# Explicit solutions of certain orientable quadratic equations in free 

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#### Abstract

For $g \geq 1$ denote by $F_{2 g}=\left\langle x_{1}, y_{1}, \ldots, x_{g}, y_{g}\right\rangle$ the free group on $2 g$ generators and let $B_{g}=\left[x_{1}, y_{1}\right] \ldots\left[x_{g}, y_{g}\right]$. For $l, c \geq 1$ and elements $w_{1}, \ldots, w_{l} \in F_{2 g}$ we study orientable quadratic equations of the form $\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=\left(B_{g}^{w_{1}}\right)^{c}\left(B_{g}^{w_{2}}\right)^{c} \ldots\left(B_{g}^{w_{l}}\right)^{c}$ with unknowns $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ and provide explicit solutions for them for the minimal possible number $h$.

In the particular case when $g=1, w_{i}=y_{1}^{i-1}$ for $i=1, \ldots, l$ and $h$ the minimal number which satisfies $h \geq l(c-1) / 2+1$ we provide two types of solutions depending on the image of the subgroup $H=\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle$ generated by the solution under the natural homomorphism $p: F_{2} \rightarrow F_{2} /\left[F_{2}, F_{2}\right]$ : the first solution, which is called a primitive solution, satisfies $p(H)=F_{2} /\left[F_{2}, F_{2}\right]$, the second solution satisfies $p(H)=\left\langle p\left(x_{1}\right), p\left(y_{1}^{l}\right)\right\rangle$.

We also provide an explicit solution of the equation $\left[u_{1}, v_{1}\right] \ldots\left[u_{k}, v_{k}\right]=\left(B_{1}\right)^{k+l}\left(B_{1}\right)^{k-l}$ for $k>l \geq 0$ in $F_{2}$, and prove that if $l \neq 0$, then every solution of this equation is primitive.

As a geometrical consequence, for every solution we obtain a map $f: S_{h} \rightarrow T$ from the orientable surface $S_{h}$ of genus $h$ to the torus $T=S_{1}$ which has the minimal number of roots among all maps from the homotopy class of $f$. Depending on the number $\left|p\left(F_{2}\right): p(H)\right|$ such maps have fundamentally different geometric properties: in some cases they satisfy the Wecken property and in other cases not.


Keywords: Orientable quadratic equation, free group, Nielsen root number, Wecken property.

## 1 Introduction and preliminaries

Let $G$ be a group and $S$ be a symmetric subset of $G$, i. e. a subset such that $1 \notin S$ and $S=S^{-1}$. For an element $a$ from $\langle S\rangle$ denote by $l_{S}(a)$ the minimal number $k$ such that $a$ is a product of $k$ elements from $S$. The number $l_{S}(a)$ is called the length of $a$ with respect to $S$. The numbers $l_{S}(a)$ and especially the value $\sup \left(l_{S}(a) \mid a \in\langle S\rangle\right)$ for different sets $S$ in different groups $G$ has been studied by various authors (see, for example, [22] and references therein).

If $S$ is the set of all nontrivial commutators in $G$, then $l_{S}(a)$ is called the commutator length (or the genus) of an element $a \in[G, G]$. The problem of determining the commutator length of an element $a \in[G, G]$ corresponds to the problem of finding the minimal $h$ for which the equation

$$
\begin{equation*}
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=a \tag{1}
\end{equation*}
$$

with unknowns $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ admits a solution in $G$. Such equation is called an orientable quadratic equation. The word "quadratic" means that every variable in the left side of the equation appears exactly twice. The word "orientable" means that every unknown variable $x$

[^0]appears once in the form $x$ and once in the form $x^{-1}$. If there exists a variable $x$ in a quadratic equation which appears twice with exponent 1 , then this equation is called non-orientable.

The problem of finding a solution of equation (1) in the free group is closely related with coincidence theory of maps between orientable surfaces, which includes the case of the study of roots. See, for example, [8, Fundamental Lemma 1.2] and [3, 5, 12].

Many works concerning the problem of finding solutions of quadratic equations and especially of equation (1) in different groups have been done from both algebraic and geometric points of view. For many years great attention was paid to equations in free groups [3, 5, $, 7,8,12,15,17,20,23,25]$. The particular case of equation (1) when $h=1$ was studied in [25] (see also [11]). For a given quadratic equation with any number of unknown variables in any free group with the right-hand side an arbitrary element an algorithm for solving the problem of the existence of a solution was given by Culler [7] using the surface method and generalizing the result of Wicks [25]. Based on different techniques, the problem has been studied by the first named author with coauthors [9-11] for parametric families of quadratic equations arising from continuous maps between closed surfaces.

The question about the existence of a solution of equation (1) can be solved in many cases. However, the majority of results are either about non-existence of a solution, or about existence only and they do not provide an algorithm how to find an explicit solution. The following result from [12, Proposition 4.2] gives one simple necessary condition for solvability of equation (1) with the right-hand side of the special form motivated by geometry.

Proposition 1.1. Let $w_{1}, \ldots, w_{l}$ be distinct elements of the free group $F_{2 g}=\left\langle x_{1}, y_{1}, \ldots, x_{g}, y_{g}\right\rangle$ and let $c_{1}, \ldots, c_{l}$ be integers which are all positive or all negative. Denote by $B_{g}=$ $\left[x_{1}, y_{1}\right] \ldots\left[x_{g}, y_{g}\right]$. If the equation

$$
\begin{equation*}
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=\left(B_{g}^{w_{1}}\right)^{c_{1}}\left(B_{g}^{w_{2}}\right)^{c_{2}} \ldots\left(B_{g}^{w_{l}}\right)^{c_{l}} \tag{2}
\end{equation*}
$$

with unknowns $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ is solvable in $F_{2 g}$, then $\left(\left|c_{1}\right|+\cdots+\left|c_{l}\right|\right)(2 g-1) \leq 2 h-2+l$.
Here and throughout the paper for elements $a, b$ we denote by $a^{b}=b a b^{-1}$ the conjugate of $a$ by $b$, and by $[a, b]=a b a^{-1} b^{-1}$ the commutator of elements $a, b$.

Orientable quadratic equations with the right-hand side as in (2) (not necessary for $c_{1}, \ldots, c_{l}$ all positive or all negative) are of special interest in geometry. If $f: S_{h} \rightarrow S_{g}$ is a continuous map between orientable surfaces, then it induces a map $f_{\#}: \pi_{1}\left(S_{h}\right) \rightarrow \pi_{1}\left(S_{g}\right)$ between fundamental groups. Denoting by $\pi_{1}\left(S_{h}\right)=\left\langle x_{1}, y_{1}, \ldots, x_{h}, y_{h} \mid\left[x_{1}, y_{1}\right] \ldots\left[x_{h}, y_{h}\right]=1\right\rangle$ and $f_{\#}\left(x_{i}\right)=u_{i}$, $f_{\#}\left(y_{i}\right)=v_{i}$ we must have $\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=1$ in $\pi_{1}\left(S_{g}\right)$, i. e. $\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]$ must be expressible as a right-side of (2). So, there is a strong connection between maps between orientable surfaces and orientable quadratic equations with the right hand side as in (2).

Note that the result of Proposition 1.1 says that if $\left(\left|c_{1}\right|+\cdots+\left|c_{l}\right|\right)(2 g-1)>2 h-2+l$, then equation (2) does not have a solution independently of elements $w_{1}, \ldots, w_{l}$. However there is no guarantee that a solution exists for $\left(\left|c_{1}\right|+\cdots+\left|c_{l}\right|\right)(2 g-1) \leq 2 h-2+l$. Moreover a solution can exist for some elements $w_{1}, \ldots, w_{l}$ but not for others. For example, if $g=1, l=2, c_{1}=c_{2}=1$, then $h=1$. Using Wicks criterion [25] it is easy to show that the equation $[u, v]=B_{1} B_{1}^{y_{1}^{2}}$ has no solutions in $F_{2}=\left\langle x_{1}, y_{1}\right\rangle$. However, the equation $[u, v]=B_{1} B_{1}^{y_{1}}$ has a solution $u=x_{1}, v=y_{1}^{2}$. So, it is reasonable to ask for which integers $c_{1}, \ldots, c_{l}$ and elements $w_{1}, \ldots, w_{l}$ equation (2) has a solution, and when it has, provide this solution. Not much is known about this problem.

In the present work we consider equation (2) in the free group $F_{2 g}$ with right parts of special forms and our goal is to provide explicit solutions for them. In turn, for $g=1$ this will provide existence of maps from the orientable surface of genus $h$ into the torus which have some features about root theory. For some cases we will find two types of solutions depending on the index of the image of the subgroup $H=\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle$ generated by the solution under the natural
homomorphism $p: F_{2 g} \rightarrow \pi_{1}\left(S_{g}\right)=F_{2 g} /\left\langle B_{g}\right\rangle^{F_{2 g}}$ in $\pi_{1}\left(S_{g}\right)$ ．In order to explain the importance of the value $\left|\pi_{1}\left(S_{g}\right): p(H)\right|$ let us recall some facts from Nielsen root theory．

Let $f: M_{1} \rightarrow M_{2}$ be a continuous map between closed manifolds $M_{1}, M_{2}$ and let $c \in M_{2}$ ． Every element from $f^{-1}(c)$ is called a root．The minimal number of roots in the homotopy class of a map $f$ is the number $M R[f]=\min _{g \simeq f}\left(\left|g^{-1}(c)\right|\right)$ ，where $\simeq$ denotes the homotopy equivalence．This number does not depend on $c$ ．Two roots $x, y \in M_{1}$ are said to belong to the same Nielsen root class if there exists a path $\gamma$ in $M_{1}$ connecting $x, y$ such that $f(\gamma)$ is contractible．For a map $f$ between two manifolds of the same dimension an index for a Nielsen root class is defined in［16］．A Nielsen root class is called essential if its index is not equal to zero．The indices of all essential Nielsen root classes coincide．The Nielsen root number $N R[f]$ is the number of essential Nielsen root classes，this number is always finite and it satisfies the inequality $N R[f] \leq M R[f]$ ．If $N R[f]=M R[f]$ ，then $f$ is said to possess the Wecken property． The map $f$ induces the map $f_{\#}: \pi_{1}\left(M_{1}\right) \rightarrow \pi_{1}\left(M_{2}\right)$ between fundamental groups．Denote by $l(f)=\left|\pi_{1}\left(M_{2}\right): f_{\#}\left(\pi_{1}\left(M_{1}\right)\right)\right|$ if $\left|\pi_{1}\left(M_{2}\right): f_{\#}\left(\pi_{1}\left(M_{1}\right)\right)\right|$ is finite，and $l(f)=0$ otherwise．If $M_{1}=S_{h}, M_{2}=S_{g}$ are closed orientable surfaces of genus $h, g$ respectively，then the map $f$ induces a homomorphism beween second homology groups $\mathbb{Z}=H_{2}\left(S_{h}\right) \rightarrow H_{2}\left(S_{g}\right)=\mathbb{Z}$ ．This map acts as a multiplication by some number $n$ ．This number is called the degree of $f$ and is denoted by $\operatorname{deg}(f)$ ．The absolute value $|\operatorname{deg}(f)|$ is denoted by $A(f)$ ．In［3，Theorem 1．1］it is proved that，if $A(f) \neq 0$ ，then

$$
\begin{equation*}
M R[f]=\max \left(l(f), \chi\left(M_{1}\right)+\left(1-\chi\left(M_{2}\right)\right) A(f)\right) \quad N R[f]=l(f) \tag{3}
\end{equation*}
$$

where $\chi$ denotes the Euler characteristic of the surface．
If $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ is some solution of equation（2），then one can construct a continuous map $f: S_{h} \rightarrow S_{g}$ which satisfies the following conditions： $\operatorname{deg}(f)=c_{1}+\cdots+c_{l},\left|f^{-1}(y)\right|=l$ for some point $y \in S_{g}$ ，the index of every Nielsen root class of $f$ is equal to $c_{i_{1}}+\cdots+c_{i_{k}}$ for some indices $i_{1}, \ldots, i_{k}$ and if $\pi_{1}\left(S_{h}\right)=\left\langle x_{1}, y_{1}, \ldots, x_{h}, y_{h} \mid\left[x_{1}, y_{1}\right] \ldots\left[x_{h}, y_{h}\right]=1\right\rangle$ ，then $f_{\#}\left(x_{i}\right)=u_{i}$ ， $f_{\#}\left(y_{i}\right)=v_{i}$ ．If $\operatorname{deg}(f)=0$ ，then $N R[f]=0$ and $\left|p\left(F_{2 g}\right): p(H)\right|$ can be either finite or infinite．If $\operatorname{deg}(f) \neq 0$ ，then $\left|p\left(F_{2 g}\right): p(H)\right|$ is finite and $N R[f]=\left|p\left(F_{2 g}\right): p(H)\right|$ ，where $p: F_{2 g} \rightarrow \pi_{1}\left(S_{g}\right)$ is a natural homomorphism and $H=\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle$ ．See details about the construction of $f$ in［12，Proposition 4．2］，here we are going to use only the properties of the constructed map．

The following result gives some information about the index of $p(H)$ in $\pi_{1}\left(S_{g}\right)$ ．
Proposition 1．2．Let $p: F_{2 g} \rightarrow \pi_{1}\left(S_{g}\right)$ be the homomorphism which sends the free generators of $F_{2 g}$ to the canonical system of generators of the fundamental group $\pi_{1}\left(S_{g}\right)$ of an orientable surface $S_{g}$ of genus $g$ ．If $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ is a solution of equation（⿴囗⿱一𧰨丶 ）with $c_{1}+c_{2}+\ldots+c_{l} \neq 0$ ， then the index of $p\left(\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle\right)$ in $\pi_{1}\left(S_{g}\right)$ is less than or equal to $l$ ．

Proof．Let $f: S_{h} \rightarrow S_{g}$ be the described before the proposition map constructed by the solution $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ ．For some point $y \in S_{g}$ the number of elements in the preimage of $y$ under $f$ is $l$ ，therefore $M R[f] \leq l$ ．Since $\operatorname{deg}(f)=c_{1}+\cdots+c_{l} \neq 0$ ，we have $\left|p\left(F_{2 g}\right): p(H)\right|=N R[f] \leq$ $M R[f] \leq l$ ．

In the present paper we will find explicit solutions for particular cases of equation（22）which are in some sense＂critical＂from the point of view of Proposition 1．2；the first solution which is called a primitive solution satisfies the equality $p(H)=\pi_{1}\left(S_{g}\right)$（the word＂primitive＂appears here naturally from the notion of primitives in free groups［21］），and the second solution satisfies $\left|\pi_{1}\left(S_{g}\right): p(H)\right|=l$ ．

In Section 2 we study the equation（2）for $c_{1}=c_{2}=\cdots=c_{l}=c$ and prove that if for $c=1$ this equation has a solution which generates a subgroup $H_{1}$ of $F_{2 g}$ ，then for every $c$ it has a solution which generates a subgroup $H_{2}$ such that $p\left(H_{1}\right)=p\left(H_{2}\right)$（Theorem［2．2）．In Section 3 we consider the particular case of this equation in the free group $F_{2}=\langle x, y\rangle$ with the right part
of the form $([x, y])^{c}\left([x, y]^{y}\right)^{c} \ldots\left([x, y]^{y^{l-1}}\right)^{c}$ for integers $c, l \geq 1$. We provide an explicit algebraic algorithm for finding the solution in a minimal subgroup (Corollary 3.2), and an algorithm for finding a primitive solution (Theorem 3.5) for such equation. In Section 4 we consider equation (22) in the free group $F_{2}=\langle x, y\rangle$ with the right part of the form $([x, y])^{k+l}\left([x, y]^{y}\right)^{k-l}$ for integers $k>l \geq 0$. We construct an explicit primitive solution for such equation (formulas (16), (19)) and prove that for $l \neq 0$ every solution of such equation is primitive (Theorem4.1). Some geometrical consequences derived from this algebraic results are formulated in Corollaries 3.3, 3.8,

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## 2 The right part has the form $\left(B_{g}^{w_{1}}\right)^{c}\left(B_{g}^{w_{2}}\right)^{c} \ldots\left(B_{g}^{w_{l}}\right)^{c}$

The purpose of this section is to give an explicit solution for equation (2) in the particular case when $c_{1}=c_{2}=\cdots=c_{l}=c$

$$
\begin{equation*}
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=\left(B_{g}^{w_{1}}\right)^{c}\left(B_{g}{ }^{w_{2}}\right)^{c} \ldots\left(B_{g}{ }^{w_{l}}\right)^{c} \tag{4}
\end{equation*}
$$

for the minimal integer $h$ which satisfies the inequality $h \geq l(c(2 g-1)-1) / 2+1$. We can assume that $c>0$ since if $c<0$, then denoting by $x_{i}^{\prime}=y_{g+1-i}, y_{i}^{\prime}=x_{g+1-i}$ for $i=1, \ldots, g$ we have $F_{2 g}=\left\langle x_{1}^{\prime}, y_{1}^{\prime}, \ldots, x_{g}^{\prime}, y_{g}^{\prime}\right\rangle$, and in these generators equation (4) has the same form where $c$ is changed by $-c$. In the case when $h<l(c(2 g-1)-1) / 2+1$ by Proposition 1.1 equation (4) does not have solutions.

At first, we need the following simple lemma.
Lemma 2.1. The word $w=a \xi_{1} b \xi_{2} c$ is a product of the commutator $\left[a \xi_{1} a^{-1}, a b a^{-1}\right]$ and the element $a b \xi_{1} \xi_{2} c$.

Proof. Straightforward calculation.
The main result of this section is the following theorem.
Theorem 2.2. Let $l, c, g \geq 1$ be integers, $h$ be the minimal integer which satisfies the inequality $h \geq l(c(2 g-1)-1) / 2+1, w_{1}, \ldots, w_{l}$ be elements of the free group $F_{2 g}=\left\langle x_{1}, y_{1}, \ldots, x_{g}, y_{g}\right\rangle$ and $B_{g}=\left[x_{1}, y_{1}\right] \ldots\left[x_{g}, y_{g}\right]$. If for $c=1$ the equation

$$
\begin{equation*}
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=\left(B_{g}^{w_{1}}\right)^{c}\left(B_{g}^{w_{2}}\right)^{c} \ldots\left(B_{g}^{w_{l}}\right)^{c} \tag{5}
\end{equation*}
$$

has a solution which generates the subgroup $H_{1}$ of $F_{2 g}$, then for an arbitrary $c \geq 1$ it has a solution (explicitly constructed from the given solution for $c=1$ ) which generates the subgroup $H_{2}$ of $F_{2 g}$ such that if $p: F_{2 g} \rightarrow \pi_{1}\left(S_{g}\right)$ is the natural homomorphism, then $p\left(H_{1}\right)=p\left(H_{2}\right)$.

Proof. We will construct the solution inductively on the variable $c$. The basis of induction $c=1$ is given as the condition of the theorem. Suppose that the statement is proved for $c=n$ and let us prove that it holds for $c=n+1$. We will consider two cases depending on the parity of $l$.

Case 1: $l$ is even. Rewrite the right-hand side of equation (5) for $c=n+1$ in the following form.

$$
\begin{align*}
\left(B_{g}^{w_{1}}\right)^{n+1}\left(B_{g}^{w_{2}}\right)^{n+1} \ldots\left(B_{g}^{w_{l}}\right)^{n+1} & =\left(B_{g}^{w_{1}}\right)\left(\left(B_{g}^{w_{1}}\right)^{n}\left(B_{g}^{w_{2}}\right)^{n} \ldots\left(B_{g}^{w_{l}}\right)^{n}\right)\left(B_{g}^{w_{1}}\right)^{-1} \\
& \cdot\left(B_{g}^{w_{1}}\right)\left(B_{g}^{w_{l}}\right)^{-n}\left(B_{g}^{w_{l}-1}\right)^{-n} \ldots\left(B_{g}^{w_{2}}\right)^{-n} \\
& \cdot\left(B_{g}^{w_{2}}\right)^{n+1}\left(B_{g}^{w_{3}}\right)^{n+1} \ldots\left(B_{g}^{w_{l}}\right)^{n+1} \tag{6}
\end{align*}
$$

By the induction hypothesis there exist $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ for $h=l(n(2 g-1)-1) / 2+1$ such that $\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=\left(B_{g}^{w_{1}}\right)^{n}\left(B_{g}^{w_{2}}\right)^{n} \ldots\left(B_{g}^{w_{l}}\right)^{n}$ and $p\left(\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle\right)=p\left(H_{1}\right)$. So, it is enough to prove that the product of two last lines of equation (6)

$$
\begin{equation*}
\left(B_{g}^{w_{1}}\right)\left(B_{g}^{w_{l}}\right)^{-n}\left(B_{g}^{w_{l-1}}\right)^{-n} \ldots\left(B_{g}^{w_{3}}\right)^{-n}\left(B_{g}^{w_{2}}\right)\left(B_{g}^{w_{3}}\right)^{n+1} \ldots\left(B_{g}^{w_{l}}\right)^{n+1} \tag{7}
\end{equation*}
$$

is the product of $l((n+1)(2 g-1)-1) / 2+1-l(n(2 g-1)-1) / 2-1=l(2 g-1) / 2$ commutators of elements images of which under $p$ belong to $p\left(H_{1}\right)$. In equation (7) denoting by $a=B_{g}^{w_{1}}$, $\xi_{1}=\left(B_{g}^{w_{l}}\right)^{-n}\left(B_{g}^{w_{l-1}}\right)^{-n}, \quad b=\left(B_{g}^{w_{l-2}}\right)^{-n} \ldots\left(B_{g}^{w_{3}}\right)^{-n}\left(B_{g}^{w_{2}}\right)\left(B_{g}^{w_{3}}\right)^{n+1} \ldots\left(B_{g}^{w_{l-2}}\right)^{n+1}\left(B_{g}^{w_{l-1}}\right)$, $\xi_{2}=\left(B_{g}^{w_{l-1}}\right)^{n}\left(B_{g}^{w_{l}}\right)^{n}, c=\left(B_{g}^{w_{l}}\right)$ and applying Lemma 2.1 we conclude that expression (7) is a product of the commutator of elements which belong to the kernel of $p$ times the element

$$
\left(B_{g}^{w_{1}}\right)\left(B_{g}^{w_{l-2}}\right)^{-n}\left(B_{g}^{w_{l-3}}\right)^{-n} \ldots\left(B_{g}^{w_{3}}\right)^{-n}\left(B_{g}^{w_{2}}\right)\left(B_{g}^{w_{3}}\right)^{n+1} \ldots\left(B_{g}^{w_{l-2}}\right)^{n+1}\left(B_{g}^{w_{l-1}}\right)\left(B_{g}^{w_{l}}\right)
$$

Repeating this idea denoting by $\xi_{1}=\left(B_{g}^{w_{l-2}}\right)^{-n}\left(B_{g}^{w_{l-3}}\right)^{-n}, \xi_{2}=\left(B_{g}^{w_{l-3}}\right)^{n}\left(B_{g}^{w_{l-2}}\right)^{n}$, we conclude that expression (7) is a product of two commutators of elements which belong to the kernel of $p$ times the element

$$
\left(B_{g}^{w_{1}}\right)\left(B_{g}^{w_{l-4}}\right)^{-n}\left(B_{g}^{w_{l-5}}\right)^{-n} \ldots\left(B_{g}^{w_{3}}\right)^{-n}\left(B_{g}^{w_{2}}\right)\left(B_{g}^{w_{3}}\right)^{n+1} \ldots\left(B_{g}^{w_{l-4}}\right)^{n+1}\left(B_{g}^{w_{l-3}}\right) \ldots\left(B_{g}^{w_{l}}\right)
$$

Repeating this procedure $(l-2) / 2$ times we conclude that expression (7) is the product of $(l-2) / 2$ commutators of elements which belong to the kernel of $p$ times the element $B_{g}^{w_{1}} B_{g}^{w_{2}} \ldots B_{g}^{w_{l}}$ which (by the induction hypothesis for $c=1$ ) is the product of $l(g-1)+1$ commutators of elements images of which under $p$ belong to $p\left(H_{1}\right)$. Therefore expression (77) is the product of $(l-2) / 2+l(g-1)+1=l(2 g-1) / 2$ commutators of elements images of which under $p$ belong to $p\left(H_{1}\right)$.

Case 2: $l$ is odd. Rewrite the right-hand side of equation (5) for $c=n+1$ in the following form.

$$
\begin{align*}
\left(B_{g}^{w_{1}}\right)^{n+1} & \left(B_{g}{ }^{w_{2}}\right)^{n+1} \ldots\left(B_{g}^{w_{l}}\right)^{n+1}= \\
= & \left(\left(B_{g}^{w_{1}}\right)\left(B_{g}{ }^{w_{2}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right) \\
& \cdot\left(\left(B_{g}^{w_{2}}\right)\left(B_{g}{ }^{w_{3}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right)^{-1}\left(\left(B_{g}^{w_{1}}\right)\left(B_{g}{ }^{w_{2}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right)\left(\left(B_{g}^{w_{2}}\right)\left(B_{g}{ }^{w_{3}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right) \\
& \cdot\left(\left(B_{g}^{w_{2}}\right)\left(B_{g}{ }^{w_{3}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right)^{-2}\left(\left(B_{g}^{w_{1}}\right)^{n-1}\left(B_{g}{ }^{w_{2}}\right)^{n-1} \ldots\left(B_{g}^{w_{l}}\right)^{n-1}\right)\left(\left(B_{g}^{w_{2}}\right)\left(B_{g}{ }^{w_{3}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right)^{2} \\
& \cdot\left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{2}}\right)^{-1}\left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{2}}\right)^{-1} \\
& \cdot\left(B_{g}{ }^{w_{l}}\right)^{-n+1} \ldots\left(B_{g}{ }^{w_{3}}\right)^{-n+1}\left(B_{g}{ }^{w_{2}}\right)^{2}\left(B_{g}{ }^{w_{3}}\right)^{n+1} \ldots\left(B_{g}{ }^{w_{l}}\right)^{n+1} \tag{8}
\end{align*}
$$

At first, we consider the particular case $n+1=2$. In this case equality (8) implies

$$
\begin{align*}
\left(B_{g}^{w_{1}}\right)^{2} & \left(B_{g}^{w_{2}}\right)^{2} \ldots\left(B_{g}^{w_{l}}\right)^{2}= \\
& =\left(\left(B_{g}^{w_{1}}\right)\left(B_{g}{ }^{w_{2}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right) \\
& \cdot\left(\left(B_{g}^{w_{2}}\right)\left(B_{g}{ }^{w_{3}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right)^{-1}\left(\left(B_{g}^{w_{1}}\right)\left(B_{g}{ }^{w_{2}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right)\left(\left(B_{g}^{w_{2}}\right)\left(B_{g}{ }^{w_{3}}\right) \ldots\left(B_{g}^{w_{l}}\right)\right) \\
& \cdot\left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{2}}\right)^{-1}\left(B_{g}^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{2}}\right)^{-1} \\
& \cdot\left(B_{g}^{w_{2}}\right)^{2}\left(B_{g}{ }^{w_{3}}\right)^{2} \ldots\left(B_{g}{ }^{w_{l}}\right)^{2} \tag{9}
\end{align*}
$$

So, if $n+1=2$, then since by induction hypothesis for $c=1$ the product of the first two lines of equation (9) is a product of $2 l(g-1)+2$ commutators, it is enough to prove that the product of two last lines in (9)

$$
\begin{equation*}
\left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}^{w_{2}}\right)^{-1}\left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{3}}\right)^{-1}\left(B_{g}{ }^{w_{2}}\right)\left(B_{g}{ }^{w_{3}}\right)^{2} \ldots\left(B_{g}{ }^{w_{l}}\right)^{2} \tag{10}
\end{equation*}
$$

is a product of $\lceil l(2(2 g-1)-1) / 2\rceil+1-2 l(g-1)-2=(l-1) / 2$ commutators. Similarly to the first case (when $l$ is even) applying Lemma 2.1 to expression (10) for $\xi_{1}=\left(B^{w_{3}}\right)^{-1}\left(B^{w_{2}}\right)^{-1}$, $\xi_{2}=\left(B^{w_{2}}\right)\left(B^{w_{3}}\right)$ we conclude that (10) is a product of a commutator of elements which belong to $\left\langle B_{g}\right\rangle^{F_{2 g}}$ times the same expression without elements $B^{w_{2}}, B^{w_{3}}$

$$
\left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{4}}\right)^{-1}\left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{5}}\right)^{-1}\left(B_{g}{ }^{w_{4}}\right)\left(B_{g}{ }^{w_{5}}\right)^{2} \ldots\left(B_{g}{ }^{w_{l}}\right)^{2} .
$$

Repeating this procedure $(l-1) / 2$ times we conclude that expression (10) is a product of $(l-1) / 2$ commutators of elements which belong to the kernel of $p$, i. e. the case $n+1=2$ is proved.

For the general case $n>1$ by the induction hypothesis for $c=1$ the product of the first two lines of (8) is the product of $2 l(g-1)+2$ commutators, by the induction hypothesis for $c=n-1$ the third line of (8) is the product of $\lceil l((n-1)(2 g-1)-1) / 2+1\rceil$ commutators of elements such that their images under $p$ generate $H_{1}$ (the case $n+1=2$ was necessary for making this inductive step). So, it is enough to prove that the product of two last lines in expression (8)

$$
\begin{align*}
& \left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{2}}\right)^{-1}\left(B_{g}{ }^{w_{l}}\right)^{-1} \ldots\left(B_{g}{ }^{w_{2}}\right)^{-1} . \\
& \quad \cdot\left(B_{g}{ }^{w_{l}}\right)^{-n+1} \ldots\left(B_{g}{ }^{w_{3}}\right)^{-n+1}\left(B_{g}{ }^{w_{2}}\right)^{2}\left(B_{g}{ }^{w_{3}}\right)^{n+1} \ldots\left(B_{g}{ }^{w_{l}}\right)^{n+1} \tag{11}
\end{align*}
$$

is a product of $\lceil l((n+1)(2 g-1)-1) / 2+1\rceil-\lceil l((n-1)(2 g-1)-1) / 2+1\rceil-2 l(g-1)-2=l-2$ commutators of elements images of which under $p$ belong to $p\left(H_{1}\right)$.

If we apply Lemma 2.1 twice to expression (11) for $\xi_{1}=\left(B_{g}^{w_{3}}\right)^{-1}\left(B_{g}^{w_{2}}\right)^{-1}, \xi_{2}=$ $\left(B_{g}^{w_{2}}\right)\left(B_{g}^{w_{3}}\right)$, then we conclude that expression (11) is a product of two commutators (of elements which belong to the kernel of $p$ ) times expression (11) without elements $\left(B_{g}^{w_{2}}\right),\left(B_{g}^{w_{3}}\right)$. If we repeat this procedure $(l-3) / 2$ times, we conclude that expression (11) is a product of $2(l-3) / 2=l-3$ commutators times the expression

$$
\left(B_{g}^{w_{l}}\right)^{-1}\left(B_{g}^{w_{l-1}}\right)^{-1}\left(B_{g}^{w_{l}}\right)^{-1}\left(B_{g}^{w_{l-1}}\right)^{-1}\left(B_{g}^{w_{l}}\right)^{-n+1}\left(B_{g}^{w_{l-1}}\right)^{2}\left(B_{g}^{w_{l}}\right)^{n+1}
$$

which is equal to the commutator $\left(B_{g}^{w_{l}}\right)^{-1}\left[\left(B_{g}^{w_{l-1}}\right)^{-1}\left(B_{g}^{w_{l}}\right)^{n-1},\left(B_{g}^{w_{l}}\right)^{-n}\left(B_{g}^{w_{l}-1}\right)^{-1}\right]\left(B_{g}^{w_{l}}\right)$, i. e. expression (11) is a product of $l-2$ commutators.

Remark 2.3. We can suppose that if $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ is a solution of equation (5) constructed in the proof of Theorem 2.2 for an arbitrary $c$, then $u_{1}, v_{1}, \ldots, u_{l(g-1)+1}, v_{l(g-1)+1}$ is the solution of equation (5) for $c=1$.

Remark 2.4. In particular case when $g=1, l=1, w_{1}=1, c=2 h-1$ the explicit solution of equation (5) is given in [12, Proposition 4.6].

The following result, which is a consequence of the remark above, is a corollary of Theorem 2.2.

Corollary 2.5. Let $G$ be a group, $a, b \in G$, and $n$ be an integer. Then $[a, b]^{n}$ can be expressed as the product of at most $\lceil(n+1) / 2\rceil$ commutators.

Proof. The equation $[u, v]=[x, y]$ has a solution $u=x, v=y$ in $F_{2}=\langle x, y\rangle$. Therefore the equation $\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=[x, y]^{n}$ has a solution in $F_{2}=\langle x, y\rangle$ for $h=\lceil(n+1) / 2\rceil$. Acting on the equality $\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=[x, y]^{n}$ by the homomorphism $\varphi: F_{2} \rightarrow G$ which is induced by $\varphi(x)=a, \varphi(y)=b$ we get the result.

## 3 The right part has the form $([x, y])^{c}\left([x, y]^{y}\right)^{c} \ldots\left([x, y]^{y^{l-1}}\right)^{c}$

In this section we will consider a particular case of equation (5) in $F_{2}=\langle x, y\rangle$. In this case $p: F_{2} \rightarrow \pi_{1}\left(S_{1}\right)=F_{2} /\left[F_{2}, F_{2}\right]$ is the abelianization map. Denote by $B=B_{1}=[x, y]$. The following statement gives a stronger version of Proposition 1.2 for $c=g=h=1$.

Proposition 3.1. Let $w_{1}, \ldots, w_{l}$ be $l$ distinct elements from $F_{2}$. If $u, v$ is a solution of the equation

$$
[u, v]=B^{w_{1}} \ldots B^{w_{l}}
$$

and $p: F_{2} \rightarrow F_{2} /\left[F_{2}, F_{2}\right]$ is the natural homomorphism, then $\left|p\left(F_{2}\right): p(\langle u, v\rangle)\right|=l$.
Proof. By the solution $u, v$ we can construct a map $f: T \rightarrow T$ such that $\operatorname{deg}(f)=l \neq 0$. Since $\operatorname{deg}(f) \neq 0$, we have $N R[f]=\left|p\left(F_{2}\right): p(\langle u, v\rangle)\right|$. From the other side, since $f$ is a map from torus to torus, $N R[f]=|\operatorname{deg}(f)|$. This follows promptly from the main result in [6] once one can identify the roots of $f$ with the fixed points of the map $g$ given by $g(x)=f(x) x$, using the multiplication of the torus.

The purpose of this section is to give explicit solutions for the particular case of equation (5)

$$
\begin{equation*}
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=([x, y])^{c}\left([x, y]^{y}\right)^{c} \ldots\left([x, y]^{]^{l-1}}\right)^{c} \tag{12}
\end{equation*}
$$

in $F_{2}=\langle x, y\rangle$ for the minimal integer $h$ which satisfies the inequality $h \geq l(c-1) / 2+1$. If $h<l(c-1) / 2+1$, then by Proposition 1.1 equation (12) does not have solutions.

In contrast to Proposition 3.1, for $c>1$ (and therefore $h>1$ ) the index $\left|p\left(F_{2}\right): p(H)\right|$, where $H=\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle$, can be different. By Proposition 1.2 this index is at most $l$. We are going to introduce two types of solution of equation (12): the first solution has the maximal possible index $\left|p\left(F_{2}\right): p(H)\right|=l$, and the second solution is primitive, i. e. it has a minimal possible index $\left|p\left(F_{2}\right): p(H)\right|=1$.

Since for $c=1$ equation (12) has a solution $u=x, v=y^{l}$, the case $\left|p\left(F_{2}\right): p(H)\right|=l$ is easy and it follows from Theorem [2.2 in the following way.

Corollary 3.2. Let $c, l \geq 1$ be integers and $h$ be the minimal number which satisfies the inequality $h \geq l(c-1) / 2+1$. Then the equation

$$
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=([x, y])^{c}\left([x, y]^{y}\right)^{c} \ldots\left([x, y]^{y^{l-1}}\right)^{c}
$$

with unknowns $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ has an explicit solution in $F_{2}$ given by recurrence in $c$ which satisfies the equality $p\left(\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle\right)=\left\langle p(x), p\left(y^{l}\right)\right\rangle$, where $p: F_{2} \rightarrow F_{2} /\left[F_{2}, F_{2}\right]$ is the natural homomorphism.

Using equality (3) and construction of the map $f$ (obtained from the solution) described in Section $\mathbb{1}$ and in details in [12, Proposition 4.2] we have the following corollary.

Corollary 3.3. Let $c>1, l \geq 1$ be integers and $h$ be a minimal number which satisfies the inequality $h \geq l(c-1) / 2+1$. Then there exists a map $f: S_{h} \rightarrow T$ with $A(f)=l c, M R[f]=l$, $N R[f]=l$ and each Nielsen root class has index c. So, the Wecken property holds for $f$.

Now we are going to construct an explicit primitive solution of equation (12). Results [2,4] about primitive branching coverings give some evidence that such solutions might exist. At first, we consider one simple particular case when $l=2, c=2$.

Lemma 3.4. The equation

$$
\left[u_{1}, v_{1}\right]\left[u_{2}, v_{2}\right]=[x, y]^{2}\left([x, y]^{y}\right)^{2}
$$

has as solution $u_{1}=x, v_{1}=y^{3}, u_{2}=y^{3} x y^{-2} x^{-1} y^{-1} x y^{2} x^{-1} y^{-3}, v_{2}=y^{2} x^{2} y^{2} x^{-1} y^{-3}$ which is primitive.

Proof. Straightforward calculation.
The general case follows.
Theorem 3.5. Let $c>1, l \geq 1$ be integers and $h$ be a minimal number which satisfies the inequality $h \geq l(c-1) / 2+1$. Then the equation

$$
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=([x, y])^{c}\left([x, y]^{y}\right)^{c} \ldots\left([x, y]^{y^{l-1}}\right)^{c}
$$

with unknowns $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ has an explicit primitive solution in $F_{2}$ given by recurrence in $l$.
Proof. We will use induction on $l$. For the basis of induction we consider two cases $l=1$ and $l=2$. The result for $l=1$ follows from Corollary 3.2. If $l=2$, then $h=c$. For $c=2$ the result follows from Lemma 3.4. Suppose that we found a solution $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ for an integer $c$ such that $u_{1}, v_{1}, u_{2}, v_{2}$ is the solution for $c=2$. Denoting by $u_{h+1}=\left(B^{y}\right)^{-c} x\left(B^{y}\right)^{c}$, $v_{h+1}=\left(B^{y}\right)^{-c} y^{2}\left(B^{y}\right)^{c}$ we have $\left[u_{h+1}, v_{h+1}\right]=\left(B^{y}\right)^{-c}\left[x, y^{2}\right]\left(B^{y}\right)^{c}$ and therefore

$$
\left[u_{1}, v_{1}\right] \ldots\left[u_{h+1}, v_{h+1}\right]=(B)^{c}\left(B^{y}\right)^{c}\left(B^{y}\right)^{-c}\left[x, y^{2}\right]\left(B^{y}\right)^{c}=(B)^{c+1}\left(B^{y}\right)^{c+1}
$$

This solution is obviously primitive since $p\left(u_{1}\right), p\left(v_{1}\right), p\left(u_{2}\right), p\left(v_{2}\right)$ generate $F_{2} /\left[F_{2}, F_{2}\right]$. The basis is proved. For the induction step we consider two similar cases depending on the parity of $l$.

Case 1: $l=2 n$ is even. In this case $h=n(c-1)+1$. We will construct a primitive solution which satisfies the condition $u_{1}=x, v_{1}=y^{l+1}$. If $l=2$, then the statement follows from the basis of induction. By the induction hypothesis we have a primitive solution $a_{1}=x$, $b_{1}=y^{l+1}, a_{2}, b_{2}, \ldots, a_{h_{1}}, b_{h_{1}}$ of equation (12) for $l=2 n, h_{1}=n(c-1)+1$. Also by induction hypothesis we have a primitive solution $r_{1}=x, s_{1}=y^{2}, r_{2}, s_{2}, \ldots, r_{h_{2}}, v_{h_{2}}$ of equation (12) for $l=2, h_{2}=c$. If we denote by

$$
\begin{align*}
u_{1} & =x & & \\
v_{1} & =y^{2 n+3} & & \\
u_{j} & =\left(B^{y^{2 n+2}}\right)^{-1}\left(B^{y^{2 n+1}}\right)^{-1} a_{j}\left(B^{y^{2 n+1}}\right)\left(B^{y^{2 n+2}}\right) & & j=2, \ldots, h_{1} \\
v_{j} & =\left(B^{y^{2 n+2}}\right)^{-1}\left(B^{y^{2 n+1}}\right)^{-1} b_{j}\left(B^{y^{2 n+1}}\right)\left(B^{y^{2 n+2}}\right) & & j=2, \ldots, h_{1} \\
u_{h_{1}+j} & =y^{2 n+1} r_{j+1} y^{-(2 n+1)} & & j=1, \ldots, h_{2}-1 \\
v_{h_{1}+j} & =y^{2 n+1} s_{j+1} y^{-(2 n+1)} & & j=1, \ldots, h_{2}-1 \tag{13}
\end{align*}
$$

and by $h=h_{1}+h_{2}-1=n(c-1)+1+c-1=(n+1)(c-1)+1$, then we have

$$
\begin{aligned}
{\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right] } & =\left(\left[x, y^{2 n+3}\right]\left(B^{y^{2 n+2}}\right)^{-1}\left(B^{y^{2 n+1}}\right)^{-1}\left[a_{2}, b_{2}\right] \ldots\left[a_{h_{1}}, b_{h_{1}}\right]\right) \\
& \cdot\left(\left(B^{y^{2 n+1}}\right)\left(B^{y^{2 n+2}}\right) y^{2 n+1}\left[r_{2}, s_{2}\right] \ldots\left[r_{h_{2}}, s_{h_{2}}\right] y^{-(2 n+1)}\right) \\
& =\left(\left[x, y^{2 n+1}\right]\left[a_{2}, b_{2}\right] \ldots\left[a_{h_{1}}, b_{h_{1}}\right]\right)\left(y^{2 n+1} B B^{y}\left[r_{2}, s_{2}\right] \ldots\left[r_{h_{2}}, s_{h_{2}}\right] y^{-(2 n+1)}\right) \\
& =\left((B)^{c}\left(B^{y}\right)^{c} \ldots\left(B^{y^{2 n}}\right)^{c}\right)\left(y^{2 n+1}(B)^{c}\left(B^{y}\right)^{c} y^{-(2 n+1)}\right) \\
& =(B)^{c}\left(B^{y}\right)^{c} \ldots\left(B^{y^{2 n+2}}\right)^{c} .
\end{aligned}
$$

and the statement is proved for $l=2 n+2$. The solution provided in (13) is primitive since the subgroup $p\left(\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle\right)$ contains $p\left(u_{1}\right)=p(x)$ and $p\left(u_{h_{1}+1}\right)=p\left(r_{2}\right)=p(y)^{-1}$.

Case 2: $l=2 n-1$ is odd. In this case $h=\lceil(2 n-1)(c-1) / 2+1\rceil$. Similarly to the first case, we will show that there exists a primitive solution such that $u_{1}=x, v_{1}=y^{l}$. For $l=1$ the result follows from Corollary 3.2 and Remark 2.3. By induction hypothesis we can suppose that we have a primitive solution $a_{1}=x, b_{1}=y^{l}, a_{2}, b_{2}, \ldots, a_{h_{1}}, b_{h_{1}}$ of equation (12) for $l=2 n-1$ and $h_{1}=\lceil(2 n-1)(c-1) / 2+1\rceil$. Also by induction hypothesis we can suppose that we have a solution $r_{1}=x, s_{1}=y^{2}, r_{2}, s_{2}, \ldots, r_{h_{2}}, v_{h_{2}}$ of equation (12) for $l=2$ and $h_{2}=c$. If we denote by

$$
\begin{align*}
u_{1} & =x & & \\
v_{1} & =y^{2 n+1} & & \\
u_{j} & =\left(B^{y^{2 n}}\right)^{-1}\left(B^{y^{2 n-1}}\right)^{-1} a_{j}\left(B^{y^{2 n-1}}\right)\left(B^{y^{2 n}}\right) & & j, \ldots, h_{1} \\
v_{j} & =\left(B^{y^{2 n}}\right)^{-1}\left(B^{y^{2 n-1}}\right)^{-1} b_{j}\left(B^{y^{2 n-1}}\right)\left(B^{y^{2 n}}\right) & & j=2, \ldots, h_{1} \\
u_{h_{1}+j} & =y^{2 n-1} r_{j+1} y^{1-2 n} & & j=1, \ldots, h_{2}-1 \\
v_{h_{1}+j} & =y^{2 n-1} s_{j+1} y^{1-2 n} & & j=1, \ldots, h_{2}-1 \tag{14}
\end{align*}
$$

and by $h=h_{1}+h_{2}-1=\lceil(2 n-1)(c-1) / 2+1\rceil+c-1=\lceil(2 n+1)(c-1) / 2+1\rceil$, then we have

$$
\begin{aligned}
{\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right] } & =\left(\left[x, y^{2 n+1}\right]\left(B^{y^{2 n}}\right)^{-1}\left(B^{y^{2 n-1}}\right)^{-1}\left[a_{2}, b_{2}\right] \ldots\left[a_{h_{1}}, b_{h_{1}}\right]\right) \\
& \cdot\left(\left(B^{y^{2 n-1}}\right)\left(B^{y^{2 n}}\right) y^{2 n-1}\left[r_{2}, s_{2}\right] \ldots\left[r_{h_{2}}, s_{h_{2}}\right] y^{1-2 n}\right) \\
& =\left(\left[x, y^{2 n-1}\right]\left[a_{2}, b_{2}\right] \ldots\left[a_{h_{1}}, b_{h_{1}}\right]\right)\left(y^{2 n-1} B B^{y}\left[r_{2}, s_{2}\right] \ldots\left[r_{h_{2}}, s_{h_{2}}\right] y^{1-2 n}\right) \\
& =\left((B)^{c}\left(B^{y}\right)^{c} \ldots\left(B^{y^{2 n-2}}\right)^{c}\right)\left(y^{2 n-1}(B)^{c}\left(B^{y}\right)^{c} y^{1-2 n}\right) \\
& =(B)^{c}\left(B^{y}\right)^{c} \ldots\left(B^{y^{2 n}}\right)^{c}
\end{aligned}
$$

and the statement is proved for $l=2 n+1$. The solution provided in (14) is primitive since the subgroup $p\left(\left\langle u_{1}, v_{1}, \ldots, u_{h}, v_{h}\right\rangle\right)$ contains $p\left(u_{1}\right)=p(x)$ and $p\left(u_{h_{1}+1}\right)=p\left(r_{2}\right)=p(y)^{-1}$.
Remark 3.6. If $c=1$, then by Proposition 3.1 the result of Theorem 3.5 holds only for $l=1$.
Remark 3.7. An old problem in the geometric group theory is the problem of determining the genus and the number $f(g)$ of Nielsen classes for a given element $g \in\left[F_{n}, F_{n}\right]$ (see [1, Section 3.3] for the definition of Nielsen classes and [1, Question 3.11] for the related question). If $n=2$, then Corollary 3.2 and Theorem 3.5 guarantee that for $g=([x, y])^{c}\left([x, y]^{y}\right)^{c} \ldots\left([x, y]^{y^{l-1}}\right)^{c}$ the number of Nielsen classes is at least 2.

Using equality (3) and construction of the map $f$ (obtained from the solution) described in Section 1 and in details in [12, Proposition 4.2] we have the following corollary.
Corollary 3.8. Let $c>1, l \geq 1$ be integers and $h$ be a minimal number which satisfies the inequality $h \geq l(c-1) / 2+1$. Then there exists a map $f: S_{h} \rightarrow T$ with $A(f)=l c, M R[f]=l$, $N R[f]=1$ and the only root class has index lc. So, the Wecken property does not hold for $f$.

## 4 The right part has the form $([x, y])^{k+l}\left([x, y]^{y}\right)^{k-l}$

The purpose of this section is to give an explicit solution for the equation

$$
\begin{equation*}
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=([x, y])^{k+l}\left([x, y]^{y}\right)^{k-l} \tag{15}
\end{equation*}
$$

for $h=k$. If $h<k$, then by Proposition 1.1 equation (15) does not have solutions. The main result of this section is the following theorem

Theorem 4.1. Let $k>l \geq 0$ be integers. Then the equation

$$
\left[u_{1}, v_{1}\right] \ldots\left[u_{h}, v_{h}\right]=([x, y])^{k+l}\left([x, y]^{y}\right)^{k-l}
$$

with unknowns $u_{1}, v_{1}, \ldots, u_{h}, v_{h}$ for $h=k$ has an explicit primitive solution. Moreover, if $l \neq 0$, then every solution of this equation is primitive.

Proof. At first, we will prove the moreover part of the theorem. Let $u_{1}, v_{1}, \ldots, u_{k}, v_{k}$ be a solution of equation (15) for $l \neq 0$ and let $H=\left\langle u_{1}, v_{1}, \ldots, u_{k}, v_{k}\right\rangle$. By Proposition 1.2, the index $\left|p\left(F_{2}\right): p(H)\right|$ is equal to 1 or 2 , and we need to prove that this index is equal to 1 . By contrary, suppose that $\left|p\left(F_{2}\right): p(H)\right|=2$. From equality (3) follows that the map $f: S_{k} \rightarrow T$ (obtained from the solution $u_{1}, v_{1}, \ldots, u_{k}, v_{k}$ ) described in Section $\square$ has two essential Nielsen root classes, one of this classes has the index $k+l$ and another one has the index $k-l$. If $l \neq 0$, then $k-l \neq k+l$, but the indices of all essential Nielsen root classes must coincide. We have a contradiction.

In order to introduce the solution of equation (15) denote by

$$
\begin{align*}
r_{i} & =y x y^{i-1} x^{-1} y^{-1} x y^{-i+1} x^{-1} y^{-1} & & i=1, \ldots, l \\
s_{i} & =y x y^{i-1} x^{-1} B^{l-i+1} y B^{k-l} y^{-i} x^{2} y^{-i+1} x^{-1} y^{-1} & & i=1, \ldots, l \\
r_{l+j} & =y x y^{(l+j)} x^{-1} y^{-1} x y^{(-l-j)} x^{-1} y^{-1} & & j=1, \ldots, k-l-1 \\
s_{l+j} & =y x y^{(l+j)} x^{-1} B^{k-l-j} y^{-l-j} x^{2} y^{(-l-j)} x^{-1} y^{-1} & & j=1, \ldots, k-l-1 \\
r_{k} & =y x y^{(k+1)} x^{-1} y^{-1} & & \\
s_{k} & =y x y^{-1} x^{-1} y x^{-1} y^{-1} & & \tag{16}
\end{align*}
$$

and let us, at first, prove some auxiliary equalities involving $r_{1}, s_{1}, \ldots, r_{k}, s_{k}$. Using induction on the number $t=1, \ldots, l$ let us prove that

$$
\begin{equation*}
\left[r_{1}, s_{1}\right] \ldots\left[r_{t}, s_{t}\right]=B^{l} y B^{k-l} y^{-1} B^{t} y^{t+1} B^{l-k} y^{-1} B^{t-1-l} x y^{-t+1} x^{-1} y^{-1} \tag{17}
\end{equation*}
$$

The basis of induction $(t=1)$ is proved in the following equality

$$
\begin{aligned}
{\left[r_{1}, s_{1}\right] } & =\left[y^{-1}, y B^{l} y B^{k-l} y^{-1} x y^{-1}\right] \\
& =y^{-1} y B^{l} y B^{k-l} y^{-1} x y^{-1} y y x^{-1} y B^{l-k} y^{-1} B^{-l} y^{-1} \\
& =B^{l} y B^{k-l} y^{-1} B y^{2} B^{l-k} y^{-1} B^{-l} y^{-1}
\end{aligned}
$$

The step of induction (omitting some detailed calculations) follows from the following equality

$$
\begin{aligned}
{\left[r_{1}, s_{1}\right] \ldots\left[r_{t+1}, s_{t+1}\right] } & =\left(\left[r_{1}, s_{1}\right] \ldots\left[r_{t}, s_{t}\right]\right)\left[r_{t+1}, s_{t+1}\right] \\
& =B^{l} y B^{k-l} y^{-1} B^{t} y^{t+1} B^{l-k} y^{-1} B^{t-1-l} x y^{-t+1} x^{-1} y^{-1} \\
& \cdot y x y^{t} x^{-1}\left[y^{-1}, B^{l-t} y B^{k-l} y^{-t-1} x\right] x y^{-t} x^{-1} y^{-1} \\
& =B^{l} y B^{k-l} y^{-1} B^{t+1} y^{t+2} B^{l-k} y^{-1} B^{t-l} x y^{-t} x^{-1} y^{-1}
\end{aligned}
$$

Similarly to equation (17) using induction on the number $t=1, \ldots, k-l-1$ we can prove the following equality.

$$
\begin{equation*}
\left[r_{l+1}, s_{l+1}\right] \ldots\left[r_{l+t}, s_{l+t}\right]=y x y^{l+1} x^{-1}\left(y^{-1} B^{k-l-1} y^{-l-1} B^{t} y^{l+t+1} B^{l+t-k}\right) x y^{-l-t} x^{-1} y^{-1} \tag{18}
\end{equation*}
$$

We will not show the proof of (18) here since it repeats the proof of (17) almost completely.
Multiplying equality (17) for $t=l$, equality (18) for $t=k-l-1$ and the value $\left[r_{k}, s_{k}\right]$ using formula (16) after some simple calculations we conclude that $\left[r_{1}, s_{1}\right] \ldots\left[r_{k}, s_{k}\right]=B^{l}\left(B^{y}\right)^{k-l} B^{k}$. From this equality follows that if for $i=1, \ldots, k$ we denote by

$$
\begin{equation*}
u_{u}=B^{k} r_{i} B^{-k}, v_{i}=B^{k} s_{i} B^{-k}, \tag{19}
\end{equation*}
$$

then $\left[u_{1}, v_{1}\right] \ldots\left[u_{k}, v_{k}\right]=B^{k+l}\left(B^{y}\right)^{k-l}$, i. e. $u_{1}, v_{1}, \ldots, u_{k}, v_{k}$ is the solution of equation (15). The images of elements $u_{1}, v_{1}, \ldots, u_{k}, v_{k}$ under the homomorphism $p: F_{2} \rightarrow F_{2} /\left[F_{2}, F_{2}\right]$ generate $p\left(F_{2}\right)$ since $p\left(u_{1}\right)=p(y)^{-1}, p\left(v_{1}\right)=p(x)$. Therefore $u_{1}, v_{1}, \ldots, u_{k}, v_{k}$ is a primitive solution of equation (15).

Remark 4.2. The same result for $k=l$ follows from Theorem 3.5.

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