words (e.g. list of subgroup generators, group relators) will be fed as input to the matrix reduction algorithm described in Thm. 3.4 of [MMNV15] and processed in the 'piecewise' manner described there, one word at a time. After this, sets of words usually only appear as full-form sequences for subgroups, the number of which is always bounded by a constant. The value n contributes a linear factor to the time complexity of this algorithm (in both uncompressed and compressed cases), but does not contribute to the space complexity.

While neither these algorithms nor the ones we describe in this paper require that the input groups G and G_1 be given by a nilpotent presentation, this form is used internally by all of the algorithms. Converting to such a presentation is accomplished as follows.

Lemma 3. Let c and r be fixed integers. There is an algorithm that, given a finitely presented nilpotent group $G = \langle X | R \rangle$ of nilpotency class at most c and with $|X| \leq r$, a finite set $Y \subset G$, and a word h over X^{\pm} guaranteed to be in the subgroup $H = \langle Y \rangle$, produces

- a consistent nilpotent presentation $\langle Y' | S \rangle$ for H, in which binary numbers are used to encode exponents in the relators S,
- a map $\phi: Y' \to (Y^{\pm 1})^*$ which extends to an isomorphism $\langle Y' | S \rangle \to H$, and
- a binary integer tuple h' giving the Mal'cev coordinates of h relative to Y'.

The algorithm runs in space logarithmic in the input length L and time quasilinear in L, or in TC^0 , and the (expanded) length of each output word is bounded

by a polynomial function of L. If compressed inputs are used (in R, Y, or h), the space requirement is O(M) and the time is $O(nM^3)$, where n is the total number of input words and M bounds the size of any single input word.

Proof. Algorithm. Begin by applying Prop. 5.1 of [MMNV15] (or Lem. 5 of [MW17] in the TC⁰ case) to compute a consistent nilpotent presentation $G = \langle X' | R' \rangle$. Here $X \subset X'$, the inclusion $X \hookrightarrow X'$ induces an isomorphism $\langle X | R \rangle \simeq \langle X' | R' \rangle$, and each element of $X' \setminus X$ is a commutator in elements of X. Use **Subgroup Presentation** to compute a nilpotent presentation $\langle Y' | S \rangle$ for H. The generating set $Y' = \{h_1, \ldots, h_s\}$ is precisely the full-form sequence for H. The relators have the form (2)-(4), and we encode the exponents appearing on the right sides in binary. To obtain ϕ , note that each element of Y' has the form $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, where $X' = \{x_1, \ldots, x_m\}$. We replace each x_i with its definition as a commutator of elements of X and encode the exponents α_i using binary numbers. Finally, use **Subgroup Membership** with input h and $\{h_1, \ldots, h_s\}$, which returns an expression $h = h_1^{\gamma_1} \cdots h_s^{\gamma_s}$, giving the Mal'cev coordinates $(\gamma_1, \ldots, \gamma_s)$.

Complexity. Follows immediately from [MMNV15] and [MW17]. Note that s is a constant depending on c and r.

We will often use this lemma in the case Y = X to convert from an arbitrary presentation of G to a nilpotent presentation. In this case, we may assume the algorithm uses $Y' = X' \supset X$. We convert all input words into their Mal'cev coordinates (relative to X') at the same time, and perform further computations directly on the Mal'cev coordinates.

Using binary numbers in the output is necessary in order to obtain quasilinear time, since writing down a word in its expanded form takes as many steps as the length of the word itself, which in this case is only bounded by a polynomial function of L.

3 Algorithmic problems

Before presenting the algorithms, let us make a few remarks regarding their complexity analysis. The analysis of most of the algorithms is similar, so we present here a general argument and fill in any additional details in the proof of each algorithm.

First, note that the nilpotency class c and maximum number of generators r of the input group(s) are constant. All other constants are expressible in terms of c and r.

At the beginning of each algorithm, we convert to a nilpotent presentation, if necessary, using Lemma 3. We denote the resulting Mal'cev basis by

 $\{a_1,\ldots,a_m\}.$

Note that m is constant. Word lengths are unchanged during this conversion (see Lemma 3). We are guaranteed by [MMNV15] Thm. 2.3 that a word of

length L has a Mal'cev form of length polynomial in L, hence its coordinates require $O(\log L)$ bits to record.

Our algorithms generally consist of a sequence of subroutine calls, using the algorithms described in §2.4 as well as those described in this section, with some minor additional processing. The complexity bounds described in §2.4 also apply to the algorithms we describe in this section, as we will see. In all cases, we prove that the total number of subroutine calls and the total number of words that must be stored in memory at any given time is constant. Consequently, the entire algorithm can, in principle, be expressed as a composition of a constant number of functions. Each such function is TC^0 -computable, hence so is the composition. Note that to 'store x in memory' in TC^0 terms means to add a parallel computation branch computing x.

Though it follows immediately that we have logspace solutions to these problems, we wish to prove that one may in fact run the algorithms on a logspace transducer in quasilinear time. To do so, we must show that each subroutine may be run directly 'in memory' on the logspace transducer.

This is achieved by invoking each subroutine in its 'compressed' form. Initially, all input words are converted into $O(\log L)$ -bit Mal'cev coordinate form. In this process, any variable-sized set of words (subgroup generators or group relators) is reduced to a constant-sized set (the full-form sequence). This size is bounded by m, and we often assume it is precisely m for notational convenience. Each subroutine is then called with a constant number of $O(\log L)$ -bit words. It will therefore run in space $O(\log L)$ and time $O(\log^3 L)$, and produce a constant number of $O(\log n)$ -bit output words.

For compressed inputs, the argument is similar. As we observed earlier, the polynomial length bound implies that the compressed size of words remains O(M) throughout the algorithm. Each subroutine therefore has space complexity O(M) and time complexity $O(M^3)$, so the overall space and time complexities are O(M) and $O(nM^3)$.

Finally, let us note that if we have a constant number elements g_1, \ldots, g_t in Mal'cev form we can, by [MMNV15] Lem. 2.10, compute the Mal'cev form of the product $g_1 \cdots g_t$ within the space and time bounds specified above, in both compressed and uncompressed cases. We use this without mention to maintain elements in coordinate form.

3.1 Subgroup conjugacy and normalizers

In this section we give an algorithm to determine whether or not two subgroups of a nilpotent group are conjugate and if so to compute a conjugating element. A natural by-product of this algorithm is the computation of subgroup normalizers.

We begin with a preliminary lemma solving the simultaneous conjugacy problem for tuples of commuting elements. In fact, commutation is not required, but we will obtain this stronger result (Theorem 7) as a corollary of the more complicated coset intersection algorithm. **Lemma 4.** Fix positive integers c, r, and l. There is an algorithm that, given a finitely generated nilpotent group $G = \langle X | R \rangle$ of nilpotency class at most cwith $|X| \leq r$ and two tuples of elements (a_1, \ldots, a_l) and (b_1, \ldots, b_l) such that $[a_i, a_j] = [b_i, b_j] = 1$ for all $1 \leq i, j \leq l$, decides if there exists $g \in G$ such that

$$a_i^g = b_i$$

for all $1 \leq i \leq l$. The algorithm produces g if one exists, returns a generating set for the centralizer of $\{b_1, \ldots, b_l\}$, and may be run in space logarithmic in the length L of the input and time quasilinear in L, or in TC^0 . The length of each output word is bounded by a polynomial function of L. If compressed inputs are used, the algorithm uses space O(M) and time $O(nM^3)$, where n = |R| and M bounds the encoded size of each input word.

Proof. Algorithm. If necessary, use **Lemma 3** to convert to a nilpotent presentation. Next, we check conjugacy of a_1 with b_1 using the **Conjugacy Algorithm**. If they are not conjugate, we may return 'No'. Otherwise, we obtain h such that $a_1^h = b_1$ and we compute a generating set for $C_G(b_1)$ using the **Centralizer Algorithm**.

If l > 1, we proceed recursively. Notice that g exists if and only if there exists $x \in G$ such that $(a_i^h)^x = b_i$ for $i = 1, \ldots, l$, since we may put $x = h^{-1}g$. Further, such x must lie in $C_G(b_1)$ since $b_1 = (a_1^h)^x = b_1^x$. Therefore it suffices to call **Lemma 4** recursively with the (commuting) tuples (a_2^h, \ldots, a_l^h) and (b_2, \ldots, b_l) and the subgroup $C_G(b_1)$ in place of G. Before making the recursive call, we use **Lemma 3** to convert to a nilpotent presentation for $C_G(b_1)$ and write each of a_i^h , b_i relative to this presentation.

If we obtain a conjugator x, we may return g = hx, using the map ϕ provided by Lemma 3 to write x in the original generators X. In addition, we obtain a generating set for the centralizer of $\{b_2, \ldots, b_l\}$ in $C_G(b_1)$, which is precisely a generating set for the centralizer of the complete set $\{b_1, \ldots, b_l\}$ in G. As above, we must use ϕ to write these words in generators X. If the recursive call returns 'No', then the tuples are not conjugate.

Complexity. The depth l of the recursion is constant and we need only store h and the (constant-sized) generating set for the centralizer at each step of the recursion, hence the general argument given at the beginning of the section applies.

We now give the algorithm for determining conjugacy of two subgroups.

Theorem 5. Fix integers c and r. There is an algorithm that, given a finitely presented nilpotent group $G = \langle X | R \rangle$ of nilpotency class at most c with $|X| \leq r$ and two subgroups H and K, determines if there exists $g \in G$ such that

 $H^g = K$

and if so finds such an element g as well as

a generating set for the normalizer $N_G(K)$

of K. The algorithm runs in space logarithmic in the total length L of the input and time quasilinear in L, or in TC^0 , and the length of every output word is bounded by a polynomial function of L. If compressed inputs are used, the space complexity is O(M) and the time complexity $O(nM^3)$ where n is the total number of input words and M bounds the encoded size of each input word.

Proof. Algorithm. Begin by converting, if necessary, to a nilpotent presentation of G using Lemma 3.

The algorithm recurses on the maximum j such that $H \cap \Gamma_j \neq 1$ and $H \cap \Gamma_{j+1} = 1$. To find j, simply compute the **Full-form Sequence** for H and observe that if the last element of the sequence begin with the letter a_k then j is the unique index such that $a_k\Gamma_{j+1}$ belongs to the generating set of Γ_j/Γ_{j+1} (see §2.1). Compute similarly the maximum j' such that $K \cap \Gamma_{j'} \neq 1$ and $K \cap \Gamma_{j'+1} = 1$. If $j \neq j'$, then H and K are not conjugate since their conjugacy would imply conjugacy of $H \cap \Gamma_i$ with $K \cap \Gamma_i$ for all i (since the Γ_i are normal subgroups), hence equality of j and j'.

Denote $H_j = H \cap \Gamma_j$ and produce the full-form sequence for this group by taking the elements of the full-form sequence for H that are in Γ_j . Proceed similarly for $K_j = K \cap \Gamma_j$. Next, we check conjugacy of H_j with K_j .

similarly for $K_j = K \cap \Gamma_j$. Next, we check conjugacy of H_j with K_j . Conjugacy of H_j with K_j . Let $\overline{}: G \to G/\Gamma_{j+1}$ be the natural homomorphism. By the definition of central series, G acts trivially by conjugation on $\overline{\Gamma_j}$. Hence if H_j and K_j are conjugate then $\overline{H_j} = \overline{K_j}$. We first check if $\overline{H_j} = \overline{K_j}$, returning 'No' if not. To do so, it suffices to compute the **Full-form Sequences** for $\overline{H_j}$ and $\overline{K_j}$, and check them for equality.

Let (h_1, \ldots, h_l) be the full-form sequence for H_j , computed above. We now produce a generating set (k_1, \ldots, k_l) for K_j such that $\overline{h_i} = \overline{k_i}$ for all i, as follows. Use the **Preimage** algorithm, with the subgroup K_j , the homomorphism $\overline{}$: $K_j \to G/\Gamma_{j+1}$, and the element $\overline{h_i}$, to produce each k_i . Since $K_j \cap \Gamma_{j+1} = 1$, (k_1, \ldots, k_l) generates K.

We claim for any $x \in G$, $H_j^x = K_j$ if and only if $h_i^x = k_i$ for i = 1, ..., l. Indeed, since the tuples $(h_1, ..., h_l)$ and $(k_1, ..., k_l)$ are generating sets their conjugacy implies H_j and K_j are conjugate. Conversely, if $H_j^x = K_j$ then $h_i^x \in K_j$ for all *i*. But $\overline{h_i^x} = \overline{h_i} = \overline{k_i}$, and since $\overline{}$ is injective on K_j , we have $h_i^x = k_i$ for all *i*. Also observe that H_j is abelian, since

$$[H_j, H_j] \le H_j \cap \Gamma_{2j} \le H_j \cap \Gamma_{j+1} = 1,$$

and similary for K_j . Hence (h_1, \ldots, h_l) and (k_1, \ldots, k_l) are both tuples of commuting elements. So to determine conjugacy of H_j with K_j it suffices to use the algorithm of **Lemma 4** to determine conjugacy of (h_1, \ldots, h_l) and (k_1, \ldots, k_l) and if so find a conjugator x and a generating set Y for $C_G(K_j)$. In fact, $C_G(K_j) = N_G(K_j)$ since if any element $y \in G$ normalizes K_j , then for each $i = 1, \ldots, l$ we have $k_i^y \in K_j$ and hence $k_i^y = k_i$, arguing as above.

Recursion. If j = 0, then $H = H_j$ and $K = K_j$ and we have already solved the problem. Otherwise, letting

$$\hat{}: N_G(K_j) \to N_G(K_j)/K_j$$

be the canonical homomorphism, we reduce the problem to conjugation of \widehat{H}^x and \widehat{K} in $N_G(K_j)/K_j$, as follows.

An element g such that $H^g = K$ exists if and only if there exists $y \in G$ such that $(H^x)^y = K$. Such an element y must lie in $N_G(K_j)$, since

$$K_j^y = (H_j^x)^y = (H \cap \Gamma_j)^{xy} = H^{xy} \cap \Gamma_j^{xy} = K \cap \Gamma_j = K_j.$$

Now $K \leq N_G(K_j)$, and $H^x \leq N_G(K_j)$ since

$$K_j = H_j^x = (H \cap \Gamma_j)^x = H^x \cap \Gamma_j \trianglelefteq H^x.$$

Finally, if $(\widehat{H^x})^{\widehat{y}} = \widehat{K}$ for some $\widehat{y} \in N_G(K_j)/K_j$, we claim that $(H^x)^y = K$. Indeed, if $k \in K$ then for some $h \in H$ and $k' \in K_j$ we have $k = y^{-1}h^x yk' = y^{-1}(h^x(k')^{y^{-1}})y$. But $y \in N_G(K_j)$ and $K_j \leq H^x$, so $h^x(k')^{y^{-1}} \in H^x$ and the inclusion $K \subseteq H^x$ follows. The reverse inclusion is proved similarly.

In order to solve the conjugation problem of $\widehat{H^x}$ and \widehat{K} in $N_G(K_j)/K_j$, we first use **Lemma 3**, with the generating set Y, to find a nilpotent presentation for $N_G(K_j)$ and to convert the generating sets for H^x , K, and K_j into coordinate form in this presentation. Add the generators of K_j to this presentation to obtain a presentation for $N_G(K_j)/K_j$, and call **Theorem 5** with this presentation and the subgroups $\widehat{H^x}$ and \widehat{K} .

It is essential to prove that the value of j decreases in the recursive call. Letting N_j denote term j of the lower central series of $N_G(K_j)$, we have that $N_j \leq \Gamma_j$, hence $K \cap N_j \leq K \cap \Gamma_j = K_j$, and the intersection is trivial modulo K_j , hence j must decrease.

The recursive call either proves that $\widehat{H^x}$ and \widehat{K} are not conjugate, in which case H and K are not conjugate, or returns a conjugator yK_j and a generating set $Z \cdot K_j$ for the normalizer of \widehat{K} in $N_G(K_j)/K_j$. Note that y (and each element of Z) is given as a word over the generating set of $N_G(K_j)$ with binary exponents. We convert back to the generating set X of G using the map ϕ provided by Lemma 3. For the conjugator, we return the word g = xy.

For the normalizer, we append to Z a generating set of K_j to obtain a generating set Z' for the normalizer of K in $N_G(K_j)$. But this is precisely the normalizer of K in G: if $K^z = K$ for some $z \in G$ then $K_j^z = K^z \cap \Gamma_j^z = K \cap \Gamma_j = K_j$ and so $N_G(K) \leq N_G(K_j)$.

Complexity. The depth of the recursion is bounded by the constant c, and the number of words to store in memory is constant.

It should be noted that while the algorithm does not compute the normalizer of K in the event that H and K are not conjugate, one may of course obtain it by running the algorithm with H = K.

3.2 Coset intersection

We describe an algorithm to compute the intersection of cosets in finitely generated nilpotent groups, and apply it to solving the simultaneous conjugacy problem. Recall that in any group, the intersection $g_1H \cap g_2K$ of two cosets is, if non-empty, a coset of the intersection $H \cap K$.

Theorem 6. Fix integers c and r. There is an algorithm that, given a finitely presented nilpotent group $G = \langle X | R \rangle$ of nilpotency class at most c with $|X| \leq r$, two subgroups H and K of G, and two elements g_1 and g_2 of G, determines if the intersection $g_1H \cap g_2K$ is non-empty and if so, produces a generating set for $H \cap K$ and an element $g' \in g_1H \cap g_2K$, hence

$$g_1H \cap g_2K = g'(H \cap K).$$

The algorithm runs in space logarithmic in the length L of the input and time quasilinear in L, or in TC^0 . If compressed inputs are used, the space complexity is O(M) and the time complexity $O(nM^3)$ where n is the total number of input words and M bounds the encoded size of each input word.

Proof. Begin by using Lemma 3 to convert to a nilpotent presentation for G, if necessary. We proceed by induction on the nilpotency class c.

Base case c=1. In this case, G is abelian. First, we will determine if the intersection is non-empty and if so find g. Writing

$$g_1 H \cap g_2 K = g_2(g_2^{-1}g_1 H \cap K),$$

it suffices to determine if there exists $h \in H$ such that $g_2^{-1}g_1h \in K$. Since G is abelian, this occurs if and only if $g_2^{-1}g_1 \in \langle H \cup K \rangle$. We use the **Membership algorithm**, with the union of the **Full-form sequences** of H and K as a generating set for $\langle H \cup K \rangle$, to determine if this is the case, returning 'No' if it is not. Otherwise, we obtain an expression of $g_2^{-1}g_1$ as a linear combination of the elements of the full-form sequence for $\langle H \cup K \rangle$. We can convert to an expression in terms of the full-form sequences for H and K, thus obtaining an expression $g_2^{-1}g_1 = hk$ for some elements $h \in H$ and $k \in K$, by following the procedure described in Cor. 3.9 of [MMNV15] (essentially, recording an expression of each matrix row in terms of the given generators during the matrix reduction process). This corollary gives only polynomial time, but Thm. 14 of [MW17] gives the corresponding result for TC⁰ (hence logspace), though we need the fact that $g_2^{-1}g_1$ and the full-form sequences of H and K are stored using only $O(\log L)$ bits. We now set $g = g_1h^{-1}$ and obtain $g_1H \cap g_2K = g(H \cap K)$.

We will now find a generating set for $H \cap K$. Let $\{u_1, \ldots, u_n\}$ be the generating set for H and consider the homomorphism $\phi : \mathbb{Z}^n \to G$ defined by

$$\phi(\alpha_1,\ldots,\alpha_n)=u_1^{\alpha_1}\cdots u_n^{\alpha_n}$$

and the composition $\phi' : \mathbb{Z}^n \to G \to G/K$. An element $u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ of H is also an element of K if and only if $(\alpha_1, \ldots, \alpha_n)$ is in the kernel of ϕ' , hence $H \cap K = \phi(\ker \phi')$. To compute the kernel, add the generators of K to the relators of G to obtain a presentation of G/K, and pass this group together with ϕ' and the standard presentation of \mathbb{Z}^n to the **Kernel algorithm**. Applying ϕ to each resulting subgroup generator, we obtain a generating set (in fact, the full-form sequence) for $H \cap K$.

Inductive case. Denote by $\overline{}: G \to G/\Gamma_c$ the canonical homomorphism. Invoke **Theorem 6** recursively in G/Γ_c with inputs $\overline{H}, \overline{K}, \overline{g_1}$, and $\overline{g_2}$. Note that it suffices to erase all generators of Γ_c to compute $\overline{}$ (more formally, one may use Lemma 3).

If the recursive call determines that $\overline{g_1}\overline{H} \cap \overline{g_2}\overline{K}$ is empty, then so is $g_1H \cap g_2K$. Otherwise, we obtain an element $\overline{\gamma} \in \overline{g_1}\overline{H} \cap \overline{g_2}\overline{K}$ and a generating set $\overline{w_1}, \ldots, \overline{w_l}$ of $\overline{H} \cap \overline{K}$, hence

$$\overline{g_1}\overline{H}\cap\overline{g_2}\overline{K}=\overline{\gamma}\langle\overline{w_1},\ldots,\overline{w_l}\rangle.$$

Denote by Λ (but do not compute) the preimage of $\overline{H} \cap \overline{K}$ under $\overline{}$. We will rewrite the intersection $g_1H \cap g_2K$ in the form

$$g_1 H \cap g_2 K = (g'(\Lambda \cap H)) \cap (g'c_0(\Lambda \cap K))$$
(5)

for certain $g' \in G$, $c_0 \in \Gamma_c$ defined below. Compute a **Preimage** x_1 of $\overline{g_1^{-1}\gamma}$ in H and a **Preimage** x_2 of $\overline{g_2^{-1}\gamma}$ in K. Let

$$g' = g_1 x_1$$

Since $\overline{g_2x_2} = \overline{\gamma} = \overline{g_1x_1}$, it follows that $(g_1x_1)^{-1}g_2x_2 = c_0 \in \Gamma_c$.

To see that (5) holds, let u be an element of the left side. Then $u = g_1 h$ and for some $h \in H$ and $\overline{u} = \overline{\gamma}\overline{\lambda}$ for some $\lambda \in \Lambda$. Then for some $c' \in \Gamma_c$, $u = g_1 x_1 \lambda c'$ hence $\lambda c' \in H$ since $x_1 \in H$. Clearly $\lambda c' \in \Lambda$, hence $u \in g'(\Lambda \cap H)$. Similarly $u \in g'c_0(\Lambda \cap K)$. Conversely, any element of the right side has the form $g'h = g_1 x_1 h$ for some $h \in H$ hence is in $g_1 H$, and has the form $g'c_0 k = g_2 x_2 k$ for some $k \in K$ hence is in $g_2 K$.

We will now find the full-form sequences for $\Lambda \cap H$ and $\Lambda \cap K$. Apply the **Preimage** algorithm to compute for each $\overline{w_i}$ preimages $u'_i \in H$ and $v'_i \in K$. Compute a generating set $y'_1, \ldots, y'_{s'}$ for $H \cap \Gamma_c$ by finding the **Full-form** sequence for H and selecting only those elements that belong to Γ_c . Similarly, compute a generating set $z'_1, \ldots, z'_{t'}$ for $K \cap \Gamma_c$. We now have

$$\Lambda \cap H = \langle u'_1, \dots, u'_l, y'_1, \dots, y'_{s'} \rangle .$$

$$\Lambda \cap K = \langle v'_1, \dots, v'_l, z'_1, \dots, z'_{t'} \rangle ,$$

Using the generating sets above, find the **Full-form sequence** $(v_1, \ldots, v_n, z_1, \ldots, z_s)$ for $\Lambda \cap K$, where z_1 denotes the first generator of the sequence that lies in Γ_c . Likewise find the **Full-form sequence** $(u_1, \ldots, u_{n'}, y_1, \ldots, y_t)$ for $\Lambda \cap H$, with y_1 being the first generator in Γ_c . Since $\Lambda \cap H$ and $\Lambda \cap K$ have the same image $\overline{H} \cap \overline{K}$ under $\overline{}$, it follows that n' = n and for all $i = 1, \ldots, n$ that $u_i = v_i c_i$ for some $c_i \in \Gamma_c$. We now have the full-form sequences

$$\Lambda \cap H = \langle v_1 c_1, \dots, v_n c_n, y_1, \dots, y_t \rangle.$$
(6)

$$\Lambda \cap K = \langle v_1, \dots, v_n, z_1, \dots, z_s \rangle, \tag{7}$$

The next step produces a generating set of $H \cap K$ and the element g. The correctness of this step is argued below. Denote $C_1 = \langle c_1, \ldots, c_n, y_1, \ldots, y_t \rangle$ and consider the intersection

$$C_1 \cap (\Lambda \cap K \cap \Gamma_c)$$

in the abelian group Γ_c . Define a homomorphism $\psi: \mathbb{Z}^{n+t} \to \Gamma_c$ by

$$\psi(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_t)=c_1^{\alpha_1}\cdots c_n^{\alpha_n}y_1^{\beta_1}\cdots y_t^{\beta_t}.$$

Using the composition $\psi' : \mathbb{Z}^{n+t} \to \Gamma_c \to \Gamma_c/(\Lambda \cap K \cap \Gamma_c)$, we may then use the **Kernel** algorithm, as in the base case, to produce a finitely generated subgroup $P = \langle p_1, \ldots, p_b \rangle \leq \mathbb{Z}^{n+t}$ such that

$$C_1 \cap \Lambda \cap K \cap \Gamma_c = \psi(P).$$

The sequence (p_1, \ldots, p_b) is the full-form sequence for P, so the corresponding matrix formed is in row-echelon form. We denote $p_i = (p_{i1}, \ldots, p_{i(n+t)})$ for $i = 1, \ldots, b$. In addition, we use the **Membership** algorithm, as described in the base case, to find

$$c_0' \in C_1 \cap c_0(\Lambda \cap K \cap \Gamma_c)$$

if such an element exists and to write c'_0 in the form

$$c_0' = c_1^{\alpha_1'} \cdots c_n^{\alpha_n'} y_1^{\beta_1'} \cdots y_t^{\beta_t'}.$$

If c'_0 does not exist, we return 'No'. Otherwise, we define

$$h = (v_1c_1)^{\alpha'_1} \cdots (v_nc_n)^{\alpha'_n} y_1^{\beta'_1} \cdots y_t^{\beta'_t}$$

and return g = g'h. For the generating set of $H \cap K$, define the *function* (it is not, in general, a homomorphism) $\theta : \mathbb{Z}^{n+t} \to \Lambda \cap H$ by

$$\theta(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_t) = (v_1c_1)^{\alpha_1}\cdots(v_nc_n)^{\alpha_n}y_1^{\beta_1}\cdots y_t^{\beta_t}$$

and return the **Full-form sequence** for the subgroup generated by the set

$$\Pi = \{\theta(p_i) \mid 1 \le i \le b\}.$$

It remains to prove the correctness of the last step. First, we prove that Π generates $H \cap K = (\Lambda \cap H) \cap (\Lambda \cap K)$. Take any $p_i = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_t)$. Then, using the fact that $C_1 \leq \Gamma_c$ is in the center of G,

$$\theta(p_i) = (v_1 c_1)^{\alpha_1} \cdots (v_n c_n)^{\alpha_n} y_1^{\beta_1} \cdots y_t^{\beta_t}$$
(8)

$$= v_1^{\alpha_1} \cdots v_n^{\alpha_n} c_1^{\alpha_1} \cdots c_n^{\alpha_n} y_1^{\beta_1} \cdots y_t^{\beta_t}.$$
 (9)

Line (8) gives $\theta(p_i) \in \Lambda \cap H$ and, since $c_1^{\alpha_1} \cdots c_n^{\alpha_n} y_1^{\beta_1} \cdots y_t^{\beta_t} = \psi(p_i) \in \psi(P) \subset \Lambda \cap K$, line (9) gives $\theta(p_i) \in \Lambda \cap K$. Hence $\langle \Pi \rangle \leq H \cap K$.

For the opposite inclusion, let $(\Lambda \cap H)_i = \langle v_i c_i, \ldots, v_n c_n, y_1, \ldots, y_t \rangle$ for $1 \leq i \leq n$ and $(\Lambda \cap H)_{n+1} = \langle y_1, \ldots, y_t \rangle$. We will prove, by induction on *i* in the reverse order $n + 1, \ldots, 1$, that

$$(\Lambda \cap H)_i \cap (\Lambda \cap K) \le \langle \Pi \rangle$$

for all i = n + 1, ..., 1 (in particular for i = 1). For the base case i = n + 1, let $q \in (\Lambda \cap H)_{n+1} \cap (\Lambda \cap K)$. Then $q = c_1^0 \cdots c_n^0 y_1^{\beta_1} \cdots y_t^{\beta_t}$ for some $(0, ..., 0, \beta_1, ..., \beta_t) = p$. Since $q \in C_1 \cap \Lambda \cap K$, we have $p \in P$. Since the matrix corresponding to P is in row echelon form, we may write p as a linear combination

$$p = \sum_{j=k}^{b} \gamma_j p_j$$

where $p_j = (0, ..., 0, p_{j(n+1)}, ..., p_{j(n+t)})$ for all $j \ge k$. Then

$$q = (y_1^{p_{k(n+1)}} \cdots y_t^{p_{k(n+t)}})^{\gamma_k} \cdots (y_1^{p_{b(n+1)}} \cdots y_t^{p_{b(n+t)}})^{\gamma_b}$$

= $\theta(p_k)^{\gamma_k} \cdots \theta(p_b)^{\gamma_b}$

hence $q \in \langle \Pi \rangle$.

For the inductive case, assume that $(\Lambda \cap H)_{i+1} \cap (\Lambda \cap K) \leq \langle \Pi \rangle$ for some $i+1 \leq n+1$ and let $q \in (\Lambda \cap H)_i \cap (\Lambda \cap K)$. Then

$$q = (v_1 c_1)^0 \cdots (v_{i-1} c_{i-1})^0 (v_i c_i)^{\alpha_i} \cdots (v_n c_n)^{\alpha_n} y_1^{\beta_1} \cdots y_t^{\beta_t}$$

for some $(0, \ldots, 0, \alpha_i, \ldots, \alpha_n, \beta_1, \ldots, \beta_t) = p'$. Since $q \in \Lambda \cap K$ and $v_i^{\alpha_i} \cdots v_n^{\alpha_n} \in \Lambda \cap K$, it follows, rewriting q as in (9), that $c_i^{\alpha_i} \cdots c_n^{\alpha_n} y_1^{\beta_1} \cdots y_t^{\beta_t} \in \Lambda \cap K$ and hence $p' \in P$. Hence

$$p = \sum_{j=k'}^{b} \gamma'_j p_j$$

for some γ'_j and $k' \ge 1$ such that $p_j = (0, \ldots, 0, p_{ji}, \ldots, p_{j(n+t)})$ for all $j \ge k'$. Now consider the element

$$q' = q\theta(p_{k'})^{-\gamma'_{k'}} \cdots \theta(p_b)^{-\gamma'_b}$$

In the word $\theta(p_{k'})^{-\gamma'_{k'}} \cdots \theta(p_b)^{-\gamma'_b}$, the generators $v_1c_1, \ldots, v_{i-1}c_{i-1}$ do not appear. Further, the total exponent sum of the generator v_ic_i is $-\alpha_i$, while in q it is α_i . Since for any $i < l \leq n$ and $1 \leq l' \leq t$ the commutators $[v_ic_i, v_lc_l]$ and $[v_ic_i, y_{l'}]$ are elements of $(\Lambda \cap H)_{i+1}$ we may collect all occurrences of v_ic_i and eliminate its occurrence in q'. Hence $q' \in (\Lambda \cap H)_{i+1} \cap (\Lambda \cap K)$. By induction, $q' \in \langle \Pi \rangle$ hence $q \in \langle \Pi \rangle$ as well.

Regarding the intersection being non-empty, observe that by (5),

$$g_1H \cap g_2K \neq \emptyset \iff (\Lambda \cap H) \cap c_0(\Lambda \cap K) \neq \emptyset.$$

Now if $g = (v_1c_1)^{\alpha_1} \cdots (v_nc_n)^{\alpha_n} y_1^{\beta_1} \cdots y_t^{\beta_t} \in \Lambda \cap H$ is also an element of $c_0(\Lambda \cap K)$ then, again rewriting as in (9), $c_1^{\alpha_1} \cdots c_n^{\alpha_n} y_1^{\beta_1} \cdots y_t^{\beta_t}$ is an element of C_1 which also lies in $c_0(\Lambda \cap K \cap \Gamma_c)$ hence c'_0 exists. Conversely, if c'_0 exists, then we have

$$c'_{0} = c_{1}^{\alpha'_{1}} \cdots c_{n}^{\alpha_{n'}} y_{1}^{\beta'_{1}} \cdots y_{t}^{\beta'_{t}} = c_{0} z_{1}^{\gamma'_{1}} \cdots z_{s}^{\gamma'_{s}}$$

hence

$$(v_1c_1)^{\alpha'_1}\cdots(v_nc_n)^{\alpha'_n}y_1^{\beta'_1}\cdots y_t^{\beta'_t} = c_0v_1^{\alpha'_1}\cdots v_n^{\alpha'_n}z_1^{\gamma'_1}\cdots z_s^{\gamma'_s}$$

is an element of $(\Lambda \cap H) \cap c_0(\Lambda \cap K)$, hence this intersection is non-empty. This proves the correctness of the decision problem.

Finally, we must show that $g \in g_1H \cap g_2K$. Since $h \in \Lambda \cap H$, we have $g = g'h \in g'(\Lambda \cap H)$. Since $c'_0 \in c_0(\Lambda \cap K)$, we have $c'_0 = c_0k$ for some $k \in \Lambda \cap K$ hence

$$g = g'h = g'v_1^{\alpha'_1} \cdots v_n^{\alpha'_n}c_0' = g'c_0(kv_1^{\alpha_1} \cdots v_n^{\alpha_n})$$

is an element of $g'c_0(\Lambda \cap K)$, as required.

Complexity. The depth of the recursion is c, which is constant, so the total number of subroutine calls is constant. The total number of group elements to record is also constant.

As an application of the intersection algorithm, we may generalize Lemma 4 to solve the simultaneous conjugation problem for tuples in nilpotent groups.

Theorem 7. Fix integers c, r, and l. There is an algorithm that, given a nilpotent group $G = \langle X | R \rangle$ of nilpotency class at most c with $|X| \leq r$ and two tuples (a_1, \ldots, a_l) and (b_1, \ldots, b_l) of elements of G, computes:

• an element $g \in G$ such that

$$g^{-1}a_ig = b_i$$

for i = 1, ..., l

• a generating set for the centralizer $C_G(b_1, \ldots, b_l)$, or determines that no such element g exists.

The algorithm runs in space logarithmic in the size L of the input and time quasilinear in L, or in TC^0 , and the length of each output word is bounded by a polynomial function of L. If compressed inputs are used, the space complexity is O(M) and the time complexity $O(nM^3)$ where n = |R| and M bounds the encoded size of each input word.

Proof. Algorithm. Begin by applying **Lemma 3** to convert to a nilpotent presentation if necessary. Next, for each i = 1, ..., l, use the **Conjugacy** algorithm to find $g_i \in G$ such that $a_i^{g_i} = b_i$. If any pair is not conjugate, then g does not exists and we may return 'No'. We also use the **Centralizer** algorithm to find, for each i, a generating set for $C_G(b_i)$.

Now for any $g \in G$ and any i, the equation

$$a_i^g = a_i^{g_i g_i^{-1}g} = b_i^{g_i^{-1}g}$$

shows that $a_i^g = b_i$ if and only if $g_i^{-1}g \in C_G(b_i)$, i.e. $g \in g_iC_G(b_i)$. Hence the set of all possible conjugators is precisely the coset intersection $\bigcap_{i=1}^l g_iC_G(b_i)$ which we may compute by iterating **Theorem 6**. As a by-product, we obtain a generating set for

$$\bigcap_{i=1}^{l} C_G(b_i) = C_G(b_1, \dots, b_l).$$

Complexity. Since l is fixed, the number of subroutine calls and elements of G to store is constant.

3.3 Torsion subgroup

In every nilpotent group G the set T consisting of all elements of finite order forms a subgroup called the torsion subgroup. We give an algorithm to compute, from a presentation of G, a generating set and presentation for T as well as its order. We follow an algorithm outlined in [KRR⁺69].

Theorem 8. Fix positive integers c and r. There is an algorithm that, given a finitely presented nilpotent group $G = \langle X | R \rangle$ of nilpotency class at most c with $|X| \leq r$, produces

- a generating set for the torsion subgroup T of G,
- a presentation for T, and
- the order of T.

The algorithm runs in space logarithmic in the size L of the given presentation and time quasilinear in L, or in TC^0 . The length of each output word is bounded by a polynomial function of L and the number of such words is bounded by a constant. If compressed inputs are used, the space complexity is O(nM) and the time complexity is $O(nM^3)$, where n = |R| and M bounds the length of each relator in R.

Proof. Define inductively a sequence T_1, T_2, \ldots of finite normal subgroups of G as follows. Let $T_1 = T(Z(G))$, which is clearly finite and normal. For i > 1 define the homomorphism $\phi_i : G \to G/T_{i-1}$ and set

$$T_i = \phi_i^{-1} \left(T \left(Z \left(G / T_{i-1} \right) \right) \right).$$

Since $Z(G/T_{i-1})$ is abelian and finitely generated, $T(Z(G/T_{i-1}))$ is finite and hence finiteness of T_i follows from that of T_{i-1} . Normality of T_i follows from normality of T_{i-1} in G and of $T(Z(G/T_{i-1}))$ in G/T_{i-1} . Since G is Noetherian, the sequence must stabilize at some T_k . But then $T(Z(G/T_k))$ is trivial, hence G/T_k is torsion-free (its torsion subgroup must otherwise intersect its center), hence $T_k = T$.

Algorithm. We compute the sequence described above. Begin by applying **Lemma 3** to compute a nilpotent presentation $G = \langle A | S \rangle$. Since Z(G) is simply the centralizer of any generating set, we may find the full-form sequence (h_1, \ldots, h_m) for Z(G) using **Theorem 7** with the set A. Since Z(G) is abelian, its torsion subgroup is generated by the set X_1 consisting of elements h_i such that $i \in \mathcal{T}$. Note that \mathcal{T} is determined by examining the relators of the form (4) is S.

Now assume, by induction, that we have a generating set X_i for T_i . Use **Theorem 7** with the nilpotent group $G/T_i = \langle A | S \cup X_i \rangle$ and the set A to find, as described in the base case, the full-form sequence $(\tau_1 T_i, \ldots, \tau_m T_i)$ for $T(Z(G/T_i))$. Then $X_i \cup \{\tau_1, \ldots, \tau_m\}$ generates T_{i+1} , and we compute the the **Full-form sequence** X_{i+1} of T_i . If $X_{i+1} = X_i$, then $T_{i+1} = T_i = T$. Otherwise, we proceed with the next step of the induction.

Once we obtain the full-form sequence (t_1, \ldots, t_n) for T, it suffices to run **Subgroup Presentation** to give a presentation for T. Denote by π_1, \ldots, π_n the pivot columns of the matrix associated with (t_1, \ldots, t_n) and by α_{ij} the (i, j)entry of this matrix. Then every element of T may be expressed uniquely in the form $t_1^{\beta_1} \cdots t_n^{\beta_n}$ where $0 \leq \beta_i < e_{\pi_i}/\alpha_{i\pi_i}$, and every such expression gives a different element. Hence the order of T is

$$|T| = \prod_{i=1}^{n} e_{\pi_i} / \alpha_{i\pi_i}$$

Complexity. First, we will prove that the depth of the recursion is bounded by c. Let $Z_i = \{h \in G \mid [h,g] \in Z_{i-1} \text{ for all } g \in G\}$ be the i^{th} term of the upper central series of G, with $Z_1 = Z(G)$. We claim that

$$T_i \supseteq Z_i \cap T$$

for all $i = 1, \ldots, c$, hence $T_c \supseteq G \cap T$ so $T_c = T$ and the depth of the recursion is bounded by c. We proceed by induction. For i = 1 we have $T_1 = T(Z(G)) =$ $Z_1 \cap T$. Now let $g \in Z_i \cap T$ and consider $\phi_i(g) = \overline{g}$. Let $\overline{h} \in G/T_{i-1}$ and consider $[\overline{g}, \overline{h}] = [\overline{g}, h]$. Since $g \in T$, we have $[g, h] = g^{-1}g^h \in T$ and since $g \in Z_i$, we have $[g, h] \in Z_{i-1}$. By the inductive assumption, $[g, h] \in T_{i-1}$ hence $[\overline{g}, \overline{h}] = 1$ hence $\overline{g} \in Z(G/T_{i-1})$. Clearly $\overline{g} \in T(Z(G/T_{i-1}))$ hence $g \in T_i$, proving the claim.

Since the depth of the recursion is constant, the total number of subroutine calls is constant, as is the number of elements kept in memory, since each X_i is a full-form sequence (hence of bounded length).

In computing the order of T, recall that the numbers e_i where $i \in \mathcal{T}$, appear as exponents in the nilpotent presentation computed by Lemma 3. Consequently, each is bounded by a polynomial function of L. Since the length n of the full-form sequence for T is bounded by a constant, the order of T is polynomially bounded.

Corollary 9. If $G = \langle X | R \rangle$ is a nilpotent group of nilpotency class c with $|X| \leq r$, then the order of the torsion subgroup of G is bounded by a polynomial function of the sum of the lengths of the relators R.

3.4 Isolator

Recall that the *isolator* of H in G is defined by

 $Is_G(H) = \{ g \in G \mid g^n \in H \text{ for some } n \neq 0 \}$

and, in nilpotent groups, forms a subgroup.

Theorem 10. Fix integers c and r. There is an algorithm that, given a finitely presented nilpotent group $G = \langle X | R \rangle$ of nilpotency class at most c with $|X| \leq r$, and a subgroup $H \leq G$, computes

a generating set for the isolator $Is_G(H)$.

The algorithm runs in space logarithmic in the length L of the input and time quasilinear in L, or in TC^0 , and the length of each generator is bounded by a polynomial function of L. If compressed inputs are used, the space complexity is O(M) and the time complexity $O(nM^3)$ where n is the number of input words and M bounds the encoded size of each input word.

Proof. Algorithm. First, apply **Lemma 3** to convert to a nilpotent presentation $G = \langle a_1, \ldots, a_m | S \rangle$.

Let $N^0 = H$ and for i > 0 define $N^i = N_G(N^{i-1})$, the normalizer of N^{i-1} in G. It is proved in [KM79] Thm. 16.2.2 that $N^c = G$. Using **Theorem 5**, we compute in turn the full-form generating sequences for each of the subgroups H, N^1, \ldots, N^c and using the **Subgroup Presentation** algorithm we compute a nilpotent presentation

$$N^i = \langle X_i \,|\, R_i \rangle$$

for each. We now proceed, by induction, to compute for each i = 0, ..., c a generating set Y_i for $Is_{N^i}(H)$. For i = 0, we have $Is_{N^0}(H) = H$ and we use the computed full-form sequence X_0 for H. Now assume that we have a generating set Y_{i-1} for $Is_{N^{i-1}}(H)$.

The subgroup N^{i-1} is normal in N^i , and we will find the torsion subgroup of N^i/N^{i-1} . Using **Lemma 3**, write each element of X_{i-1} in its X_i -coordinates. Appending these elements to R_i we obtain a presentation of N^i/N^{i-1} , which we pass to **Theorem 8** to obtain a generating set $\{\tau_1 N^{i-1}, \ldots, \tau_m N^{i-1}\}$ for the torsion subgroup. Then $\operatorname{Is}_{N^i}(N^{i-1})$ is generated by $X_{i-1} \cup \{\tau_1, \ldots, \tau_r\}$. Converting these elements back to generators of G, we then compute the **Full-form sequence** Z_i for $\operatorname{Is}_{N^i}(N^{i-1})$ and, using **Lemma 3** the corresponding nilpotent presentation

$$\operatorname{Is}_{N^i}(N^{i-1}) = \langle Z_i \,|\, S_i \rangle.$$

We claim that $\operatorname{Is}_{N^{i-1}}(H) \leq \operatorname{Is}_{N^i}(N^{i-1})$. Indeed, the property that $g^n \in H$ for some n > 0 is unchanged under conjugation, and since N^i normalizes N^{i-1}

all conjugates remain in N^{i-1} . Using **Lemma 3**, write each element of Y_{i-1} in terms of the generators Z_i and append these words to S_i to obtain a presentation of $\operatorname{Is}_{N^i}(N^{i-1})/\operatorname{Is}_{N^{i-1}}(H)$. Now apply **Theorem 8** to compute a generating set $\{\sigma_1 \operatorname{Is}_{N^{i-1}}(H), \ldots, \sigma_r \operatorname{Is}_{N^{i-1}}(H)\}$ for the torsion subgroup.

Set $Y_i = \{\sigma_1, \ldots, \sigma_r\} \cup Y_{i-1}$, using the two prior calls to Lemma 3 to write each σ_i in generators of G. We claim that Y_i generates $\operatorname{Is}_{N^i}(H)$. Clearly $Y_{i-1} \subset \operatorname{Is}_{N^{i-1}}(H) \subset \operatorname{Is}_{N^i}(H)$. For each σ_i there exists $p_i > 0$ such that $\sigma_i^{p_i} \in \operatorname{Is}_{N^{i-1}}(H)$, therefore there exists $q_i > 0$ such that $\sigma_i^{p_i q_i} \in H$. Hence $Y_i \subset \operatorname{Is}_{N^i}(H)$. Now if $g \in \operatorname{Is}_{N^i}(H)$ then $g^n \in H$ for some n > 0 and $g \in \operatorname{Is}_{H_i}(H_{i-1})$, hence $g \cdot \operatorname{Is}_{H_{i-1}}(H)$ lies in the torsion subgroup of $\operatorname{Is}_{H_i}(H_{i-1})/\operatorname{Is}_{H_{i-1}}(H)$, and so g is an element of the subgroup generated by Y_i .

Finally, return the **Full-form sequence** for Y_n .

Complexity. Since the number of elements in a full-form sequence is bounded by m, the total number of elements in the sequences for N^i , $i = 0, \ldots, c$, is constant, as is the number of relators in each R_i and the number of elements in the generating set for the torsion subgroups. The depth of the recursion is bounded by c. Hence the total number of elements to store and the number of subroutine calls is constant.

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