BINOMIAL EDGE IDEALS OF GENERALIZED BLOCK GRAPHS

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ABSTRACT. We classify generalized block graphs whose binomial edge ideals admit a unique extremal Betti number. We prove that the Castelnuovo-Mumford regularity of binomial edge ideals of generalized block graphs is bounded below by m(G) + 1, where m(G) is the number of minimal cut sets of the graph G and obtain an improved upper bound for the regularity in terms of the number of maximal cliques and pendant vertices of G.

1. INTRODUCTION

Let $R = K[x_1, \ldots, x_m]$ be a standard graded polynomial ring over an arbitrary field K, and M be a finitely generated graded R-module. Let

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}^R(M)} \xrightarrow{\phi_p} \cdots \xrightarrow{\phi_1} \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}^R(M)} \xrightarrow{\phi_0} M \longrightarrow 0$$

be the minimal graded free resolution of M, where $p \leq m$ and R(-j) is the free R-module of rank 1 generated in degree j. The number $\beta_{i,j}^R(M)$ is called the (i, j)-th graded Betti number of M. The projective dimension and Castelnuovo-Mumford regularity (henceforth called regularity) are two invariants associated with M that can be read off from the minimal graded free resolution of M. The regularity of M, denoted by $\operatorname{reg}(M)$, is defined as

 $\operatorname{reg}(M) := \max\{j - i \mid \beta_{i,j}^R(M) \neq 0\},\$

and the projective dimension of M, denoted by $pd_R(M)$, is defined as

$$\mathrm{pd}_R(M) := \max\{i : \beta_{i,j}^R(M) \neq 0\}.$$

A Betti number $\beta_{i,j}^R(M) \neq 0$ is called an *extremal Betti number* if $\beta_{r,s}^R(M) = 0$ for all pairs $(r, s) \neq (i, j)$ with $r \geq i$, and $s \geq j$. Observe that M admits a unique extremal Betti number if and only if $\beta_{p,p+r}^R(M) \neq 0$, where $p = pd_R(M)$ and r = reg(M). For a graded R-module M, we denote the *Betti polynomial* of M by

$$B_M(s,t) = \sum_{i,j} \beta_{i,j}^R(M) s^i t^j$$

Herzog et al. in [9] and independently Ohtani in [20] introduced the notion of binomial edge ideal corresponding to a finite simple graph. Let G be a simple graph on [n]. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where K is a field. The binomial edge ideal of G is $J_G =$ $(x_iy_j - x_jy_i : \{i, j\} \in E(G), i < j)$. Researchers have found exact formulas or bounds for algebraic invariants of J_G , such as codimension, depth, Betti numbers and regularity, in terms of combinatorial invariants of the underlying graph G, see e.g., [1, 2, 3, 5, 15, 17, 19, 25, 26]. The study of regularity and Betti numbers of homogeneous ideals has attracted a lot of

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attention in the recent past due to its algebraic and geometric importance. In [19], Matsuda and Murai proved that if G is a graph on the vertex set [n], then $\ell(G) \leq \operatorname{reg}(S/J_G) \leq n-1$, where $\ell(G)$ is the length of a longest induced path in G. In particular, they conjectured, and later proved by Kiani and Saaedi Madani [15], that $\operatorname{reg}(S/J_G) = n-1$ if and only if G is a path on n vertices. Another conjectured upper bound for $\operatorname{reg}(S/J_G)$ is given by $\operatorname{cl}(G)$, the number of maximal cliques of G [26]. The latter conjecture has been recently proved for chordal graphs by Rouzbahani Malayeri et al. [24], extending results of [6, 12, 25], and for some classes of non-chordal graphs in [16].

An interesting class of chordal graphs consists of *block graphs*, which are connected graphs whose *blocks* (i.e., maximal subgraphs that cannot be disconnected by removing a vertex) are cliques. Recently, in [11], Herzog and Rinaldo improved the lower bound for the regularity of binomial edge ideals of block graphs and classified block graphs whose binomial edge ideal admits a unique extremal Betti number. In this article, we extend these results to the class of generalized block graphs that contains block graphs. We also obtain improved lower and upper bounds for the regularity of binomial edge ideals of these graphs.

The article is organized as follows. In the second section, we recall some results on graphs and binomial edge ideals. In the third section, we characterize generalized block graphs whose binomial edge ideal admits a unique extremal Betti number (Theorem 3.11). In particular, we prove that $\beta_{p(G),p(G)+m(G)+1}^S(S/J_G)$ is an extremal Betti number of S/J_G if Gis a connected generalized block graph (Theorem 3.7), where $p(G) = \text{pd}_S(S/J_G)$ and m(G)is the number of minimal cut sets of G. As a consequence, we obtain $\text{reg}(S/J_G) \ge m(G) + 1$ (Corollary 3.12). In the fourth section, we obtain improved upper bound for the regularity of binomial edge ideals of generalized block graphs in terms of the number of maximal cliques and pendant vertices of G (Theorem 4.5).

2. Preliminaries

In this section, we recall some notation and terminology from graph theory, and some important results about binomial edge ideals.

Let G be a finite simple graph with vertex set V(G) and edge set E(G). For $A \subseteq V(G)$, G[A] denotes the *induced subgraph* of G on the vertex set A, that is the subgraph with edge set $E(G[A]) = \{\{i, j\} \in E(G) : i, j \in A\}$. For a vertex $v, G \setminus v$ denotes the induced subgraph of G on the vertex set $V(G) \setminus \{v\}$. A vertex $v \in V(G)$ is said to be a *cut vertex* if $G \setminus v$ has more connected components than G. For $T \subset [n]$, let $\overline{T} = [n] \setminus T$, $c_G(T)$ be the number of connected components of $G[\overline{T}]$ and c_G the number of connected components of G. We say that a subset $T \subset [n]$ is a *cut set* of G if $c_G(T) > c_G$. A cut set of G is said to be a *minimal cut set* if it is minimal under inclusion. A subset U of V(G) is said to be a *clique* if G[U] is a complete graph.

Let Δ be a simplicial complex on the vertex set [n] and $\mathcal{F}(\Delta)$ be the set of its facets. A facet F of Δ is called a *leaf* if either F is the only facet of Δ or else there exists a facet G, called a *branch* of F, such that for each facet H of Δ with $H \neq F$, $H \cap F \subseteq G \cap F$.

The simplicial complex Δ is called a *quasi-forest* if its facets can be ordered as F_1, \ldots, F_s such that for all i > 1, the facet F_i is a leaf of the simplicial complex with facets F_1, \ldots, F_{i-1} . Such an order of the facets is called a *leaf order*. The simplicial complex whose facets are the maximal cliques of a graph G is called the *clique complex* of G and denoted by $\Delta(G)$. By [8, Theorem 9.2.12], G is chordal if and only if $\Delta(G)$ is a quasi-forest. A vertex v of G is said to be a *free vertex* if v belongs to exactly one maximal clique of G. A vertex v is said to be an *internal vertex* of G if it is not a free vertex. The set $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ is called the *neighborhood* of v, and G_v denotes the graph on the vertex set V(G) and edge set $E(G_v) = E(G) \cup \{\{u, w\} : u, w \in N_G(v)\}$. Observe that, if v is a free vertex, then $G_v = G$.

Let $G_1, \ldots, G_{c_G(T)}$ be the connected components of $G[\overline{T}]$. For each i, let $\widetilde{G_i}$ denote the complete graph on $V(G_i)$ and $P_T(G) = (\bigcup_{i \in T} \{x_i, y_i\}, J_{\widetilde{G_1}}, \ldots, J_{\widetilde{G_{c_G(T)}}})$. In [9], it was shown by Herzog et al. that $J_G = \bigcap_{T \subseteq [n]} P_T(G)$. For each $i \in T$, if i is a cut vertex of the graph $G[\overline{T} \cup \{i\}]$, then we say that T has the *cut point property*. Set $\mathcal{C}(G) = \{\emptyset\} \cup \{T : T \text{ has the cut point property}\}$. It follows from [9, Corollary 3.9] that $T \in \mathcal{C}(G)$ if and only if $P_T(G)$ is a minimal prime of J_G . Hence, by [9, Theorem 3.2] and [9, Corollary 3.9], we have $J_G = \bigcap_{T \in \mathcal{C}(G)} P_T(G)$.

3. Extremal Betti number of generalized block graphs

In this section, we study the extremal Betti number $\beta_{p(G),p(G)+j}^S(S/J_G)$ of binomial edge ideals of generalized block graphs, where $p(G) = \text{pd}_S(S/J_G)$. A maximal connected subgraph of G with no cut vertex is called a *block*. A graph G is called a *block graph* if each block of Gis a clique. In other words, a block graph is a chordal graph such that every pair of blocks of G intersects in at most one vertex. Block graphs were extensively studied by many authors, see [5], [6], [11], [13].

Generalized block graphs are the generalization of block graphs and were introduced in [14]. A chordal graph G is said to be a generalized block graph if $F_i, F_j, F_k \in \mathcal{F}(\Delta(G))$ such that $F_i \cap F_j \cap F_k \neq \emptyset$, then $F_i \cap F_j = F_i \cap F_k = F_j \cap F_k$. One could see that all block graphs are generalized block graphs. By definition of generalized block graph, it is clear that a subset A of vertices of G is a minimal cut set if and only if there exist $F_{t_1}, \ldots, F_{t_q} \in \mathcal{F}(\Delta(G))$ such that $\bigcap_{j=1}^q F_{t_j} = A$, and for all other facets F of $\Delta(G), F \cap A = \emptyset$. Note that if A is a minimal cut set, then A is a clique. For a minimal cut set A, we denote by G_A the graph obtained from G by replacing the cliques F_{t_1}, \ldots, F_{t_q} with the clique on the vertex set $\bigcup_{j \in [q]} F_{t_j}$.

Lemma 3.1. Let G be a graph on the vertex set [n] and A be a minimal cut set of G. Then $A \in \mathcal{C}(G)$. Moreover, if G is a generalized block graph, then for every $T \in \mathcal{C}(G)$, either $A \subseteq T$ or $A \cap T = \emptyset$.

Proof. For the first claim, suppose that $A \notin C(G)$. Then there exists $v \in A$ such that $c_G(A \setminus \{v\}) = c_G(A)$. Since A is a minimal cut set, we know that $c_G(A) > c_G$, and hence, $c_G(A \setminus \{v\}) > c_G$. This means that $A \setminus \{v\}$ is a cut set and is properly contained in A, against the minimality of A. Thus, $A \in C(G)$. The second claim clearly holds when |A| = 1. Let $|A| \ge 2$ and $T \in C(G)$. If $T \cap A = \emptyset$, then there is nothing to prove. Assume that $T \cap A \neq \emptyset$ and let $v \in T \cap A$. Then v is a cut vertex of $G[\overline{T} \cup \{v\}]$. Suppose that there exists $w \in A \setminus T$. We want to show that $N_G(v) = N_G(w)$. Suppose that there exists $u \in N_G(w) \setminus N_G(v)$ and let F be a facet of $\Delta(G)$ containing w and u. Since A is a cut set, there are at least two facets F_1, F_2 containing v, w but not u. Then, we have $F_1 \cap F_2 \cap F \neq \emptyset$, $v \in F_1 \cap F_2$ but $v \notin F_1 \cap F$ and $v \notin F_2 \cap F$, against the fact that G is a generalized block graph. Thus, $N_G(w) \subset N_G(v)$. Similarily, $N_G(v) \subset N_G(w)$, and hence, $N_G(v) = N_G(w)$. Consequently, v is not a cut vertex of $G[\overline{T} \cup \{v\}]$, which is a contradiction. Hence, $A \subseteq T$.

We now give an example of a chordal graph G that is not a generalized block graph for which the second claim of Lemma 3.1, does not hold.

Example 3.2. Let G be a graph as shown in Fig. 1. Then, it can be seen that G is a chordal graph that is not a generalized block graph. The sets $A = \{2, 3\}$ and $B = \{3, 4\}$ are minimal cut sets of G, thus $A, B \in \mathcal{C}(G)$, and $A \cap B \neq \emptyset$. However, $A \not\subseteq B$ and $B \not\subseteq A$.



Figure 1

Let G be a generalized graph on [n]. Let A be a minimal cut set of G. Set

$$Q_1 = \bigcap_{\substack{T \subseteq [n] \\ A \cap T = \emptyset}} P_T(G) \quad , \quad Q_2 = \bigcap_{\substack{T \subseteq [n] \\ A \subseteq T}} P_T(G)$$

By [9, Theorem 3.2, Corollary 3.9], $J_{G_A} = \bigcap_{T \subset [n]} P_T(G_A) = \bigcap_{T \in \mathcal{C}(G_A)} P_T(G_A)$. It follows from [22, Proposition 2.1] that $T \in \mathcal{C}(G_A)$ if and only if $A \cap T = \emptyset$ and $T \in \mathcal{C}(G_A[\overline{A}])$. If $A \cap T = \emptyset$, then $P_T(G) = P_T(G_A)$. Consequently, $Q_1 = J_{G_A}$. Note that

$$Q_2 = (x_i, y_i : i \in A) + \bigcap_{T \setminus A \subset [n] \setminus A} P_{T \setminus A}(G[\overline{A}]) = (x_i, y_i : i \in A) + J_{G[\overline{A}]},$$

where the last equality follows from [9, Theorem 3.2]. Thus,

$$Q_1 + Q_2 = (x_i, y_i : i \in A) + J_{G_A[\overline{A}]}$$

By virtue of Lemma 3.1, $J_G = Q_1 \cap Q_2$. This gives us the following short exact sequence,

$$0 \longrightarrow \frac{S}{J_G} \longrightarrow \frac{S}{Q_1} \oplus \frac{S}{Q_2} \longrightarrow \frac{S}{Q_1 + Q_2} \longrightarrow 0.$$
(1)

The following example illustrates that, in general, $Q_1 \neq J_{G_A}$ for a minimal set A of G.

Example 3.3. Let G be a graph as shown in Fig. 1. Then, $T \in \mathcal{C}(G)$ if and only if $T \in \{\emptyset, \{2,3\}, \{3,4\}\}$. Set $A = \{2,3\}$ and $B = \{3,4\}$. Note that A is a minimal cut set of G and $G_A = G \cup \{1,4\}$. Let $T \subseteq [5]$ such that $A \cap T = \emptyset$. Then, $T \subseteq \{1,4,5\}$, and hence, $P_{\emptyset}(G) \subseteq P_T(G)$. Thus,

$$Q_1 = \bigcap_{T \subseteq \{1,4,5\}} P_T(G) = P_{\emptyset}(G) = J_{K_5} \neq J_{G_A}.$$

However,

$$Q_{2} = \bigcap_{T \subseteq [5], A \subset T} P_{T}(G) = (x_{2}, y_{2}, x_{3}, y_{3}) + \bigcap_{T \subseteq \{1, 4, 5\}} P_{T}(G[\{1, 4, 5\}])$$
$$= (x_{2}, y_{2}, x_{3}, y_{3}, x_{4}y_{5} - x_{5}y_{4}) = P_{A}(G).$$

Since $J_G = P_{\emptyset}(G) \cap P_A(G) \cap P_B(G)$, we have $J_G \neq Q_1 \cap Q_2$.

Let p(G) denote the projective dimension of S/J_G . Then $pd_S(S/Q_1) = p(G_A)$. Let $T = K[x_1, \ldots, x_n]$. Let $0 < m < n, I \subset R = K[x_1, \ldots, x_m]$ and $J \subset R' = K[x_{m+1}, \ldots, x_n]$ be homogeneous ideals. Then, the minimal graded free resolution of T/(I + J) is the tensor product of the minimal free resolutions of R/I and R'/J. For $A \subset [n]$, set $S_A = K[x_i, y_i : i \notin A]$. Hence, $pd_S(S/Q_2) = 2|A| + pd_{S_A}(S_A/J_{G[\overline{A}]}) = 2|A| + p(G[\overline{A}])$ and $pd_S(S/(Q_1 + Q_2)) = 2|A| + pd(S_A/J_{G_A[\overline{A}]}) = 2|A| + p(G_A[\overline{A}])$.

In [14], Kiani and Saeedi Madani obtained the depth of binomial edge ideals of generalized block graphs, and hence, their projective dimension by the Auslander-Buchsbaum formula. Recall that the *clique number* of a graph G, denoted $\omega(G)$, is the maximum size of the maximal cliques of G. Let G be a generalized block graph on [n]. For each $i = 1, \ldots, \omega(G) - 1$, we set

$$\mathcal{A}_i(G) = \{A \subseteq [n] : |A| = i, A \text{ is a minimal cut set of } G\}$$

and $a_i(G) = |\mathcal{A}_i(G)|$. Observe that a generalized block graph G is a block graph if and only if $a_i(G) = 0$, for all i > 1. Let m(G) denote the number of minimal cut sets of G. Then, $m(G) = \sum_{i=1}^{\omega(G)-1} a_i(G)$.

By [14, Theorem 3.2] and the Auslander-Buchsbaum formula, it follows that:

Theorem 3.4. Let G be a generalized block graph on [n]. Then,

$$p(G) = n - c_G + \sum_{i=2}^{\omega(G)-1} (i-1)a_i(G),$$

where c_G is the number of connected components of G.

We recall the notion of decomposability from [22, Section 2] and [23]. A graph G is called *decomposable* if there exist subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v\}$ and v is a free vertex of both G_1 and G_2 .

A graph G is called *indecomposable* if it is not decomposable. Up to ordering, G has a unique decomposition into indecomposable subgraphs, i.e., there exist G_1, \ldots, G_r indecomposable induced subgraphs of G with $G = G_1 \cup \cdots \cup G_r$ such that for each $i \neq j$, either $V(G_i) \cap V(G_j) = \emptyset$ or $V(G_i) \cap V(G_j) = \{v\}$ and v is a free vertex of both G_i and G_j .

In [11, Proposition 3], Herzog and Rinaldo proved that:

Proposition 3.5. [11, Proposition 1.3] Let $G = G_1 \cup G_2$ be a decomposable graph. Let $S_i = K[x_j, y_j : j \in V(G_i)]$, for i = 1, 2. Then,

$$B_{S/J_G}(s,t) = B_{S_1/J_{G_1}}(s,t)B_{S_2/J_{G_2}}(s,t).$$

It follows from Proposition 3.5 that if $G = G_1 \cup \cdots \cup G_r$ is a decomposition of G into indecomposable graphs, then $\operatorname{reg}(S/J_G) = \sum_{i \in [r]} \operatorname{reg}(S_i/J_{G_i})$ and $p(G) = \sum_{i \in [r]} p(G_i)$. Also, if for each $i, \beta_{p(G_i), p(G_i) + j_i}^{S_i}(S_i/J_{G_i})$ is an extremal Betti number of S_i/J_{G_i} , then

$$\beta_{p(G),p(G)+j}^{S}(S/J_G) = \prod_{i \in [r]} \beta_{p(G_i),p(G_i)+j_i}^{S_i}(S_i/J_{G_i})$$

is an extremal Betti number of S/J_G , where $j = j_1 + \cdots + j_r$. Therefore, it is enough to find the position of the extremal Betti number $\beta_{p(G),p(G)+i}^S(S/J_G)$ for indecomposable graphs. **Lemma 3.6.** Let G be a connected indecomposable generalized block graph and let F_1, \ldots, F_r be a leaf order of $\mathcal{F}(\Delta(G))$. Denote by F_{t_1}, \ldots, F_{t_q} all the branches of the leaf F_r . Set $A = F_r \cap \bigcap_{i=1}^q F_{t_i}$ and $\alpha = |A|$. Then,

(a) the graphs G_A , $G_A[\overline{A}]$ and $G[\overline{A}]$ are generalized block graphs.

(b) for $i \neq \alpha$, $a_i(G_A) = a_i(G)$ and $a_\alpha(G_A) = a_\alpha(G) - 1$. In particular,

$$m(G_A) = m(G) - 1$$
 and $p(G_A) = p(G) - \alpha + 1$.

(c) for $i \neq \alpha$, $a_i(G_A[\overline{A}]) = a_i(G)$ and $a_\alpha(G_A[\overline{A}]) = a_\alpha(G) - 1$. In particular, $m(G_A[\overline{A}]) = m(G) - 1$ and $p(G_A[\overline{A}]) = p(G) - 2\alpha + 1$.

(d) for
$$i \neq \alpha$$
, $a_i(G[\overline{A}]) \leq a_i(G)$ and $a_\alpha(G[\overline{A}]) \leq a_\alpha(G) - 1$. In particular $m(G[\overline{A}]) \leq m(G) - 1$ and $p(G[\overline{A}]) = p(G) - 2\alpha - q + 1$.

Proof. (a) This easily follows by the fact that G is a generalized block graph.

(b) Notice that for $i \neq \alpha, \mathcal{A}_i(G_A) = \mathcal{A}_i(G)$ and $\mathcal{A}_\alpha(G_A) = \mathcal{A}_\alpha(G) \setminus \{A\}$. Thus, by Proposition 3.4, $p(G_A) = n - 1 + \sum_{i=2}^{\omega(G_A)-1} (i-1)a_i(G_A) = p(G) - \alpha + 1$ and $m(G_A) = m(G) - 1$.

(c) Notice that for $i \neq \alpha$, $\mathcal{A}_i(G_A[\overline{A}]) = \mathcal{A}_i(G)$ and $\mathcal{A}_\alpha(G_A[\overline{A}]) = \mathcal{A}_\alpha(G) \setminus \{A\}$. Thus, by Proposition 3.4, $p(G_A[\overline{A}]) = (n - \alpha) - 1 + \sum_{i=2}^{\omega(G_A[\overline{A}]) - 1} (i - 1)a_i(G_A[\overline{A}]) = p(G) - 2\alpha + 1$ and $m(G_A[\overline{A}]) = \sum_{i=1}^{\omega(G_A[\overline{A}]) - 1} a_i(G_A[\overline{A}]) = m(G) - 1.$ (d) Let *B* be a minimal cut set of *G*[\overline{A}]. Since *G*[\overline{A}] is an induced subgraph of *G* and

(d) Let *B* be a minimal cut set of *G*[*A*]. Since *G*[*A*] is an induced subgraph of *G* and $B \cap A = \emptyset$, *B* is a minimal cut set of *G*. Therefore, for $i \neq \alpha$, $\mathcal{A}_i(G[\overline{A}]) \subseteq \mathcal{A}_i(G)$ and $\mathcal{A}_\alpha(G[\overline{A}]) \subseteq \mathcal{A}_\alpha(G) \setminus \{A\}$. Thus, $m(G[\overline{A}]) = \sum_{i=1}^{\omega(G[\overline{A}])-1} a_i(G[\overline{A}]) \leq m(G) - 1$ and by Proposition 3.4, $p(G[\overline{A}]) = (n-\alpha) - (q+1) + \sum_{i=2}^{\omega(G[\overline{A}])-1} (i-1)a_i(G[\overline{A}]) \leq p(G) - 2\alpha - q + 1$. \Box

Recall that a vertex v is said to be an *internal vertex* of G if it is not a free vertex. For $v \in V(G)$, let $\operatorname{cdeg}_G(v)$ denote the number of maximal cliques of G which contains v. The number of free vertices of G is denoted by f(G).

Theorem 3.7. Let G be a connected indecomposable generalized block graph on the vertex set [n]. Then, $\beta_{p(G),p(G)+m(G)+1}^{S}(S/J_G)$ is an extremal Betti number of S/J_G . Moreover, if G is a complete graph or for every internal vertex v, $\operatorname{cdeg}_G(v) > 2$, then $\beta_{p(G),p(G)+m(G)+1}^{S}(S/J_G) = f(G) - 1$.

Proof. We prove this assertion by induction on m(G). If m(G) = 0, then G is a complete graph. Therefore, the claim follows by the Eagon-Northcott resolution [4]. Assume that m(G) > 0. Since G is a chordal graph, by [8, Theorem 9.2.12], $\Delta(G)$ is a quasi-forest. Let F_1, \ldots, F_r be a leaf order of $\mathcal{F}(\Delta(G))$. Let F_{t_1}, \ldots, F_{t_q} be all the branches of the leaf F_r . Note that $q \ge 1$. Since G is a generalized block graph, $F_r \cap F_{t_i} = F_{t_j} \cap F_{t_k}$ for every pair of $i, j, k \in [q]$ with $j \ne k$ and for all $l \ne t_1, \ldots, t_q, F_r \cap F_l = \emptyset$. Let $A = F_r \cap F_{t_1} = \bigcap_{i=1}^q F_{t_i} \cap F_r$ and $\alpha = |A|$. Since A is a minimal cut set, by the discussion after Lemma 3.1, $J_G = Q_1 \cap Q_2$, where $Q_1 = J_{G_A}$ and $Q_2 = (x_i, y_i : i \in A) + J_{G[\overline{A}]}$.

By Lemma 3.6, G_A , $G_A[\overline{A}]$ and $G[\overline{A}]$ are generalized block graphs. We have the following cases:

Case (1): If $\alpha = 1$, then it follows from Theorem 3.4 and Lemma 3.6 that $p(G_A) = p(G)$, $p(G_A[\overline{A}]) = p(G) - 1$ and $p(G[\overline{A}]) \leq p(G) - q - 1$ (where $G[\overline{A}]$ has q + 1 connected components). Note that $G[\overline{A}]$ is not necessarily indecomposable, but we can split it into

smaller indecomposable graphs. Since G is an indecomposable graph, $q \ge 2$, and hence, $\operatorname{pd}_S(S/Q_2) = 2 + p(G[\overline{A}]) \le p(G) - 1$. Therefore,

$$\operatorname{Tor}_{i}^{S}\left(\frac{S}{Q_{2}}, K\right) = 0, \text{ for } i \ge p(G).$$

Thus, for each $j \ge 0$, the exact sequence (1) yields the long exact sequence of Tor sequence:

$$0 \to \operatorname{Tor}_{p(G)+1,p(G)+j}^{S}\left(\frac{S}{Q_{1}+Q_{2}},K\right) \to \operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{J_{G}},K\right) \to \\ \to \operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{J_{G_{A}}},K\right) \to \dots$$

$$(2)$$

Since $Q_1 + Q_2 = (x_i, y_i : i \in A) + J_{G_A[\overline{A}]}$, we have that

$$\operatorname{Tor}_{p(G)+1,p(G)+j}^{S}\left(\frac{S}{Q_{1}+Q_{2}},K\right) \cong \operatorname{Tor}_{p(G_{A}[\overline{A}]),p(G_{A}[\overline{A}])+(j-1)}^{S_{A}}\left(\frac{S_{A}}{J_{G_{A}}[\overline{A}]},K\right)$$
(3)

where $S_A = K[x_i, y_i : i \notin A]$. It follows from induction that

$$\operatorname{Tor}_{p(G_{A}[\overline{A}]),p(G_{A}[\overline{A}])+(j-1)}^{S_{A}}\left(\frac{S}{J_{G_{A}[\overline{A}]}},K\right) = 0 \quad \text{for} \quad j > m(G_{A}[\overline{A}]) + 2 = m(G) + 1, \quad (4)$$

and

$$\operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{J_{G_{A}}},K\right) = 0 \quad \text{for} \quad j > m(G_{A}) + 1 = m(G)$$

Now, (2), (3) and (4) imply that

$$\operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{J_{G}},K\right) = 0 \quad \text{for} \quad j > m(G) + 1,$$
(5)

and

$$\operatorname{Tor}_{p(G_{A}[\overline{A}]),p(G_{A}[\overline{A}])+m(G_{A}[\overline{A}])+1}\left(\frac{S_{A}}{J_{G_{A}[\overline{A}]}},K\right) \cong \operatorname{Tor}_{p(G),p(G)+m(G)+1}^{S}\left(\frac{S}{J_{G}},K\right).$$
(6)

By induction, $\beta_{p(G_A[\overline{A}]), p(G_A[\overline{A}])+m(G_A[\overline{A}])+1}^{S_A}(S_A/J_{G_A[\overline{A}]}) \neq 0$ is an extremal Betti number. Now, Eq. (6) implies

$$\beta_{p(G),p(G)+m(G)+1}^{S}(S/J_G) \neq 0,$$

and by Eq. (5), we get that $\beta_{p(G),p(G)+m(G)+1}^{S}(S/J_G)$ is an extremal Betti number. **Case** (2): If $\alpha \geq 2$, then by virtue of Theorem 3.4 and Lemma 3.6, $p(G_A) = p(G) - \alpha + 1$, $p(G_A[\overline{A}]) = p(G) - 2\alpha + 1$. Therefore,

$$\operatorname{Tor}_{i}^{S}\left(\frac{S}{Q_{1}},K\right) = \operatorname{Tor}_{i}^{S}\left(\frac{S}{J_{G_{A}}},K\right) = 0, \text{ for } i \ge p(G).$$

Note that $G[\overline{A}]$ has q+1 connected components. By Theorem 3.4 and Lemma 3.6, we have that $p(G[\overline{A}]) \leq p(G) - 2\alpha - q + 1$. Therefore, $pd_S(S/Q_2) = 2\alpha + p(G[\overline{A}]) \leq p(G) - q + 1$. Thus, for each $j \ge 0$, the exact sequence (1) yields the long exact sequence of Tor sequence:

$$0 \to \operatorname{Tor}_{p(G)+1,p(G)+j}^{S}\left(\frac{S}{Q_{1}+Q_{2}},K\right) \to \operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{J_{G}},K\right) \to \\ \to \operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{Q_{2}},K\right) \to \dots$$

$$(7)$$

We now distinguish between two sub-cases.

Case (2.1): If $pd_S(S/Q_2) = 2\alpha + p(G[\overline{A}]) \le p(G) - 1$, then

$$\operatorname{Tor}_{p(G)}^{S}\left(\frac{S}{Q_{2}},K\right) = 0$$

For each $j \ge 0$, (7) yields that

$$\operatorname{Tor}_{p(G)+1,p(G)+j}^{S}\left(\frac{S}{Q_{1}+Q_{2}},K\right) \cong \operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{J_{G}},K\right).$$
(8)

Now, Eqs. (3), (4) and (8) imply

$$\operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{J_{G}},K\right) = 0 \quad \text{for} \quad j > m(G) + 1,$$
(9)

and

$$\operatorname{Tor}_{p(G_{A}[\overline{A}]),p(G_{A}[\overline{A}])+m(G_{A}[\overline{A}])+1}^{S_{A}}\left(\frac{S_{A}}{J_{G_{A}}[\overline{A}]},K\right) \cong \operatorname{Tor}_{p(G),p(G)+m(G)+1}^{S}\left(\frac{S}{J_{G}},K\right).$$
(10)

By induction, $\beta_{p(G_A[\overline{A}]), p(G_A[\overline{A}]) + m(G_A[\overline{A}]) + 1}^{S_A}(S_A/J_{G_A[\overline{A}]}) \neq 0$ is an extremal Betti number. Hence, Eq. (10) implies

$$\beta_{p(G),p(G)+m(G)+1}^{S}(S/J_G) \neq 0,$$

and by Eq. (9), we get that $\beta_{p(G),p(G)+m(G)+1}^{S}(S/J_G)$ is an extremal Betti number.

Case (2.2): If $pd_S(S/Q_2) = p(G)$, then q = 1. Let H_1 and H_2 be connected components of $G[\overline{A}]$. Then, $m(G[\overline{A}]) = m(H_1) + m(H_2)$. For i = 1, 2, set $S_{H_i} = K[x_j, y_j : j \in V(H_i)]$. If H_2 is an isolated vertex, then $pd_S(S/Q_2) = 2\alpha + p(H_1) = 2\alpha + p(G[\overline{A}])$ and $m(H_1) = m(G[\overline{A}]) \leq m(G) - 1$. If H_2 is a non-trivial graph, then $pd_S(S/Q_2) = 2\alpha + p(H_1) + p(H_2) = 2\alpha + p(G[\overline{A}])$ and $m(H_1) + m(H_2) + 2 = m(G[\overline{A}]) + 2 \leq m(G) + 1$. By induction,

$$\operatorname{Tor}_{p(G[\overline{A}]),p(G[\overline{A}])+j}^{S_A}\left(\frac{S}{J_{G[\overline{A}]}},K\right) = 0 \quad \text{for} \quad j > m(G) + 1 \ge m(G[\overline{A}]) + 2.$$

Thus, $\operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{Q_{2}},K\right) = 0$, for j > m(G) + 1. Now, Eqs. (3), (4) and (7) imply

$$\operatorname{Tor}_{p(G),p(G)+j}^{S}\left(\frac{S}{J_{G}},K\right) = 0 \quad \text{for} \quad j > m(G) + 1.$$
(11)

By induction, $\beta_{p(G_A[\overline{A}]),p(G_A[\overline{A}])+m(G_A[\overline{A}])+1}^{S_A}(S_A/J_{G_A[\overline{A}]}) \neq 0$. Consequently, by Eq. (7),

$$\beta_{p(G),p(G)+m(G)+1}^{S}(S/J_G) \neq 0,$$

and together with Eq. (11), we get that $\beta_{p(G),p(G)+m(G)+1}^S(S/J_G)$ is an extremal Betti number.

If G is a complete graph, then p(G) = n - 1 and m(G) = 0. It follows from [10, Corollary 4.3] that $\beta_{n-1,n}^S(S/J_G) = n - 1 = f(G) - 1$. We now assume that for every internal vertex v, $\operatorname{cdeg}_G(v) > 2$. Therefore, $q \ge 2$, and as before we conclude that

$$\operatorname{Tor}_{p(G_A[\overline{A}]), p(G_A[\overline{A}]) + m(G_A[\overline{A}]) + 1}^{S_A} \left(\frac{S_A}{J_{G_A[\overline{A}]}}, K \right) \cong \operatorname{Tor}_{p(G), p(G) + m(G) + 1}^{S} \left(\frac{S}{J_G}, K \right)$$

Now, by induction, $\beta_{p(G_A[\overline{A}]), p(G_A[\overline{A}]) + m(G_A[\overline{A}]) + 1}^{S_A}(S_A/J_{G_A[\overline{A}]}) = f(G_A[\overline{A}]) - 1$. Since $f(G) = f(G_A[\overline{A}])$, we conclude that $\beta_{p(G), p(G) + m(G) + 1}^S(S/J_G) = f(G) - 1$.

Corollary 3.8. Let G be a generalized block graph for which $G = G_1 \cup \cdots \cup G_s$ is the decomposition of G into indecomposable graphs. Then, $\beta_{p(G),p(G)+m(G)+1}^S(S/J_G)$ is an extremal Betti number of S/J_G .

Proof. Note that $m(G) = m(G_1) + \cdots + m(G_s) + s - 1$. Now, the assertion follows from Proposition 3.5 and Theorem 3.7.

As of now, the only lower bound known for regularity of binomial edge ideals of generalized block graphs is $\ell(G)$, which is a general lower bound given by Matsuda and Murai. If H is a longest induced path of a generalized block graph G, then $\ell(G) = \ell(H) = m(H) + 1 \leq m(G) + 1$. Thus, as an immediate consequence of Theorem 3.7, we obtain an improved lower bound for the regularity of binomial edge ideals of generalized block graphs.

Corollary 3.9. Let G be a generalized block graph on [n] with c_G connected components. Then, $\operatorname{reg}(S/J_G) \ge m(G) + c_G$.

Proof. Let G_1, \ldots, G_{c_G} be connected components of G. For $1 \leq i \leq c_G$, set $S_i = K[x_j, y_j : j \in V(G_i)]$. Then, for $1 \leq i \leq c_G$, by Corollary 3.8, $\operatorname{reg}(S_i/J_{G_i}) \geq m(G_i) + 1$. Note that $m(G) = m(G_1) + \cdots + m(G_{c_G})$ and $S/J_G \simeq S_1/J_{G_1} \otimes \cdots \otimes S_{c_G}/J_{G_{c_G}}$. Thus, the minimal graded free resolution of S/J_G is the tensor product of the minimal free resolutions of $S_1/J_{G_1}, \ldots, S_{c_G}/J_{G_{c_G}}$. Hence, $\operatorname{reg}(S/J_G) = \sum_{i=1}^{c_G} \operatorname{reg}(S_i/J_{G_i}) \geq m(G) + c_G$.

We now give an example of a connected chordal graph G that is not a generalized block graph for which $\operatorname{reg}(S/J_G) < m(G) + 1$.

Example 3.10. Let G be a graph as shown in Fig. 2. Then, it can be seen that G is a chordal graph that is not a generalized block graph. The minimal cut sets of G are $\{2,3\}, \{2,5\}, \{3,5\}$. Therefore, m(G) = 3. Using Macaulay2 [7], it can be seen that $\operatorname{reg}(S/J_G) = 3 < m(G) + 1 = 4$.



FIGURE 2

Our aim is to classify generalized block graphs whose binomial edge ideals admit a unique extremal Betti number. Equivalently, we want to classify the generalized block graphs G for which $\operatorname{reg}(S/J_G) = m(G) + 1$. For that recall the definition of flower graph, introduced by Mascia and Rinaldo in [18]: a *flower* graph $F_{h,k}(v)$ is a connected graph obtained by gluing each of h copies of the complete graph K_3 and k copies of the star graph $K_{1,3}$ at a common vertex v, that is free in each of them. Now, we characterize generalized block graphs whose binomial edge ideals admit a unique extremal Betti number.

We denote by iv(G) the number of internal vertices of G.

Theorem 3.11. Let G be a connected indecomposable generalized block graph. Then, the following are equivalent:

- (1) S/J_G admits a unique extremal Betti number.
- (2) For any $v \in V(G)$, $F_{h,k}(v)$ is not an induced subgraph of G for every $h, k \ge 0$ with $h+k \ge 3$.

In this case, $\operatorname{reg}(S/J_G) = m(G) + 1$.

Proof. (1) \implies (2) : Suppose that for some $v \in V(G)$ and $h, k \geq 0$ with $h + k \geq 3$, $F_{h,k}(v)$ is an induced subgraph of G. It is enough to prove $\operatorname{reg}(S/J_G) > m(G) + 1$. Let Hbe an induced subgraph of G obtained in the following way: for every minimal cut set Awith $|A| \geq 2$, remove |A| - 1 elements of A from G. Note that H is a block graph with $\operatorname{iv}(H) = m(G)$ by [22, Proposition 2.1] and $F_{h,k}(v)$ is an induced subgraph of H. It follows from [11, Theorem 8] that $\operatorname{reg}(S/J_H) > \operatorname{iv}(H) + 1$. Now, by virtue of [19, Corollary 2.2], $\operatorname{reg}(S/J_G) \geq \operatorname{reg}(S/J_H) > m(G) + 1$.

(2) \implies (1) : By Corollary 3.9, it is enough to prove $\operatorname{reg}(S/J_G) \leq m(G) + 1$. We prove this by induction on m(G). If m(G) = 0, then G is a complete graph and the assertion is obvious. Assume that m(G) > 0. Let F_1, \ldots, F_r be a leaf order of $\mathcal{F}(\Delta(G))$. Let A be the minimal cut set defined in the proof of Theorem 3.7. Then $G_A, G_A[\overline{A}]$ and $G[\overline{A}]$ are generalized block graphs. Note that G_A and $G_A[\overline{A}]$ are generalized block graphs satisfying the hypothesis with $m(G_A) = m(G_A[\overline{A}]) = m(G) - 1$. By induction, we have $\operatorname{reg}(S/J_{G_A}) = \operatorname{reg}(S/J_{G_A[\overline{A}]}) \leq m(G)$. As in the proof of Theorem 3.7, $G[\overline{A}]$ has q + 1 connected components, say H_1, \ldots, H_{q+1} . Since G has no induced $F_{h,k}$ with $h + k \geq 3$, at least q - 1 components are isolated vertices. The two remaining components are a clique and a generalized block graph, say H_1 , satisfying the assumption with $m(H_1) \leq m(G) - 1$. Applying induction we obtain that $\operatorname{reg}(S/J_{G[\overline{A}]}) \leq m(H_1) + 2 \leq m(G) + 1$. Now, the assertion follows from the exact sequence (1) and [21, Corollary 18.7].

As an immediate consequence of Proposition 3.5 and Theorem 3.11, we have the following results:

Corollary 3.12. Let G be a connected generalized block graph for which $G = G_1 \cup \cdots \cup G_r$ is the decomposition of G into indecomposable graphs. Then, S/J_G admits a unique extremal Betti number if and only if for each i, S_i/J_{G_i} admits a unique extremal Betti number. Moreover, in this case $\operatorname{reg}(S/J_G) = m(G) + 1$.

Recall that a *caterpillar* is a tree in which the removal of all pendant vertices leaves a path graph.

Corollary 3.13. Let T be an indecomposable tree on [n]. Then, S/J_T admits a unique extremal Betti number if and only if T is a caterpillar.

Example 3.14. Let $G = F_{h,k}(v)$ be a flower graph with $h + k \ge 3$. Then, it follows from [18, Corollary 3.5] that $\operatorname{reg}(S/J_G) = m(G) + h + k - 1 > m(G) + 1$.

4. Regularity upper bound for generalized block graph

In this section, we give an improved upper bound for the regularity of binomial edge ideals of generalized block graphs. Let $u, v \in V(G)$ be such that $e = \{u, v\} \notin E(G)$, then we denote by G_e , the graph on the vertex set V(G) and edge set $E(G_e) = E(G) \cup \{\{x, y\} : x, y \in N_G(u) \text{ or } x, y \in N_G(v)\}$. An edge e is said to be a *cut edge* of G if the number of connected components of $G \setminus e$ is larger than that of G.

We now recall a result from [15] that will be used repeatedly in this section.

Lemma 4.1. [15, Proposition 2.1] Let G be a graph and e be a cut edge of G. Then,

$$\operatorname{reg}(S/J_G) \le \max\{\operatorname{reg}(S/J_{G\setminus e}), \operatorname{reg}(S/J_{(G\setminus e)_e}) + 1\}.$$

The degree of a vertex v of G is $\deg_G(v) = |N_G(v)|$. A vertex v is said to be a pendant vertex if $\deg_G(v) = 1$. For $v \in V(G)$, let $\deg_G(v)$ denote the number of maximal cliques of G which contains v, and $\operatorname{pdeg}_G(v)$ denote the number of pendant vertices adjacent to v. Note that for every $v \in V(G)$, $\operatorname{pdeg}_G(v) \leq \operatorname{cdeg}_G(v)$.

Remark 4.2. Let G be a connected indecomposable generalized block graph which is not a star graph. If $e = \{u, v\}$ is an edge with pendant vertex u, then $(G \setminus e)_e = (G \setminus u)_v \sqcup \{u\}$, $\operatorname{cl}(G \setminus u) = \operatorname{cl}(G) - 1$, $\operatorname{cl}((G \setminus u)_v) = \operatorname{cl}(G) - \operatorname{cdeg}_G(v) + 1$, $J_{G \setminus e} = J_{G \setminus u}$ and $J_{(G \setminus e)_e} = J_{(G \setminus u)_v}$. Also, $(G \setminus e)_e$ and $G \setminus e$ are generalized block graphs other than star graphs.

A vertex $v \in V(G)$ with $\operatorname{pdeg}_G(v) \geq 1$ is said to be of $type \ 1$ if $\operatorname{cdeg}_G(v) = \operatorname{pdeg}_G(v) + 1$, and of $type \ 2$ if $\operatorname{cdeg}_G(v) \geq \operatorname{pdeg}_G(v) + 2$. We denote by $\alpha(G)$, the number of vertices of type 1 in G and by $\operatorname{pv}(G)$, the number of pendant vertices of G.

Lemma 4.3. If G is a connected indecomposable graph on [n] with pv(G) > 0, then $pv(G) - \alpha(G) > 0$.

Proof. First, we assume that $\alpha(G) = 0$, then the claim follows. Now, assume that $\alpha(G) = r > 0$. Let v_1, \ldots, v_r be all type 1 vertices of G. Since G is an indecomposable graph, $\operatorname{cdeg}_G(v_i) = \operatorname{pdeg}_G(v_i) + 1 \ge 3$, for $i \in [r]$. Thus, $\operatorname{pv}(G) \ge 2\alpha(G)$ which completes the proof.

Proposition 4.4. Let $G = G_1 \cup G_2$ be a decomposable graph such that G_1 and G_2 are indecomposable and let $S_i = K[x_j, y_j : j \in V(G_i)]$ for i = 1, 2. Suppose that one of the following conditions is satisfied:

- (a) G_1 and G_2 are star graphs, or
- (b) G_1 is a star graph, G_2 is not a star graph and $\operatorname{reg}(S_2/J_{G_2}) \leq \operatorname{cl}(G_2) + \alpha(G_2) \operatorname{pv}(G_2)$, or

(c) for $i = 1, 2, G_i$ is not a star graph and $\operatorname{reg}(S_i/J_{G_i}) \leq \operatorname{cl}(G_i) + \alpha(G_i) - \operatorname{pv}(G_i)$. Then $\operatorname{reg}(S/J_G) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$.

Proof. (a) Since G_1 and G_2 are indecomposable star graphs, $\alpha(G) = 2$, pv(G) = cl(G) - 2. By Proposition 3.5 and [27, Theorem 4.1(a)], $reg(S/J_G) = 4 = cl(G) + \alpha(G) - pv(G)$. Hence, the claim follows.

(b) Let $V(G_1) \cap V(G_2) = \{u\}$. First, assume that u is not a pendant vertex of G_2 . Therefore, $\alpha(G) = \alpha(G_2) + 1$ and $pv(G) = pv(G_1) + pv(G_2) - 1$. Note that $cl(G) = cl(G_1) + cl(G_2)$ and $cl(G_1) = pv(G_1)$. By [27, Theorem 4.1(a)], $reg(S_1/J_{G_1}) = 2$. Hence, Proposition 3.5 yields that $reg(S/J_G) \leq cl(G) + \alpha(G) - pv(G)$. We now assume that u is a pendant vertex of G_2 . Note that $pv(G) = pv(G_1) + pv(G_2) - 2$ and $cl(G_1) = pv(G_1)$. Let $v \in N_{G_2}(u)$. If v is of type 1 in G_2 , then $\alpha(G) = \alpha(G_2)$, and hence, the claim follows from Proposition 3.5. If v is of type 2 in G_2 , then $\alpha(G) = \alpha(G_2) + 1$, and hence, by Proposition 3.5, $reg(S/J_G) \leq cl(G) + \alpha(G) - pv(G)$.

(c) Let $V(G_1) \cap V(G_2) = \{u\}$. Observe that, $cl(G) = cl(G_1) + cl(G_2)$. If $deg_{G_1}(u), deg_{G_2}(u) > 1$, then $\alpha(G) = \alpha(G_1) + \alpha(G_2)$ and $pv(G) = pv(G_1) + pv(G_2)$. Thus, by Proposition 3.5, $reg(S/J_G) = reg(S_1/J_{G_1}) + reg(S_2/J_{G_2}) \le cl(G) + \alpha(G) - pv(G)$.

Assume that u is a pendant vertex of G_1 , let $N_{G_1}(u) = \{u_1\}$ and $\deg_{G_2}(u) > 1$ (or vice versa). Then $pv(G) = pv(G_1) + pv(G_2) - 1$. If u_1 is of type 1 in G_1 , then we have $\alpha(G) = \alpha(G_1) + \alpha(G_2) - 1$. If u_1 is of type 2 in G_1 , then we have $\alpha(G) = \alpha(G_1) + \alpha(G_2)$. Hence, by Proposition 3.5, $reg(S/J_G) = reg(S/J_{G_1}) + reg(S/J_{G_2}) \leq cl(G) + \alpha(G) - pv(G)$.

Now, assume that for i = 1, 2, u is a pendant vertex of G_i , let $N_{G_i}(u) = \{u_i\}$. Then, $pv(G) = pv(G_1) + pv(G_2) - 2$. If both u_1 and u_2 are of type 1 in G_1 and G_2 , respectively, then we have $\alpha(G) = \alpha(G_1) + \alpha(G_2) - 2$. If both u_1 and u_2 are of type 2 in G_1 and G_2 , respectively, then $\alpha(G) = \alpha(G_1) + \alpha(G_2)$. If u_1 is of type 1 in G_1 and u_2 is of type 2 in G_2 (or vice versa), then we have $\alpha(G) = \alpha(G_1) + \alpha(G_2) - 1$. Hence, Proposition 3.5 yields $reg(S/J_G) = reg(S_1/J_{G_1}) + reg(S_2/J_{G_2}) \leq cl(G) + \alpha(G) - pv(G)$.

We now obtain a tight upper bound for the regularity of binomial edge ideals of connected indecomposable generalized block graphs.

Theorem 4.5. Let G be a connected indecomposable generalized block graph on [n] which is not a star graph. Then, $\operatorname{reg}(S/J_G) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$.

Proof. Let $k(G) = |\{v : \text{pdeg}_G(v) \ge 1\}|$. We proceed by induction on $k(G) + m(G) \ge 0$. For k(G) = 0, $\text{pv}(G) = \alpha(G) = 0$, and hence, the result is immediate from [24, Theorem 3.5] or [16, Theorem 3.15]. If m(G) = 0, then G is a complete graph and the assertion is obvious. Assume that k = k(G) > 0, m(G) > 0 and the assertion is true up to k(G) + m(G) - 1.

Let $v_1, \ldots, v_k \in V(G)$ be such that for each $i = 1, \ldots, k$, $pdeg(v_i) = r_i \ge 1$. For each $i = 1, \ldots, k$, let $e_{i,1} = \{v_i, w_{i,1}\}, \ldots, e_{i,r_i} = \{v_i, w_{i,r_i}\}$ be pendant edges incident to v_i . Since G is an indecomposable graph, $cdeg_G(v_i) = s_i \ge 3$.

We proceed by induction on r_k . If $r_k = 1$, then v_k is of type 2. Notice that $k((G \setminus w_{k,1})_{v_k}) = k(G) - 1$ and $m((G \setminus w_{k,1})_{v_k}) = m(G) - 1$. Thus, by induction on k(G) + m(G) and Remark 4.2, we have

$$\operatorname{reg}(S/J_{(G\setminus e_{k,1})_{e_{k,1}}}) = \operatorname{reg}(S/J_{(G\setminus w_{k,1})_{v_k}})$$

$$\leq \operatorname{cl}((G\setminus w_{k,1})_{v_k}) + \alpha((G\setminus w_{k,1})_{v_k}) - \operatorname{pv}((G\setminus w_{k,1})_{v_k})$$

$$= \operatorname{cl}(G) - \operatorname{cdeg}_G(v_k) + \alpha(G) - \operatorname{pv}(G) + 2$$

$$\leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G) - 1.$$

If $\operatorname{cdeg}_G(v_k) = 3$, then $G \setminus w_{k,1} = G_1 \cup G_2$ is a decomposable graph. If G_i is not a star graph, then $k(G_i) \leq k(G)$ and $m(G_i) < m(G)$, and hence, by induction on k(G) + m(G), we have $\operatorname{reg}(S_i/J_{G_i}) \leq \operatorname{cl}(G_i) + \alpha(G_i) - \operatorname{pv}(G_i)$. Note that G satisfies the assumption of Proposition 4.4. It follows from Proposition 4.4 that

$$\operatorname{reg}(S/J_{G\setminus e_{k,1}}) = \operatorname{reg}(S/J_{G\setminus w_{k,1}})$$

$$\leq \operatorname{cl}(G\setminus w_{k,1}) + \alpha(G\setminus w_{k,1}) - \operatorname{pv}(G\setminus w_{k,1})$$

$$= \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G).$$

If $\operatorname{cdeg}_G(v_k) > 3$, then $G \setminus w_{k,1}$ is an indecomposable generalized block graph with $k(G \setminus w_{k,1}) = k(G) - 1$ and $m(G \setminus w_{k,1}) = m(G)$. Thus, by induction on k(G) + m(G) and Remark 4.2, we have

$$\operatorname{reg}(S/J_{G\setminus e_{k,1}}) = \operatorname{reg}(S/J_{G\setminus w_{k,1}})$$

$$\leq \operatorname{cl}(G\setminus w_{k,1}) + \alpha(G\setminus w_{k,1}) - \operatorname{pv}(G\setminus w_{k,1})$$

$$= \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G).$$

In both the cases, we get $\operatorname{reg}(S/J_{G\setminus e_{k,1}}) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$. Hence, by Lemma 4.1, we have

$$\operatorname{reg}(S/J_G) \le \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G).$$

Assume now that $r_k > 1$. By Remark 4.2, we get $k((G \setminus w_{k,r_k})_{v_k}) = k(G) - 1$, $pv((G \setminus w_{k,r_k})_{v_k}) = pv(G) - r_k$, $cl((G \setminus w_{k,r_k})_{v_k}) = cl(G) - cdeg_G(v_k) + 1$ and $m((G \setminus w_{k,r_k})_{v_k}) < m(G)$. Case (1): v_k is of type 2 in G.

Thus, $\alpha((G \setminus w_{k,r_k})_{v_k}) = \alpha(G)$, $\operatorname{cdeg}_G(v_k) - r_k \ge 2$, and hence, by Remark 4.2 and induction on k(G) + m(G), we have

$$\operatorname{reg}(S/J_{(G \setminus e_{k,r_k})_{e_{k,r_k}}}) = \operatorname{reg}(S/J_{(G \setminus w_{k,r_k})_{v_k}})$$

$$\leq \operatorname{cl}((G \setminus w_{k,r_k})_{v_k}) + \alpha((G \setminus w_{k,r_k})_{v_k}) - \operatorname{pv}((G \setminus w_{k,r_k})_{v_k})$$

$$= \operatorname{cl}(G) - \operatorname{cdeg}_G(v_k) + 1 + \alpha(G) - \operatorname{pv}(G) + r_k$$

$$\leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G) - 1.$$

Note that v_k is of type 2 in $G \setminus w_{k,r_k}$ and $G \setminus w_{k,r_k}$ is indecomposable. It follows from Remark 4.2, and induction on r_k that

$$\operatorname{reg}(S/J_{G\setminus e_{k,r_k}}) = \operatorname{reg}(S/J_{G\setminus w_{k,r_k}})$$

$$\leq \operatorname{cl}(G\setminus w_{k,r_k}) + \alpha(G\setminus w_{k,r_k}) - \operatorname{pv}(G\setminus w_{k,r_k})$$

$$= \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G).$$

Thus, by Lemma 4.1, $\operatorname{reg}(S/J_G) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$.

Case (2): v_k is of type 1 in G.

Thus, $\alpha((G \setminus w_{k,r_k})_{v_k}) = \alpha(G) - 1$, $\operatorname{cdeg}_G(v_k) - r_k = 1$, and hence, by Remark 4.2 and induction on k(G) + m(G),

$$\operatorname{reg}(S/J_{(G \setminus e_{k,r_{k}})e_{k,r_{k}}}) = \operatorname{reg}(S/J_{(G \setminus w_{k,r_{k}})v_{k}})$$

$$\leq \operatorname{cl}((G \setminus w_{k,r_{k}})v_{k}) + \alpha((G \setminus w_{k,r_{k}})v_{k}) - \operatorname{pv}((G \setminus w_{k,r_{k}})v_{k})$$

$$= \operatorname{cl}(G) - \operatorname{cdeg}_{G}(v_{k}) + \alpha(G) - \operatorname{pv}(G) + r_{k}$$

$$= \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G) - 1.$$

Note that v_k is of type 1 in $G \setminus w_{k,r_k}$. If $r_k = 2$, then $G \setminus w_{k,r_k}$ is a decomposable graph with decomposition $(G \setminus \{w_{k,1}, w_{k,2}\}) \cup \{e_{k,1}\}$. Set $H = G \setminus \{w_{k,1}, w_{k,2}\}$. If H is not a star graph, then $k(H) \leq k(G)$ and m(H) < m(G). Thus, by induction on k(G) + m(G),



FIGURE 3

 $\operatorname{reg}(S/J_H) \leq \operatorname{cl}(H) + \alpha(H) - \operatorname{pv}(H). \text{ Note that } \operatorname{cl}(H) = \operatorname{cl}(G) - 2. \text{ If } \operatorname{deg}_H(v_k) = 1, \text{ then } \operatorname{pv}(H) = p(G) - 1 \text{ and } \alpha(H) \leq \alpha(G), \text{ and if } \operatorname{deg}_H(v_k) > 1, \text{ then } \operatorname{pv}(H) = \operatorname{pv}(G) - 2 \text{ and } \alpha(H) = \alpha(G) - 1. \text{ Therefore, in both cases, } \operatorname{reg}(S/J_H) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G) - 1. \text{ Now, it follows from Proposition 3.5 that } \operatorname{reg}(S/J_{G\setminus e_{k,r_k}}) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G). \text{ If } G \setminus \{w_{k,1}, w_{k,2}\} \text{ is a star graph, then } \alpha(G) = 2 \text{ and } \operatorname{cl}(G) - \operatorname{pv}(G) = 1. \text{ By Proposition 3.5, } \operatorname{reg}(S/J_{G\setminus e_{k,r_k}}) = 3. \text{ Thus, } \operatorname{reg}(S/J_{G\setminus e_{k,r_k}}) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G).$

If $r_k > 2$, then $G \setminus w_{k,r_k}$ is indecomposable. By induction on r_k , we have

$$\operatorname{reg}(S/J_{G\setminus e_{k,r_{k}}}) = \operatorname{reg}(S/J_{G\setminus w_{k,r_{k}}})$$

$$\leq \operatorname{cl}(G\setminus w_{k,r_{k}}) + \alpha(G\setminus w_{k,r_{k}}) - \operatorname{pv}(G\setminus w_{k,r_{k}})$$

$$= \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G) - 1.$$

Thus, for $r_k \geq 2$, $\operatorname{reg}(S/J_{G\setminus e_{k,r_k}}) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$. Consequently, by Lemma 4.1, $\operatorname{reg}(S/J_G) \leq \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$. Hence, the assertion follows.

The following example illustrates that the upper bound is not always attained in Theorem 4.5.

Example 4.6. Let G be a tree as shown in Fig. 3. Notice that cl(G) = 13, $\alpha(G) = 4$ and pv(G) = 8. Thus, $cl(G) + \alpha(G) - pv(G) = 9$. Using Macaulay2 [7], we get $reg(S/J_G) = 8$.

As an immediate consequence, we have the following:

Corollary 4.7. Let T be an indecomposable tree on [n] which is not a star graph. Then, $\operatorname{reg}(S/J_T) \leq \operatorname{cl}(T) + \alpha(T) - \operatorname{pv}(T)$.

Finally we show two classes of block graphs that attain the upper bound $cl(G) + \alpha(G) - pv(G)$.

Corollary 4.8. If G is an indecomposable caterpillar that is not a star graph or a flower graph, then $\operatorname{reg}(S/J_G) = \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$.

Proof. First, let G be an indecomposable caterpillar that is not a star graph. Then $m(G) + 1 = \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$. Therefore, by Corollaries 3.13 and 4.7, $\operatorname{reg}(S/J_G) = \operatorname{cl}(G) + \alpha(G) - \operatorname{pv}(G)$.

Now, let $G = F_{h,k}(v)$ be a flower graph. Then $cl(G) + \alpha(G) - pv(G) = m(G) + cdeg_G(v) - 1 = h + 2k$. Thus, by virtue of [18, Corollary 3.5], $reg(S/J_G) = cl(G) + \alpha(G) - pv(G)$.

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