ADO THEOREM FOR NILPOTENT HOM-LIE ALGEBRAS

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ABSTRACT. We prove an analog of the Ado theorem – the existence of a finite-dimensional faithful representation – for a certain kind of finite-dimensional nilpotent Hom-Lie algebras.

0. Introduction

During the last decade, a new class of algebras has been studied extensively in the literature: the so-called Hom-Lie algebras, anticommutative algebras with the Jacobi identity "twisted" by an endomorphism (for a precise definition, see §1). The impetus came from mathematical physics, where particular instances of such structures appeared as symmetries of some physical models, and as σ -deformations. As it happens, after that people started to study Hom-Lie (and related) algebras on their own. A lot of interesting results were obtained recently, we mention here just a few: structure theory of simple and semisimple Hom-Lie algebras ([CH]); a Hom-analog of classification of filiform Lie algebras ([MM]); cohomology and deformation theory of various classes of Hom-algebras à la Gerstenhaber, with applications to Hom-analogs of the known constructions related to the Witt algebra ([ABM, MS10]); Hom-Lie structures on Kac-Moody algebras ([MZ]); Hom-analogs of some interesting identities in alternative algebras ([Y12]).

When one tries to extend Lie-algebraic constructions and results to the Hom-Lie case, the innocently-looking, at the first glance, twist often leads to substantial difficulties of combinatorial character, and the theory of Hom-Lie algebras appears to be, generally, more difficult than its "ordinary" Lie analog. For example, presently it is not known whether any Hom-Lie algebra embeds into its universal enveloping algebra, or, more generally, whether an analog of the Poincaré–Birkhoff–Witt theorem for Hom-Lie algebras holds (see a discussion in [HMS, §2.3]; the Poincaré–Birkhoff–Witt theorem is known to be true only in a very particular case of involutive Hom-Lie algebras – i.e. those for which the square of the twist map is identity – see [GZZ]).

In this paper we consider in the Hom-Lie setting another classical Lie-algebraic result, closely connected with the Poincaré–Birkhoff–Witt theorem: the Ado theorem which claims the existence of a finite-dimensional faithful representation of any finite-dimensional Lie algebra. The classical proof of the Ado theorem involves universal enveloping algebras. The analogous construction in the Hom-Lie setting is, apparently, much more complicated than in the Lie case, and presently is not very well understood (see, again, [HMS], and also [CG] and [LMT]). That is why we choose to follow another route, presented in [Z], which avoids usage of universal enveloping algebras and stays entirely inside the category of finite-dimensional algebras. The drawback of this approach that is works only for nilpotent algebras and in characteristic zero, so, strictly speaking – and from the historical perspective – our Hom-Lie result is an analog not of the Ado theorem, but of the earlier result: the Birkhoff theorem, [Bi], guaranteeing the existence of a finite-dimensional faithful representation of a *nilpotent* finite-dimensional Lie algebra.

The contents of the paper are as follows. The first four sections are preliminary in nature, and treat elementary concepts related to the Hom-Lie theory: representations (§§1 and 2), nilpotent algebras (§3), and free algebras (§4). Most of this is based on a material existing in the literature, or a variation thereof;

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but because the Hom-Lie theory is still in the formative period, we found it necessary to fix the basic concepts and the terminology. Among other things, we define the tensor product in the category of representations of a multiplicative Hom-Lie algebra (Proposition 1.1); prove that, similarly with the Lie case, the existence of a faithful representation is equivalent to the embedding into a Hom-associative algebra (Theorem 1); and prove that two notions of nilpotency coincide for multiplicative Hom-Lie algebras over algebraically closed fields (Proposition 3.1). Finally, in §5 we prove the Ado theorem for a certain class of nilpotent finite-dimensional Hom-Lie algebras – the so-called multiplicative non-degenerate algebras (for the precise definitions, see §1), defined over an algebraically closed field of characteristic zero (Theorem 2). All the restrictions on the structure of algebra are stipulated by our chosen approach, and it remains a challenging problem to establish the Ado theorem for a broader class of finite-dimensional Hom-Lie algebras – not necessarily nilpotent, or multiplicative, or nondegenerate, or defined over an arbitrary ground field.

Throughout the paper, "algebra" means a not necessarily Lie, or associative, or satisfying any other distinguished identity, algebra. $\operatorname{End}(V)$ denotes the vector space of linear maps from a vector space V to itself. Direct sums, denoted by \oplus , are understood in the category of vector spaces. The ground field is assumed arbitrary, unless stated explicitly otherwise. $\mathbb N$ denotes the set of all positive integers.

1. Hom-Lie algebras and their representations

In this section we recall the basic definitions and facts related to Hom-Lie algebras and their representations; for the most part, we just meticulously record all variations of the relevant concepts, the more so there is still some ambiguity in that regard in the literature, and divide everything into small trivial steps. Perhaps, the only result which can be called new here, if only conditionally, is the construction of the tensor product of representations of a Hom-Lie algebra (Proposition 1.1), which is a straightforward generalization of the Lie case; but it is implicitly contained in [LMT] (and, in the regular case, in [CG]), where it is proved that the universal enveloping algebra of a multiplicative Hom-Lie algebra is Hom-Hopf, suitably defined.

Definition 1.1. A pair (V, α) , where V is a vector space, and $\alpha : V \to V$ is a linear map, is called a *Hom-vector space*, and the map α is called a *twist map*. A Hom-vector space with a nondegenerate (i.e., having zero kernel) twist map is called *nondegenerate*. A *Hom-subspace* of the Hom-vector space (V, α) is a vector subspace of V invariant under α .

Definition 1.2. A triple (A, \cdot, α) , where (A, \cdot) is an algebra, and (A, α) is a Hom-vector space, is called a *Hom-algebra*. If α is a homomorphism of the algebra structure, i.e.

$$\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$$

for any $x, y \in A$, then the Hom-algebra (A, \cdot, α) is called *multiplicative*. If (A, α) is a nondegenerate Hom-vector space, then the Hom-algebra (A, \cdot, α) is called *nondegenerate*.

In the literature there exists a close notion of a *regular* Hom-algebra, namely, a Hom-algebra which is multiplicative and whose twist map is invertible. That is, in the finite-dimensional situation, which is our main concern here, regular Hom-algebras are exactly those which are, in our terms, multiplicative *and* nondegenerate. But when treating free algebras in § 4, it is more convenient to deal with nondegeneracy rather than with invertibility, for Hom-algebras with a nondegenerate twist map form a quasivariety, while Hom-algebras with an invertible twist map do not. (Still, occasionally we need to refer to the situation when the twist map is invertible and not merely nondegenerate, as in Lemmas 2.3 and 2.5).

Also, we want to have multiplicativity and nondegeneracy as complementary notions, rather than later to be included in the former, for two reasons: first, it better corresponds to the similar notions for representations (see Definitions 1.8 and 2.3), and, second, the desire to clearly isolate places where we need that or another restriction on the twist map in our proof of a variant of the Ado theorem in §5.

Definition 1.3. A *subalgebra*, respectively *ideal*, of the Hom-algebra (A, \cdot, α) is a subalgebra, respectively ideal of the algebra (A, \cdot) which is simultaneously a Hom-subspace of (A, α) .

The quotient of a Hom-vector space by a Hom-subspace, the quotient of a Hom-algebra by an ideal, and extension of Hom-algebras are defined in the standard way.

Below, when speaking about the algebra or the ideal generated by a given set, we will mean generation in the Hom-algebra sense, i.e. the minimal algebra or ideal containing the given set and closed under multiplication *and* the twist map.

By abuse of notation, and as it is customary in the case of ordinary (i.e., not Hom) algebras, if the algebra multiplication and the twist map are not needed to be specified explicitly, or are clear from the context, we will omit them from the Hom-algebra notation, and denote the Hom-algebra just by the underlying vector space, A.

Definition 1.4. A Hom-algebra $(L, [\cdot, \cdot], \alpha)$ is called a *Hom-Lie algebra*, if L is anticommutative as an algebra, i.e.

$$[y, x] = -[x, y],$$

and the following Hom-Jacobi identity holds:

$$[[x, y], \alpha(z)] + [[z, x], \alpha(y)] + [[y, z], \alpha(x)] = 0$$

for any $x, y, z \in L$.

Various examples of Hom-Lie algebras can be found, among others, in [BM], [HMS], [MM], [MS08], and [MS10].

Definition 1.5. A linear map $\varphi: A_1 \to A_2$ is a homomorphism of a Hom-algebra (A_1, \cdot_1, α_1) to a Hom-algebra (A_2, \cdot_2, α_2) , if φ is a homomorphism of algebras, i.e.

$$\varphi(x \cdot_1 y) = \varphi(x) \cdot_2 \varphi(y)$$

for any $x, y \in A_1$, and $\varphi \circ \alpha_1 = \alpha_2 \circ \varphi$.

An interesting construction, provided in [Y09], allows to build a Hom-Lie algebra starting from a Lie algebra and an algebra homomorphism. Namely, if $(L, [\cdot, \cdot])$ is a Lie algebra, and α a homomorphism of L, then $(L, \alpha[\cdot, \cdot], \alpha)$ is a Hom-Lie algebra, called the *Yau twist* of L. Hom-Lie algebras obtained in this way are called algebras of Lie type. In particular, if $(L, [\cdot, \cdot], \alpha)$ is a multiplicative nondegenerate Hom-Lie algebra, then $(L, \alpha^{-1}[\cdot, \cdot])$ is a Lie algebra.

Definition 1.6. Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra, and A an associative commutative algebra. A *current Hom-Lie algebra* is a Hom-Lie algebra $(L \otimes A, [\cdot, \cdot], \widehat{\alpha})$ defined on the tensor product $L \otimes A$, where the Lie bracket (denoted by the same symbol by abuse of notation) is defined by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for $x, y \in L$, $a, b \in A$, and the twist map is defined by

$$\widehat{\alpha}(x \otimes a) = \alpha(x) \otimes a.$$

(Note parenthetically that this current algebra construction can be generalized to the case where A is also a Hom-algebra, and, further, to algebras over operads Koszul dual in some extended Hom sense, but we will not need this generality here; hopefully, it will be treated elsewhere).

Lemma 1.1. Let L be a Hom-Lie algebra, and A an associative commutative algebra. If L is multiplicative or nondegenerate, then so is the current Hom-Lie algebra $L \otimes A$.

Definition 1.7. Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. A linear map $D: L \to L$ is called an α -derivation of L, if

(1.1)
$$D([x, y]) = [D(x), \alpha(y)] + [\alpha(x), D(y)]$$

holds for any $x, y \in L$.

If, additionally, L is multiplicative, then, in addition to (1.1), we require that

$$D \circ \alpha = \alpha \circ D$$
.

In the literature there exist several different definitions of derivation of a Hom-Lie algebra. Say, in [MS10] one defines 1-coboundaries (i.e., in fact, derivations) as ordinary derivations of the algebra structure, thus ignoring the twist map α ; and in [S] one defines the so-called α^k -derivations, obtained by replacing in formula (1.1) α by its kth power. While these definitions have their merits in appropriate contexts, the definition adopted here is stipulated, like the definition of representation below, by the structural interpretation: for a (multiplicative, nondegenerate) Hom-Lie algebra L, a linear map $D: L \to L$ is an α -derivation, if and only if the direct sum $L \oplus KD$, where the multiplication between elements of L and D is determined by the action of the former on the latter, and $\alpha(D) = D$, is a (multiplicative, nondegenerate) Hom-Lie algebra.

Similarly, there are few slightly different definitions in the literature of what is a representation of a Hom-Lie algebra (see, for example, [BM], [S], [SX]). To see what is the "right" definition, let us employ an old principle which goes back to Eilenberg: for an algebra L in a given variety of algebras, two L-actions on a vector space V, from the left and from the right, are declared a birepresentation of L, if the semidirect sum $L \oplus V$, where multiplication between elements of L and V is determined by the actions of L on V, and multiplication on V is zero, belongs to the same variety.

So, let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra, $\rho: L \to \operatorname{End}(V)$ a linear map providing an action of L on V (as in the case of ordinary Lie algebras, because L is anticommutative, the left and right actions differ only by sign, so we can speak on – say, left – representations instead of birepresentations), and an anticommutative multiplication on the semidirect sum $L \oplus V$ is defined by $[x, v] = \rho(x)(v)$ for $x \in L$ and $v \in V$, and by [V, V] = 0. To make from it a Hom-Lie algebra, we have to extend the twist map α from L to $L \oplus V$. According to the Eilenberg principle, it is reasonable to assume that the image of the restriction of α to V lies in V; let us denote this restriction by $\beta \in \operatorname{End}(V)$. Since the multiplication on V is zero, the Hom-Jacobi identity gives nothing for triples where at least two elements belong to V; for the triple $x, y \in L$, $v \in V$ it is equivalent to

$$\rho([x, y])(\beta(v)) = \rho(\alpha(x))(\rho(y)(v)) - \rho(\alpha(y))(\rho(x)(v)).$$

If $(L, [\cdot, \cdot], \alpha)$ is multiplicative, then α , extended to the semidirect sum $L \oplus V$, is a homomorphism of the algebra structure if and only if

$$\rho(\alpha(x))(\beta(v)) = \beta(\rho(x)(v))$$

for any $x \in L$ and $v \in V$.

Thus we arrive at

Definition 1.8. Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra, and (V, β) a Hom-vector space. A linear map $\rho: L \to \operatorname{End}(V)$ is called a *representation of L in V*, if

(1.2)
$$\rho([x, y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x)$$

holds for any $x, y \in L$.

The representation ρ is called *multiplicative* if, in addition to (1.2),

$$\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x)$$

holds for any $x \in L$.

The representation is called *nondegenerate* if the Hom-vector space (V, β) is nondegenerate.

In all these situations, V is called a (multiplicative, nondegenerate) module over L.

Note that though the main utility of multiplicative representation is to serve as the "right" notion of representation in the category of multiplicative Hom-Lie algebras, nothing prevents us to consider multiplicative representations of not necessarily multiplicative Hom-Lie algebras (see, for example, Proposition 1.1 for the situation in which this may make sense).

The notions of the kernel of a representation, the direct sum of representations, and subrepresentation and submodule are defined in the obvious standard ways. Note that, in general, the kernel of a representation is not a subalgebra, but the kernel of a multiplicative and nondegenerate representation is a subalgebra.

The center is defined exactly as in the case of Lie algebras, ignoring the twist map:

Definition 1.9. The *center* of a Hom-Lie algebra L, denoted as Z(L), is $\{z \in L \mid [z, L] = 0\}$.

This definition is justified by

Lemma 1.2. Let L be a (multiplicative) Hom-Lie algebra. The map $ad : L \to End(L)$, where ad(x)(y) = [x, y], is a (multiplicative) representation of L, called the adjoint representation. The kernel of the adjoint representation coincides with Z(L).

Proof. For $\rho = ad$, the condition (1.2) is equivalent to the Hom-Jacobi identity, and the condition (1.3) is equivalent to the condition that the twist map is the algebra homomorphism. The statement about the kernel is obvious.

Definition 1.10. A representation ρ of a Hom-Lie algebra L is called *nilpotent*, if the associative algebra of linear transformations generated by $\rho(L)$ in $\operatorname{End}(L)$ is nilpotent. That is to say, there exists $n \in \mathbb{N}$ such that $\rho(x_1) \circ \cdots \circ \rho(x_n) = 0$ for any $x_1, \ldots, x_n \in L$. The minimal such n is called the *index of nilpotency*, or *nilindex*, of ρ .

Definition 1.11. Let L be a Hom-Lie algebra, and $\rho: L \to \operatorname{End}(V)$, $\tau: L \to \operatorname{End}(W)$ be two representations of L in the Hom-vector spaces (V,β) and (W,γ) respectively. Then the map $\rho \otimes \tau: L \to \operatorname{End}(V \otimes W)$ defined as

$$(\rho \otimes \tau)(x) = \rho(x) \otimes \gamma + \beta \otimes \tau(x)$$

for $x \in L$, is a called the *tensor product* of representations ρ and τ .

Proposition 1.1. The tensor product of two multiplicative representations of a Hom-Lie algebra L in Hom-vector spaces (V,β) and (W,γ) , is a multiplicative representation of L in the Hom-vector space $(V \otimes W, \beta \otimes \gamma)$.

Proof. To check (1.2): for any $x, y \in L$ we have

$$(\rho \otimes \tau)([x,y]) \circ (\beta \otimes \gamma)$$

$$\stackrel{(1)}{=} (\rho([x,y]) \circ \beta) \otimes \gamma^2 + \beta^2 \otimes (\tau([x,y]) \circ \gamma)$$

$$\stackrel{(2)}{=} (\rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x)) \otimes \gamma^2 + \beta^2 \otimes (\tau(\alpha(x)) \circ \tau(y) - \tau(\alpha(y)) \circ \rho(x))$$

$$+ ((\beta \circ \rho(y)) \otimes (\tau(\alpha(x)) \circ \gamma) - (\rho(\alpha(y)) \circ \beta) \otimes (\gamma \circ \tau(x)))$$

$$+ ((\rho(\alpha(x)) \circ \beta) \otimes (\gamma \circ \tau(y)) - (\beta \circ \rho(x)) \otimes (\tau(\alpha(y)) \circ \gamma))$$

$$\stackrel{(3)}{=} ((\rho(\alpha(x)) \circ \rho(y) \otimes \gamma^2 + (\beta \circ \rho(y)) \otimes (\tau(\alpha(x)) \circ \gamma)$$

$$+ (\rho(\alpha(x)) \circ \beta) \otimes (\gamma \circ \tau(y)) + \beta^2 \otimes \tau(\alpha(x)) \circ \tau(y))$$

$$- ((\rho(\alpha(y)) \circ \rho(x)) \otimes \gamma^2 + (\beta \circ \rho(x)) \otimes (\tau(\alpha(y)) \circ \gamma)$$

$$+ (\rho(\alpha(y)) \circ \beta) \otimes (\gamma \circ \tau(x)) + \beta^2 \otimes \tau(\alpha(y)) \circ \rho(x))$$

$$\stackrel{(4)}{=} (\rho \otimes \tau)(\alpha(x)) \circ (\rho(y) \otimes \gamma + \beta \otimes \tau(y)) - (\rho \otimes \tau)(\alpha(y)) \circ (\rho(x) \otimes \gamma + \beta \otimes \tau(x))$$

$$\stackrel{(5)}{=} (\rho \otimes \tau)(\alpha(x)) \circ (\rho \otimes \tau)(y) - (\rho \otimes \tau)(\alpha(y)) \circ (\rho \otimes \tau)(x).$$

Here in the equality (1) we use the definition of $\rho \otimes \tau$, in (2) we use the fact that ρ and τ are representations (formula (1.2)), and add terms which cancel out due to (1.3), in (3) we rearrange terms, and in (4) and (5) we use again repeatedly the definition of $\rho \otimes \tau$.

To check (1.3): for any $x \in L$ we have

$$(\rho \otimes \tau)(\alpha(x)) \circ (\beta \otimes \gamma) \stackrel{(1)}{=} (\rho(\alpha(x)) \circ \beta) \otimes \gamma^2 + \beta^2 \otimes (\tau(\alpha(x)) \circ \gamma)$$

$$\stackrel{(2)}{=} (\beta \circ \rho(x)) \otimes \gamma^2 + \beta^2 \otimes (\gamma \circ \tau(x)) = (\beta \otimes \gamma) \circ (\rho(x) \otimes \gamma + \beta \otimes \tau(x)) \stackrel{(3)}{=} (\beta \otimes \gamma) \circ (\rho \otimes \tau)(x).$$

Here in the equalities (1) and (3) we use the definition of $\rho \otimes \tau$, and in (2) we use twice the formula (1.3).

Note that it is no longer true that the tensor product of not necessarily multiplicative representations, even of a multiplicative Hom-Lie algebra, is a representation.

Lemma 1.3. The tensor product of two nilpotent multiplicative representations of a Hom-Lie algebra is nilpotent.

Proof. Let ρ , τ be two nilpotent multiplicative representations of a Hom-Lie algebra L, with indices of nilpotency n and m, respectively. For each $x_1, \ldots, x_{n+m-1} \in L$, the expression

$$(1.4) (\rho \otimes \tau)(x_1) \circ \cdots \circ (\rho \otimes \tau)(x_{n+m-1})$$

is equal to the sum of elements of the form $B \otimes C$, where the first tensor factor B is of the form

$$\beta^{r_1} \circ \rho(x_{i_1}) \circ \beta^{r_2} \circ \rho(x_{i_2}) \circ \cdots \circ \beta^{r_k} \circ \rho(x_{i_k}) \circ \beta^{r_{k+1}}$$

for some (possibly empty) subset of indices $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n+m-1\}$, and some (possibly zero) $r_1, \ldots, r_{k+1} \in \mathbb{N}$, the second tensor factor C has the same form with β replaced by γ , and ρ by τ , and the respective subsets of indices occurring in B and C are complementary in $\{1, \ldots, n+m-1\}$ (so the sum of the number of $\rho(x_i)$'s in the first tensor factor and in the second tensor factor is equal to n+m-1).

Using the relation (1.3), one can put in the expression (1.5) all β 's "to the right", getting "at the left" expression of the form

$$(1.6) \qquad \qquad \rho(\alpha^{s_1}(x_{i_1})) \circ \rho(\alpha^{s_2}(x_{i_2})) \circ \cdots \circ \rho(\alpha^{s_k}(x_{i_k}))$$

for some (possibly zero) $s_1, \ldots, s_k \in \mathbb{N}$. Similarly, in the second tensor factor C we can get "at the left" expression of the form

(1.7)
$$\tau(\alpha^{t_1}(x_{j_1})) \circ \tau(\alpha^{t_2}(x_{j_2})) \circ \cdots \circ \tau(\alpha^{t_\ell}(x_{j_\ell}))$$

for some $t_1, \ldots, t_\ell \in \mathbb{N}$.

Since $k + \ell = n + m - 1$, we have either $k \ge n$, or $\ell \ge m$, and at least one of expressions (1.6) and (1.7) vanishes, thus the whole expression (1.4) vanishes, which shows that $\rho \otimes \tau$ is nilpotent of index $\le n + m - 1$.

2. Hom-associative algebras and their representations

In this section we recall basic definitions and facts related to Hom-associative algebras (see [MS08], [Y09]), similarly how it was done for Hom-Lie algebras in the previous section. Though Hom-associative algebras are not our primary concern in this paper, it is necessary to consider them in relation to Hom-Lie algebras. Our exposition culminates in Theorem 1 which establishes an equivalence of the two statements which can be considered as "Ado theorem for Hom-Lie algebras" – existence of a faithful representation, and embedding into a Hom-associative algebra – similarly with the ordinary Lie case.

Definition 2.1. A Hom-algebra (A, \cdot, α) is called a *Hom-associative algebra*, if the following *Hom-associative identity* holds:

$$(x \cdot y) \cdot \alpha(z) = \alpha(x) \cdot (y \cdot z)$$

for any $x, y, z \in A$.

An element $u \in A$ is called a *unit* if $x \cdot u = u \cdot x = \alpha(x)$ for any $x \in A$.

Definition 2.2. If (A, \cdot, α) is a Hom-algebra, then the Hom-algebra defined on the same Hom-vector space (A, α) , and with multiplication $[\cdot, \cdot]$ defined by the commutator:

$$[x, y] = x \cdot y - y \cdot x$$

for $x, y \in A$, is called an algebra associated to A, and is denoted by $(A^{(-)}, [\cdot, \cdot], \alpha)$.

Proposition 2.1. A Hom-algebra associated to a Hom-associative algebra A, is Hom-Lie. If A is multiplicative (respectively, nondegenerate), then the associated Hom-algebra $A^{(-)}$ is multiplicative (respectively, nondegenerate).

Proof. An easy computation, similar to its (classical) non-Hom counterpart. The statement about multiplicativity and nondegeneracy is evident.

Performing the same Eilenberg game as in §1 in the class of Hom-associative algebras, we arrive at

Definition 2.3. Let (A, \cdot, α) be a Hom-associative algebra, and (V, β) a Hom-vector space.

A linear map $\rho_L: A \to \text{End}(V)$ is called a *left representation of A in V*, if

(2.1)
$$\rho_L(x \cdot y) \circ \beta = \rho_L(\alpha(x)) \circ \rho_L(y)$$

holds for any $x, y \in A$.

A linear map $\rho_R: A \to \text{End}(V)$ is called a right representation of A in V, if

(2.2)
$$\beta \circ \rho_R(x \cdot y) = \rho_R(x) \circ \rho_R(\alpha(y))$$

holds for any $x, y \in A$.

A pair of linear maps (ρ_L, ρ_R) , where $\rho_L, \rho_R : A \to \operatorname{End}(V)$, is called a *birepresentation of A in V*, if ρ_L is a left representation, ρ_R is a right representation, and, additionally, the following compatibility condition

(2.3)
$$\rho_L(\alpha(x)) \circ \rho_R(y) = \rho_L(x) \circ \rho_R(\alpha(y))$$

holds for any $x, y \in A$

The left representation ρ_L is called *multiplicative* if, in addition to (2.1),

$$\rho_L(\alpha(x)) \circ \beta = \beta \circ \rho_L(x)$$

holds for any $x, y \in A$; the right representation ρ_R is called *multiplicative* if, in addition to (2.2),

$$\rho_R(\alpha(x)) \circ \beta = \beta \circ \rho_R(x)$$

holds for any $x, y \in A$; and the birepresentation ρ_L, ρ_R is called *multiplicative*, if ρ_L is a left multiplicative representation, and ρ_R is a right multiplicative representation.

The representation or birepresentation is called *nondegenerate* if the Hom-vector space (V, β) is non-degenerate.

In all these situations, V is called a (left, right, bi, multiplicative, nondegenerate) module over A.

This definition extends those adopted in [MS10] (where left representations are considered), and is a particular case of those in [GMMP] (where representations of BiHom-algebras – in which the associative identity is deformed by two twist maps instead of one – are considered).

Note that as in the ordinary associative case, we cannot define in any reasonable way the tensor product of two representations of a Hom-associative algebra, multiplicative or not.

The following is a Hom-associative analog of Lemma 1.2.

Lemma 2.1. Let A be a Hom-associative algebra. Then the map sending an element of A to the left (respectively, right) multiplication on that element, is a left (respectively, right) representation of A in itself, and together they form a birepresentation of A in itself. The kernel of this left (respectively, right) representation coincide with the left annulator

$$Ann_L(A) = \{x \in A \mid xA = 0\},\$$

and with the right annulator

$$Ann_R(A) = \{x \in A \mid Ax = 0\},\$$

respectively.

Proof. All the three conditions defining a birepresentation – of left representation (2.1), of right representation (2.2), and the compatibility condition (2.3) – are equivalent to the Hom-associative identity. The statement about the kernels is obvious.

Lemma 2.2. Let A be an associative algebra with unit, and $a \in A$ an invertible element. Then (A, \cdot, Ad_a) , where the new multiplication \cdot in A is defined by

$$x \cdot y = axa^{-1}ya^{-1}$$

for $x, y \in A$, and $Ad_a : A \to A$ is the adjoint (in the group sense) map:

$$Ad_a(x) = axa^{-1}$$

for $x \in A$, is a multiplicative nondegenerate Hom-associative algebra.

Proof. Straightforward computation, see [SX, Proposition 4.1], or [GMMP, Lemma 4.3] for a more general BiHom-associative case. The nondegeneracy is evident. □

Lemma 2.3. Let A be a Hom-associative algebra, and (V,β) a Hom-vector space with β invertible. A linear map $\rho: A \to \operatorname{End}(V)$ is a left (respectively, right) multiplicative representation of A, if and only if ρ is a homomorphism from the Hom-associative algebra A to the Hom-associative algebra $(\operatorname{End}(V), \circ, \operatorname{Ad}_{\beta^{-1}})$.

Proof. Straightforward computation, see [SX, Theorem 4.2] for a similar computation in the Hom-Lie case, or [GMMP, Proposition 4.4] for a more general BiHom-associative case. □

The following could be dubbed as "Ado theorem for Hom-associative algebras". Of course, its associative analog is trivial (adjoin a unit, and consider the representation in itself), and the proof below just mimics this triviality in the Hom case.

Lemma 2.4. Any (finite-dimensional) multiplicative nondegenerate Hom-associative algebra admits a faithful (finite-dimensional) left (or right) nondegenerate representation.

Proof. Let (A, \cdot, α) be a multiplicative nondegenerate Hom-associative algebra. Adjoin to A a unit (in the Hom sense, see Definition 2.1); namely, consider the direct sum $\widehat{A} = A \oplus K1_{\alpha}$, and extend to it the multiplication and the twist map as follows:

$$x \cdot 1_{\alpha} = 1_{\alpha} \cdot x = \alpha(x)$$

for any $x \in \widehat{A}$, and

$$\alpha(1_{\alpha}) = 1_{\alpha}$$
.

Then $(\widehat{A}, \cdot, \alpha)$ is a multiplicative nondegenerate Hom-associative algebra, containing A as a subalgebra. By Lemma 2.1, the left (right) representation of \widehat{A} in itself has zero kernel, and its restriction to A provides the desired representation.

Lemma 2.5. Let L be a Hom-Lie algebra, and (V,β) a Hom-vector space with β invertible. A linear map $\rho: L \to \operatorname{End}(V)$ is a multiplicative representation of L, if and only if ρ is a homomorphism from the Hom-Lie algebra L to the Hom-Lie algebra $(\operatorname{End}(V)^{(-)}, [\cdot, \cdot], \operatorname{Ad}_{\beta})$ (where $[\cdot, \cdot]$ is a commutator with respect to composition of linear maps on V).

Proof. This is Theorem 4.2 from [SX], or a particular case of Proposition 4.10 from [GMMP].

Theorem 1. For a Hom-Lie algebra L, the following are equivalent:

- (i) L admits a finite-dimensional faithful multiplicative nondegenerate representation;
- (ii) L is embedded into a Hom-Lie algebra of the form $A^{(-)}$, where A is a finite-dimensional multiplicative nondegenerate Hom-associative algebra.

Proof. Let us introduce another auxiliary condition:

(iii) *L* is embedded into a Hom-Lie algebra of the form $(\operatorname{End}(V)^{(-)}, [\,\cdot\,,\cdot\,], \operatorname{Ad}_{\beta})$ for some finite-dimensional nondegenerate Hom-vector space (V,β) .

Then (iii) \Rightarrow (ii) is obvious, and (i) \Leftrightarrow (iii) follows from Lemma 2.5.

(ii) \Rightarrow (iii): By Lemma 2.4, A admits a faithful (say, left) representation in a finite-dimensional nondegenerate Hom-vector space (V,β) , and then by Lemma 2.3, A is embedded into a Hom-associative algebra of the form $(\operatorname{End}(V), \circ, \operatorname{Ad}_{\beta})$. Consequently, $A^{(-)}$, and hence L, is embedded into the Hom-Lie algebra $(\operatorname{End}(V)^{(-)}, [\cdot, \cdot], \operatorname{Ad}_{\beta})$.

3. NILPOTENT ALGEBRAS

According to [MM], the notion of nilpotency is carried over to the Hom-Lie case verbatim:

Definition 3.1. A Hom-Lie algebra $(L, [\cdot, \cdot], \alpha)$ is called *nilpotent* if it is nilpotent as an algebra; that is, $L^n = 0$ for some $n \in \mathbb{N}$, where L^n , the members of the lower central series of L, are defined inductively: $L^1 = L$, and $L^n = [L^{n-1}, L]$ for n > 1. The minimal number n such that $L^n = 0$ is called *the degree of nilpotency*, or *nilindex*. A nilpotent Hom-Lie algebra of degree 2 (i.e., with zero multiplication) is called *abelian*.

Note that for a multiplicative Hom-Lie algebra L we have $\alpha(L^n) \subseteq L^n$ for any n.

Note also that the Yau twist of a nilpotent Lie algebra $(L, [\cdot, \cdot])$ along a Lie algebra homomorphism $\alpha: L \to L$, is a nilpotent Hom-Lie algebra; that is, $(L, [\cdot, \cdot]_{\alpha}, \alpha)$ is a nilpotent Hom-Lie algebra, where $[x, y]_{\alpha} = [\alpha(x), \alpha(y)]$. Conversely, if $(L, [\cdot, \cdot], \alpha)$ is a multiplicative nondegenerate nilpotent Hom-Lie algebra, then its untwist $(L, \alpha^{-1}[\cdot, \cdot])$ is a nilpotent Lie algebra.

Definition 3.2. A Hom-Lie algebra L is called *strongly nilpotent*, if there is a chain of ideals of L, starting with 0 and ending with L:

$$(3.1) 0 = I_n \triangleleft I_{n-1} \triangleleft \cdots \triangleleft I_2 \triangleleft I_1 = L$$

such that for each 1 < i < n, dim $I_i/I_{i+1} = 1$ and $[L, I_i] \subseteq I_{i+1}$.

Lemma 3.1. A finite-dimensional Hom-Lie algebra L is strongly nilpotent if and only if it satisfies the following property: every nonzero ideal I of L contains an ideal J of codimension 1 in I such that $[L, I] \subseteq J$.

Proof. The "if" part: starting with $I_1 = L$, we build inductively the chain of ideals such that I_{i+1} is of codimension 1 in I_i , and $[L, I_i] \subseteq I_{i+1}$. Since L is finite-dimensional, this chain will terminate at 0.

The proof of the "only if" part repeats verbatim the proof of [Z, Lemma 2.8], with "Lie" being replaced by "Hom-Lie", and "nilpotent" by "strongly nilpotent".

Lemma 3.2. A strongly nilpotent Hom-Lie algebra is nilpotent.

Proof. The proof is exactly the same as in the Lie case (see, for example, [Bo, §4, Proposition 1]). Let L be a strongly nilpotent Hom-Lie algebra with the chain of ideals (3.1). Then by induction we have $L^i \subseteq I_i$ for any $1 \le i \le n$; in particular, $L^n \subseteq I_n = 0$.

It is well known that in the class of Lie algebras the converse is true, i.e., the notions of nilpotency and strong nilpotency coincide. This is not so in the class of Hom-Lie algebras: a trivial example is provided by an abelian Hom-Lie algebra L of dimension > 1. Ideals in L are exactly Hom-subspaces, so if we choose the twist map α in such a way that there exists no proper α -invariant subspaces, L will be not strongly nilpotent. Of course, this requires the ground field to be not algebraically closed. However, over algebraically closed fields, and in the class of finite-dimensional multiplicative Hom-Lie algebras, the notions of nilpotency and strong nilpotency, like in the Lie case, do coincide. To prove this, we need the following elementary linear-algebraic fact.

Lemma 3.3. For any two Hom-subspaces $A \subsetneq B$ of a finite-dimensional Hom-vector space over an algebraically closed field, there is a third Hom-subspace C such that $A \subseteq C \subset B$, and C is of codimension 1 in B.

(In essence, this says that a finite-dimensional abelian Hom-Lie algebra over an algebraically closed field is strongly nilpotent).

Proof. In fact, for any integer n such that $\dim A \leq n \leq \dim B$, there exists a Hom-subspace C of dimension n sitting between A and B. This is proved by induction on n: the induction step consists of taking the quotient B/C by the n-dimensional Hom-subspace C containing A, taking eigenvector (i.e., an 1-dimensional Hom-subspace) of the twist map in this quotient, and passing back to its preimage in B.

Proposition 3.1. A finite-dimensional multiplicative Hom-Lie algebra over an algebraically closed field is nilpotent if and only if it is strongly nilpotent.

Proof. The "if" part is covered by Lemma 3.2, so let us prove the "only if" part.

Let I be a nonzero ideal in a finite-dimensional nilpotent multiplicative Hom-Lie algebra L defined over an algebraically closed field. First note that $[L, I] \neq I$. Indeed, if [L, I] = I, then for any $n \in \mathbb{N}$, $[L, \ldots, [L, [L, I]] \ldots] = I$ (where L occurs n times). But the left-hand side in the last equality lies in L^{n+1} , and hence vanishes for some n, a contradiction.

Since L is multiplicative, $\alpha([L, I]) \subseteq [\alpha(L), \alpha(I)] \subseteq [L, I]$, and hence [L, I] is an ideal in L. By Lemma 3.3, there is a Hom-subspace J of L such that $[L, I] \subseteq J \subset I$, and J is of codimension 1 in I. Then $[L, J] \subseteq [L, I] \subseteq J$, so J is an ideal.

Thus we see that any nonzero ideal I of L contains an ideal J of codimension 1 in I such that $[L, I] \subseteq J$, and by Lemma 3.1 L is strongly nilpotent.

4. Free algebras

In this section we review the construction and elementary properties of free and free nilpotent Hom-Lie algebras. As the previous sections, it does not contain anything really new; everything here is either implicitly available in the literature (see, for example, [R, §49.2] which treats the more general case of free linear algebras with several multiary operations), or is a straightforward application of the general universal-algebraic machinery.

The main outcome of this section is the fact – sounding almost tautologically trivial – that any finite-dimensional nilpotent (multiplicative, nondegenerate) Hom-Lie algebra is a homomorphic image of a – suitably defined – finite-dimensional \mathbb{N} -graded free nilpotent (multiplicative, nondegenerate) Hom-Lie algebra.

We start with an obvious

Definition 4.1. Let G be an abelian semigroup. A Hom-Lie algebra $(L, [\cdot, \cdot], \alpha)$ is called G-graded if it is G-graded as an algebra, i.e. L is decomposed as the direct sum $L = \bigoplus_{g \in G} L_g$ with $[L_g, L_h] \subseteq L_{g+h}$ for any $g, h \in G$, and each homogeneous component is stable under α : $\alpha(L_g) \subseteq L_g$ for any $g \in G$.

By abstract universal algebraic nonsense, free Hom-Lie algebras can be defined as just free algebraic systems in the variety of Hom-Lie algebras considered as vector spaces with one binary and one unary operation.

A more concrete construction can run as follows. Let X be a set. Consider all (nonassociative) words in X, and all possible "applications" of α on subwords in those words. More precisely, define recursively

$$(4.1) F_1(X) = \{\alpha^{\ell}(x) \mid x \in X, \ \ell \in \mathbb{N}\}$$

(we assume that $\alpha^0(x) = x$), and, for n > 1,

(4.2)
$$F_n(X) = \bigcup_{k=1}^{n-1} \{ \alpha^{\ell}(uv) \mid u \in F_k(X), v \in F_{n-k}(X), \ \ell \in \mathbb{N} \}.$$

The multiplication between elements of $F(X) = \bigcup_{n\geq 1} F_n(X)$ is performed by concatenation; thus, $F_n(X)F_m(X) \subseteq F_{n+m}(X)$. The twist map α maps a word $u \in F(X)$ to the word $\alpha(u)$, and applying the relation $\alpha(\alpha^{\ell}(v)) = \alpha^{\ell+1}(v)$ for any $\ell \in \mathbb{N}$ and $v \in F(X)$; thus, $\alpha(F_n(X)) \subseteq F_n(X)$. Then F(X) forms a magma with a unary operation α , and the magma algebra KF(X) over the ground field K, with α

extended by linearity, forms the free Hom-algebra with the generating set X. It is obvious that KF(X) is an \mathbb{N} -graded Hom-algebra:

(4.3)
$$KF(X) = \bigoplus_{n \in \mathbb{N}} KF_n(X).$$

The free Hom-Lie algebra $\mathcal{L}(X)$ freely generated by a set X is obtained as a quotient of KF(X) by the ideal generated by elements of the form uv + vu and $(uv)\alpha(w) + (wu)\alpha(v) + (vw)\alpha(u)$, where $u, v, w \in KF(X)$. Since this ideal is homogeneous with respect to the grading (4.3), $\mathcal{L}(X)$ is \mathbb{N} -graded too.

Of course, this is equivalent to constructing $\mathcal{L}(X)$ directly by the inductive process (4.1)–(4.2), where instead of juxtaposition uv one takes the bracket [u, v] (which is assumed to be anticommutative and satisfying the Hom-Jacobi identity). Clearly, the inductively constructed elements in this case will be no longer linearly independent, and the question of constructing a basis of a free Hom-Lie algebra, similar to one of the known bases of free Lie algebras, seems to be a difficult one.

In [HMS, §3] the elements of free Hom-algebras (Lie and associative) are represented as labeled binary trees: branching, as usual, corresponds to the binary multiplication, and labels are equal to the exponent ℓ in "application" of the power α^{ℓ} to the given vertex.

According to the general universal-algebraic principles, it would be natural to define the free nilpotent Hom-Lie algebra $\mathcal{N}_n(X)$ of degree n and freely generated by a set X, as the quotient of $\mathcal{L}(X)$ by the ideal generated by $\mathcal{L}(X)^n$. This definition is, however, unsatisfactory for our purposes: below, when proving an analog of the Ado theorem for nilpotent Hom-Lie algebras, we want to stay in the category of finite-dimensional algebras, but $\mathcal{N}_n(X)$, unlike its ordinary Lie-algebraic counterpart, is obviously infinite-dimensional, as for every its element u it contains all the powers $\alpha^{\ell}(u)$, all of them are linearly independent. This can be remedied in the following way. Observe that for a finite-dimensional Hom-Lie algebra, the twist map α , being a linear map on a finite-dimensional vector space, satisfies some polynomial equation

$$f(\alpha) = 0.$$

We fix this equation and add it – or, rather, the identity $f(\alpha)(x) = 0$ for any $x \in L$ – to the defining relations of the corresponding free algebra, and define $\mathcal{N}_{n,f}(X)$ as the quotient of $\mathcal{L}(X)$ by the ideal generated by $\mathcal{L}(X)^n$ and $f(\alpha)(\mathcal{L}(X))$ (or, what is the same, as the quotient of $\mathcal{N}_n(X)$ by the ideal generated by $f(\alpha)(\mathcal{N}_n(X))$). The ensuing algebra $\mathcal{N}_{n,f}(X)$ is, obviously, finite-dimensional. The ideal $\mathcal{L}(X)^n$ of $\mathcal{L}(X)$ is homogeneous, and, since α preserves the \mathbb{N} -grading of $\mathcal{L}(X)$, the ideal generated by $f(\alpha)(\mathcal{L}(X))$ is homogeneous too, so $\mathcal{N}_{n,f}(X)$ remains to be \mathbb{N} -graded. Any finite-dimensional nilpotent Hom-Lie algebra with a generating set X, of degree n, and satisfying the condition (4.4), is a quotient of $\mathcal{N}_{n,f}(X)$.

In the multiplicative variant of all these, we need additionally factorize by the ideal generated by elements of the form

$$\alpha([u,v]) - [\alpha(u),\alpha(v)],$$

where u, v are elements from the respective free Hom-algebra. Roughly speaking, that means that we may move all α 's to the "innermost positions". In terms of the inductive process (4.1)–(4.2) that means that keeping (4.1), we may define the free multiplicative Hom-algebra as a free (nonassociative) algebra freely generated by the set $F_1(X)$. In terms of the labeled trees used in [HMS] that means that we label only the terminal vertices. Let us denote the multiplicative analog of $\mathcal{N}_{n,f}(X)$, i.e. the quotient of the latter algebra by the ideal generated by the elements of the form (4.5), by $\mathcal{M}_{n,f}(X)$. Obviously, $\mathcal{M}_{n,f}(X)$ remains to be \mathbb{N} -graded. Any finite-dimensional nilpotent multiplicative Hom-Lie algebra with a generating set X, of degree n, and satisfying the condition (4.4), is a quotient of $\mathcal{M}_{n,f}(X)$.

Note that while finite-dimensional nondegenerate Hom-algebras are closed with respect to homomorphic images, nondegenerate Hom-algebras in general are not closed, so they do not form a variety; but they form a quasivariety, and hence we can still speak about free nondegenerate Hom-Lie algebras. In a finite-dimensional situation the nondegeneracy of α can be expressed in terms of the polynomial f: indeed, we may take f to be the characteristic or the minimal polynomial of α , and then the nondegeneracy

of α is equivalent to the nonvanishing of the free term of f. Thus, free (multiplicative) nondegenerate nilpotent Hom-Lie algebras are merely $\mathcal{N}_{n,f}(X)$ (or $\mathcal{M}_{n,f}(X)$) with f having the nonvanishing free term.

It is clear that all the free Hom-algebras considered here do not depend on the set X itself, but merely on its cardinality |X|. In particular, instead of $\mathcal{N}_{n,f}(X)$ and $\mathcal{M}_{n,f}(X)$, we will write $\mathcal{N}_{k,n,f}$ and $\mathcal{M}_{k,n,f}$ respectively, where k = |X|.

5. Ado theorem for nilpotent Hom-Lie algebras

As explained in the Introduction, we follow the scheme of [Z].

It is obvious that if a Hom-Lie algebra has a finite-dimensional faithful (multiplicative, nondegenerate) representation, then so does any its subalgebra.

Lemma 5.1. A finite-dimensional \mathbb{N} -graded Hom-Lie algebra L is embedded into the current Hom-Lie algebra $L \otimes tK[t]/(t^n)$ for some $n \in \mathbb{N}$.

(Note that a finite-dimensional N-graded algebra is necessarily nilpotent).

Proof. Verbatim repetition of the proof of [Z, Lemma 2.1].

Lemma 5.2. If a finite-dimensional (nilpotent, multiplicative, nondegenerate) Hom-Lie algebra has a nondegenerate α -derivation, then it has a finite-dimensional faithful (nilpotent, multiplicative, nondegenerate) representation.

Proof. Let L be a finite-dimensional Hom-Lie algebra having a nondegenerate α -derivation D. The desired representation is the action of L on the ambient Hom-Lie algebra $L \oplus KD$. If L is nilpotent, or multiplicative, or nondegenerate, then this action is respectively nilpotent, or multiplicative, or nondegenerate too.

Note that a classical result of Jacobson, [J], says that a finite-dimensional Lie algebra over a field of characteristic zero having a nondegenerate derivation, is necessarily nilpotent. It appears to be an interesting question whether the same is true for Hom-Lie algebras; if true, the condition of nilpotency in Lemma 5.2 is redundant.

One can generalize Lemma 5.2 by considering 1-cocycles in an arbitrary module instead of α -derivations, similarly to the ordinary Lie case ([Z, Lemma 2.3]). However, this will require us to analyze various definitions in the literature of cohomology of Hom-Lie algebras, even more diverse then those of derivations or representations; we want to avoid this task here, and confine ourselves with Lemma 5.2 which is enough for our purposes.

Lemma 5.3. A finite-dimensional (multiplicative) nondegenerate \mathbb{N} -graded Hom-Lie algebra over a field of characteristic zero has a finite-dimensional faithful nilpotent (multiplicative) nondegenerate representation.

Proof. Let $(L, [\cdot, \cdot], \alpha)$ be such a Hom-Lie algebra. By Lemma 5.1, L is embedded into the nilpotent current Hom-Lie algebra $L \otimes tK[t]/(t^n)$. By Lemma 1.1, the latter Hom-Lie algebra is nondegenerate, and is multiplicative if L is so. The map $\alpha \otimes t \frac{d}{dt}$ is a nondegenerate α -derivation of $L \otimes tK[t]/(t^n)$, and hence by Lemma 5.2 $L \otimes tK[t]/(t^n)$ has a finite-dimensional faithful nilpotent nondegenerate (and multiplicative, if L is multiplicative) representation; and so does its subalgebra L.

Both conditions of nondegeneracy and zero characteristic are needed here to ensure nondegeneracy of $\alpha \otimes t \frac{\mathrm{d}}{\mathrm{d}t}$: the nondegeneracy of α ensures that it acts nondegenerately on the first tensor factor L, and the zero characteristic ensures that the Euler derivation $t \frac{\mathrm{d}}{\mathrm{d}t}$ acts nondegenerately on the second tensor factor $tK[t]/(t^n)$: $t^i \mapsto it^i$ for $i \in \mathbb{N}$. We conjecture Lemma 5.3 (in fact, the whole Ado theorem) remains to be true without those restrictions, but a proof of this will require a different approach (see discussion in [Z, §3] most of which may be applicable also to the Hom-Lie case).

Lemma 5.4. Suppose the ground field is of characteristic zero. For any $k, n \in \mathbb{N}$, and any polynomial f with the nonzero free term:

- (i) the free nilpotent Hom-Lie algebra $\mathcal{N}_{k,n,f}$ has a finite-dimensional faithful nilpotent nondegenerate representation;
- (ii) the free nilpotent multiplicative Hom-Lie algebra $\mathcal{M}_{k,n,f}$ has a finite-dimensional faithful nilpotent multiplicative nondegenerate representation.

Proof. As noted in §4, the algebras in question are finite-dimensional, \mathbb{N} -graded, and nondegenerate. Apply Lemma 5.3.

The following lemma shows that instances of "local faithfulness" can be assembled to a "global" one.

Lemma 5.5. Let L be a finite-dimensional Hom-Lie algebra such that for any nonzero $x \in L$ there is a finite-dimensional (nilpotent, multiplicative, nondegenerate) representation ρ_x of L such that $\rho_x(x) \neq 0$. Then L has a finite-dimensional faithful (nilpotent, multiplicative, nondegenerate) representation.

Proof. The proof repeats verbatim those of [Z, Lemma 2.7]. If each ρ_x is nilpotent, or multiplicative, or nondegenerate, the resulting representation, being assembled as a direct sum of nilpotent, or multiplicative, or nondegenerate representations, is itself nilpotent, or multiplicative, or nondegenerate.

The following lemma, of combinatorial character, is an almost verbatim repetition of [Z, Lemma 2.10] which treats the Lie case, and shows that one can always distinguish elements of a finite-dimensional nilpotent Hom-Lie algebra by kernels of suitable representations.

Lemma 5.6. Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra over a field K of characteristic $\neq 2$, having a finite-dimensional faithful nilpotent multiplicative nondegenerate representation. Then for any two linearly independent elements $x, y \in L$ such that $\alpha(x) = \lambda x$ for some nonzero $\lambda \in K$, there is a finite-dimensional nilpotent multiplicative nondegenerate representation ρ of L such that $\text{Ker } \rho(x) \not\subset \text{Ker } \rho(y)$.

Proof. Suppose the contrary: there are two linearly independent elements $x, y \in L$ such that $\alpha(x) = \lambda x$ for some nonzero $\lambda \in K$, and for any finite-dimensional nilpotent multiplicative nondegenerate representation $\rho: L \to \operatorname{End}(V)$, it holds that $\operatorname{Ker} \rho(x) \subseteq \operatorname{Ker} \rho(y)$. Then there is a linear map $h_\rho: V \to V$ (depending on ρ) such that

(5.1)
$$\rho(y) = h_{\rho} \circ \rho(x).$$

Let $\rho: L \to \operatorname{End}(V)$ and $\tau: L \to \operatorname{End}(W)$ be two finite-dimensional nilpotent multiplicative nondegenerate representations of L in Hom-vector spaces (V,β) and (W,γ) , and let n and m be the indices of nilpotency of the linear maps $\rho(x)$ and $\tau(x)$, respectively. According to Proposition 1.1 and Lemma 1.3, the tensor product $\rho \otimes \tau$ is also multiplicative and nilpotent, and it is obviously nondegenerate. Writing the condition (5.1) for $\rho \otimes \tau$, with both sides applied to elements $\rho(x)^{n-2}(v) \otimes \tau(x)^{m-1}(\gamma^{-1}(w))$ and $\rho(x)^{n-1}(\beta^{-1}(v)) \otimes \tau(x)^{m-2}(w)$, where $v \in V$ and $w \in W$, and taking into account the same condition for ρ and for τ , we get respectively:

(5.2)
$$h_{\rho}(\rho(x)^{n-1}(v)) \otimes \gamma(\tau(x)^{m-1}(\gamma^{-1}(w))) = h_{\rho \otimes \tau}(\rho(x)^{n-1}(v) \otimes \gamma(\tau(x)^{m-1}(\gamma^{-1}(w))))$$

and

(5.3)
$$\beta(\rho(x)^{n-1}(\beta^{-1}(v))) \otimes h_{\tau}(\tau(x)^{m-1}(w)) = h_{\rho \otimes \tau}(\beta(\rho(x)^{n-1}(\beta^{-1}(v))) \otimes \tau(x)^{m-1}(w)).$$

Since $\alpha(x) = \lambda x$, and due to the condition of multiplicativity (1.3), we have

(5.4)
$$\beta \circ \rho(x)^k = \lambda^k \rho(x)^k \circ \beta$$

for any $k \in \mathbb{N}$, and similarly for τ and γ , so the equalities (5.2)–(5.3) can be rewritten as

$$h_{\rho}(\rho(x)^{n-1}(v)) \otimes \tau(x)^{m-1}(w) = h_{\rho \otimes \tau}(\rho(x)^{n-1}(v) \otimes \tau(x)^{m-1}(w))$$

and

$$\rho(x)^{n-1}(v) \otimes h_{\tau}\Big(\tau(x)^{m-1}(w)\Big) = h_{\rho \otimes \tau}\Big(\rho(x)^{n-1}(v) \otimes \tau(x)^{m-1}(w)\Big)$$

for any $v \in V$ and $w \in W$, respectively. This implies that the linear maps $h_{\rho} \otimes \operatorname{id}$ and $\operatorname{id} \otimes h_{\tau}$ coincide on the vector space $\rho(x)^{n-1}(V) \otimes \tau(x)^{m-1}(W)$, whence

(5.5)
$$h_{\rho}(\rho(x)^{n-1}(v)) = \mu \rho(x)^{n-1}(v)$$

and

$$h_{\tau}(\tau(x)^{m-1}(w)) = \mu \tau(x)^{m-1}(w)$$

for some $\mu \in K$. Since this holds for any pair of representations ρ , τ , we get that (5.5) holds for any finite-dimensional nilpotent multiplicative nondegenerate representation ρ of L for some uniform value of μ .

Further, writing the condition (5.1) for the tensor product $\rho \otimes \tau$ applied to elements $\rho(x)^{n-3}(\beta(v)) \otimes \tau(x)^{m-1}(\gamma^{-1}(w))$, $\rho(x)^{n-1}(\beta^{-1}(v)) \otimes \tau(x)^{m-3}(\gamma(w))$, and $\rho(x)^{n-2}(v) \otimes \tau(x)^{m-2}(w)$, and taking into account (5.5), we get respectively:

$$h_{\rho}(\rho(x)^{n-2}(\beta(v))) \otimes \tau(x)^{m-1}(w) = h_{\rho \otimes \tau}(\rho(x)^{n-2}(\beta(v)) \otimes \tau(x)^{m-1}(w)),$$

$$\rho(x)^{n-1}(v) \otimes h_{\tau}\left(\tau(x)^{m-2}(\gamma(w))\right) = h_{\rho \otimes \tau}\left(\rho(x)^{n-1}(v) \otimes \tau(x)^{m-2}(\gamma(w))\right),$$

and

$$\mu \Big(\lambda^{m-2} \rho(x)^{n-1}(v) \otimes \tau(x)^{m-2} (\gamma(w)) + \lambda^{n-2} \rho(x)^{n-2} (\beta(v)) \otimes \tau(x)^{m-1}(w) \Big)$$

$$= h_{\rho \otimes \tau} \Big(\lambda^{m-2} \rho(x)^{n-1}(v) \otimes \tau(x)^{m-2} (\gamma(w)) + \lambda^{n-2} \rho(x)^{n-2} (\beta(v)) \otimes \tau(x)^{m-1}(w) \Big).$$

Taking a linear combination of the first two of these equalities with coefficients λ^{n-2} and λ^{m-2} , and subtracting the third one, we get that the linear map

$$(5.6) \qquad ((h_{\rho} - \mu \operatorname{id}) \circ \beta) \otimes \tau(x) + \rho(x) \otimes ((h_{\tau} - \mu \operatorname{id}) \circ \gamma)$$

is identically zero on the vector space $\rho(x)^{n-2}(V) \otimes \tau(x)^{m-2}(W)$, whence

$$(h_{\rho} - \mu \operatorname{id}) \circ \beta = \eta_{\rho,\tau} \rho(x)$$

and

$$(h_{\tau} - \mu \operatorname{id}) \circ \gamma = -\eta_{\rho,\tau} \tau(x)$$

for some $\eta_{\rho,\tau} \in K$, as linear maps on $\rho(x)^{n-2}(V)$ and on $\tau(x)^{m-2}(W)$, respectively. Note that this holds for any pair of representations ρ, τ . Taking $\rho = \tau$, we get $\eta_{\rho,\rho} = -\eta_{\rho,\rho}$, whence $\eta_{\rho,\rho} = 0$ (this is the place where we need the assumption that characteristic of the ground field is different from 2), and

$$h_{\rho}(\beta(\rho(x)^{n-2}(v))) = \mu\beta(\rho(x)^{n-2}(v))$$

for any $v \in V$. Using here again (5.4), and taking $\beta^{-1}(v)$ instead of v, we get

$$h_{\rho}(\rho(x)^{n-2}(v)) = \mu \rho(x)^{n-2}(v)$$

for any finite-dimensional nilpotent multiplicative nondegenerate representation ρ of L^{\dagger} .

Repeating this procedure by considering at the kth step the condition (5.1) for the tensor product of two representations ρ and τ , and applying it to all elements of the form $\rho(x)^i(v) \otimes \tau(x)^j(w)$ with i + j equal to n + m - (k + 2), where n and m are the indices of nilpotency of $\rho(x)$ and $\tau(x)$ respectively, and heavily using (5.4), we consecutively arrive at the equalities

$$h_{\rho}(\rho(x)^{n-k}(v)) = \mu \rho(x)^{n-k}(v)$$

[†] The corresponding reasoning in the proof of [Z, Lemma 2.10] (p. 678 in the published version, and p. 4 in the arXiv version) is in error: at one place, the sign went wrong – the linear maps $(h_{\rho} - \lambda \operatorname{id}) \otimes \tau(x)$ and $\rho(x) \otimes (h_{\tau} - \lambda \operatorname{id}) \otimes \tau(x)$ do not coincide, but sum up to zero (compare with the expression (5.6) here). The correct reasoning is obtained from one in the present paper by assuming all the twist maps are equal to identity map. In fact, the correct reasoning is slightly simpler than the original one, and avoids the necessity to assume non-vanishing of certain binomial coefficients. This is significant, as it eliminates one of the places where the characteristic zero assumption is needed (see the discussion in [Z, §3]).

for each k = 1, 2, ..., n. For k = n this means $h_{\rho} = \mu$ id, and hence $\rho(y - \mu x) = 0$ for any finite-dimensional nilpotent multiplicative nondegenerate representation ρ of L, what implies $y - \mu x = 0$, a contradiction.

It seems that the conclusion of Lemma 5.6 is valid for arbitrary linearly independent elements x, y of the Hom-Lie algebra, without the assumption $\alpha(x) = \lambda x$. However, this assumption greatly simplifies the calculations, and we will not venture into attempting to give a proof without it; Lemma 5.6 as stated is enough for our purposes here.

Finally, we arrive at our main result, whose proof is assembled from the previous lemmas exactly in the same way as the proof of [Z, Theorem 2.11].

Theorem 2. A finite-dimensional nilpotent multiplicative nondegenerate Hom-Lie algebra over an algebraically closed field of characteristic zero has a finite-dimensional faithful nilpotent multiplicative nondegenerate representation.

Proof. Let us present a finite-dimensional nilpotent multiplicative nondegenerate Hom-Lie algebra L defined over an algebraically closed field K of characteristic zero, as a quotient $\mathcal{M}_{k,n,f}/I$ for suitable $k, n \in \mathbb{N}$ and polynomial f (with the nonzero free term). We will proceed by induction on the dimension of I. The case I = 0 is covered by Lemma 5.4(ii).

Suppose I is nonzero. By Proposition 3.1 $\mathcal{M}_{k,n,f}$ is strongly nilpotent, and by Lemma 3.1 there is an ideal J of $\mathcal{M}_{k,n,f}$ such that $J \subset I$, dim I/J = 1, and $[\mathcal{M}_{k,n,f}, I] \subseteq J$. Consequently, $\widetilde{L} = \mathcal{M}_{k,n,f}/J$ is an extension of L by an one-dimensional central ideal, say, $K\widetilde{z}$. Since $K\widetilde{z}$ is an ideal in the Hom-Lie algebra $(\widetilde{L}, [\cdot, \cdot], \alpha)$, we have $\alpha(\widetilde{z}) = \lambda \widetilde{z}$ for some $\lambda \in K$; and since $\mathcal{M}_{k,n,f}$ is finite-dimensional and nondegenerate, so is \widetilde{L} , what implies $\lambda \neq 0$.

Take an arbitrary nonzero $x \in L$, and consider its preimage \tilde{x} in \widetilde{L} . By the induction assumption, \widetilde{L} has a finite-dimensional faithful nilpotent multiplicative nondegenerate representation, and by Lemma 5.6 there is a finite-dimensional nilpotent multiplicative nondegenerate representation $\rho: \widetilde{L} \to \operatorname{End}(V)$ such that

(5.7)
$$\operatorname{Ker} \rho(\tilde{z}) \not\subset \operatorname{Ker} \rho(\tilde{x}).$$

Since \tilde{z} lies in the center of \widetilde{L} , for any $\tilde{y} \in \widetilde{L}$ we have

$$\rho(\alpha(\tilde{y})) \circ \rho(\tilde{z}) = \lambda \rho(\tilde{z}) \circ \rho(\tilde{y}),$$

and hence the vector space $\operatorname{Ker} \rho(\widetilde{z})$ is an \widetilde{L} -submodule of V, and thus also carries a natural structure of an L-module on which x acts nontrivially, due to (5.7). Since the initial representation ρ of \widetilde{L} is nilpotent, multiplicative, and nondegenerate, the ensuing representation τ of L is nilpotent, multiplicative, and nondegenerate too.

Thus we see that for any nonzero $x \in L$ there is a finite-dimensional nilpotent multiplicative nondegenerate representation τ of L such that $\tau(x) \neq 0$, and by Lemma 5.5 L has a finite-dimensional faithful nilpotent multiplicative nondegenerate representation.

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REFERENCES

- [ABM] A. Arfa, N. Ben Fraj, and A. Makhlouf, *Morphisms cohomology and deformations of Hom-algebras*, J. Nonlin. Math. Phys. **25** (2018), no. 4, 589–603; arXiv:1710.07599.
- [BM] S. Benayadi and A. Makhlouf, *Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms*, J. Geom. Phys. **76** (2014), 38–60; arXiv:1009.4226.
- [Bi] G. Birkhoff, *Representability of Lie algebras and Lie groups by matrices*, Ann. Math. **38** (1937), no. 2, 526–532; reprinted in *Selected Papers on Algebra and Topology*, Birkhäuser, 1987, 332–338.
- [Bo] N. Bourbaki, Groupes et algèbres de Lie. Chapitre 1, Hermann, Paris, 1972; reprinted by Springer, 2007.
- [CG] S. Caenepeel and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra **39** (2011), no. 6, 2216–2240; arXiv:0907.0187.

- [CH] X. Chen and W. Han, *Classification of multiplicative simple Hom-Lie algebras*, J. Lie Theory **26** (2016), no. 3, 767–775.
- [GMMP] G. Graziani, A. Makhlouf, C. Menini, and F. Panaite, *BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras*, SIGMA **11** (2015), 086; arXiv:1505.00469.
- [GZZ] L. Guo, B. Zhang, and S. Zheng, *Universal enveloping algebras and Poincaré-Birkhoff-Witt theorem for involutive Hom-Lie algebras*, J. Lie Theory **28** (2018), no. 3, 735–756; arXiv:1607.05973.
- [HMS] L. Hellström, A. Makhlouf, and S.D. Silvestrov, *Universal algebra applied to Hom-associative algebras, and more*, Algebra, Geometry and Mathematical Physics (ed. A. Makhlouf et al.), Springer Proc. Math. Stat. **85** (2014), 157–199; arXiv:1404.2516.
- [J] N. Jacobson, *A note on automorphisms and derivations of Lie algebras*, Proc. Amer. Math. Soc. **6** (1955), no. 2, 281–283; reprinted in *Collected Mathematical Papers*, Vol. 2, Birkhäuser, 1989, 251–253.
- [LMT] C. Laurent-Gengoux, A. Makhlouf, and J. Teles, *Universal algebra of a Hom-Lie algebra and group-like elements*, J. Pure Appl. Algebra **222** (2018), no. 5, 1139–1163; arXiv:1505.02439.
- [MM] A. Makhlouf and M. Mehidi, On classification of filiform Hom-Lie algebras; arXiv:1807.08324.
- [MZ] A. Makhlouf and P. Zusmanovich, *Hom-Lie structures on Kac-Moody algebras*, J. Algebra **515** (2018), 278–297; arXiv:1805.00187.
- [MS08] A. Makhlouf and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), no. 2, 51–64; arXiv:math/0609501.
- [MS10] A. Makhlouf and S. Silvestrov, *Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras*, Forum Math. **22** (2010), no. 4, 715–739; arXiv:0712.3130.
- [R] Yu.P. Razmyslov, Identities of Algebras and Their Representations, Nauka, 1989 (in Russian); AMS, 1994 (English translation).
- [S] Y. Sheng, *Representations of Hom-Lie algebras*, Algebras Repr. Theory **15** (2012), no. 6, 1081–1098; arXiv:1005.0140.
- [SX] Y. Sheng and Z. Xiong, *On Hom-Lie algebras*, Lin. Multilin. Algebra **63** (2015), no. 12, 2379–2395; arXiv:1411.6839.
- [Y09] D. Yau, *Hom-algebras and homology*, J. Lie Theory **19** (2009), no. 2, 409–421; arXiv:0712.3515.
- [Y12] D. Yau, The Mikheev identity in right Hom-alternative algebras; arXiv:1205.0148.
- [Z] P. Zusmanovich, Yet another proof of the Ado theorem, J. Lie Theory 26 (2016), no. 3, 673–681; arXiv:1507.02233.

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