

The algebraic and geometric classification of nilpotent binary Lie algebras ^{1,2}

Hani Abdelwahab^a, Antonio Jesús Calderón^b & Ivan Kaygorodov^c

^a Department of Mathematics, Faculty of Sciences, Mansoura University, Mansoura, Egypt

^b Department of Mathematics, Faculty of Sciences, University of Cadiz, Cadiz, Spain

^c CMCC, Universidade Federal do ABC. Santo André, Brasil

E-mail addresses:

Hani Abdelwahab (haniamar1985@gmail.com)

Antonio Jesús Calderón (ajesus.calderon@uca.es)

Ivan Kaygorodov (kaygorodov.ivan@gmail.com)

Abstract: *We give a complete algebraic classification of nilpotent binary Lie algebras of dimension at most 6 over an arbitrary base field of characteristic not 2 and a complete geometric classification of nilpotent binary Lie algebras of dimension 6 over \mathbb{C} . As an application, we give an algebraic and geometric classification of nilpotent anticommutative \mathfrak{CD} -algebras of dimension at most 6.*

Keywords: *Nilpotent algebras, binary Lie algebras, Malcev algebras, \mathfrak{CD} -algebras, Lie algebras, algebraic classification, geometric classification, degeneration.*

MSC2010: 17D10, 17D30.

INTRODUCTION

There are many results related to both the algebraic and geometric classification of small dimensional algebras in the varieties of Jordan, Lie, Leibniz, Zinbiel algebras; for algebraic results see, for example, [2, 5, 8, 14, 16]; for geometric results see, for example, [6, 9, 11, 12, 14]. Here we give an algebraic and geometric classification of low dimensional nilpotent binary Lie algebras.

Malcev defined binary Lie algebras as algebras such that *every two-generated subalgebra is a Lie algebra* [17]. Identities of the variety of binary Lie algebras were described by Gainov [4]. Note that every Lie algebra

¹ The authors thank Prof. Dr. Yury Volkov for constructive discussions about degenerations of algebras and Prof. Dr. Pasha Zusmanovich for discussions about \mathfrak{CD} -algebras; two referees and Prof. Dr. Eamonn O'Brien for detailed reading of this work and for suggestions which improved the final version of the paper. The work was supported by RFBR 18-31-20004.

²Corresponding Author: kaygorodov.ivan@gmail.com

is a Malcev algebra and every Malcev algebra is a binary Lie algebra. The systematic study of Malcev and binary Lie algebras began with the work of Sagle [18]. Properties of binary Lie algebras were studied by Filippov, Kaygorodov, Kuzmin, Popov, Shirshov, Volkov and many others [5, 12, 16]. Another interesting subclass of binary Lie algebras is anticommutative \mathfrak{CD} -algebras. The idea of the definition of \mathfrak{CD} -algebras is to generalize a certain property of Jordan and Lie algebras — *every commutator of two multiplication operators is a derivation*. Commutative \mathfrak{CD} -algebras (sometimes called *Lie triple algebras*) were considered in [13, 19].

Our method of classification of nilpotent binary Lie algebras is based on calculation of central extensions of smaller nilpotent algebras from the same variety. The algebraic study of central extensions of Lie and non-Lie algebras has a long history [7, 20]. Skjelbred and Sund [20] used central extensions of Lie algebras for a classification of nilpotent Lie algebras. After using the method of [20] all non-Lie central extensions of all 4-dimensional Malcev algebras [7], all anticommutative central extensions of 3-dimensional anticommutative algebras [1] and some others were described. Also, all 4-dimensional nilpotent associative algebras, all 4-dimensional nilpotent Novikov algebras, all 4-dimensional nilpotent bi-commutative algebras, all 5-dimensional nilpotent Jordan algebras, all 5-dimensional nilpotent restricted Lie algebras, all 6-dimensional nilpotent Lie algebras, all 6-dimensional nilpotent Malcev algebras and some others were described (see, [2, 3, 8, 10]).

1. THE ALGEBRAIC CLASSIFICATION OF BINARY LIE ALGEBRAS

1.1. Definitions and notation. Throughout the paper, \mathbb{F} denotes a field of characteristic not 2 and the multiplication of an algebra is specified by giving only the nonzero products among the basis elements.

In an anticommutative algebra $(\mathbf{A}, [-, -])$ we define the Jacobian $\mathcal{J}(x, y, z)$ of elements x, y, z in \mathbf{A} in the following way:

$$\mathcal{J}(x, y, z) := [[x, y], z] + [[y, z], x] + [[z, x], y].$$

It is clear that the Jacobian $\mathcal{J}(x, y, z)$ is skew-symmetric in its arguments.

Definition 1. Let $(\mathbf{A}, [-, -])$ be an anticommutative algebra. Then $(\mathbf{A}, [-, -])$ is a:

- Lie algebra if

$$\mathcal{J}(x, y, z) = 0, \text{ for all } x, y, z \in \mathbf{A}.$$

- Malcev algebra if

$$(1.1) \quad \mathcal{J}(x, y, [x, z]) = [\mathcal{J}(x, y, z), x], \text{ for all } x, y, z \in \mathbf{A}.$$

- *Binary Lie algebra if*

$$(1.2) \quad \mathcal{J}([x, y], x, y) = 0, \text{ for all } x, y \in \mathbf{A}.$$

Every Lie algebra is a Malcev algebra and every Malcev algebra is a binary Lie algebra.

The linearization of the identity (1.1) is

$$[[w, y], [x, z]] = [[w, x], y], z] + [[[x, y], z], w] + [[y, z], w], x] + [[z, w], x], y],$$

for all $x, y, z, w \in \mathbf{A}$ (see [15, 18]). Further, the linearization of the identity (1.2) is

$$(1.3) \quad \mathcal{J}([x, y], z, t) + \mathcal{J}([x, t], z, y) + \mathcal{J}([z, y], x, t) + \mathcal{J}([z, t], x, y) = 0,$$

for all $x, y, z, t \in \mathbf{A}$ (see [15, 18]).

We define inductively $\mathbf{A}^1 = \mathbf{A}$ and

$$\mathbf{A}^{n+1} = [\mathbf{A}^n, \mathbf{A}^1] + [\mathbf{A}^{n-1}, \mathbf{A}^2] + \dots + [\mathbf{A}^1, \mathbf{A}^n].$$

The algebra \mathbf{A} is *nilpotent* if $\mathbf{A}^n = 0$.

We state main results of the first part of this paper.

Theorem 2. *Let $\mathcal{N}_{\text{BL}}^6(\mathbb{F})$ denote the number of 6-dimensional nilpotent binary Lie algebras over \mathbb{F} . Then*

$$\mathcal{N}_{\text{BL}}^6(\mathbb{F}) = 41 + 2|\mathbb{F}^*| + 5|\mathbb{F}^*/(\mathbb{F}^*)^2|$$

where $|\mathbb{F}^*|$ and $|\mathbb{F}^*/(\mathbb{F}^*)^2|$ denote the, possibly infinite, cardinality of the multiplicative group \mathbb{F}^* and the quotient group of \mathbb{F}^* by the subgroup $(\mathbb{F}^*)^2 = \{x^2 : x \in \mathbb{F}^*\}$, respectively.

Theorem 3. *Every 6-dimensional nilpotent non-Malcev binary Lie algebra over \mathbb{F} is isomorphic to one of the following algebras:*

- $\mathbf{B}_{6,1}^\alpha : [e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = \alpha e_6, [e_4, e_5] = e_6;$
- $\mathbf{B}_{6,2} : [e_1, e_2] = e_3, [e_3, e_4] = e_5, [e_4, e_5] = e_6;$
- $\mathbf{B}_{6,3} : [e_1, e_2] = e_3, [e_3, e_4] = e_5, [e_1, e_3] = e_6, [e_4, e_5] = e_6.$

Among these algebras there are precisely the following isomorphisms:

- $\mathbf{B}_{6,1}^\alpha \cong \mathbf{B}_{6,1}^\beta$ if and only if there is an $\lambda \in \mathbb{F}^*$ such that $\beta = \lambda^2 \alpha$.

The proofs of Theorem 2 and Theorem 3 follow from the algebraic classification of 6-dimensional nilpotent Lie algebras over \mathbb{F} given in [2], the algebraic classification of 6-dimensional nilpotent non-Lie Malcev algebras over \mathbb{F} given in [8] and the algebraic classification of 6-dimensional non-Malcev binary Lie algebras over \mathbb{F} (see Section 1.4).

1.2. A method for the algebraic classification of nilpotent algebras. Let \mathbf{A} be a binary Lie algebra over \mathbb{F} and let \mathbf{V} be a vector space over \mathbb{F} . Then the \mathbb{F} -linear space $Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$ is defined as the set of all skew-symmetric bilinear maps $\theta : \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{V}$ such that

$$(1.4) \quad \theta([x, y], x, y) = \theta([x, y], y, x)$$

for all $x, y \in \mathbf{A}$. For a linear map f from \mathbf{A} to \mathbf{V} , if we write $\delta f : \mathbf{A} \times \mathbf{A} \rightarrow \mathbf{V}$ by $\delta f(x, y) = f([x, y])$, then $\delta f \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$. We define $B^2(\mathbf{A}, \mathbf{V}) = \{\theta = \delta f : f \in \text{Hom}(\mathbf{A}, \mathbf{V})\}$. One can easily check that $B^2(\mathbf{A}, \mathbf{V})$ is a linear subspace of $Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$. We define the set $H_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$ as the quotient space $Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})/B^2(\mathbf{A}, \mathbf{V})$. The equivalence class of $\theta \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$ is denoted by $[\theta] \in H_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$.

Let $\text{Aut}(\mathbf{A})$ be the automorphism group of the binary Lie algebra \mathbf{A} and let $\phi \in \text{Aut}(\mathbf{A})$. For $\theta \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$ define $\phi\theta(x, y) = \theta(\phi(x), \phi(y))$. Now $\phi\theta \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$, so $\text{Aut}(\mathbf{A})$ acts on $Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$. It is easy to verify that $B^2(\mathbf{A}, \mathbf{V})$ is invariant under the action of $\text{Aut}(\mathbf{A})$ and so $\text{Aut}(\mathbf{A})$ acts on $H_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$.

Let \mathbf{A} be a binary Lie algebra of dimension $m < n$ over \mathbb{F} . Let \mathbf{V} be an \mathbb{F} -vector space of dimension $n - m$. For every skew-symmetric bilinear map $\theta : \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{V}$ define on the linear space $\mathbf{A}_\theta := \mathbf{A} \oplus \mathbf{V}$ the bilinear product “ $[-, -]_{\mathbf{A}_\theta}$ ” by $[x + x', y + y']_{\mathbf{A}_\theta} = [x, y] + \theta(x, y)$ for all $x, y \in \mathbf{A}, x', y' \in \mathbf{V}$. It is easy to see that the algebra \mathbf{A}_θ is a binary Lie algebra if and only if $\theta \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$. It is also clear that if \mathbf{A} is nilpotent, then so is \mathbf{A}_θ . If $\theta \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$, we call \mathbf{A}_θ an $(n - m)$ -dimensional central extension of \mathbf{A} by \mathbf{V} . We also call $\theta^\perp = \{x \in \mathbf{A} : \theta(x, \mathbf{A}) = 0\}$ the annihilator of θ .

We recall that the annihilator of an algebra \mathbf{A} is defined as the ideal $\text{Ann}(\mathbf{A}) = \{x \in \mathbf{A} : [x, \mathbf{A}] = 0\}$. It is easy to verify that

$$\text{Ann}(\mathbf{A}_\theta) = (\theta^\perp \cap \text{Ann}(\mathbf{A})) \oplus \mathbf{V}.$$

As in [7, Lemma 5], we can also prove that every binary Lie algebra of dimension n with $\text{Ann}(\mathbf{A}) \neq 0$ can be expressed in the form \mathbf{A}_θ for a m -dimensional binary Lie algebra \mathbf{A} , where $m < n$, and a vector space \mathbf{V} of dimension $n - m$ (here $\theta \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$).

To solve the isomorphism problem we need to study the action of $\text{Aut}(\mathbf{A})$ on $H_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$. To do that, let us fix e_1, \dots, e_s a basis of \mathbf{V} , and $\theta \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$. Then θ can be uniquely written as $\theta(x, y) = \sum_{i=1}^s \theta_i(x, y) e_i$,

where $\theta_i \in Z_{\text{BL}}^2(\mathbf{A}, \mathbb{F})$. Moreover, $\theta^\perp = \theta_1^\perp \cap \theta_2^\perp \cdots \cap \theta_s^\perp$. Further, $\theta \in B^2(\mathbf{A}, \mathbf{V})$ if and only if every $\theta_i \in B^2(\mathbf{A}, \mathbb{F})$.

Given a binary Lie algebra \mathbf{A} , if $\mathbf{A} = \mathbf{B} \oplus \mathbb{F}x$ is a direct sum of two ideals, then $\mathbb{F}x$ is an *annihilator component* of \mathbf{A} . It is not difficult to prove, (see [7, Lemma 13]), that given a binary Lie algebra \mathbf{A}_θ , if we write $\theta(x, y) = \sum_{i=1}^s \theta_i(x, y) e_i \in Z_{\text{BL}}^2(\mathbf{A}, \mathbf{V})$ and $\theta^\perp \cap \text{Ann}(\mathbf{A}) = 0$, then \mathbf{A}_θ has an annihilator component if and only if $[\theta_1], [\theta_2], \dots, [\theta_s]$ are linearly dependent in $H_{\text{BL}}^2(\mathbf{A}, \mathbb{F})$.

Let \mathbf{A} be a binary Lie algebra over \mathbb{F} and let \mathbf{V} be a vector space over \mathbb{F} . The *Grassmannian* $G_k(\mathbf{V})$ is the set of all k -dimensional linear subspaces of \mathbf{V} . Let $G_s(H_{\text{BL}}^2(\mathbf{A}, \mathbb{F}))$ be the Grassmannian of subspaces of dimension s in $H_{\text{BL}}^2(\mathbf{A}, \mathbb{F})$. There is a natural action of $\text{Aut}(\mathbf{A})$ on $G_s(H_{\text{BL}}^2(\mathbf{A}, \mathbb{F}))$. Let $\phi \in \text{Aut}(\mathbf{A})$. For $\mathbf{W} = \langle [\theta_1], [\theta_2], \dots, [\theta_s] \rangle \in G_s(H_{\text{BL}}^2(\mathbf{A}, \mathbb{F}))$ define $\phi\mathbf{W} = \langle [\phi\theta_1], [\phi\theta_2], \dots, [\phi\theta_s] \rangle$. Then $\phi\mathbf{W} \in G_s(H_{\text{BL}}^2(\mathbf{A}, \mathbb{F}))$. We denote the orbit of $\mathbf{W} \in G_s(H_{\text{BL}}^2(\mathbf{A}, \mathbb{F}))$ under the action of $\text{Aut}(\mathbf{A})$ by $\text{Orb}(\mathbf{W})$. Let

$$\mathbf{W}_1 = \langle [\theta_1], \dots, [\theta_s] \rangle, \mathbf{W}_2 = \langle [\vartheta_1], \dots, [\vartheta_s] \rangle \in G_s(H_{\text{BL}}^2(\mathbf{A}, \mathbb{F})).$$

If $\mathbf{W}_1 = \mathbf{W}_2$, then $\bigcap_{i=1}^s \theta_i^\perp \cap \text{Ann}(\mathbf{A}) = \bigcap_{i=1}^s \vartheta_i^\perp \cap \text{Ann}(\mathbf{A})$. So

$$\mathbf{T}_s(\mathbf{A}) = \left\{ \langle [\theta_1], \dots, [\theta_s] \rangle \in G_s(H_{\text{BL}}^2(\mathbf{A}, \mathbb{F})) : \bigcap_{i=1}^s \theta_i^\perp \cap \text{Ann}(\mathbf{A}) = 0 \right\},$$

which is stable under the action of $\text{Aut}(\mathbf{A})$.

Now, let \mathbf{V} be an s -dimensional linear space and let us denote by $\mathbf{E}(\mathbf{A}, \mathbf{V})$ the set of all binary Lie algebras without annihilator components which are s -dimensional central extensions of \mathbf{A} by \mathbf{V} and have s -dimensional annihilator. Let

$$\mathbf{E}(\mathbf{A}, \mathbf{V}) = \left\{ \mathbf{A}_\theta : \theta(x, y) = \sum_{i=1}^s \theta_i(x, y) e_i, \langle [\theta_1], \dots, [\theta_s] \rangle \in \mathbf{T}_s(\mathbf{A}) \right\}.$$

Lemma 4. Let $\mathbf{A}_\theta, \mathbf{A}_\vartheta \in \mathbf{E}(\mathbf{A}, \mathbf{V})$. Suppose that $\theta(x, y) = \sum_{i=1}^s \theta_i(x, y) e_i$ and $\vartheta(x, y) = \sum_{i=1}^s \vartheta_i(x, y) e_i$. Then the binary Lie algebras \mathbf{A}_θ and \mathbf{A}_ϑ are isomorphic if and only if

$$\text{Orb} \langle [\theta_1], [\theta_2], \dots, [\theta_s] \rangle = \text{Orb} \langle [\vartheta_1], [\vartheta_2], \dots, [\vartheta_s] \rangle.$$

Proof. The proof is similar to [7, Lemma 17]. □

Hence, there exists a one-to-one correspondence between the set of $\text{Aut}(\mathbf{A})$ -orbits on $\mathbf{T}_s(\mathbf{A})$ and the set of isomorphism classes of $\mathbf{E}(\mathbf{A}, \mathbf{V})$. Consequently we have a procedure that, given the (nilpotent) binary Lie algebras \mathbf{A}' of dimension $n - s$, allows us to construct all of the (nilpotent) binary Lie algebras \mathbf{A} of dimension n with no annihilator components and with s -dimensional annihilator. This procedure is the following:

- (1) For a given (nilpotent) binary Lie algebra \mathbf{A}' of dimension $n - s$, determine $\mathbf{T}_s(\mathbf{A}')$ and $\text{Aut}(\mathbf{A}')$.
- (2) Determine the set of $\text{Aut}(\mathbf{A}')$ -orbits on $\mathbf{T}_s(\mathbf{A}')$.
- (3) For each orbit, construct the binary Lie algebra corresponding to a representative of it.

The above method gives all (Malcev and non-Malcev) binary Lie algebras. But we also are interested in developing this method in such a way that it only gives non-Malcev binary Lie algebras. Clearly, every central extension of a non-Malcev binary Lie algebra is non-Malcev. So, we only have to study the central extensions of Malcev algebras. Let \mathbf{M} be a Malcev algebra and $\theta \in Z_{\text{BL}}^2(\mathbf{M}, \mathbb{F})$. Then \mathbf{M}_θ is a Malcev algebra if and only if

$$\begin{aligned} \theta([w, y], [x, z]) &= \theta([w, x], [y], z) + \theta([x, y], [z], w) + \\ &\quad \theta([y, z], [w], x) + \theta([z, w], [x], y), \end{aligned}$$

for all $x, y, z, w \in \mathbf{M}$. Define a subspace $Z_{\mathbf{M}}^2(\mathbf{M}, \mathbb{F})$ of $Z_{\text{BL}}^2(\mathbf{M}, \mathbb{F})$ by

$$Z_{\mathbf{M}}^2(\mathbf{M}, \mathbb{F}) = \left\{ \theta \in Z_{\text{BL}}^2(\mathbf{M}, \mathbb{F}) \left| \begin{array}{l} \theta([w, y], [x, z]) = \\ \theta([w, x], [y], z) + \theta([x, y], [z], w) + \\ \theta([y, z], [w], x) + \theta([z, w], [x], y) \end{array} \right. \right\}.$$

Define $H_{\mathbf{M}}^2(\mathbf{M}, \mathbb{F}) = Z_{\mathbf{M}}^2(\mathbf{M}, \mathbb{F}) / B^2(\mathbf{M}, \mathbb{F})$. Therefore, $H_{\mathbf{M}}^2(\mathbf{M}, \mathbb{F})$ is a subspace of $H_{\text{BL}}^2(\mathbf{M}, \mathbb{F})$. Define

$$\mathbf{R}_s(\mathbf{M}) = \{ \mathbf{W} \in \mathbf{T}_s(\mathbf{M}) : \mathbf{W} \in G_s(H_{\mathbf{M}}^2(\mathbf{M}, \mathbb{F})) \},$$

$$\mathbf{U}_s(\mathbf{M}) = \{ \mathbf{W} \in \mathbf{T}_s(\mathbf{M}) : \mathbf{W} \notin G_s(H_{\mathbf{M}}^2(\mathbf{M}, \mathbb{F})) \}.$$

Then $\mathbf{T}_s(\mathbf{M}) = \mathbf{R}_s(\mathbf{M}) \cup \mathbf{U}_s(\mathbf{M})$. The sets $\mathbf{R}_s(\mathbf{M})$ and $\mathbf{U}_s(\mathbf{M})$ are stable under the action of $\text{Aut}(\mathbf{M})$. Thus the binary Lie algebras corresponding to the representatives of $\text{Aut}(\mathbf{M})$ -orbits on $\mathbf{R}_s(\mathbf{M})$ are Malcev algebras while those corresponding to the representatives of $\text{Aut}(\mathbf{M})$ -orbits on $\mathbf{U}_s(\mathbf{M})$ are not. Hence, given those binary Lie algebras \mathbf{A}' of dimension $n - s$, we may construct all non-Malcev algebras \mathbf{A} of dimension n with s -dimensional annihilator which have no annihilator components, in the following way:

- (1) For a given binary Lie algebra \mathbf{A}' of dimension $n - s$, if \mathbf{A}' is non-Malcev then apply the procedure described above.

(2) Otherwise, do the following:

- (a) Determine $U_s(A')$ and $\text{Aut}(A')$.
- (b) Determine the set of $\text{Aut}(A')$ -orbits on $U_s(A')$.
- (c) For each orbit, construct the binary Lie algebra corresponding to a representative of it.

Finally, let us introduce notation. Let A be a binary Lie algebra algebra with basis e_1, e_2, \dots, e_n . By Δ_{ij} we denote the skew-symmetric bilinear form

$$\Delta_{ij} : A \times A \longrightarrow \mathbb{F}$$

with $\Delta_{ij}(e_i, e_j) = -\Delta_{ij}(e_j, e_i) = 1$ and $\Delta_{ij}(e_l, e_m) = 0$ if $\{i, j\} \neq \{l, m\}$. Then $\{\Delta_{ij} : 1 \leq i < j \leq n\}$ is a basis for the linear space of skew-symmetric bilinear forms on A . Every $\theta \in Z_{\text{BL}}^2(A, \mathbb{F})$ can be uniquely written as $\theta = \sum_{1 \leq i < j \leq n} c_{ij} \Delta_{ij}$, where $c_{ij} \in \mathbb{F}$. Further, let $\theta = \sum_{1 \leq i < j \leq n} c_{ij} \Delta_{ij}$

be a skew-symmetric bilinear form on A . Then $\theta \in Z_{\text{BL}}^2(A, \mathbb{F})$ if and only if the c_{ij} 's satisfy property (1.4). We can decide this by computer. Note that property (1.4) is not linear in x, y ; it is better to linearize it. For that we have the following lemma.

Lemma 5. *Let A be a binary Lie algebra and $\theta \in Z_{\text{BL}}^2(A, \mathbb{F})$. Then*

(1.5)

$$\psi_\theta([x, y], z, t) + \psi_\theta([x, t], z, y) + \psi_\theta([z, y], x, t) + \psi_\theta([z, t], x, y) = 0,$$

where $\psi_\theta(x, y, z) := \theta([x, y], z) + \theta([y, z], x) + \theta([z, x], y)$.

Proof. Let $\theta \in Z_{\text{BL}}^2(A, \mathbb{F})$. Then A_θ is a binary Lie algebra. We denote the Jacobian of elements x, y, z in A_θ by $\mathcal{J}_{A_\theta}(x, y, z)$. Now consider $x, y, z, t \in A$.

$$\begin{aligned} \mathcal{J}_{A_\theta}([x, y], z, t) &= \mathcal{J}([x, y], z, t) + \psi_\theta([x, y], z, t); \\ \mathcal{J}_{A_\theta}([x, t], z, y) &= \mathcal{J}([x, t], z, y) + \psi_\theta([x, t], z, y); \\ \mathcal{J}_{A_\theta}([z, y], x, t) &= \mathcal{J}([z, y], x, t) + \psi_\theta([z, y], x, t); \\ \mathcal{J}_{A_\theta}([z, t], x, y) &= \mathcal{J}([z, t], x, y) + \psi_\theta([z, t], x, y). \end{aligned}$$

By the identity (1.3),

$$\begin{aligned} \mathcal{J}_{A_\theta}([x, y], z, t) + \mathcal{J}_{A_\theta}([x, t], z, y) + \\ \mathcal{J}_{A_\theta}([z, y], x, t) + \mathcal{J}_{A_\theta}([z, t], x, y) = 0; \end{aligned}$$

$$\mathcal{J}([x, y], z, t) + \mathcal{J}([x, t], z, y) + \mathcal{J}([z, y], x, t) + \mathcal{J}([z, t], x, y) = 0;$$

we deduce that

$$\psi_\theta([x, y], z, t) + \psi_\theta([x, t], z, y) + \psi_\theta([z, y], x, t) + \psi_\theta([z, t], x, y) = 0,$$

as desired. \square

Note that (1.4) can be obtained from (1.5) by taking $z = x$, $t = y$ in (1.5) since the characteristic of \mathbb{F} is not 2.

1.3. Nilpotent binary Lie algebras of dimensions at most 5. In this section the classification of nilpotent binary Lie algebras of dimension at most 5 is given. Throughout the paper we use some notational conventions:

$\mathbf{L}_{i,j}$: the j -th nilpotent Lie algebra of dimension i ;

$\mathbf{M}_{i,j}$: the j -th nilpotent non-Lie Malcev algebra of dimension i ;

$\mathbf{B}_{i,j}$: the j -th nilpotent non-Malcev binary Lie algebra of dimension i ;

and the basis elements of an algebra of dimension i are denoted by e_1, e_2, \dots, e_i .

It is known from [16] that every nilpotent binary Lie algebra of dimension at most 4 over \mathbb{F} is a nilpotent Lie algebra and thus $H_{BL}^2(\mathbf{A}, \mathbb{F}) = H_M^2(\mathbf{A}, \mathbb{F})$ for every nilpotent binary Lie algebra \mathbf{A} of dimension at most 3 since otherwise we have a 4-dimensional nilpotent binary Lie algebra which is neither a Lie algebra nor a Malcev algebra.

Theorem 6. *Every nilpotent binary Lie algebra of dimension n at most 4 is isomorphic to one of the pairwise nonisomorphic algebras in Table 1.*

Table 1. Nilpotent binary Lie algebras of dimension up to 4

\mathbf{A}	Multiplication	$H_M^2(\mathbf{A}, \mathbb{F})$	$H_{BL}^2(\mathbf{A}, \mathbb{F})$	$\text{Ann}(\mathbf{A})$
$\mathbf{L}_{1,1}$	—	0	$H_M^2(\mathbf{L}_{1,1}, \mathbb{F})$	$\mathbf{L}_{1,1}$
$\mathbf{L}_{2,1}$	—	$\langle [\Delta_{12}] \rangle$	$H_M^2(\mathbf{L}_{2,1}, \mathbb{F})$	$\mathbf{L}_{2,1}$
$\mathbf{L}_{3,1}$	—	$\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{23}] \rangle$	$H_M^2(\mathbf{L}_{3,1}, \mathbb{F})$	$\mathbf{L}_{2,1}$
$\mathbf{L}_{3,2}$	$[e_1, e_2] = e_3$	$\langle [\Delta_{13}], [\Delta_{23}] \rangle$	$H_M^2(\mathbf{L}_{3,2}, \mathbb{F})$	$\mathbf{L}_{2,1}$
$\mathbf{L}_{4,1}$	—	$\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{14}], [\Delta_{23}], [\Delta_{24}], [\Delta_{34}] \rangle$	$H_M^2(\mathbf{L}_{4,1}, \mathbb{F})$	$\mathbf{L}_{4,1}$
$\mathbf{L}_{4,2}$	$[e_1, e_2] = e_3$	$\langle [\Delta_{13}], [\Delta_{14}], [\Delta_{23}], [\Delta_{24}], [\Delta_{34}] \rangle$	$H_M^2(\mathbf{L}_{4,2}, \mathbb{F})$	$\langle e_3, e_4 \rangle$
$\mathbf{L}_{4,3}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$	$\langle [\Delta_{14}], [\Delta_{23}] \rangle$	$H_M^2(\mathbf{L}_{4,3}, \mathbb{F})$	$\langle e_4 \rangle$

Lemma 7. *Let \mathbf{A} be an n -dimensional nilpotent binary Lie algebra.*

- (1) *If $n \leq 4$, then $H_{BL}^2(\mathbf{A}, \mathbb{F}) = H_M^2(\mathbf{A}, \mathbb{F})$ and so $\mathbf{U}_s(\mathbf{A}) = \emptyset$ for $s \geq 1$.*
- (2) *If \mathbf{A} is non-Malcev, then $\dim \text{Ann}(\mathbf{A}) \leq n - 5$.*
- (3) *If $n = 5$, then \mathbf{A} is a Malcev algebra.*

Proof. (1) It follows from Table 1.

(2) Suppose to the contrary that $\dim \text{Ann}(\mathbf{A}) > n - 5$. Then $\dim \mathbf{A}/\text{Ann}(\mathbf{A}) \leq 4$ and so $\mathbf{A}/\text{Ann}(\mathbf{A})$ is a Lie algebra. Further, \mathbf{A} can be viewed as $(n - 4)$ -dimensional extension of $\mathbf{A}/\text{Ann}(\mathbf{A})$. Since $\dim \mathbf{A}/\text{Ann}(\mathbf{A}) \leq 4$, $H_{BL}^2(\mathbf{A}/\text{Ann}(\mathbf{A}), \mathbb{F}) = H_M^2(\mathbf{A}/\text{Ann}(\mathbf{A}), \mathbb{F})$ and hence $\mathbf{U}_{n-4}(\mathbf{L}) = \emptyset$. Therefore \mathbf{A} is a Malcev algebra, which is a contradiction.

(3) It follows from (1). Also, since \mathbf{A} is nilpotent, $\dim \text{Ann}(\mathbf{A}) \geq 1$ and therefore, by (2), \mathbf{A} is a Malcev algebra. \square

Theorem 8. *Every 5-dimensional nilpotent binary Lie algebras is a Malcev algebra and isomorphic to one of the pairwise nonisomorphic algebras in Table 2.*

Table 2. Nilpotent binary Lie algebras of dimension 5

\mathbf{A}	Multiplication	$H_M^2(\mathbf{A}, \mathbb{F})$	$H_{BL}^2(\mathbf{A}, \mathbb{F})$
$\mathbf{L}_{5,1}$	—	$H_{BL}^2(\mathbf{L}_{4,1}, \mathbb{F}) \oplus \langle [\Delta_{15}], [\Delta_{25}], [\Delta_{35}], [\Delta_{45}] \rangle$	$H_M^2(\mathbf{L}_{5,1}, \mathbb{F})$
$\mathbf{L}_{5,2}$	$[e_1, e_2] = e_3$	$H_{BL}^2(\mathbf{L}_{4,2}, \mathbb{F}) \oplus \langle [\Delta_{15}], [\Delta_{25}], [\Delta_{35}], [\Delta_{45}] \rangle$	$H_M^2(\mathbf{L}_{5,2}, \mathbb{F})$
$\mathbf{L}_{5,3}$	$[e_1, e_2] = e_3,$ $[e_1, e_3] = e_4$	$H_{BL}^2(\mathbf{L}_{4,3}, \mathbb{F}) \oplus \langle [\Delta_{15}], [\Delta_{25}], [\Delta_{35}] \rangle$	$H_M^2(\mathbf{L}_{5,3}, \mathbb{F})$
$\mathbf{L}_{5,4}$	$[e_1, e_2] = e_5,$ $[e_3, e_4] = e_5$	$\langle [\Delta_{13}], [\Delta_{14}], [\Delta_{23}], [\Delta_{24}], [\Delta_{34}], [\Delta_{15}], [\Delta_{25}], [\Delta_{35}], [\Delta_{45}] \rangle$	$H_M^2(\mathbf{L}_{5,4}, \mathbb{F})$
$\mathbf{L}_{5,5}$	$[e_1, e_2] = e_3,$ $[e_1, e_3] = e_5,$ $[e_2, e_4] = e_5$	$\langle [\Delta_{13}], [\Delta_{14}], [\Delta_{23}], [\Delta_{34}], [\Delta_{15}] \rangle$	$H_M^2(\mathbf{L}_{5,5}, \mathbb{F})$
$\mathbf{L}_{5,6}$	$[e_1, e_2] = e_3,$ $[e_1, e_3] = e_4,$ $[e_1, e_4] = e_5,$ $[e_2, e_3] = e_5$	$\langle [\Delta_{14}], [\Delta_{15}] - [\Delta_{24}], [\Delta_{25}] - [\Delta_{34}] \rangle$	$H_M^2(\mathbf{L}_{5,6}, \mathbb{F})$
$\mathbf{L}_{5,7}$	$[e_1, e_2] = e_3,$ $[e_1, e_3] = e_4,$ $[e_1, e_4] = e_5$	$\langle [\Delta_{15}], [\Delta_{23}], [\Delta_{25}] - [\Delta_{34}] \rangle$	$H_M^2(\mathbf{L}_{5,7}, \mathbb{F})$
$\mathbf{L}_{5,8}$	$[e_1, e_2] = e_4,$ $[e_1, e_3] = e_5$	$\langle [\Delta_{14}], [\Delta_{15}], [\Delta_{23}], [\Delta_{24}], [\Delta_{34}], [\Delta_{25}], [\Delta_{35}] \rangle$	$H_M^2(\mathbf{L}_{5,8}, \mathbb{F}) \oplus \langle [\Delta_{45}] \rangle$
$\mathbf{L}_{5,9}$	$[e_1, e_2] = e_3,$ $[e_1, e_3] = e_4,$ $[e_2, e_3] = e_5$	$\langle [\Delta_{14}], [\Delta_{15}] + [\Delta_{24}], [\Delta_{25}] \rangle$	$H_M^2(\mathbf{L}_{5,9}, \mathbb{F})$
$\mathbf{M}_{5,1}$	$[e_1, e_2] = e_3,$ $[e_3, e_4] = e_5$	$\langle [\Delta_{13}], [\Delta_{14}], [\Delta_{23}], [\Delta_{24}] \rangle$	$H_M^2(\mathbf{M}_{5,1}, \mathbb{F}) \oplus \langle [\Delta_{45}] \rangle$

$$\begin{aligned}
\text{Ann}(\mathbf{L}_{5,1}) &= \mathbf{L}_{5,1}, & \text{Ann}(\mathbf{L}_{5,2}) &= \langle e_3, e_4, e_5 \rangle, \\
\text{Ann}(\mathbf{L}_{5,3}) &= \langle e_4, e_5 \rangle, & \text{Ann}(\mathbf{L}_{5,4}) &= \langle e_5 \rangle, \\
\text{Ann}(\mathbf{L}_{5,5}) &= \langle e_5 \rangle, & \text{Ann}(\mathbf{L}_{5,6}) &= \langle e_5 \rangle, \\
\text{Ann}(\mathbf{L}_{5,7}) &= \langle e_5 \rangle, & \text{Ann}(\mathbf{L}_{5,8}) &= \langle e_4, e_5 \rangle, \\
\text{Ann}(\mathbf{L}_{5,9}) &= \langle e_4, e_5 \rangle, & \text{Ann}(\mathbf{M}_{5,1}) &= \langle e_5 \rangle
\end{aligned}$$

$$U_1(\mathbf{L}_{5,8}) \neq 0, U_1(\mathbf{M}_{5,1}) \neq 0, U_1(\mathbf{L}_{5,i}) = 0, i \in \{1, 2, 3, 4, 5, 6, 7, 9\}.$$

1.4. Nilpotent binary Lie algebras of dimension 6. In this section we give a complete classification of all 6-dimensional nilpotent binary Lie algebras over \mathbb{F} . Nilpotent Lie algebras of dimension 6 over \mathbb{F} were classified in [2]; nilpotent non-Lie Malcev algebras were classified in [8]. Therefore we only classify nilpotent binary Lie algebras which are not Malcev

algebras. Every nilpotent binary Lie algebras of dimension 5 is a Malcev algebra. Therefore 6-dimensional nilpotent non-Malcev binary Lie algebras with annihilator components do not exist. Next we classify 6-dimensional nilpotent non-Malcev binary Lie algebra without any annihilator component. By Theorem 8, for a 5-dimensional nilpotent binary Lie algebra \mathbf{A} , $\mathbf{U}_1(\mathbf{A}) \neq \emptyset$ if and only if $\mathbf{A} \cong \mathbf{L}_{5,8}$ or $\mathbf{A} \cong \mathbf{M}_{5,1}$.

1.4.1. *The binary Lie algebras corresponding to the representatives of $\text{Aut}(\mathbf{L}_{5,8})$ -orbits on $\mathbf{U}_1(\mathbf{L}_{5,8})$.* The automorphism group of $\mathbf{L}_{5,8}$ consists of invertible matrices of the form

$$\phi = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}a_{22} & a_{11}a_{23} \\ a_{51} & a_{52} & a_{53} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}.$$

Choose an arbitrary subspace $\mathbf{W} \in \mathbf{U}_1(\mathbf{L}_{5,8})$. From Table 2, such a subspace is spanned by

$$[\theta] = C_{14}[\Delta_{14}] + C_{15}[\Delta_{15}] + C_{23}[\Delta_{23}] + C_{24}[\Delta_{24}] + C_{25}[\Delta_{25}] + C_{34}[\Delta_{34}] + C_{35}[\Delta_{35}] + C_{45}[\Delta_{45}]$$

such that $C_{45} \neq 0$. Let $\phi = (a_{ij}) \in \text{Aut}(\mathbf{L}_{5,8})$. Write

$$[\phi\theta] = C'_{14}[\Delta_{14}] + C'_{15}[\Delta_{15}] + C'_{23}[\Delta_{23}] + C'_{24}[\Delta_{24}] + C'_{25}[\Delta_{25}] + C'_{34}[\Delta_{34}] + C'_{35}[\Delta_{35}] + C'_{45}[\Delta_{45}].$$

Then

$$\begin{aligned} C'_{14} &= a_{11}(C_{14}a_{11}a_{22} + C_{15}a_{11}a_{32} + C_{24}a_{21}a_{22} + C_{25}a_{21}a_{32} + C_{34}a_{22}a_{31} + C_{35}a_{31}a_{32} - C_{45}a_{22}a_{51} + C_{45}a_{32}a_{41}), \\ C'_{15} &= a_{11}(C_{14}a_{11}a_{23} + C_{15}a_{11}a_{33} + C_{24}a_{21}a_{23} + C_{25}a_{21}a_{33} + C_{34}a_{31}a_{23} + C_{35}a_{31}a_{33} - C_{45}a_{23}a_{51} + C_{45}a_{41}a_{33}), \\ C'_{23} &= C_{23}a_{22}a_{33} - C_{23}a_{23}a_{32} + C_{24}a_{22}a_{43} - C_{24}a_{23}a_{42} + C_{25}a_{22}a_{53} - C_{25}a_{23}a_{52} + C_{34}a_{32}a_{43} - C_{34}a_{33}a_{42} + C_{35}a_{32}a_{53} - C_{35}a_{33}a_{52} + C_{45}a_{42}a_{53} - C_{45}a_{43}a_{52}, \\ C'_{24} &= a_{11}(C_{24}a_{22}^2 + C_{35}a_{32}^2 + C_{25}a_{22}a_{32} + C_{34}a_{22}a_{32} - C_{45}a_{22}a_{52} + C_{45}a_{32}a_{42}), \\ C'_{25} &= a_{11}(C_{24}a_{22}a_{23} + C_{25}a_{22}a_{33} + C_{34}a_{23}a_{32} + C_{35}a_{32}a_{33} - C_{45}a_{23}a_{52} + C_{45}a_{33}a_{42}), \\ C'_{34} &= a_{11}(C_{24}a_{22}a_{23} + C_{25}a_{23}a_{32} + C_{34}a_{22}a_{33} + C_{35}a_{32}a_{33} - C_{45}a_{22}a_{53} + C_{45}a_{32}a_{43}), \\ C'_{35} &= a_{11}(C_{24}a_{23}^2 + C_{35}a_{33}^2 + C_{25}a_{23}a_{33} + C_{34}a_{23}a_{33} - C_{45}a_{23}a_{53} + C_{45}a_{33}a_{43}), \\ C'_{45} &= a_{11}^2(a_{22}a_{33} - a_{23}a_{32})C_{45}. \end{aligned}$$

Set $\delta = C_{23}C_{45} - C_{24}C_{35} + C_{25}C_{34}$ and $\delta' = C'_{23}C'_{45} - C'_{24}C'_{35} + C'_{25}C'_{34}$. Easy computations show that $\delta' = a_{11}^2 (a_{22}a_{33} - a_{23}a_{32})^2 \delta$. Thus $\text{Orb}(\langle [\theta] : \delta \neq 0 \rangle) \cap \text{Orb}(\langle [\theta] : \delta = 0 \rangle) = \emptyset$ and hence $\text{Aut}(\mathbf{L}_{5,8})$ has at least two orbits on $\mathbf{U}_1(\mathbf{L}_{5,8})$.

• CASE 1. $\delta \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & C_{45}^{-1} & 0 & 0 \\ -C_{45}^{-1}C_{15} & -C_{45}^{-1}C_{25} & -C_{45}^{-2}C_{35} & 1 & 0 \\ C_{45}^{-1}C_{14} & C_{45}^{-1}C_{24} & C_{45}^{-2}C_{34} & 0 & C_{45}^{-1} \end{bmatrix}.$$

Then $\phi\mathbf{W} = \langle C_{45}^{-2}\delta[\Delta_{23}] + [\Delta_{45}] \rangle$. Set $\alpha = C_{45}^{-2}\delta$. Then $\alpha \neq 0$ and so we get the representatives $\mathbf{W}_\alpha = \langle \alpha[\Delta_{23}] + [\Delta_{45}] : \alpha \in \mathbb{F}^* \rangle$. We claim that $\text{Orb}(\mathbf{W}_\alpha) = \text{Orb}(\mathbf{W}_\beta)$ if and only if there is an $\lambda \in \mathbb{F}^*$ such that $\beta = \lambda^2\alpha$. Hence the number of possible orbits among such representatives is $|\mathbb{F}^* / (\mathbb{F}^*)^2|$. To see this, suppose that $\text{Orb}(\mathbf{W}_\alpha) = \text{Orb}(\mathbf{W}_\beta)$. Then there exist $\phi = (a_{ij}) \in \text{Aut}(\mathbf{L}_{5,8})$ and $\lambda \in \mathbb{F}^*$ such that $\phi(\beta[\Delta_{23}] + [\Delta_{45}]) = \lambda(\alpha[\Delta_{23}] + [\Delta_{45}])$. Consequently, we obtain the following polynomial equations:

$$\begin{aligned} a_{11}(a_{32}a_{41} - a_{22}a_{51}) &= 0; & a_{11}(a_{41}a_{33} - a_{23}a_{51}) &= 0; \\ a_{11}(a_{32}a_{42} - a_{22}a_{52}) &= 0; & a_{11}(a_{33}a_{42} - a_{23}a_{52}) &= 0; \\ a_{11}(a_{32}a_{43} - a_{22}a_{53}) &= 0; & a_{11}(a_{33}a_{43} - a_{23}a_{53}) &= 0; \\ a_{11}^2(a_{22}a_{33} - a_{23}a_{32}) &= \lambda; & a_{42}a_{53} - a_{43}a_{52} + \beta(a_{22}a_{33} - a_{23}a_{32}) &= \lambda\alpha. \end{aligned}$$

Since $\det \phi \neq 0$ if and only if $a_{11}(a_{22}a_{33} - a_{23}a_{32}) \neq 0$, we can easily see that $a_{42}a_{53} - a_{43}a_{52} = 0$. We obtain from the last two equations that $\beta = a_{11}^2\alpha$. Conversely, suppose that $\beta = \lambda^2\alpha$ for some $\lambda \in \mathbb{F}^*$. Let ϕ be the diagonal matrix with the entries $(\lambda, 1, 1, \lambda, \lambda)$ in the diagonal. Then $\phi\mathbf{W}_\beta = \langle \beta[\Delta_{23}] + \lambda^2[\Delta_{45}] \rangle = \langle \lambda^2(\alpha[\Delta_{23}] + [\Delta_{45}]) \rangle = \mathbf{W}_\alpha$. This completes the proof of the claim. Hence we get the following algebras:

$$\mathbf{B}_{6,1}^{\alpha \neq 0} : [e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = \alpha e_6, [e_4, e_5] = e_6.$$

Moreover, the algebras $\mathbf{B}_{6,1}^{\alpha \neq 0}$ and $\mathbf{B}_{6,1}^{\beta \neq 0}$ are isomorphic if and only if there is an $\lambda \in \mathbb{F}^*$ such that $\beta = \lambda^2\alpha$. So the number of non-isomorphic algebras among the family $\mathbf{B}_{6,1}^{\alpha \neq 0}$ is $|\mathbb{F}^* / (\mathbb{F}^*)^2|$.

- **CASE 2.** $\delta = 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & C_{45}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -C_{45}^{-1}C_{15} & -C_{45}^{-2}C_{25} & -C_{45}^{-1}C_{35} & C_{45}^{-1} & 0 \\ C_{45}^{-1}C_{14} & C_{45}^{-2}C_{24} & C_{45}^{-1}C_{34} & 0 & 1 \end{bmatrix}.$$

Then $\phi\mathbf{W} = \langle [\Delta_{45}] \rangle$. So we get the algebra:

$$\mathbf{B}_{6,1}^0 : [e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_4, e_5] = e_6.$$

1.4.2. The binary Lie algebras corresponding to the representatives of $\text{Aut}(\mathbf{M}_{5,1})$ -orbits on $\mathbf{U}_1(\mathbf{M}_{5,1})$. The automorphism group of $\mathbf{M}_{5,1}$ consists of invertible matrices of the form

$$\phi = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{11}a_{22} - a_{12}a_{21} & a_{34} & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ a_{51} & a_{52} & 0 & a_{54} & a_{44}(a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}.$$

Choose an arbitrary subspace $\mathbf{W} \in \mathbf{U}_1(\mathbf{M}_{5,1})$. From Table 2, such a subspace is spanned by

$$[\theta] = C_{13} [\Delta_{13}] + C_{14} [\Delta_{14}] + C_{23} [\Delta_{23}] + C_{24} [\Delta_{24}] + C_{45} [\Delta_{45}]$$

where $C_{45} \neq 0$. Let $\phi = (a_{ij}) \in \text{Aut}(\mathbf{M}_{5,1})$. Write

$$[\phi\theta] = C'_{13} [\Delta_{13}] + C'_{14} [\Delta_{14}] + C'_{23} [\Delta_{23}] + C'_{24} [\Delta_{24}] + C'_{45} [\Delta_{45}].$$

Then

$$\begin{aligned} C'_{13} &= (C_{13}a_{11} + C_{23}a_{21})(a_{11}a_{22} - a_{12}a_{21}), \\ C'_{14} &= C_{13}a_{11}a_{34} + C_{14}a_{11}a_{44} + C_{23}a_{21}a_{34} + C_{24}a_{21}a_{44} - C_{45}a_{51}a_{44}, \\ C'_{23} &= (C_{13}a_{12} + C_{23}a_{22})(a_{11}a_{22} - a_{12}a_{21}), \\ C'_{24} &= C_{13}a_{12}a_{34} + C_{14}a_{12}a_{44} + C_{23}a_{22}a_{34} + C_{24}a_{22}a_{44} - C_{45}a_{52}a_{44}, \\ C'_{45} &= C_{45}a_{44}^2(a_{11}a_{22} - a_{12}a_{21}). \end{aligned}$$

It is clear that if $C_{13} = C_{23} = 0$ then $C'_{13} = C'_{23} = 0$. From here,

$$\text{Orb}(\langle [\theta] : (C_{13}, C_{23}) = (0, 0) \rangle) \cap \text{Orb}(\langle [\theta] : (C_{13}, C_{23}) \neq (0, 0) \rangle) = \emptyset$$

and hence $\text{Aut}(\mathbf{M}_{5,1})$ has at least two orbits on $\mathbf{U}_1(\mathbf{M}_{5,1})$.

- **CASE 1.** $(C_{13}, C_{23}) = (0, 0)$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} C_{45}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & C_{45}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ C_{45}^{-2}C_{14} & C_{45}^{-1}C_{24} & 0 & 0 & C_{45}^{-1} \end{bmatrix}.$$

Then $\phi\mathbf{W} = \langle [\Delta_{45}] \rangle$. So we get the algebra:

$$\mathbf{B}_{6,2} : [e_1, e_2] = e_3, [e_3, e_4] = e_5, [e_4, e_5] = e_6.$$

- **CASE 2.** $(C_{13}, C_{23}) \neq (0, 0)$. Suppose first that $C_{13} \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} C_{13}C_{45}^{-1} & -C_{13}^{-4}C_{23}C_{45}^2 & 0 & 0 & 0 \\ 0 & C_{13}^{-3}C_{45}^2 & 0 & 0 & 0 \\ 0 & 0 & C_{13}^{-2}C_{45}^{-1}C_{45}^2 & 0 & 0 \\ 0 & 0 & 0 & C_{13}C_{45}^{-1} & 0 \\ C_{13}C_{14}C_{45}^{-2} & \phi_{25} & 0 & 0 & C_{13}^{-1} \end{bmatrix},$$

where

$$\phi_{25} = C_{24}C_{45}C_{13}^{-3} - C_{13}^{-4}C_{14}C_{23}C_{45}.$$

Then $\phi\mathbf{W} = \langle [\Delta_{13}] + [\Delta_{45}] \rangle$. Hence we get a representative $\langle [\Delta_{13}] + [\Delta_{45}] \rangle$. Assume now that $C_{13} = 0$. Then $C_{23} \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} 0 & -C_{23}^{-3}C_{45}^2 & 0 & 0 & 0 \\ C_{23}C_{45}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{23}^{-2}C_{45} & 0 & 0 \\ 0 & 0 & 0 & C_{23}C_{45}^{-1} & 0 \\ C_{23}C_{24}C_{45}^{-2} & -C_{14}C_{23}^{-3}C_{45} & 0 & 0 & C_{23}^{-1} \end{bmatrix}.$$

Then we get again a representative $\langle [\Delta_{13}] + [\Delta_{45}] \rangle$. This shows that if $(C_{13}, C_{23}) \neq (0, 0)$, then we get only one algebra:

$$\mathbf{B}_{6,3} : [e_1, e_2] = e_3, [e_3, e_4] = e_5, [e_1, e_3] = e_6, [e_4, e_5] = e_6.$$

2. THE GEOMETRIC CLASSIFICATION OF NILPOTENT BINARY LIE ALGEBRAS

2.1. Definitions and notation. Given an n -dimensional complex vector space \mathbf{V} , the set $\text{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V}) \cong \mathbf{V}^* \otimes \mathbf{V}^* \otimes \mathbf{V}$ is a vector space of dimension n^3 . This space has a structure of the affine variety \mathbb{C}^{n^3} . Fix a basis e_1, \dots, e_n of \mathbf{V} . Every $\mu \in \text{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V})$ is determined by the n^3 structure constants $c_{i,j}^k \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum_{k=1}^n c_{i,j}^k e_k$. A subset of

$\text{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V})$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables $c_{i,j}^k$ ($1 \leq i, j, k \leq n$).

Let T be a set of polynomial identities. All algebra structures on \mathbf{V} satisfying polynomial identities from T form a Zariski-closed subset of the variety $\text{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V})$. We denote this subset by $\mathbb{L}(T)$. The general linear group $\text{GL}(\mathbf{V})$ acts on $\mathbb{L}(T)$ by conjugations:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in \mathbf{V}, \mu \in \mathbb{L}(T) \subset \text{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V})$ and $g \in \text{GL}(\mathbf{V})$. Thus, $\mathbb{L}(T)$ is decomposed into $\text{GL}(\mathbf{V})$ -orbits that correspond to the isomorphism classes of algebras. Let $O(\mu)$ denote the orbit of $\mu \in \mathbb{L}(T)$ under the action of $\text{GL}(\mathbf{V})$ and let $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$.

Let \mathbf{A} and \mathbf{B} be two n -dimensional algebras satisfying identities from T and $\mu, \lambda \in \mathbb{L}(T)$ represent \mathbf{A} and \mathbf{B} respectively. We say that \mathbf{A} degenerates to \mathbf{B} and write $\mathbf{A} \rightarrow \mathbf{B}$ if $\lambda \in \overline{O(\mu)}$. In this case $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of μ or λ . If $\mathbf{A} \not\rightarrow \mathbf{B}$, then the assertion $\mathbf{A} \rightarrow \mathbf{B}$ is a *proper degeneration*. We write $\mathbf{A} \nrightarrow \mathbf{B}$ if $\lambda \notin \overline{O(\mu)}$.

Let \mathbf{A} be represented by $\mu \in \mathbb{L}(T)$. Then \mathbf{A} is *rigid* in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is an *irreducible component*. It is well known that every affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra \mathbf{A} is rigid in $\mathbb{L}(T)$ if and only if $\overline{O(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

2.2. Degenerations of algebras. We use the methods applied to Lie algebras in [6]. First of all, if $\mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{A} \not\rightarrow \mathbf{B}$, then $\dim \mathfrak{Der}(\mathbf{A}) < \dim \mathfrak{Der}(\mathbf{B})$, where $\mathfrak{Der}(\mathbf{A})$ is the Lie algebra of derivations of \mathbf{A} . We will compute the dimensions of algebras of derivations and will check the assertion $\mathbf{A} \rightarrow \mathbf{B}$ only for such \mathbf{A} and \mathbf{B} that $\dim \mathfrak{Der}(\mathbf{A}) < \dim \mathfrak{Der}(\mathbf{B})$.

To prove degenerations, we will construct families of matrices parametrized by t . Namely, let \mathbf{A} and \mathbf{B} be two algebras represented by the structures μ and λ from $\mathbb{L}(T)$ respectively. Let e_1, \dots, e_n be a basis of \mathbf{V} and let $c_{i,j}^k$ ($1 \leq i, j, k \leq n$) be the structure constants of λ in this basis.

If there exist $a_i^j(t) \in \mathbb{C}$ ($1 \leq i, j \leq n, t \in \mathbb{C}^*$) such that $E_i^t = \sum_{j=1}^n a_i^j(t)e_j$ ($1 \leq i \leq n$) form a basis of \mathbf{V} for every $t \in \mathbb{C}^*$, and the structure constants of μ in the basis E_1^t, \dots, E_n^t are polynomials $c_{i,j}^k(t) \in \mathbb{C}[t]$ such that $c_{i,j}^k(0) = c_{i,j}^k$, then $\mathbf{A} \rightarrow \mathbf{B}$. In this case E_1^t, \dots, E_n^t is a *parametrized basis* for $\mathbf{A} \rightarrow \mathbf{B}$.

2.3. The geometric classification of 6-dimensional nilpotent binary Lie algebras. The geometric classification of 6-dimensional nilpotent binary Lie algebras is based on the description of all degenerations of 6-dimensional Malcev algebras. Thanks to [12], the variety of 6-dimensional nilpotent Malcev algebras has only two irreducible components defined by the following algebras:

$$\begin{aligned} g_6 & : \begin{aligned} [e_1, e_2] &= e_3, & [e_1, e_3] &= e_4, & [e_1, e_4] &= e_5, \\ [e_2, e_3] &= e_5, & [e_2, e_5] &= e_6, & [e_3, e_4] &= -e_6, \end{aligned} \\ \mathbf{M}_6^\epsilon & : \begin{aligned} [e_1, e_2] &= e_3, & [e_1, e_3] &= e_5, & [e_1, e_5] &= e_6, \\ [e_2, e_4] &= \epsilon e_5, & [e_3, e_4] &= e_6. \end{aligned} \end{aligned}$$

The main result of the present section is the following theorem.

Theorem 9. *The variety of 6-dimensional nilpotent binary Lie algebras over \mathbb{C} has two irreducible components defined by the rigid algebras $\mathbf{B}_{6,3}$ and g_6 .*

Proof. Note that $\dim \mathfrak{Der}(\mathbf{B}_{6,3}) = \dim \mathfrak{Der}(g_6) = 8$ and there is no degeneration between these algebras. All other algebras degenerate to one of $\mathbf{B}_{6,3}$ and g_6 ; the latter algebras cannot degenerate to each other because of the dimensions of the derivation spaces; therefore there must be two components in which the orbits of the given algebras are open. Now we construct some degenerations to prove that all non-Lie Malcev and all non-Malcev binary Lie algebras lie in the irreducible component defined by the algebra $\mathbf{B}_{6,3}$.

- The parametrized basis formed by

$$\begin{aligned} E_1^t &= te_1 - ite_4, & E_2^t &= e_2 - \epsilon e_3 + (\epsilon^2 - \epsilon)ie_5, \\ E_3^t &= te_3 - i\epsilon te_5 + (\epsilon^2 - 2\epsilon)te_6, & E_4^t &= -t^2e_1, \\ E_5^t &= it^2e_5 + (1 - \epsilon)t^2e_6, & E_6^t &= t^3e_6 \end{aligned}$$

gives the degeneration $\mathbf{B}_{6,3} \rightarrow \mathbf{M}_6^\epsilon$.

- The parametrized basis formed by

$$E_1^t = te_1, \quad E_2^t = t^{-1}e_2, \quad E_3^t = e_3, \quad E_4^t = e_4, \quad E_5^t = e_5, \quad E_6^t = e_6$$

gives the degeneration $\mathbf{B}_{6,3} \rightarrow \mathbf{B}_{6,2}$.

- The parametrized basis formed by

$$\begin{aligned} E_1^t &= te_1 - e_3 - te_5, & E_2^t &= e_2 + te_5, & E_3^t &= e_3 - e_4, \\ E_4^t &= te_4, & E_5^t &= e_5, & E_6^t &= te_6 \end{aligned}$$

gives the degeneration $\mathbf{B}_{6,3} \rightarrow \mathbf{B}_{6,1}^1$.

- The parametrized basis formed by

$$\begin{aligned} E_1^t &= t^{-1}e_1, & E_2^t &= e_2, & E_3^t &= e_3, \\ E_4^t &= t^{-1}e_4, & E_5^t &= t^{-1}e_5, & E_6^t &= t^{-2}e_6 \end{aligned}$$

gives the degeneration $B_{6,1}^1 \rightarrow B_{6,1}^0$.

The listed degenerations imply that $B_{6,4} \rightarrow M_6^\epsilon, B_{6,1}^1, B_{6,1}^0, B_{6,2}$, and from the description of all degenerations of the Malcev part of this variety [12], we see that the variety of 6-dimensional nilpotent binary Lie algebras has only two irreducible components defined by $B_{6,3}$ and g_6 . \square

3. APPLICATION: CLASSIFICATION OF ANTICOMMUTATIVE \mathfrak{CD} -ALGEBRAS

The class of non-associative \mathfrak{CD} -algebras is defined by a certain property of Jordan and Lie algebras:

every commutator of two multiplication operators is a derivation.

Namely, an algebra A is a \mathfrak{CD} -algebra if and only if

$$T_x T_y - T_y T_x \in \mathfrak{Der}(A), \text{ for all } x, y \in A, T_z \in \{R_z, L_z\}.$$

It is easy to see that the class of \mathfrak{CD} -algebras is defined by three identities of degree 4. In the case of commutative and anticommutative \mathfrak{CD} -algebras, there is only one defined identity:

$$(3.1) \quad [[x, y], a], b] - [[[x, y], b], a] = \\ [[x, a], b], y] - [[[x, b], a], y] + [x, [[y, a], b]] - [x, [[y, b], a]].$$

If we set $a = y$ and $b = x$ in (3.1) then $[[[x, y], y], x] = [[[x, y], x], y]$. We conclude that every anticommutative \mathfrak{CD} -algebra is a binary Lie algebra. So the variety of anticommutative \mathfrak{CD} -algebras is between Lie and binary-Lie algebras. It is clear that if an algebra A satisfies $A^4 = 0$, then A is a \mathfrak{CD} -algebra. So every 5-dimensional nilpotent binary Lie algebra is a \mathfrak{CD} -algebra. By some easy checking of identity (3.1) for all 6-dimensional nilpotent binary Lie algebras, we obtain the following result.

Theorem 10. *Let A be a 6-dimensional nilpotent anticommutative \mathfrak{CD} -algebra over \mathbb{F} . Then A is isomorphic to a Malcev algebra or to $B_{6,1}^\alpha$. Every 6-dimensional nilpotent Malcev algebra over \mathbb{F} is a \mathfrak{CD} -algebra.*

As a corollary of Theorem 10, we obtain the following result.

Theorem 11. *The variety of 6-dimensional nilpotent anticommutative \mathfrak{CD} -algebras over \mathbb{C} has three irreducible components defined by the family of algebras M_6^ϵ and the rigid algebras $B_{6,1}^1, g_6$.*

Proof. Using the algebraic classification of 6-dimensional nilpotent anticommutative \mathfrak{CD} -algebras (Theorem 10), the geometric classification of 6-dimensional nilpotent binary Lie algebras (Theorem 9), and the description of all degenerations of 6-dimensional nilpotent Malcev algebras [12], we

obtain that g_6 is rigid. Recalling the degeneration $B_{6,1}^1 \rightarrow B_{6,1}^0$ we deduce that $B_{6,1}^1$ is rigid. Irreducible components defined by the family of Malcev algebras M_6^ϵ and the rigid algebra $B_{6,1}^1$ have the same dimension and they are different.

□

REFERENCES

- [1] Calderón A., Fernández Ouaridi A., Kaygorodov I., The classification of n -dimensional anticommutative algebras with $(n - 3)$ -dimensional annihilator, *Comm. Algebra*, 47 (2019), 1, 173–181.
- [2] De Graaf W., Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2, *J. Algebra*, 309 (2007), 2, 640–653.
- [3] De Graaf W., Classification of nilpotent associative algebras of small dimension, *Internat. J. Algebra Comput.*, 28 (2018), 1, 133–161.
- [4] Gainov A., Identical relations for binary Lie rings (Russian), *Uspehi Mat. Nauk N.S.*, 12 (1957), 3 (75), 141–146.
- [5] Gainov A., Binary Lie algebras of lower ranks (Russian), *Algebra i Logika Sem.*, 2 (1963), 4, 21–40.
- [6] Grunewald F., O’Halloran J., A Characterization of orbit closure and applications, *J. Algebra*, 116 (1988), 163–175.
- [7] Hegazi A., Abdelwahab H., Calderón Martín A., The classification of n -dimensional non-Lie Malcev algebras with $(n - 4)$ -dimensional annihilator, *Linear Algebra Appl.* 505 (2016), 32–56.
- [8] Hegazi A., Abdelwahab H., Calderón Martín A., Classification of nilpotent Malcev algebras of small dimensions over arbitrary fields of characteristic not 2, *Algebr. Represent. Theory*, 21 (2018), 1, 19–45.
- [9] Ismailov N., Kaygorodov I., Volkov Yu., The geometric classification of Leibniz algebras, *Internat. J. Math.*, 29 (2018), 5, 1850035.
- [10] Kaygorodov I., Pérez-Guillán P., Voronin V., The algebraic and geometric classification of nilpotent bicommutative algebras, *arXiv:1903.08997*
- [11] Kaygorodov I., Popov Yu., Pozhidaev A., Volkov Yu., Degenerations of Zinbiel and nilpotent Leibniz algebras, *Linear Multilinear Algebra*, 66 (2018), 4, 704–716.
- [12] Kaygorodov I., Popov Yu., Volkov Yu., Degenerations of binary-Lie and nilpotent Malcev algebras, *Comm. Algebra*, 46 (2018), 11, 4929–4941.
- [13] Kaygorodov I., Pozhidaev A., Saraiva P., On a ternary generalization of Jordan algebras, *Linear Multilinear Algebra*, 2018, DOI: 10.1080/03081087.2018.1443426.
- [14] Kaygorodov I., Volkov Yu., The variety of 2-dimensional algebras over an algebraically closed field, *Canad. J. Math.*, 2018, DOI: 10.4153/S0008414X18000056
- [15] Kuzmin E., Malcev algebras and their representations, *Algebra i Logika*, 7 (1968), 233–244.
- [16] Kuzmin E., Binary Lie algebras of small dimensions, *Algebra and Logic*, 37 (1998), 3, 181–186.
- [17] Malcev A., Analytic loops (Russian), *Mat. Sb. N.S.*, 36, (1955), 569–576.
- [18] Sagle A., Malcev algebras, *Trans. Amer. Math. Soc.*, 101 (1961), 426–458.
- [19] Sidorov A., Lie triple algebras, *Algebra i Logika*, 87 (1981), 72–78.
- [20] Skjelbred T., Sund T., Sur la classification des algèbres de Lie nilpotentes, *C. R. Acad. Sci. Paris Ser. A-B*, 286 (1978), 5, 241–242.