A CRITERION FOR KOLCHIN SUBGROUPS OF $\mathrm{Out}(F_r)$

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Abstract. This article provides a decidable criterion for when a subgroup of $\mathrm{Out}(F_r)$ generated by two Dehn twists consists entirely of polynomially growing elements, answering an earlier question of the author.

1. Introduction

Outer automorphims of a free group divide into two categories, polynomially growing and exponentially growing, according to the behaviour of word lengths under iteration. Subgroups of $\mathrm{Out}(F_r)$ that consist of only polynomially growing outer automorphisms are known as *Kolchin* subgroups and are the $\mathrm{Out}(F_r)$ analog of unipotent subgroups of a linear group.

Clay and Pettet [3] and Gultepe [6] give sufficient conditions for when a subgroup of $\mathrm{Out}(F_r)$ generated by two Dehn twists contains an exponentially growing outer automorphism. This article gives an algorithmic criterion that characterizes when a subgroup of $\mathrm{Out}(F_r)$ generated by two Dehn twists is Kolchin, in terms of a combinatorial invariant of the gnerators known as the *edge-twist* directed graph (see Definition 2.8).

Theorem 1.1 (Main Theorem)**.** *Suppose $\sigma, \tau \in \mathrm{Out}(F_r)$ are Dehn twists. The subgroup $\langle \sigma, \tau\rangle$ is Kolchin if and only if the edge twist digraph of the defining graphs of groups is directed acyclic.*

2. Background

2.1. **Graphs of groups and Dehn twists.** A *graph* Γ is a collection of vertices $V(\Gamma)$, edges $E(\Gamma)$, initial and terminal vertex maps $o, t : E \to V$ and an involution $\bar{\cdot} : E \to E$ satisfying $\bar{e} \neq e$ and $o(\bar{e}) = t(e)$. A *directed graph (digraph)* omits the involution.

Definition 2.1. *A* graph of groups *is a pair (G, Γ) where Γ is a connected graph and G is an assignment of groups to the vertices and edges of Γ satisfying $G_e = G_{\bar{e}}$ and injections $\iota_e : G_e \to G_{t(e)}$. The assignment will often be suppressed and Γ_v, Γ_e used instead.*

Definition 2.2. *The* fundamental groupoid $\pi_1(\Gamma)$ *of a graph of groups Γ is the groupoid with vertex set $V(\Gamma)$ generated by the path groupoid of Γ and the groups G_v subject to the following relations. We require that for each $v \in V(\Gamma)$ the group G_v is a subgroupoid based at v and that the group and groupoid structures agree. Further, for all $e \in E(\Gamma)$ and $g \in G_e$ we have $\bar{e}\iota_{\bar{e}}(g)e = \iota_e(g)$.*

The *fundamental group* of Γ based at v, $\pi_1(\Gamma, v)$ is the vertex subgroup of $\pi_1(\Gamma)$ based at v. It is standard that changing the basepoint gives an isomorphic group [7, 8].

Let $(e_1, \ldots, e_n)$ be a possibly empty edge path in Γ starting at v and $(g_0, \ldots, g_n)$ be a sequence of elements $g_i \in G_{t(e_i)}$ with $g_0 \in G_v$. These data represent an arrow of $\pi_1(\Gamma)$ by the groupoid product

$$g_0 e_1 g_1 \cdots e_n g_n.$$

A non-identity element of $\pi_1(\Gamma)$ expressed this way is *reduced* if either $n = 0$ and $g_0 \neq \mathrm{id}$, or $n > 0$ ans for all i such that $e_i = \bar{e}_{i+1}$, $g_i \notin \iota_{e_i}(G_{e_i})$. The edges appearing in a reduced arrow are uniquely determined. Further, if $t(e_n) = o(e_1)$ the arrow is cyclically reduced if either $e_n \neq \bar{e}_0$ or $e_n = \bar{e}_0$ and $g_n g_0 \notin \iota_{e_n}(G_{e_n})$. For an element $g \in \pi_1(\Gamma, v)$, the edges appearing in a cyclically reduced arrow conjugate to g in $\pi_1(\Gamma)$ is a conjugacy class invariant.[1]

Definition 2.3. *Given a graph of groups Γ a subset of edges $E' \subseteq E(\Gamma)$ and edge-group elements $\{z_e\}_{e \in E'}$ satisfying $z_e \in Z(\Gamma_e)$ and $z_{\bar{e}} = z_e^{-1}$, the* Dehn twist *of Γ about E' by $\{z_e\}$ is the fundamental groupoid automorphism D_z given on the generators by*

$$\begin{aligned} D_z(e) &= e z_e & \qquad & e \in E' \\ D_z(e) &= e & & e \notin E' \\ D_z(g) &= g & & g \in \Gamma_v \end{aligned}$$

As this groupoid automorphism preserves vertex subgroups it induces a well-defined outer automorphism class $D_z \in \mathrm{Out}(\pi_1(\Gamma, v))$, which we will also refer to as a Dehn twist.

For a group G we say $\sigma \in \mathrm{Out}(G)$ is a *Dehn twist* if it can be realized as a Dehn twist about some graph of groups Γ with $\pi_1(\Gamma, v) \cong G$.

Specializing to $\mathrm{Out}(F_r)$, when $\sigma \in \mathrm{Out}(F_r)$ is a Dehn twist there are many graphs of groups Γ with $\pi_1(\Gamma, v) \cong F_r$ that can be used to realize σ. However, Cohen and Lustig [4] define the notion of an *efficient graph of groups* representative of a Dehn twist and show that each Dehn twist in $\mathrm{Out}(F_r)$ has a unique efficient representative. For a fixed σ let $\mathcal{G}(\sigma)$ denote the graph of groups of its efficient representative; edge groups of $\mathcal{G}(\sigma)$ are infinite cyclic [4].

Remark 2.4. *If $\sigma, \tau \in \mathrm{Out}(F_r)$ are Dehn twists with a common power, then $\mathcal{G}(\sigma) = \mathcal{G}(\tau)$.*

2.2. **Topological representatives and the Kolchin theorem.** Given a graph Γ the *topological realization* of Γ is a simplicial complex with zero-skeleton $V(\Gamma)$ and one-cells joining $o(e)$ and $t(e)$ for each edge in a set of $\bar{\cdot}$ orbit representatives. It will not cause confusion to use Γ for both a graph and its topological representative. If $\gamma \subset \Gamma$ is a based loop, denote the associated element of $\pi_1(\Gamma)$ by γ^*. Given $\sigma \in \mathrm{Out}(F_r)$, a *topological realization* is a homotopy equivalence $\hat{\sigma} : \Gamma \to \Gamma$ so that $\hat{\sigma}_* : \pi_1(\Gamma, v) \to \pi_1(\Gamma, \hat{\sigma}(v))$ is a representative of σ. A homotopy equivalence $\hat{\sigma} : \Gamma \to \Gamma$ is *filtered* if there is a filtration $\emptyset = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_k = \Gamma$ preserved by $\hat{\sigma}$.

Definition 2.5. *A filtered homotopy equivalence $\hat{\sigma} : \Gamma \to \Gamma$ is* upper triangular *if*

(1) *$\hat{\sigma}$ fixes the vertices of Γ,*

[1] These edges are covered by the axis of g in the Bass-Serre tree of Γ.

(2) *Each stratum of the filtration* $\Gamma_i \setminus \Gamma_{i-1} = e_i$ *is a single topological edge,*
(3) *Each edge* e_i *has a preferred orientation and with this orientation there is a closed path* $u_i \subseteq \Gamma_{i-1}$ *based at* $t(e_i)$ *so that* $\hat{\sigma}(e_i) = e_i u_i$.

The path u_i is called the suffix associated to u_i. A filtration assigns each edge a height, the i such that $e \in \Gamma_i \setminus \Gamma_{i-1}$, and taking a maximum this definition extends to edge paths.

Every Dehn twist in $\mathrm{Out}(F_r)$ has an upper-triangular representative [2, 4]. In a previous paper [1] I describe how to construct $\mathcal{G}(\sigma)$ from an upper-triangular representative, following a similar construction of Bestvina, Feighn, and Handel [2]. The following is an immediate consequence of my construction.

Lemma 2.6. *Suppose* Γ *is a filtered graph and* $\sigma \in \mathrm{Out}(F_r)$ *is a Dehn twist that is upper triangular with respect to* Γ. *Then*

(1) *there is a height function* $ht : E(\mathcal{G}(\sigma)) \to \mathbb{N}$ *so that for any loop* $\gamma \subseteq \Gamma_i$ *the height of the edges in a cyclically reduced representative of the conjugacy class of* γ^* *in* $\pi_1(\mathcal{G}(\sigma))$ *is at most* i,
(2) *For each edge* $e \in E(\mathcal{G}(\sigma))$, $ht(e) = ht(\bar{e})$, *and the edge group* $\mathcal{G}(\sigma)_e$ *is a conjugate of a maximal cyclic subgroup of* F_r *that contains* u_i^* *for some suffix* u_i, *and* $ht(e) > \min_{\gamma \sim u_i}\{ht_\Gamma(\gamma)\}$, *where* γ *ranges over loops freely homotopic to* u_i.

Bestvina, Feighn, and Handel proved an $\mathrm{Out}(F_r)$ analog of the classical Kolchin theorem for $\mathrm{Out}(F_r)$, which provides simultaneous upper-triangular representatives for Kolchin-type subgroups of $\mathrm{Out}(F_r)$.

Theorem 2.7 ([2])**.** *Suppose* $H \leq \mathrm{Out}(F_r)$ *is a Kolchin subgroup. Then there is a finite index subgroup* $H' \leq H$ *and a filtered graph* Γ *so that each* $\sigma \in H'$ *is upper triangular with respect to* Γ.

2.3. **Twists and polynomial growth.** In a previous paper [1] I introduced the edge-twist digraph of two Dehn twists and used it to provide a sufficient condition for a subgroup of $\mathrm{Out}(F_r)$ generated by two Dehn twists to be Kolchin.

Definition 2.8 ([1])**.** *The edge-twist digraph* $\mathcal{ET}(A, B)$ *of two graphs of groups* A, B *with isomorphic fundamental groups and infinite cyclic edge stabilizers is the digraph with vertex set*

$$V(\mathcal{ET}(A, B)) = \{(e, \bar{e}) | e \in E(A)\} \cup \{(f, \bar{f}) | f \in E(B)\}$$

directed edges $((e, \bar{e}), (f, \bar{f}))$ $e \in E(A), f \in E(B)$ *when a generator of* A_e *uses* f *or* $\bar{f}$ *in its cyclically reduced representation in* $\pi_1(B)$, *and directed edges* $((f, \bar{f}), (e, \bar{e}))$, $f \in E(B), e \in E(A)$ *when a generator of* B_f *uses* e *or* $\bar{e}$ *in a cyclically reduced representation in* $\pi_1(A)$.

Remark 2.9. *This is well-defined, using an edge is a conjugacy invariant, and using an edge or its reverse is preserved under taking inverses.*

Lemma 2.10 ([1])**.** *If* $\sigma, \tau \in \mathrm{Out}(F_r)$ *are Dehn twists and* $\mathcal{ET}(\mathcal{G}(\sigma), \mathcal{G}(\tau))$ *is directed acyclic, then* $\langle \sigma, \tau \rangle$ *is Kolchin.*

3. Proof of the main theorem

Proof. It suffices to prove the converse to Lemma 2.10. Suppose $\langle \sigma, \tau \rangle$ is Kolchin. By Theorem 2.7, there is a finite index subgroup $H \leq \langle \sigma, \tau \rangle$ where every element of H is upper triangular with respect respect to a fixed filtered graph Γ. Since H is finite index, there are powers m, n so that $\sigma^m, \tau^n \in H$, so that σ^m and τ^n are upper triangular with respect to Γ.

By Lemma 2.6, the filtration of Γ induces height functions on $E(\mathcal{G}(\sigma^m))$ and $E(\mathcal{G}(\tau^n))$, combining these gives a height function on the vertices of $\mathcal{ET}(\mathcal{G}(\sigma^m), \mathcal{G}(\tau^n))$. Every directed edge $((e, \bar{e}), (f, \bar{f}))$ satisfies $ht(e) > ht(f)$. Indeed, suppose $(e, \bar{e}) \in E(\mathcal{G}(\sigma^m))$. Let $[g] \subset F_r$ be the conjugacy class of a generator of $\mathcal{G}(\sigma^m)_e$. By Lemma 2.6 (ii), there is a representative $g \in [g]$ such that $g^k = u_i^*$ for some σ-suffix u_i. Take a minimum height loop γ representing $[g]$, so that $\gamma \subseteq \Gamma_{ht(\gamma)}$. Again by Lemma 2.6 (ii), $ht(e) > ht(\gamma)$. Finally, by Lemma 2.6 (i), each edge f in a cyclically reduced $\pi_1(\mathcal{G}(\tau^n))$ representative of $[g]$ satisfies $ht(f) \leq ht(\gamma)$. Thus $ht(e) > ht(f)$ for each directed edge with origin $(e, \bar{e})$, as required. The argument for generators of the edge groups of $\mathcal{G}(\tau^n)$ is symmetric. Therefore any directed path in $\mathcal{ET}(\mathcal{G}(\sigma^m), \mathcal{G}(\tau^n))$ has monotone decreasing vertex height, which implies that $\mathcal{ET}$ is directed acyclic. To conclude, observe that by Remark 2.4 $\mathcal{ET}(\mathcal{G}(\sigma^m), \mathcal{G}(\tau^n)) = \mathcal{ET}(\mathcal{G}(\sigma), \mathcal{G}(\tau))$. □

Cohen and Lustig [5] give an algorithm to find efficient representatives. Computing the edge-twist digraph and testing if it is acyclic are straightforward computations, so the criterion in the main theorem is algorithmic.

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