

# ON THE STRUCTURES OF HIVE ALGEBRAS AND TENSOR PRODUCT ALGEBRAS FOR GENERAL LINEAR GROUPS OF LOW RANK

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ABSTRACT. The tensor product algebra  $\mathrm{TA}(n)$  for the complex general linear group  $\mathrm{GL}(n)$ , introduced by Howe et al., describes the decomposition of tensor products of irreducible polynomial representations of  $\mathrm{GL}(n)$ . Using the hive model for the Littlewood-Richardson coefficients, we provide a finite presentation of the algebra  $\mathrm{TA}(n)$  for  $n = 2, 3, 4$  in terms of generators and relations, thereby giving a description of highest weight vectors of irreducible representations in the tensor products. We also compute the generating function of certain sums of Littlewood-Richardson coefficients.

## 1. INTRODUCTION

For the complex general linear group  $\mathrm{GL}(n) = \mathrm{GL}_n(\mathbb{C})$ , the group of  $n \times n$  invertible matrices over the field  $\mathbb{C}$  of complex numbers, we let  $V_n^\lambda$  denote its polynomial irreducible representation labeled by a Young diagram  $\lambda$ . The *Littlewood-Richardson (LR) coefficient* for  $\mathrm{GL}(n)$  is the multiplicity  $c_{\mu\nu}^\lambda$  of the irreducible representation  $V_n^\lambda$  occurring in the decomposition of the tensor product of two irreducible representations  $V_n^\mu$  and  $V_n^\nu$

$$(1.1) \quad V_n^\mu \otimes V_n^\nu = \bigoplus_{\lambda} \left( V_n^\lambda \right)^{\oplus c_{\mu\nu}^\lambda}.$$

Since the Schur polynomials in  $n$  variables are the characters of polynomial irreducible representations of  $\mathrm{GL}(n)$ , the LR coefficients can be also defined from the decomposition of the product of two Schur polynomials

$$s_n^\mu s_n^\nu = \sum_{\lambda} c_{\mu\nu}^\lambda s_n^\lambda.$$

The LR coefficients are usually described in terms of a family of combinatorial objects such as LR tableaux. For various combinatorial objects counting the LR coefficients, see, for example, [6, 21, 22, 24, 25]. A very different approach was proposed by Howe et al. in [10, 11, 12]. Using classical invariant theory, they constructed a multi-graded algebra, which we call *the  $\mathrm{GL}(n)$  tensor product algebra* and denote by  $\mathrm{TA}(n)$ . The dimension of its  $(\lambda, \mu, \nu)$ -homogeneous component  $\mathrm{TA}(n)_{\mu\nu}^\lambda$  is exactly the LR coefficient  $c_{\mu\nu}^\lambda$ , and for each space  $\mathrm{TA}(n)_{\mu\nu}^\lambda$  they gave a description of  $\mathbb{C}$ -basis elements labeled by LR tableaux on  $\lambda/\mu$  with content  $\nu$  (see [10, §2]). From a representation theoretic point of view, this approach provides a lot more refined information than many combinatorial ones in that the space  $\mathrm{TA}(n)_{\mu\nu}^\lambda$  consists of the highest weight vectors of the isomorphic copies of  $V_n^\lambda$  occurring in the decomposition of  $V_n^\mu \otimes V_n^\nu$ . From the existence of a finite SAGBI basis,

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it is also shown that the algebra  $\text{TA}(n)$  is a flat deformation of its initial algebra (see [11, §2.4-2.5]).

In this paper, for each  $n = 2, 3, 4$ , we give a finite presentation of the  $\text{GL}(n)$  tensor product algebra  $\text{TA}(n)$  in terms of generators and relations. This will provide a method, different from the one given in [10], to construct highest weight vectors appearing in the decomposition of the tensor products (1.1). An explicit example illustrating how to compute highest weight vectors corresponding to hives or LR tableaux is given in §3.2.3. Also, by applying a technique in projective algebraic geometry to the hive model for the LR coefficients, we compute a closed form formula for the series

$$\mathfrak{HP}_n(t) = \sum_{d \geq 0} m_d t^d$$

where  $m_d$  is the sum of LR coefficients  $c_{\mu\nu}^\lambda$  over the triples  $(\lambda, \mu, \nu)$  of Young diagrams such that  $d = |\lambda| = |\mu| + |\nu|$ .

We remark that for the special linear group  $G = \text{SL}_n(\mathbb{C})$  and its standard maximal unipotent subgroup  $U$ , Grosshans studied in [9] the algebra of the invariants of  $G$  acting on  $G/U \times G/U \times G/U$  by left translation. In particular, for  $n = 2, 3, 4$ , he gave the invariants and all relations among them, and showed how such results are related to the tensor product decomposition problem for the representations of  $G$ . See also [10, §4] and references therein for the related works on the case  $n = 4$ .

## 2. HIVE ALGEBRA AND ITS HP SERIES

In this section, we impose a monoid structure on the collection of hives for  $\text{GL}(n)$ , investigate the structure of the associated monoid algebra, and then compute its Hilbert-Poincare(HP) series. These results will be extended to the  $\text{GL}(n)$  tensor product algebras in §3.

**2.1. LR tableaux.** First let us recall that a LR tableau  $T$  for  $\text{GL}(n)$  is a filling of skew Young diagram with  $1, 2, \dots, n$  satisfying the semistandard condition and the Yamanouchi word, or reverse lattice word, condition. For example, the following tableaux

					1	1	1	1	1	1
			1	2	2	2				
	2	3	3	3						
2	4	4								

is a LR tableau for  $\text{GL}(4)$  on a skew Young diagram  $(11, 7, 5, 3)/(5, 3, 1, 0)$  with content  $(7, 5, 3, 2)$ . It is well known that for Young diagrams  $\lambda, \mu$ , and  $\nu$  having at most  $n$  rows, the number of LR tableaux on the skew Young diagram  $\lambda/\mu$  with content  $\nu$  is equal to the LR coefficient  $c_{\mu\nu}^\lambda$  for  $\text{GL}(n)$ . See, for example, [7, 12, 19, 23, 25].

**2.2. Hives.** Now we recall the hive model for the LR coefficients introduced by Knutson and Tao [16]. A sequence  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n) \in \mathbb{Z}^n$  is *non-negative dominant*, if

$$k_1 \geq k_2 \geq \dots \geq k_n \geq 0.$$

We write  $|\kappa|$  for the sum  $\kappa_1 + \dots + \kappa_n$ . Note that we can identify a non-negative dominant sequence  $\kappa$  with the Young diagram having  $\kappa_i$  boxes in its  $i$ -th row counting from top to bottom.



We remark that if  $|\lambda| \neq |\mu| + |\nu|$  then  $c_{\mu\nu}^\lambda = 0$ ; if  $(h_{ij})_{1 \leq j \leq i \leq n+1}$  is a hive for  $\mathrm{GL}(n)$  whose boundary is  $(\lambda, \mu, \nu)$  then we have

$$(2.1) \quad h_{n+1, n+1} = |\lambda| = |\mu| + |\nu|.$$

**2.3. Hive algebra and HP series.** Let us impose a monoid structure on the set of hives for  $\mathrm{GL}(n)$ . Note that for two hives  $h = (h_{ij})$  and  $h' = (h'_{ij})$  for  $\mathrm{GL}(n)$ , as elements of the free abelian group  $\mathbb{Z}^{(n+1)(n+2)/2}$ , the entries of their sum

$$h + h' = (h_{ij} + h'_{ij})_{1 \leq j \leq i \leq n+1}$$

also satisfy the rhombus conditions, and the boundary of their sum

$$(\underline{a}_h + \underline{a}_{h'}, \underline{b}_h + \underline{b}_{h'}, \underline{c}_h + \underline{c}_{h'})$$

is also non-negative dominant.

**Definition 2.3** (Hive cone and Hive algebra).

- (1) The hive cone  $\mathcal{H}(n)$  for  $\mathrm{GL}(n)$  is the submonoid of  $\mathbb{Z}^{(n+1)(n+2)/2}$  consisting of all the hives for  $\mathrm{GL}(n)$ .
- (2) The hive algebra  $\mathrm{HA}(n)$  for  $\mathrm{GL}(n)$  is the subalgebra of the polynomial algebra

$$\mathrm{HA}(n) \subset \mathbb{C}[z_{ij} : 1 \leq j \leq i \leq n+1]$$

over  $\mathbb{C}$  generated by the monomials  $\mathbf{z}^h = \prod_{i,j} z_{ij}^{h_{ij}}$  for all  $h = (h_{ij}) \in \mathcal{H}(n)$ .

From  $\mathbf{z}^h \cdot \mathbf{z}^{h'} = \mathbf{z}^{h+h'}$ , it follows that the hive algebra  $\mathrm{HA}(n)$  is isomorphic to the monoid algebra of the hive cone  $\mathcal{H}(n)$ . With the subspace  $\mathrm{HA}(n)_{\mu\nu}^\lambda$  of  $\mathrm{HA}(n)$  spanned by the monomials  $\mathbf{z}^h$  corresponding to all the hives  $h$  with boundary  $(\lambda, \mu, \nu)$ , the hive algebra  $\mathrm{HA}(n)$  is multi-graded by the triples  $(\lambda, \mu, \nu)$  of non-negative dominant sequences.

Using (2.1), we can also consider the  $\mathbb{Z}$ -grading structure of the hive algebra  $\mathrm{HA}(n)$  with respect to the degree of  $z_{n+1, n+1}$ . That is,

$$(2.2) \quad \mathrm{HA}(n) = \bigoplus_{(\lambda, \mu, \nu)} \mathrm{HA}(n)_{\mu\nu}^\lambda = \bigoplus_{d \geq 0} \bigoplus_{\substack{(\lambda, \mu, \nu): \\ |\lambda| = d}} \mathrm{HA}(n)_{\mu\nu}^\lambda.$$

With this  $\mathbb{Z}$ -grading, we can consider the HP series  $\mathfrak{HP}_n(t)$  of the hive algebra  $\mathrm{HA}(n)$

$$\mathfrak{HP}_n(t) = \sum_{d \geq 0} m_d t^d$$

where  $m_d$  is the dimension of the  $d$ -homogeneous space of  $\mathrm{HA}(n)$ , or equivalently, the number of hives with boundary  $(\lambda, \mu, \nu)$  such that  $|\lambda| = |\mu| + |\nu| = d$ . Since the number of hives with boundary  $(\lambda, \mu, \nu)$  is equal to the LR coefficient  $c_{\mu\nu}^\lambda$  and if  $|\lambda| \neq |\mu| + |\nu|$  then  $c_{\mu\nu}^\lambda = 0$ , we have

**Proposition 2.4.** The coefficient  $m_d$  of the HP series  $\mathfrak{HP}_n(t)$  of  $\mathrm{HA}(n)$  is the sum

$$m_d = \sum_{\substack{(\lambda, \mu, \nu): \\ |\lambda| = d}} c_{\mu\nu}^\lambda$$

of the LR coefficients  $c_{\mu\nu}^\lambda$  for  $\mathrm{GL}(n)$  over  $(\lambda, \mu, \nu)$  with  $d = |\lambda| = |\mu| + |\nu|$ .

Now for each  $n = 2, 3, 4$ , we give a finite presentation of the hive algebra  $\mathrm{HA}(n)$  and compute its HP series.

**Theorem 2.5** (Hive algebra for  $GL(2)$ ).

(1) *The hive algebra  $HA(2)$  is isomorphic to the polynomial algebra in five indeterminates.*

$$HA(2) \cong \mathbb{C}[x_1, x_2, \dots, x_5].$$

(2) *The HP series  $\mathfrak{H}\mathfrak{P}_2(t)$  of the hive algebra  $HA(2)$  is*

$$\begin{aligned} \mathfrak{H}\mathfrak{P}_2(t) &= \frac{1}{(1-t)^2(1-t^2)^3} \\ &= 1 + 2t + 6t^2 + 10t^3 + 20t^4 + 30t^5 + 50t^6 + 70t^7 + 105t^8 + 140t^9 + \dots \end{aligned}$$

*Proof.* For Statement (1), by a direct computation or by a software tool for computing lattice points, for example Normaliz [2], we can easily obtain the following Hilbert basis of the hive cone  $\mathcal{H}(2)$ .

$$\begin{aligned} h_1 &= \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & 0 & 1 & 2 \end{bmatrix}, \quad h_2 = \begin{bmatrix} & & 0 & \\ & 1 & & 1 \\ 1 & & 2 & 2 \end{bmatrix}, \quad h_3 = \begin{bmatrix} & & & 0 \\ & 1 & & 1 \\ 1 & & 1 & 1 \end{bmatrix}, \\ h_4 &= \begin{bmatrix} & & 0 & \\ & 1 & & 1 \\ 2 & & 2 & 2 \end{bmatrix}, \quad h_5 = \begin{bmatrix} & & & 0 \\ & 0 & & 1 \\ 0 & & 1 & 1 \end{bmatrix}. \end{aligned}$$

The monomials  $\mathbf{z}^{h_i} \in HA(2)$  corresponding to the generators  $h_i$  of  $\mathcal{H}(2)$

$$z_{22}z_{32}z_{33}^2, \quad z_{21}z_{22}z_{31}z_{32}^2z_{33}^2, \quad z_{21}z_{22}z_{31}z_{32}z_{33}, \quad z_{21}z_{22}z_{31}^2z_{32}^2z_{33}^2, \quad z_{22}z_{32}z_{33}$$

generate the algebra  $HA(2)$ . Since these generators are algebraically independent,  $HA(2)$  is isomorphic to the polynomial algebra in five indeterminates.

For Statement (2), with the grading (2.2), note that the degrees of the generators are

$$\deg(\mathbf{z}^{h_1}) = 2, \quad \deg(\mathbf{z}^{h_2}) = 2, \quad \deg(\mathbf{z}^{h_3}) = 1, \quad \deg(\mathbf{z}^{h_4}) = 2, \quad \deg(\mathbf{z}^{h_5}) = 1.$$

Then, the hive algebra  $HA(2)$  is isomorphic to the weighted polynomial ring corresponding to the weighted projective space  $\mathbb{P}(2, 2, 1, 2, 1)$ . Every monomial in  $\mathbb{C}[x_1, x_2, \dots, x_5]$  appears in

$$\prod_{i=1}^5 \frac{1}{1-x_i} = \sum_{(m_1, \dots, m_5)} x_1^{m_1} x_2^{m_2} \dots x_5^{m_5},$$

and therefore, by replacing  $x_i$  with  $t^{d_i}$  where  $d_i = \deg(\mathbf{z}^{h_i})$  for  $1 \leq i \leq 5$ , we obtain the HP series of  $HA(2)$ .  $\square$

We note that there are ten LR tableaux for  $GL(2)$  on skew diagrams whose outer diagrams have three boxes.

$$(2.3) \quad \begin{array}{cccccc} \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}, & \begin{array}{|c|c|} \hline & \\ \hline \end{array}, & \begin{array}{|c|c|c|} \hline & & 1 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline & \\ \hline 1 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline & 1 \\ \hline & \\ \hline \end{array}, \\ \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline & 1 \\ \hline 1 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline & 1 \\ \hline 2 \\ \hline \end{array}, & \begin{array}{|c|c|c|} \hline & & 1 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \hline \end{array}. \end{array}$$

As we observed in Proposition 2.4, this agrees with the coefficient of  $t^3$  in the above series  $\mathfrak{H}\mathfrak{P}_2(t)$ .

**Theorem 2.6** (Hive algebra for  $GL(3)$ ).

- (1) The hive algebra  $\text{HA}(3)$  is isomorphic to the quotient of the polynomial algebra in ten indeterminates  $x_i$  by the ideal generated by  $x_1x_6x_7 - x_5x_{10}$ .

$$\text{HA}(3) \cong \mathbb{C}[x_1, \dots, x_{10}] / \langle x_1x_6x_7 - x_5x_{10} \rangle.$$

- (2) The HP series  $\mathfrak{HP}_3(t)$  of  $\text{HA}(3)$  is

$$\begin{aligned} \mathfrak{HP}_3(t) &= \frac{1 - t^6}{(1 - t)^2(1 - t^2)^3(1 - t^3)^4(1 - t^4)} \\ &= 1 + 2t + 6t^2 + 14t^3 + 29t^4 + 56t^5 + 105t^6 + 182t^7 + 308t^8 + 502t^9 + \dots \end{aligned}$$

*Proof.* For Statement (1), first we recall that the hive cone  $\mathcal{H}(3)$  is the collection of all the integral points

$$(h_{11}, h_{21}, h_{22}, h_{31}, h_{32}, h_{33}, h_{41}, h_{42}, h_{43}, h_{44})$$

satisfying the conditions in Definition 2.1. Using a computer program such as Normaliz [2], it is straightforward to verify that the following ten elements form the Hilbert basis of the hive cone  $\mathcal{H}(3)$ :

$$\begin{aligned} h_1 &= (0, 0, 1, 0, 1, 1, 0, 1, 1, 1), & h_2 &= (0, 0, 1, 0, 1, 2, 0, 1, 2, 3), \\ h_3 &= (0, 0, 1, 0, 1, 2, 0, 1, 2, 2), & h_4 &= (0, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\ h_5 &= (0, 1, 1, 1, 2, 2, 1, 2, 2, 2), & h_6 &= (0, 1, 1, 1, 2, 2, 1, 2, 3, 3), \\ h_7 &= (0, 1, 1, 2, 2, 2, 2, 2, 2, 2), & h_8 &= (0, 1, 1, 2, 2, 2, 2, 3, 3, 3), \\ h_9 &= (0, 1, 1, 2, 2, 2, 3, 3, 3, 3), & h_{10} &= (0, 1, 2, 2, 3, 3, 2, 3, 4, 4). \end{aligned}$$

On the other hand, by computing the kernel of the map from the polynomial algebra  $\mathbb{C}[x_1, \dots, x_{10}]$  to the hive algebra  $\text{HA}(3)$  sending  $x_i$  to  $\mathbf{z}^{h_i}$  (by using a computer program such as Macaulay 2 [8]), we find the following basic relation:

$$h_1 + h_6 + h_7 = h_5 + h_{10} \quad \text{and therefore} \quad \mathbf{z}^{h_1} \cdot \mathbf{z}^{h_6} \cdot \mathbf{z}^{h_7} = \mathbf{z}^{h_5} \cdot \mathbf{z}^{h_{10}}$$

in  $\mathcal{H}(3)$  and  $\text{HA}(3)$  respectively.

For Statement (2), note that since the  $\mathbb{Z}$ -grading of the algebra  $\text{HA}(3)$  is given by the degree of  $z_{44}$  (see (2.2)), the degrees of our generators are

$$\begin{aligned} \deg(\mathbf{z}^{h_1}) &= 1, \quad \deg(\mathbf{z}^{h_2}) = 3, \quad \deg(\mathbf{z}^{h_3}) = 2, \quad \deg(\mathbf{z}^{h_4}) = 1, \quad \deg(\mathbf{z}^{h_5}) = 2, \\ \deg(\mathbf{z}^{h_6}) &= 3, \quad \deg(\mathbf{z}^{h_7}) = 2, \quad \deg(\mathbf{z}^{h_8}) = 3, \quad \deg(\mathbf{z}^{h_9}) = 3, \quad \deg(\mathbf{z}^{h_{10}}) = 4. \end{aligned}$$

Then, the hive algebra  $\text{HA}(3)$  is isomorphic to the ring of a hypersurface determined by the homogeneous polynomial  $f = x_1x_6x_7 - x_5x_{10}$  in the weighted projective space

$$\mathbb{P}(1, 3, 2, 1, 2, 3, 2, 3, 3, 4).$$

Using the same argument given in the proof of Theorem 2.5 (2), we can compute the HP series of the above weighted projective space, which is

$$H(t) = \frac{1}{(1 - t)^2(1 - t^2)^3(1 - t^3)^4(1 - t^4)}.$$

Next, we note that the degree of  $f$  is 6. By comparing the space of the degree  $d$  elements in  $\mathbb{C}[x_1, \dots, x_{10}]$  and the space of the degree  $d$  elements of the form  $f \cdot g$  for  $g \in \mathbb{C}[x_1, \dots, x_{10}]$ , we obtain the HP series of the quotient  $\mathbb{C}[x_1, \dots, x_{10}] / \langle f \rangle$

$$(1 - t^6) \cdot H(t),$$

which is the HP series of  $\text{HA}(3)$  in the statement. One can also use some general results on the HP series of a graded ring corresponding to a complete intersection in a weighted projective space. See for example [5, §3.4].  $\square$

There are fourteen LR tableaux for  $\text{GL}(3)$  on skew diagrams whose outer diagrams have three boxes.

$$(2.4) \quad \begin{array}{cccccccc} \boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{\phantom{0}}, & \boxed{\phantom{0}}\boxed{\phantom{0}}, & \boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{\phantom{0}}, & \boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{1}, & \boxed{1}\boxed{\phantom{0}}\boxed{\phantom{0}}, & \boxed{\phantom{0}}\boxed{1}\boxed{\phantom{0}}, & \boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{1}, \\ \boxed{\phantom{0}}\boxed{1}\boxed{1}, & \boxed{\phantom{0}}\boxed{1}\boxed{1}, & \boxed{1}\boxed{1}\boxed{1}, & \boxed{1}\boxed{1}\boxed{1}, & \boxed{1}\boxed{1}\boxed{1}, & \boxed{1}\boxed{1}\boxed{1}, & \boxed{1}\boxed{2}\boxed{3}. \end{array}$$

As in Proposition 2.4, this agrees with the coefficient of  $t^3$  in  $\mathfrak{HP}_3(t)$ .

**Theorem 2.7** (Hive algebra for  $\text{GL}(4)$ ).

- (1) *The hive algebra  $\text{HA}(4)$  is isomorphic to the quotient of the polynomial algebra  $\mathbb{C}[x_1, \dots, x_{20}]$  by the ideal generated by the following fifteen basic relations*

$$\begin{array}{ll} r_1 : x_1x_7x_9 - x_6x_{15}, & r_2 : x_1x_8x_9 - x_6x_{16}, \\ r_3 : x_1x_{11}x_{12} - x_{10}x_{18}, & r_4 : x_6x_{11}x_{12} - x_{10}x_{20}, \\ r_5 : x_2x_8x_{10} - x_7x_{17}, & r_6 : x_2x_8x_{12} - x_7x_{19}, \\ r_7 : x_6x_{18} - x_1x_{20}, & r_8 : x_7x_{16} - x_8x_{15}, \\ r_9 : x_{10}x_{19} - x_{12}x_{17}, & r_{10} : x_{15}x_{17} - x_2x_{10}x_{16}, \\ r_{11} : x_{15}x_{19} - x_2x_{12}x_{16}, & r_{12} : x_{15}x_{20} - x_7x_9x_{18}, \\ r_{13} : x_{16}x_{20} - x_8x_9x_{18}, & r_{14} : x_{17}x_{20} - x_6x_{11}x_{19}, \\ r_{15} : x_{17}x_{18} - x_1x_{11}x_{19}. & \end{array}$$

- (2) *The HP series of  $\text{HA}(4)$  is  $\mathfrak{HP}_4(t) = f(t)/g(t)$  where*

$$\begin{aligned} f(t) &= 1 - 2t - 2t^2 + 10t^3 - 2t^4 - 24t^5 + 22t^6 + 32t^7 - 54t^8 \\ &\quad - 18t^9 + 80t^{10} - 14t^{11} - 72t^{12} + 34t^{13} + 44t^{14} - 18t^{15} - 25t^{16} \\ &\quad - 18t^{17} + 44t^{18} + 34t^{19} - 72t^{20} - 14t^{21} + 80t^{22} - 18t^{23} - 54t^{24} \\ &\quad + 32t^{25} + 22t^{26} - 24t^{27} - 2t^{28} + 10t^{29} - 2t^{30} - 2t^{31} + t^{32}; \\ g(t) &= (1-t)^4(1-t^2)^6(1-t^{12})^4, \end{aligned}$$

which is

$$\mathfrak{HP}_4(t) = 1 + 2t + 6t^2 + 14t^3 + 34t^4 + 68t^5 + 142t^6 + 268t^7 + 508t^8 + 902t^9 + \dots$$

We will verify these statements with the aid of a computer. The Normaliz codes we used are given in §4. First, let us recall that the hive monoid  $\mathcal{H}(4)$  for  $\text{GL}(4)$  is the collection of all the triangular arrays of integers in Figure 1 or the integral points

$$(h_{11}, h_{21}, h_{22}, h_{31}, h_{32}, h_{33}, h_{41}, h_{42}, h_{43}, h_{44}, h_{51}, h_{52}, h_{53}, h_{54}, h_{55})$$

satisfying the conditions in Definition 2.1. Then, we can compute the Hilbert basis of the hive cone  $\mathcal{H}(4)$  by using Normaliz [2]:

$$\begin{aligned}
h_1 &= (0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1), & h_2 &= (0, 0, 1, 0, 1, 2, 0, 1, 2, 2, 0, 1, 2, 2, 2), \\
h_3 &= (0, 0, 1, 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 3), & h_4 &= (0, 0, 1, 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 4), \\
h_5 &= (0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), & h_6 &= (0, 1, 1, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2), \\
h_7 &= (0, 1, 1, 1, 2, 2, 1, 2, 3, 3, 1, 2, 3, 3, 3), & h_8 &= (0, 1, 1, 1, 2, 2, 1, 2, 3, 3, 1, 2, 3, 4, 4), \\
h_9 &= (0, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2), & h_{10} &= (0, 1, 1, 2, 2, 2, 2, 3, 3, 3, 2, 3, 3, 3, 3), \\
h_{11} &= (0, 1, 1, 2, 2, 2, 3, 3, 3, 2, 3, 4, 4, 4, 4), & h_{12} &= (0, 1, 1, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3), \\
h_{13} &= (0, 1, 1, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4), & h_{14} &= (0, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4), \\
h_{15} &= (0, 1, 2, 2, 3, 3, 2, 3, 4, 4, 2, 3, 4, 4, 4), & h_{16} &= (0, 1, 2, 2, 3, 3, 2, 3, 4, 4, 2, 3, 4, 5, 5), \\
h_{17} &= (0, 1, 2, 2, 3, 4, 2, 4, 5, 5, 2, 4, 5, 6, 6), & h_{18} &= (0, 1, 2, 2, 3, 3, 3, 4, 4, 4, 3, 4, 5, 5, 5), \\
h_{19} &= (0, 1, 2, 2, 3, 4, 3, 4, 5, 5, 3, 4, 5, 6, 6), & h_{20} &= (0, 2, 2, 3, 4, 4, 4, 5, 5, 5, 4, 5, 6, 6, 6).
\end{aligned}$$

Now, by computing, with for example Macaulay 2 [8], the kernel of the map from the polynomial algebra  $\mathbb{C}[x_1, \dots, x_{20}]$  to the hive algebra  $\text{HA}(4)$  sending  $x_i$  to  $\mathbf{z}^{h_i}$ , we obtain the following 15 basic relations:

$$\begin{aligned}
r'_1 : h_1 + h_7 + h_9 &= h_6 + h_{15}, & r'_2 : h_1 + h_8 + h_9 &= h_6 + h_{16}, \\
r'_3 : h_1 + h_{11} + h_{12} &= h_{10} + h_{18}, & r'_4 : h_6 + h_{11} + h_{12} &= h_{10} + h_{20}, \\
r'_5 : h_2 + h_8 + h_{10} &= h_7 + h_{17}, & r'_6 : h_2 + h_8 + h_{12} &= h_7 + h_{19}, \\
r'_7 : h_6 + h_{18} &= h_1 + h_{20}, & r'_8 : h_7 + h_{16} &= h_8 + h_{15}, \\
r'_9 : h_{10} + h_{19} &= h_{12} + h_{17}, & r'_{10} : h_{15} + h_{17} &= h_2 + h_{10} + h_{16}, \\
r'_{11} : h_{15} + h_{19} &= h_2 + h_{12} + h_{16}, & r'_{12} : h_{15} + h_{20} &= h_7 + h_9 + h_{18}, \\
r'_{13} : h_{16} + h_{20} &= h_8 + h_9 + h_{18}, & r'_{14} : h_{17} + h_{20} &= h_6 + h_{11} + h_{19}, \\
r'_{15} : h_{17} + h_{18} &= h_1 + h_{11} + h_{19}.
\end{aligned}$$

With (2.2), the degrees of the generators are given by  $h_{55}$ , and therefore we have

$$\begin{aligned}
\deg(\mathbf{z}^{h_1}) &= 1, \deg(\mathbf{z}^{h_2}) = 2, \deg(\mathbf{z}^{h_3}) = 3, \deg(\mathbf{z}^{h_4}) = 4, \deg(\mathbf{z}^{h_5}) = 1, \\
\deg(\mathbf{z}^{h_6}) &= 2, \deg(\mathbf{z}^{h_7}) = 3, \deg(\mathbf{z}^{h_8}) = 4, \deg(\mathbf{z}^{h_9}) = 2, \deg(\mathbf{z}^{h_{10}}) = 3, \\
\deg(\mathbf{z}^{h_{11}}) &= 4, \deg(\mathbf{z}^{h_{12}}) = 3, \deg(\mathbf{z}^{h_{13}}) = 4, \deg(\mathbf{z}^{h_{14}}) = 4, \deg(\mathbf{z}^{h_{15}}) = 4, \\
\deg(\mathbf{z}^{h_{16}}) &= 5, \deg(\mathbf{z}^{h_{17}}) = 6, \deg(\mathbf{z}^{h_{18}}) = 5, \deg(\mathbf{z}^{h_{19}}) = 6, \deg(\mathbf{z}^{h_{20}}) = 6.
\end{aligned}$$

Now by using the software Normaliz [2], we can compute the HP series of the hive cone  $\mathcal{H}(4)$ , which is  $f(t)/g(t)$  given in the statement. See §4 for more details.

### 3. GENERATORS OF HIGHEST WEIGHT VECTORS

In this section, we first review the construction of the  $\text{GL}(n)$  tensor product algebra and its properties given in [10, 11]. Then, for each  $n = 2, 3, 4$ , we give a finite presentation of the algebra  $\text{TA}(n)$  in terms of generators and relations. This will give a method to construct highest weight vectors appearing in the decomposition of the tensor product of two irreducible polynomial representations of  $\text{GL}(n)$ .

**3.1.  $\mathrm{GL}(n)$  tensor product algebra.** Let us recall the  $\mathrm{GL}(n)$  tensor product algebra<sup>1</sup> introduced by Howe et al. in [10, 11]. See also [12, 18]. We consider two copies of the space  $M_n$  of  $n \times n$  complex matrices, and use the coordinates  $(x_{ij})$  and  $(y_{ij})$  respectively. Therefore, a typical element  $(m_1, m_2)$  in the space  $M_n \oplus M_n$  is

$$\begin{bmatrix} x_{11} & \cdots & x_{1n} & y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} & y_{n1} & \cdots & y_{nn} \end{bmatrix}.$$

The three copies of  $\mathrm{GL}(n)$  act on the algebra  $\mathbb{C}[M_n \oplus M_n]$  of polynomials on  $M_n \oplus M_n$  as follows:

$$(g \cdot f)(m_1, m_2) = f(g_1^t m_1 g_2, g_1^t m_2 g_3)$$

for  $g = (g_1, g_2, g_3) \in \mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(n)$  and  $f \in \mathbb{C}[M_n \oplus M_n]$ .

We write  $U_n$  for the maximal unipotent subgroup of  $\mathrm{GL}(n)$  consisting of all the upper triangular matrices with 1's on the diagonal, and  $A_n$  for the maximal torus of  $\mathrm{GL}(n)$  consisting of all the diagonal matrices.

**Definition 3.1.** *The  $\mathrm{GL}(n)$  tensor product algebra  $\mathrm{TA}(n)$  is the algebra of polynomials in  $\mathbb{C}[M_n \oplus M_n]$  invariant under the subgroup  $U_n \times U_n \times U_n$  of  $\mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(n)$*

$$\begin{aligned} \mathrm{TA}(n) &= \mathbb{C}[M_n \oplus M_n]^{U_n \times U_n \times U_n} \\ &= \{f \in \mathbb{C}[M_n \oplus M_n] : g \cdot f = f \text{ for all } g \in U_n \times U_n \times U_n\}. \end{aligned}$$

Since  $A_n$  normalizes  $U_n$ , the space  $\mathrm{TA}(n)$  consists of weight vectors under the action of  $A_n \times A_n \times A_n$ . In [10], it is shown that, as an algebra,  $\mathrm{TA}(n)$  is graded by the triples  $(\lambda, \mu, \nu)$  of non-negative dominant sequences; the  $(\lambda, \mu, \nu)$ -homogeneous component  $\mathrm{TA}(n)_{\mu\nu}^\lambda$  consists of the highest weight vectors of the isomorphic copies of  $V_n^\lambda$  occurring in the decomposition of the tensor product  $V_n^\mu \otimes V_n^\nu$ ; the dimension of the space  $\mathrm{TA}(n)_{\mu\nu}^\lambda$  is equal to the LR coefficient  $c_{\mu\nu}^\lambda$ . Moreover, for each space  $\mathrm{TA}(n)_{\mu\nu}^\lambda$ , they constructed explicit  $\mathbb{C}$ -basis elements  $f_T$  associated with LR tableaux  $T$  on  $\lambda/\mu$  with content  $\nu$ .

**Lemma 3.2.** [10, 11]

- (1) *There is a  $\mathbb{C}$ -basis for the space  $\mathrm{TA}(n)_{\mu\nu}^\lambda$*

$$\mathcal{B}_n(\lambda, \mu, \nu) = \{f_T \in \mathrm{TA}(n) : \text{LR tableaux } T \text{ on } \lambda/\mu \text{ with content } \nu\}$$

*such that (with respect to a certain monomial order) the initial monomial  $\mathrm{in}(f_T)$  of  $f_T$  is*

$$\mathrm{in}(f_T) = \prod_i x_{ii}^{\mu_i} \cdot \prod_{i,j} y_{ij}^{t_{ij}} \in \mathbb{C}[M_n \oplus M_n]$$

*where  $t_{ij}$  is the number of boxes in the  $i$ th row of  $T$  containing  $j$ . In particular, the initial monomials of these basis elements are distinct.*

- (2) *The initial algebra  $\mathrm{in}(\mathrm{TA}(n))$  of  $\mathrm{TA}(n)$  is generated by  $\mathrm{in}(f_T)$  for all the LR tableaux  $T$  for  $\mathrm{GL}(n)$ .*

$$\begin{aligned} \mathrm{in}(\mathrm{TA}(n)) &= \{\mathrm{in}(f) : f \in \mathrm{TA}(n)\} \\ &= \{\mathrm{in}(f_T) : \text{for all the LR tableaux } T \text{ for } \mathrm{GL}(n)\}. \end{aligned}$$

<sup>1</sup>Howe et al. investigated a family of algebras parameterized by three positive integers  $n$ ,  $p$ , and  $q$ . In this paper, we will focus on the case  $n = p = q$  related to the most general form of the LR coefficients.

From this, it is also shown that there is a flat one parameter family of algebras whose general fiber is  $\text{TA}(n)$  and special fiber is  $\text{in}(\text{TA}(n))$ . See [10, 11] for more details.

**Proposition 3.3.** *The initial algebra  $\text{in}(\text{TA}(n))$  of  $\text{TA}(n)$  is isomorphic to the hive algebra  $\text{HA}(n)$ .*

*Proof.* One can easily show that the bijection between hives  $h$  and LR tableaux  $T$  (Lemma 2.2) and the bijection between LR tableaux  $T$  and initial monomials  $\text{in}(f_T)$  of basis elements  $f_T$  (Lemma 3.2) give rise to a monoid isomorphism between  $\mathcal{H}(n)$  and  $\text{in}(\text{TA}(n))$ . Alternatively, one can use the similar result given in terms of LR triangles in [11, §2.4] combined with a bijection between hives and LR triangles [21].  $\square$

**3.2. Presentation of  $\text{TA}(n)$ .** Now, we investigate a finite presentation of the  $\text{GL}(n)$  tensor product algebra, thereby giving a way to express highest weight vectors with generators of the algebra  $\text{TA}(n)$ .

The following notation will be useful to describe our generators of the algebra  $\text{TA}(n)$ . It will also make it easier to notice the relation between the products of generators and the LR tableaux corresponding to them.

**Notation 3.4.** *The determinant of the  $(i+k) \times (i+k)$  submatrix of*

$$\begin{bmatrix} x_{11} & \cdots & x_{1n} & y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} & y_{n1} & \cdots & y_{nn} \end{bmatrix}$$

*obtained by choosing rows  $1, 2, \dots, i+k$  and columns  $1, 2, \dots, i, n+j_1, n+j_2, \dots, n+j_k$  will be denoted by a script-sized Young tableau with a single column having  $i$  empty boxes followed by boxes with entries  $j_\ell$ 's for  $1 \leq j_1 < j_2 < \dots < j_k \leq n$ . Then, its initial monomial is  $\prod_{a=1}^i x_{aa} \prod_{b=1}^k y_{bj_b}$  with respect to a diagonal term order (see [20]).*

With this notation, the product of column tableaux is the product of the corresponding determinants. For example,

$$\begin{array}{|c|} \hline \square \\ \hline p \\ \hline q \\ \hline \end{array} \cdot \begin{array}{|c|} \hline r \\ \hline s \\ \hline \end{array} \cdot \square = \det \begin{bmatrix} x_{11} & y_{1p} & y_{1q} \\ x_{21} & y_{2p} & y_{2q} \\ x_{31} & y_{3p} & y_{3q} \end{bmatrix} \times \det \begin{bmatrix} y_{1r} & y_{1s} \\ y_{2r} & y_{2s} \end{bmatrix} \times \det [x_{11}]$$

and its initial monomial is  $x_{11}^2 y_{2p} y_{3p} y_{1r} y_{2s}$ .

**3.2.1. Tensor product algebra for  $\text{GL}(2)$ .** Now we give a finite presentation of the tensor product algebra for  $\text{GL}(2)$ .

**Theorem 3.5.** *The  $\text{GL}(2)$  tensor product algebra  $\text{TA}(2)$  is generated by*

$$g_1 = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \quad g_2 = \begin{array}{|c|} \hline \square \\ \hline 1 \\ \hline \end{array}, \quad g_3 = \square, \quad g_4 = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad g_5 = \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

*and these generators are algebraically independent. Therefore,*

$$\text{TA}(2) \cong \mathbb{C}[z_1, z_2, \dots, z_5].$$

*Proof.* We first note that all these determinants  $g_i$  are invariant under  $U_n \times U_n \times U_n$  and therefore they belong to the tensor product algebra  $\text{TA}(2)$ . We also note that they are algebraically independent. Computing their weights under  $A_n \times A_n \times A_n$ , we see that  $g_i$  is a highest weight vector of  $V_2^\lambda$  in the decomposition of  $V_2^\mu \otimes V_2^\nu$  where

$i$	$\lambda$	$\mu$	$\nu$
1	(1, 1)	(0, 0)	(1, 1)
2	(1, 1)	(1, 0)	(1, 0)
3	(1, 0)	(1, 0)	(0, 0)
4	(1, 1)	(1, 1)	(0, 0)
5	(1, 0)	(0, 0)	(1, 0)

Because these tensor products are multiplicity free, our highest weight vector of  $V_2^\lambda$  in the decomposition of  $V_2^\mu \otimes V_2^\nu$  is unique up to constant. We note that the column tableaux labeling the determinants  $g_i$  are LR tableaux for  $GL(2)$ . In fact, one can check that  $g_i$  is equal to  $f_{T_i}$  given in Lemma 3.2 where  $T_i$  is the tableau labeling  $g_i$  by Notation 3.4. Moreover, these LR tableaux  $T_i$  correspond to the Hilbert basis elements  $h_i$  listed in the proof of Theorem 2.5 by Lemma 2.2. Since  $h_i$  generate the hive cone  $\mathcal{H}(2)$ , by Proposition 3.3, the monomials  $in(g_i)$  generate the initial algebra of  $TA(2)$ . Therefore  $g_i$  generate the algebra  $TA(2)$ .  $\square$

We remark that, with the tableau notation of determinants (Notation 3.4), every LR tableau for  $GL(2)$  can be matched with a concatenation of the tableaux for  $g_i$ . For example, the LR tableaux listed in (2.3) can be matched with the concatenation of the tableaux labeling  $g_i$ , or the following products of  $g_i$ 's:

$$g_3^3, g_3g_4, g_3^2g_5, g_2g_3, g_4g_5, \\ g_3g_5^2, g_2g_5, g_1g_3, g_5^3, g_1g_5.$$

Here we remark that  $g_1g_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \square$  can be rewritten as

$$(3.1) \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \square = \begin{bmatrix} \square \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \end{bmatrix} - \begin{bmatrix} 1 \\ \square \end{bmatrix} \cdot \begin{bmatrix} 2 \end{bmatrix},$$

by using the polynomial identity

$$(3.2) \quad \det \begin{bmatrix} y_{11} & y_{12} \\ y_{22} & y_{22} \end{bmatrix} \cdot x_{11} = \det \begin{bmatrix} x_{11} & y_{12} \\ x_{21} & y_{22} \end{bmatrix} \cdot y_{11} - \det \begin{bmatrix} x_{11} & y_{11} \\ x_{21} & y_{21} \end{bmatrix} \cdot y_{12}.$$

Note that in (3.1), we obtain a LR tableau by aligning the columns  $\begin{bmatrix} \square \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \end{bmatrix}$  to the top, and this LR tableau is the one matched with the product  $g_1g_3$ .

3.2.2. *Tensor product algebra for  $GL(3)$ .* Next we describe the  $GL(3)$  tensor product algebra in terms of generators and relations.

**Theorem 3.6.** *The  $GL(3)$  tensor product algebra  $TA(3)$  is generated by*

$$g_1 = \begin{bmatrix} 1 \\ \square \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad g_4 = \square, \quad g_5 = \begin{bmatrix} \square \\ 1 \end{bmatrix}, \\ g_6 = \begin{bmatrix} \square \\ 1 \\ 2 \end{bmatrix}, \quad g_7 = \begin{bmatrix} \square \\ \square \end{bmatrix}, \quad g_8 = \begin{bmatrix} \square \\ \square \\ 1 \end{bmatrix}, \quad g_9 = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}, \quad g_{10} = \begin{bmatrix} \square \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \end{bmatrix} - \begin{bmatrix} 1 \\ \square \end{bmatrix} \cdot \begin{bmatrix} 2 \end{bmatrix},$$

and there is one basic relation  $g_1g_6g_7 - g_5g_{10} + g_3g_4g_8 = 0$  among them. Therefore,

$$TA(3) \cong \mathbb{C}[z_1, z_2, \dots, z_{10}] / \langle z_1z_6z_7 - z_5z_{10} + z_3z_4z_8 \rangle.$$

*Proof.* We note that all these polynomials  $g_i$  are invariant under the action of  $U_n \times U_n \times U_n$ , and therefore they belong to the tensor product algebra  $\text{TA}(3)$ . Computing their weights under  $A_n \times A_n \times A_n$ , we see that  $g_i$  is a highest weight vector of  $V_3^\lambda$  in the decomposition of  $V_3^\mu \otimes V_3^\nu$  where

$i$	$\lambda$	$\mu$	$\nu$
1	(1, 0, 0)	(0, 0, 0)	(1, 0, 0)
2	(1, 1, 1)	(0, 0, 0)	(1, 1, 1)
3	(1, 1, 0)	(0, 0, 0)	(1, 1, 0)
4	(1, 0, 0)	(1, 0, 0)	(0, 0, 0)
5	(1, 1, 0)	(1, 0, 0)	(1, 0, 0)
6	(1, 1, 1)	(1, 0, 0)	(1, 1, 0)
7	(1, 1, 0)	(1, 1, 0)	(0, 0, 0)
8	(1, 1, 1)	(1, 1, 0)	(1, 0, 0)
9	(1, 1, 1)	(1, 1, 1)	(0, 0, 0)
10	(2, 1, 1)	(1, 1, 0)	(1, 1, 0)

In each case, since  $V_3^\mu$  or  $V_3^\nu$  is a fundamental representation of  $\text{GL}(3)$ , the tensor product  $V_3^\mu \otimes V_3^\nu$  is multiplicity free. Hence, our highest weight vectors are unique up to constant. Let  $T_i$  be the column tableau labeling the determinant  $g_i$  for  $1 \leq i \leq 9$  by Notation 3.4. For  $g_{10}$ , we let  $T_{10}$  be the concatenation of two columns in the first term of  $g_{10}$ . Then, all these tableaux  $T_i$  are LR tableaux, and  $g_i$  is indeed equal to  $f_{T_i}$  given in Lemma 3.2. Moreover, these LR tableaux, by Lemma 2.2, correspond to the Hilbert basis elements  $h_i$  listed in the proof of Theorem 2.6. Since these  $h_i$  generate the hive cone  $\mathcal{H}(3)$ , by Proposition 3.3, the monomials  $\text{in}(g_i)$  generate the initial algebra of  $\text{TA}(3)$ . Therefore, the polynomials  $g_i$  generate the algebra  $\text{TA}(3)$ .

Now, for the basic relation between these generators, by lifting the relation between hives obtained in Theorem 2.6, we can find the polynomial identity in the statement.  $\square$

We remark that using the above result every LR tableau for  $\text{GL}(3)$  can be matched with a concatenation of the LR tableaux  $T_i$  associated with  $g_i$ , or simply the products of  $g_i$ 's. For example, the LR tableaux for  $\text{GL}(3)$  listed in (2.4) can be matched with

$$g_4^3, g_4g_7, g_9, g_1g_4^2, g_4g_5, g_1g_7, g_8, \\ g_1^2g_4, g_1g_5, g_3g_4, g_6, g_1^3, g_1g_3, g_2.$$

Here, as in (3.1), the product  $g_3g_4 = \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$  is equal to  $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ , and we see that the first term in the new expression indeed matches with the LR tableau  $T_1$  below once the columns  $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$  and  $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$  are top-aligned.

$$T_1 = \begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix} \quad \text{and} \quad T_2 = \begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix}$$

Similarly, the product  $g_6g_7 = \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$  is equal to  $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$  by the corresponding polynomial identity, and therefore  $g_6g_7$  can be matched with the LR tableau  $T_2$  when

the columns  $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$  and  $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$  in the new expression of  $g_6g_7$  are top-aligned.

3.2.3. *Example.* For  $\text{GL}(3)$ , if  $\mu = \nu = (2, 1, 0)$  and  $\lambda = (3, 2, 1)$ , then there are exactly two copies of  $V_3^\lambda$  occurring in the decomposition of the tensor product  $V_3^\mu \otimes V_3^\nu$ . That is, the LR coefficient  $c_{\mu\nu}^\lambda$  is 2.

This multiplicity can be computed by counting all the LR tableaux on skew diagram  $\lambda/\mu$  with content  $\nu$ . They are

$$T = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \\ \hline 1 & & \\ \hline \end{array} \quad \text{and} \quad T' = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline 2 & & \\ \hline \end{array}.$$

Alternatively, we can count all the hives with boundary  $(\lambda, \mu, \nu)$ .

$$h = \begin{bmatrix} & & 0 & & \\ & & 2 & 3 & \\ & 3 & 4 & 5 & \\ 3 & 5 & 6 & 6 & \end{bmatrix} \quad \text{and} \quad h' = \begin{bmatrix} & & 0 & & \\ & & 2 & 3 & \\ & 3 & 5 & 5 & \\ 3 & 5 & 6 & 6 & \end{bmatrix}.$$

To find explicit highest weight vectors of the isomorphic copies of  $V_3^\lambda$  corresponding to  $h$  and  $h'$  (or  $T$  and  $T'$ ) using our generators  $g_1, \dots, g_{10}$  of  $\text{TA}(3)$ , we first express the hives  $h$  and  $h'$  with our Hilbert basis of the cone  $\mathcal{H}(3)$

$$h = h_3 + h_4 + h_8 \quad \text{and} \quad h' = h_1 + h_6 + h_7,$$

which implies

$$\mathbf{z}^h = \mathbf{z}^{h_3} \cdot \mathbf{z}^{h_4} \cdot \mathbf{z}^{h_8} \quad \text{and} \quad \mathbf{z}^{h'} = \mathbf{z}^{h_1} \cdot \mathbf{z}^{h_6} \cdot \mathbf{z}^{h_7}$$

in the hive algebra  $\text{HA}(3) \cong \text{in}(\text{TA}(n))$ . By lifting these expressions to the tensor product algebra  $\text{TA}(4)$ , we obtain the highest weight vectors corresponding to  $h$  and  $h'$  respectively.

$$(3.3) \quad \begin{aligned} f &= g_3 \cdot g_4 \cdot g_8 = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} \\ &= \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \times x_{11} \times \det \begin{bmatrix} x_{11} & x_{12} & y_{11} \\ x_{21} & x_{22} & y_{21} \\ x_{31} & x_{32} & y_{31} \end{bmatrix} \\ &= \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} - \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 2 \\ \hline \end{array}. \end{aligned}$$

$$(3.4) \quad \begin{aligned} f' &= g_1 \cdot g_6 \cdot g_7 = \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} \\ &= y_{11} \times \det \begin{bmatrix} x_{11} & y_{11} & y_{12} \\ x_{21} & y_{21} & y_{22} \\ x_{31} & x_{31} & y_{32} \end{bmatrix} \times \det \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \\ &= \begin{array}{|c|} \hline \\ \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} - \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array}. \end{aligned}$$

We note that these  $f$  and  $f'$  are indeed elements in the  $(\lambda, \mu, \nu)$ -homogeneous component  $\text{TA}(3)_{\mu\nu}^\lambda$ . With the tableau notation for determinants, by top-aligning the columns

$\begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline \\ \hline 2 \\ \hline \end{array}$ , and  $\begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array}$  in the first term of the last expression of  $f$  in (3.3), we can match  $f$  with

the LR tableau  $T$ . Also, by top-aligning the columns  $\begin{smallmatrix} \square \\ \square \\ 2 \end{smallmatrix}$ ,  $\begin{smallmatrix} \square \\ \square \\ 1 \end{smallmatrix}$ , and  $\begin{smallmatrix} \square \\ 1 \end{smallmatrix}$  in the first term of the last expression of  $f'$  in (3.4), we can match  $f'$  with the LR tableau  $T'$ . Note that one can check that  $f$  and  $f'$  have distinct initial monomials

$$\text{in}(f) = x_{11}^2 x_{22} y_{11} y_{22} y_{31} \quad \text{and} \quad \text{in}(f') = x_{11}^2 x_{22} y_{11} y_{21} y_{32}$$

with respect to the monomial order given in [10], and these initial monomials can be matched with the LR tableaux  $T$  and  $T'$  as in Lemma 3.2.

Finally, we remark that the highest weight vectors corresponding to  $T$  and  $T'$  by the formula in [10] are different from our highest weight vectors  $f$  and  $f'$ . In fact, they are  $f$  and  $-(f + f')$ . See the computations in [18, §8.6].

3.2.4. *Tensor product algebra for  $\text{GL}(4)$ .* Next we investigate the  $\text{GL}(4)$  tensor product algebra.

**Theorem 3.7.** *The  $\text{GL}(4)$  tensor product algebra  $\text{TA}(4)$  is generated by the following twenty polynomials*

$$\begin{aligned} g_1 &= \begin{smallmatrix} \square \\ 1 \end{smallmatrix}, & g_2 &= \begin{smallmatrix} \square \\ 2 \end{smallmatrix}, & g_3 &= \begin{smallmatrix} \square \\ 1 \\ 3 \end{smallmatrix}, & g_4 &= \begin{smallmatrix} \square \\ 1 \\ 2 \\ 3 \\ 4 \end{smallmatrix}, & g_5 &= \square, & g_6 &= \begin{smallmatrix} \square \\ 1 \end{smallmatrix}, & g_7 &= \begin{smallmatrix} \square \\ 1 \\ 2 \end{smallmatrix}, \\ v_8 &= \begin{smallmatrix} \square \\ 1 \\ 2 \\ 3 \end{smallmatrix}, & g_9 &= \begin{smallmatrix} \square \\ \square \end{smallmatrix}, & g_{10} &= \begin{smallmatrix} \square \\ \square \\ 1 \end{smallmatrix}, & g_{11} &= \begin{smallmatrix} \square \\ \square \\ 1 \\ 2 \end{smallmatrix}, & g_{12} &= \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}, & g_{13} &= \begin{smallmatrix} \square \\ \square \\ \square \\ 1 \end{smallmatrix}, & g_{14} &= \begin{smallmatrix} \square \\ \square \\ \square \\ \square \end{smallmatrix}, \\ g_{15} &= \begin{smallmatrix} \square \\ \square \\ 2 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 1 \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \\ 1 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 2 \end{smallmatrix}, & g_{16} &= \begin{smallmatrix} \square \\ \square \\ 2 \\ 3 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 1 \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \\ 1 \\ 3 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 2 \end{smallmatrix} + \begin{smallmatrix} \square \\ \square \\ 1 \\ 2 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 3 \end{smallmatrix}, & g_{17} &= \begin{smallmatrix} \square \\ \square \\ 1 \\ 3 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 2 \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \\ 1 \\ 2 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 3 \end{smallmatrix}, \\ g_{18} &= \begin{smallmatrix} \square \\ \square \\ \square \\ 2 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 1 \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \\ \square \\ 1 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 2 \end{smallmatrix}, & g_{19} &= \begin{smallmatrix} \square \\ \square \\ \square \\ 3 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 2 \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \\ \square \\ 2 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 3 \end{smallmatrix} + \begin{smallmatrix} \square \\ \square \\ \square \\ 1 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 3 \end{smallmatrix}, & g_{20} &= \begin{smallmatrix} \square \\ \square \\ \square \\ 2 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 1 \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \\ \square \\ 1 \end{smallmatrix} \cdot \begin{smallmatrix} \square \\ 2 \end{smallmatrix} \end{aligned}$$

with the fifteen basic relations  $\hat{r}_j = 0$  among them where

$$\begin{aligned} \hat{r}_1 &= g_1 g_7 g_9 - g_6 g_{15} + g_2 g_5 g_{10}, & \hat{r}_2 &= g_1 g_8 g_9 - g_6 g_{16} + g_5 g_{17}, \\ \hat{r}_3 &= g_1 g_{11} g_{12} - g_{10} g_{18} + g_{13} g_{15}, & \hat{r}_4 &= g_6 g_{11} g_{12} - g_{10} g_{20} + g_7 g_9 g_{13}, \\ \hat{r}_5 &= g_2 g_8 g_{10} - g_7 g_{17} + g_3 g_6 g_{11}, & \hat{r}_6 &= g_2 g_8 g_{12} - g_7 g_{19} + g_3 g_{20}, \\ \hat{r}_7 &= g_6 g_{18} - g_1 g_{20} + g_2 g_5 g_{13}, & \hat{r}_8 &= g_7 g_{16} - g_8 g_{15} - g_3 g_5 g_{11}, \\ \hat{r}_9 &= g_{10} g_{19} - g_{12} g_{17} - g_3 g_9 g_{13}, & \hat{r}_{10} &= g_{15} g_{17} - g_2 g_{10} g_{16} - g_1 g_3 g_9 g_{11}, \\ \hat{r}_{11} &= g_{15} g_{19} - g_2 g_{12} g_{16} - g_3 g_9 g_{18}, & \hat{r}_{12} &= g_{15} g_{20} - g_7 g_9 g_{18} - g_2 g_5 g_{11} g_{12}, \\ \hat{r}_{13} &= g_{16} g_{20} - g_5 g_{11} g_{19} - g_8 g_9 g_{18}, & \hat{r}_{14} &= g_{17} g_{20} - g_6 g_{11} g_{19} - g_2 g_8 g_9 g_{13}, \\ \hat{r}_{15} &= g_{17} g_{18} - g_1 g_{11} g_{19} - g_2 g_{13} g_{16}. \end{aligned}$$

We remark that, as in the previous cases, with our tableau notation of determinants, the first terms of our generators can be, once their columns are top-aligned, identified with LR tableaux. We first note that all these polynomials  $g_i$  are invariant under the action of  $U_n \times U_n \times U_n$ , and therefore belong to the tensor product algebra  $\text{TA}(4)$ . Computing their weights under  $A_n \times A_n \times A_n$ , we see that they are indeed highest weight vectors

appearing in the decomposition of tensor products:  $g_i$  is a highest weight vector of  $V_4^\lambda$  in the decomposition of  $V_4^\mu \otimes V_4^\nu$  where

$i$	$\lambda$	$\mu$	$\nu$
1	(1, 0, 0, 0)	(0, 0, 0, 0)	(1, 0, 0, 0)
2	(1, 1, 0, 0)	(0, 0, 0, 0)	(1, 1, 0, 0)
3	(1, 1, 1, 0)	(0, 0, 0, 0)	(1, 1, 1, 0)
4	(1, 1, 1, 1)	(0, 0, 0, 0)	(1, 1, 1, 1)
5	(1, 0, 0, 0)	(1, 0, 0, 0)	(0, 0, 0, 0)
6	(1, 1, 0, 0)	(1, 0, 0, 0)	(1, 0, 0, 0)
7	(1, 1, 1, 0)	(1, 0, 0, 0)	(1, 1, 0, 0)
8	(1, 1, 1, 1)	(1, 0, 0, 0)	(1, 1, 1, 0)
9	(1, 1, 0, 0)	(1, 1, 0, 0)	(0, 0, 0, 0)
10	(1, 1, 1, 0)	(1, 1, 0, 0)	(1, 0, 0, 0)
11	(1, 1, 1, 1)	(1, 1, 0, 0)	(1, 1, 0, 0)
12	(1, 1, 1, 0)	(1, 1, 1, 0)	(0, 0, 0, 0)
13	(1, 1, 1, 1)	(1, 1, 1, 0)	(1, 0, 0, 0)
14	(1, 1, 1, 1)	(1, 1, 1, 1)	(0, 0, 0, 0)
15	(2, 1, 1, 0)	(1, 1, 0, 0)	(1, 1, 0, 0)
16	(2, 1, 1, 1)	(1, 1, 0, 0)	(1, 1, 1, 0)
17	(2, 2, 1, 1)	(1, 1, 0, 0)	(2, 1, 1, 0)
18	(2, 1, 1, 1)	(1, 1, 1, 0)	(1, 1, 0, 0)
19	(2, 2, 1, 1)	(1, 1, 1, 0)	(1, 1, 1, 0)
20	(2, 2, 1, 1)	(2, 1, 1, 0)	(1, 1, 0, 0)

We note that  $V_4^\mu$  or  $V_4^\nu$  is a fundamental representation of  $\mathrm{GL}(4)$ , and therefore  $V_4^\mu \otimes V_4^\nu$  is multiplicity-free. Therefore, the highest weight vector of  $V_4^\lambda$  in the decomposition of  $V_4^\mu \otimes V_4^\nu$  is unique. Now with the same arguments we used for  $\mathrm{GL}(2)$  and  $\mathrm{GL}(3)$ , by lifting the computations done for  $\mathcal{H}(4) \cong \mathrm{in}(\mathrm{TA}(4))$  to  $\mathrm{TA}(4)$ , we see that  $\mathrm{in}(g_i)$  generate the initial algebra of  $\mathrm{TA}(4)$ , and therefore  $g_i$  generate  $\mathrm{TA}(4)$ . The relations between  $g_j$  can be also obtained easily by lifting the relations of hives computed in Theorem 2.7.

**3.3. Remarks on the generators and relations.** The algebra of invariants of the standard maximal unipotent subgroup  $U_p$  of  $\mathrm{GL}(p)$  under left multiplication on the space  $M_{p,q}$  of  $p \times q$  matrices is generated by certain minors over  $M_{p,q}$ . This is a well-known result from classical invariant theory. See, for example, [7, §9] and [20, §14]. Hence it is not so surprising that the elements of the  $\mathrm{GL}(n)$  tensor product algebra  $\mathrm{TA}(n)$  can be expressed in terms of certain minors over  $M_{n,2n} \cong M_n \oplus M_n$  and their relations can be realized in terms of classical determinantal identities such as the quadratic relations of Sylvester type given in [7, §8].

Here we give some interesting examples of such identities. Recall that the Lewis Carroll identity [4], also known as the Desnanot-Jacobi identity, for a  $n \times n$  matrix  $A$  is

$$\det A \det A' = \det A_n^n \det A_1^1 - \det A_n^1 \det A_1^n$$

where  $A'$  is the  $(n-2) \times (n-2)$  submatrix of  $A$  obtained by erasing the first and last rows of  $A$  and the first and last columns of  $A$ ;  $A_p^q$  is the  $(n-1) \times (n-1)$  submatrix of  $A$  obtained by erasing the  $p$ th row and  $q$ th column of  $A$ . Then, the relations  $\hat{r}_1 = 0$ ,  $\hat{r}_3 = 0$ , and  $\hat{r}_5 = 0$  for  $\mathrm{TA}(4)$  in Theorem 3.7 are essentially the Lewis Carroll identity

applied to the following matrices

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ y_{11} & x_{11} & x_{12} & y_{12} \\ y_{21} & x_{21} & x_{22} & y_{22} \\ y_{31} & x_{31} & x_{32} & y_{32} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ y_{11} & x_{11} & x_{12} & x_{13} & y_{12} \\ y_{21} & x_{21} & x_{22} & x_{23} & y_{22} \\ y_{31} & x_{31} & x_{32} & x_{33} & y_{32} \\ y_{41} & x_{41} & x_{42} & x_{43} & y_{42} \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ y_{12} & x_{11} & y_{11} & x_{12} & y_{13} \\ y_{22} & x_{21} & y_{21} & x_{22} & y_{23} \\ y_{32} & x_{31} & y_{31} & x_{32} & y_{33} \\ y_{42} & x_{41} & y_{41} & x_{42} & y_{43} \end{bmatrix}$$

respectively. We refer the interested reader to [13, §3] and [14, §5] for similar observations.

#### 4. APPENDIX: NORMALIZ CODES FOR $\mathcal{H}(4)$

In this section, we provide the input codes for jNormaliz (a Java-based graphical interface for Normaliz) we used to analyze the hive cone  $\mathcal{H}(4)$  together with their outputs. For the interpretation of these outputs other than the ones mentioned in Theorem 2.7, we refer the readers to Normaliz documents [2].

**4.1. Hilbert basis.** We first define the hive cone  $\mathcal{H}(4)$  using inequalities.

```

30
15
-1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 /*the boundary is nonnegative dominant*/
0 -1 0 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 -1 0 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 -1 0 0 0 1 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 -1 1 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 -1 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 -1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 -1 1
-1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 -1 0 0 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 -1 0 0 0 1 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 -1 0 0 0 0 0 1
-1 1 1 0 -1 0 0 0 0 0 0 0 0 0 0 0 /* the rhombus conditions */
0 -1 0 1 1 0 0 -1 0 0 0 0 0 0 0 0
0 0 -1 0 1 1 0 0 -1 0 0 0 0 0 0 0
0 0 0 -1 0 0 1 1 0 0 0 -1 0 0 0 0
0 0 0 0 -1 0 0 1 1 0 0 0 -1 0 0 0
0 0 0 0 0 -1 0 0 1 1 0 0 0 -1 0 0
0 -1 1 0 1 -1 0 0 0 0 0 0 0 0 0 0
0 0 0 -1 1 0 0 1 -1 0 0 0 0 0 0 0
0 0 0 0 -1 1 0 0 1 -1 0 0 0 0 0 0
0 0 0 0 0 0 -1 1 0 0 0 1 -1 0 0 0
0 0 0 0 0 0 0 -1 1 0 0 0 1 -1 0 0
0 0 0 0 0 0 0 0 -1 1 0 0 0 1 -1
0 1 -1 -1 1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 1 -1 0 -1 1 0 0 0 0 0 0 0 0
0 0 0 0 1 -1 0 -1 1 0 0 0 0 0 0 0
0 0 0 0 0 0 1 -1 0 0 -1 1 0 0 0 0
0 0 0 0 0 0 0 1 -1 0 0 -1 1 0 0 0
0 0 0 0 0 0 0 0 1 -1 0 0 -1 1 0
inequalities

```

```

1
15
1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 /* h_11 = 0 */
equations

```

---

Its output gives a Hilbert basis for  $\mathcal{H}(4)$  among others.

```

20 Hilbert basis elements
20 extreme rays
20 support hyperplanes

embedding dimension = 15
rank = 14
external index = 1

size of partial triangulation = 0
resulting sum of |det|s = 0

No implicit grading found

rank of class group = 6
class group is free

```

\*\*\*\*\*

```

20 Hilbert basis elements:
0 0 1 0 1 1 0 1 1 1 0 1 1 1 1
0 0 1 0 1 2 0 1 2 2 0 1 2 2 2
0 0 1 0 1 2 0 1 2 3 0 1 2 3 3
0 0 1 0 1 2 0 1 2 3 0 1 2 3 4
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 1 1 1 2 2 1 2 2 2 1 2 2 2 2
0 1 1 1 2 2 1 2 3 3 1 2 3 3 3
0 1 1 1 2 2 1 2 3 3 1 2 3 4 4
0 1 1 2 2 2 2 2 2 2 2 2 2 2 2
0 1 1 2 2 2 2 3 3 3 2 3 3 3 3
0 1 1 2 2 2 2 3 3 3 2 3 4 4 4
0 1 1 2 2 2 3 3 3 3 3 3 3 3 3
0 1 1 2 2 2 3 3 3 3 3 4 4 4 4
0 1 1 2 2 2 3 3 3 3 4 4 4 4 4
0 1 2 2 3 3 2 3 4 4 2 3 4 4 4
0 1 2 2 3 3 2 3 4 4 2 3 4 5 5
0 1 2 2 3 3 3 4 4 4 3 4 5 5 5
0 1 2 2 3 4 2 4 5 5 2 4 5 6 6
0 1 2 2 3 4 3 4 5 5 3 4 5 6 6
0 2 2 3 4 4 4 5 5 5 4 5 6 6 6

```

```

20 extreme rays:
0 0 1 0 1 1 0 1 1 1 0 1 1 1 1
0 0 1 0 1 2 0 1 2 2 0 1 2 2 2
0 0 1 0 1 2 0 1 2 3 0 1 2 3 3
0 0 1 0 1 2 0 1 2 3 0 1 2 3 4

```

```

0 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 1 1 1 2 2 1 2 2 2 1 2 2 2 2
0 1 1 1 2 2 1 2 3 3 1 2 3 3 3
0 1 1 1 2 2 1 2 3 3 1 2 3 4 4
0 1 1 2 2 2 2 2 2 2 2 2 2 2 2
0 1 1 2 2 2 2 3 3 3 2 3 3 3 3
0 1 1 2 2 2 2 3 3 3 2 3 4 4 4
0 1 1 2 2 2 3 3 3 3 3 3 3 3 3
0 1 1 2 2 2 3 3 3 3 3 4 4 4 4
0 1 1 2 2 2 3 3 3 3 4 4 4 4 4
0 1 2 2 3 3 2 3 4 4 2 3 4 4 4
0 1 2 2 3 3 2 3 4 4 2 3 4 5 5
0 1 2 2 3 3 3 4 4 4 3 4 5 5 5
0 1 2 2 3 4 2 4 5 5 2 4 5 6 6
0 1 2 2 3 4 3 4 5 5 3 4 5 6 6
0 2 2 3 4 4 4 5 5 5 4 5 6 6 6

```

20 support hyperplanes :

```

0 -1 0 1 1 0 0 -1 0 0 0 0 0 0 0
0 -1 1 0 1 -1 0 0 0 0 0 0 0 0 0
0 0 -1 0 1 1 0 0 -1 0 0 0 0 0 0
0 0 0 -1 0 0 1 1 0 0 0 -1 0 0 0
0 0 0 -1 1 0 0 1 -1 0 0 0 0 0 0
0 0 0 0 -1 0 0 1 1 0 0 0 -1 0 0
0 0 0 0 -1 1 0 0 1 -1 0 0 0 0 0
0 0 0 0 0 -1 0 0 1 1 0 0 0 -1 0
0 0 0 0 0 0 -1 0 0 0 1 0 0 0 0
0 0 0 0 0 0 -1 1 0 0 0 1 -1 0 0
0 0 0 0 0 0 0 -1 1 0 0 0 1 -1 0
0 0 0 0 0 0 0 0 -1 1 0 0 0 1 -1
0 0 0 0 0 0 0 0 0 0 0 0 -1 1
0 0 0 0 0 0 0 0 1 -1 0 0 -1 1 0
0 0 0 0 0 0 0 1 -1 0 0 -1 1 0 0
0 0 0 0 0 0 1 -1 0 0 -1 1 0 0 0
0 0 0 0 1 -1 0 -1 1 0 0 0 0 0 0
0 0 0 1 -1 0 -1 1 0 0 0 0 0 0 0
0 1 -1 -1 1 0 0 0 0 0 0 0 0 0 0
0 1 1 0 -1 0 0 0 0 0 0 0 0 0 0

```

---

4.2. **HP series.** To obtain the HP series of  $\mathcal{H}A(4)$  and other properties of the hive cone  $\mathcal{H}(4)$ , we redefine the cone using the above Hilbert basis elements and specify their degrees.

amb\_space 15

cone 20

```

0 0 1 0 1 1 0 1 1 1 0 1 1 1 1
0 0 1 0 1 2 0 1 2 2 0 1 2 2 2
0 0 1 0 1 2 0 1 2 3 0 1 2 3 3
0 0 1 0 1 2 0 1 2 3 0 1 2 3 4
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 1 1 1 2 2 1 2 2 2 1 2 2 2 2
0 1 1 1 2 2 1 2 3 3 1 2 3 3 3

```

```

0 1 1 1 2 2 1 2 3 3 1 2 3 4 4
0 1 1 2 2 2 2 2 2 2 2 2 2 2 2
0 1 1 2 2 2 2 3 3 3 2 3 3 3 3
0 1 1 2 2 2 2 3 3 3 2 3 4 4 4
0 1 1 2 2 2 3 3 3 3 3 3 3 3 3
0 1 1 2 2 2 3 3 3 3 3 4 4 4 4
0 1 1 2 2 2 3 3 3 3 4 4 4 4 4
0 1 2 2 3 3 2 3 4 4 2 3 4 4 4
0 1 2 2 3 3 2 3 4 4 2 3 4 5 5
0 1 2 2 3 3 3 4 4 4 3 4 5 5 5
0 1 2 2 3 4 2 4 5 5 2 4 5 6 6
0 1 2 2 3 4 3 4 5 5 3 4 5 6 6
0 2 2 3 4 4 4 5 5 5 4 5 6 6 6

```

grading

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
```

---

The following is a part of the output obtained by running the previous input file. It gives among others the HP series of the monoid algebra of the hive cone  $\mathcal{H}(4)$ .

20 extreme rays

20 support hyperplanes

embedding dimension = 15

rank = 14

external index = 1

internal index = 1

size of triangulation = 16

resulting sum of |det|s = 16

grading:

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
```

degrees of extreme rays:

```
1: 2  2: 3  3: 4  4: 6  5: 2  6: 3
```

multiplicity = 1/147456

Hilbert series:

```
1 -2 -2 10 -2 -24 22 32 -54 -18 80 -14 -72 34 44 -18 -25
-18 44 34 -72 -14 80 -18 -54 32 22 -24 -2 10 -2 -2 1
```

denominator with 14 factors:

```
1: 4  2: 6  12: 4
```

degree of Hilbert Series as rational function = -32

The numerator of the Hilbert Series is symmetric.

Hilbert series with cyclotomic denominator:

```
1 0 -1 2 2 0 1 0 2 2 -1 0 1
```

cyclotomic denominator:

## REFERENCES

- [1] A. S. Buch, The saturation conjecture (after A. Knutson and T. Tao). With an appendix by William Fulton. *Enseign. Math.* (2) 46 (2000), no. 1-2, 43–60.
- [2] W. Bruns, B. Ichim, T. Römer, and C. Söger, Normaliz. Algorithms for rational cones and affine monoids. Available at <https://www.normaliz.uni-osnabrueck.de/>
- [3] A. Conca, J. Herzog, and G. Valla, SAGBI basis with applications to blow-up algebras, *J. Reine Angew. Math.*, 474 (1996), pp. 113–138
- [4] C. L. Dodgson, Condensation of determinants, being a new and brief method for computing their arithmetical values, *Proceedings of the Royal Society* 15 (1866), pp. 150–155.
- [5] I. Dolgachev, Weighted projective varieties. Group actions and vector fields (Vancouver, B.C., 1981), 34–71, *Lecture Notes in Math.*, 956, Springer, Berlin, 1982.
- [6] P. Doolan and S. Kim, The Littlewood-Richardson rule and Gelfand-Tsetlin patterns. *Algebra Discrete Math.* 22 (2016), no. 1, 21–47.
- [7] W. Fulton, Young tableaux. With applications to representation theory and geometry. London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.
- [8] D. R. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2/>
- [9] F. D. Grosshans, Invariants on  $G/U \times G/U \times G/U$ ,  $G = \mathrm{SL}(4, \mathbf{C})$ . *Invariant methods in discrete and computational geometry* (Curaçao, 1994), 257–277, *Kluwer Acad. Publ.*, Dordrecht, 1995.
- [10] R. Howe, E.-C. Tan, and J. F. Willenbring, A basis for the  $GL_n$  tensor product algebra. *Adv. Math.* 196 (2005), no. 2, 531–564.
- [11] R. Howe, S. Jackson, S. T. Lee, E.-C. Tan, and J. Willenbring, Toric degeneration of branching algebras. *Adv. Math.* 220 (2009), no. 6, 1809–1841.
- [12] R. Howe and S. T. Lee, Why should the Littlewood-Richardson rule be true? *Bull. Amer. Math. Soc. (N.S.)* 49 (2012), no. 2, 187–236.
- [13] S. Kim, A presentation of the double Pieri algebra. *J. Pure Appl. Algebra* 222 (2018), no. 2, 368–381.
- [14] S. Kim and S. Yoo, Pieri and Littlewood-Richardson rules for two rows and cluster algebra structure. *J. Algebraic Combin.* 45 (2017), no. 3, 887–909.
- [15] R. C. King, C. Tollu, and F. Toumazet, The hive model and the polynomial nature of stretched Littlewood-Richardson coefficients. *Sém. Lothar. Combin.* 54A (2005/07), Art. B54Ad, 19 pp.
- [16] A. Knutson and T. Tao, The honeycomb model of  $GL_n(\mathbf{C})$  tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.* 12 (1999), no. 4, 1055–1090.
- [17] A. Knutson, T. Tao, and C. Woodward, A positive proof of the Littlewood-Richardson rule using the octahedron recurrence. *Electron. J. Combin.* 11 (2004), no. 1, Research Paper 61, 18 pp.
- [18] S. T. Lee, Branching rules and branching algebras for the complex classical groups, COE Lecture Note, 47. Math-for-Industry (MI) Lecture Note Series. Kyushu University, Faculty of Mathematics, Fukuoka, 2013.
- [19] I. G. Macdonald, *Symmetric functions and Hall polynomials*. Second edition. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
- [20] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.
- [21] I. Pak and E. Vallejo, Combinatorics and geometry of Littlewood-Richardson cones. *European J. Combin.* 26 (2005), no. 6, 995–1008.
- [22] K. Purbhoo, Puzzles, tableaux, and mosaics. *J. Algebraic Combin.* 28 (2008), no. 4, 461–480.
- [23] R. P. Stanley, *Enumerative combinatorics*. Vol. 2. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.
- [24] H. Thomas and A. Yong, An  $S_3$ -symmetric Littlewood-Richardson rule. *Math. Res. Lett.* 15 (2008), no. 5, 1027–1037.

- [25] M. A. A. van Leeuwen, The Littlewood-Richardson rule, and related combinatorics. Interaction of combinatorics and representation theory, 95–145, MSJ Mem., 11, Math. Soc. Japan, Tokyo, 2001.

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