Deformed higher rank Heisenberg-Virasoro algebras

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Abstract

In this paper, we study a class of infinitesimal deformations of the centerless higher rank Heisenberg-Virasoro algebras. Explicitly, the universal central extensions, derivations and isomorphism classes of these algebras are determined.

Keywords: Heisenberg-Virasoro algebra, deformed Heisenberg-Virasoro algebra, higher rank Heisenberg-Virasoro algebra, infinitesimal deformation.

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1 Introduction

Let G be an additive subgroup of \mathbb{C} and $\lambda \in \mathbb{C}$. The Lie algebra $\mathfrak{g}(G,\lambda)$ we study in this paper has a basis $\{L_a, I_a \mid a \in G\}$ subjecting to the following Lie brackets

$$[L_a, L_b] = (b - a)L_{a+b}; \quad [L_a, I_b] = (b - \lambda a)I_{a+b}; \quad [I_a, I_b] = 0.$$
 (1.1)

We will simply denote $\mathfrak{g} = \mathfrak{g}(G, \lambda)$ if no confusion. Since for any nonzero $\epsilon \in \mathbb{C}$, there is a Lie algebra isomorphism between $\mathfrak{g}(G, \lambda)$ and $\mathfrak{g}(\epsilon G, \lambda)$ defined by

$$L_a \mapsto \epsilon^{-1} L_{\epsilon a}; \quad I_a \mapsto \epsilon^{-1} I_{\epsilon a},$$

we always assume in this paper that $\mathbb{Z} \subseteq G$ and $\frac{1}{k} \notin G$ for any integer k > 1.

The algebra \mathfrak{g} is formed through semi-product of the higher rank Virasoro algebra $\mathfrak{v} = \operatorname{span}_{\mathbb{C}}\{L_a \mid a \in G\}$ and one of its module of tensor fields $V(\lambda) = \operatorname{span}_{\mathbb{C}}\{v_a \mid a \in G\}$ defined by

$$L_a v_b = (b - \lambda a) v_{a+b}.$$

Notice that the algebra $\mathfrak{g}(G,0)$ is the centerless higher rank Heisenberg-Virasoro algebra [8], and $\mathfrak{g}(G,\lambda)$ is an infinitesimal deformation of $\mathfrak{g}(G,0)$. Therefore, we call $\mathfrak{g}(G,\lambda)$ a centerless deformed higher rank Heisenberg-Virasoro algebra. It also has close connections with the Heisenberg-Virasoro algebra and deformed Heisenberg-Virasoro algebras. Taking $G = \mathbb{Z}$, we get the centerless deformed Heisenberg-Virasoro algebra $\mathfrak{g}(\mathbb{Z},\lambda)$, which was given in [7] through the second cohomology group of the centerless Heisenberg-Virasoro algebra $\mathfrak{g}(\mathbb{Z},\lambda)$ with coefficients in the adjoint representation. Hence the deformed Heisenberg-Virasoro algebras and higher rank Heisenberg-Virasoro algebras are generalizations of the Heisenberg-Virasoro algebra, which was first introduced in [1]. Representations of these three algebras were studied in [5, 6, 9] and references therein.

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Due to the connection with these algebras, it seems reasonable and important to study the algebra \mathfrak{g} . Our main purpose of this paper is to determine the universal central extensions, derivations and isomorphism classes of all $\mathfrak{g}(G,\lambda)$. Such computation is a priority in the structure theory of Lie algebra, and has been done for many Lie algebras, such as generalized Witt algebras [3], generalized Schrodinger-Virasoro algebras [13, 12], higher rank Heisenberg-Virasoro algebras [8], deformed Heisenberg-Virasoro algebras [7], solenoidal Lie algebras over a quantum torus [14], and so on.

The following is the outline of this paper. In section 2, we compute the universal central extension $\overline{\mathfrak{g}}$ of \mathfrak{g} , which we call deformed higher rank Heisenberg-Virasoro algebra. Section 3 is devoted to derivations of \mathfrak{g} and the lifts of these derivations to derivations of $\overline{\mathfrak{g}}$. In the last section isomorphism classes and automorphism groups of all the algebras $\mathfrak{g}(G,\lambda)$ are determined, and finally the explicit form of the unique lift of any outer automorphism of \mathfrak{g} to $\overline{\mathfrak{g}}$ is given.

Throughout this paper, the symbols $\mathbb{Z}, \mathbb{C}, \mathbb{C}^{\times}$ refer to the set of integers, complex numbers and nonzero complex numbers respectively. We denote by G^* the set of characters of G, and by $\text{Hom}(G,\mathbb{C})$ the set of all additive group homomorphisms.

2 Universal central extensions

In this section we compute the universal central extensions of \mathfrak{g} for $\lambda \neq -1$, G being free and of rank $n \geq 1$. We fix a \mathbb{Z} -basis $\epsilon_1 = 1, \epsilon_2, \dots, \epsilon_n$ of G.

Let $\varphi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ be a 2-cocycle, hence

$$\varphi([x,y],z) + \varphi([y,z],x) + \varphi([z,x],y) = 0;$$

$$\varphi(x,y) = -\varphi(y,x), \text{ for any } x,y,z \in \mathfrak{g}.$$
(2.1)

Define a linear function f on \mathfrak{g} by

$$f(L_a) = \frac{1}{a}\varphi(L_0, L_a); \quad f(I_a) = \frac{1}{a}\varphi(L_0, I_a) \text{ for } a \neq 0;$$

$$f(L_0) = \frac{1}{2}\varphi(L_{-1}, L_1); \quad f(I_0) = \frac{1}{\lambda + 1}\varphi(L_{-1}, I_1),$$

and a 2-coboundary φ_f by $\varphi_f(x,y) = f([x,y])$. Replacing φ by $\varphi - \varphi_f$, we may assume

$$\varphi(L_0, L_a) = \varphi(L_0, I_a) = \varphi(L_{-1}, L_1) = \varphi(L_{-1}, I_1) = 0$$
 for any $a \neq 0$.

Lemma 2.1. $\varphi(L_a, L_b) = \frac{1}{12}(a^3 - a)C_L\delta_{a+b,0}$ for some $C_L \in \mathbb{C}$.

Proof. One reference for this computation is [11].

Lemma 2.2. $\varphi(I_a, I_b) = aC_I\delta_{a+b,0}\delta_{\lambda,0}$ for some $C_I \in \mathbb{C}$.

Proof. Consider the triple $(x, y, z) = (L_a, I_b, I_c)$ in (2.1) and we have

$$(b - \lambda a)\varphi(I_{a+b}, I_c) + (c - \lambda a)\varphi(I_b, I_{a+c}) = 0.$$
(2.2)

Set a = 0 in (2.2) and we get

$$\varphi(I_b, I_c) = 0 \text{ if } b + c \neq 0.$$

Take c = -a - b in (2.2),

$$(b - \lambda a)\varphi(I_{a+b}, I_{-a-b}) = ((\lambda + 1)a + b)\varphi(I_b, I_{-b}).$$
(2.3)

Set a = 1 - b in (2.3), we obtain

$$-(\lambda b - \lambda - 1)\varphi(I_b, I_{-b}) = ((\lambda + 1)b - \lambda)\varphi(I_1, I_{-1}), \tag{2.4}$$

which implies the lemma for $\lambda = 0$ by setting $C_I = \varphi(I_1, I_{-1})$.

Now we only need to prove that $\varphi(I_a, I_{-a}) = 0$ for any $a \in G$ if $\lambda \neq 0, -1$. If $\lambda^{-1} \in \mathbb{Z}$, then set $a = \frac{b}{\lambda} \in G$ in (2.3) and one gets

$$b(\frac{1}{\lambda} + 2)\varphi(I_b, I_{-b}) = 0,$$

which shows that $\varphi(I_b, I_{-b}) = 0$ if $\lambda \neq -\frac{1}{2}$. Suppose $\lambda = -\frac{1}{2}$ and by (2.4) we see

$$\varphi(I_b, I_{-b}) = \varphi(I_1, I_{-1}) \text{ if } b \neq -1.$$

Choose $b \neq \pm 1$ and let a = -1 - b in (2.3), we get

$$\frac{b-1}{2}\varphi(I_{-1},I_1) = \frac{b-1}{2}\varphi(I_b,I_{-b}) = \frac{b-1}{2}\varphi(I_1,I_{-1}),$$

which forces that $\varphi(I_1, I_{-1}) = 0$ by skew-symmetry. So $\varphi(I_b, I_{-b}) = 0$. Suppose that $\lambda^{-1} \notin \mathbb{Z}$, then $\lambda b + \lambda + 1 \neq 0$ for any $b \in G$. So (2.4) gives

$$\varphi(I_a, I_{-a}) = -\frac{(\lambda + 1)a - \lambda}{\lambda a - \lambda - 1} \varphi(I_1, I_{-1}).$$

Put it into (2.3) and set b = 0, then we get

$$\lambda(\lambda+1)(2\lambda+1)a(a-1)\varphi(I_1,I_{-1})=0$$
 for any $a\in G$,

which forces $\varphi(I_1, I_{-1}) = 0$, and hence $\varphi(I_a, I_{-a}) = 0$. We complete the proof.

Lemma 2.3.
$$\varphi(L_a, I_b) = \delta_{a+b,0} \left(C_{LI}^{(0)}(a^2 + a) \delta_{\lambda,0} + \frac{1}{12}(a^3 - a) C_{LI}^{(1)} \delta_{\lambda,1} + \sum_{i=2}^n a_i C_{LI}^{(i)} \delta_{\lambda,-2} \right),$$

where $C_{LI}^{(i)} \in \mathbb{C}, 0 \le i \le n$ and $a = \sum_{i=1}^n a_i \epsilon_i$.

Proof. Consider the triple $(x, y, z) = (L_a, L_b, I_c)$ in (2.1), we have

$$(b-a)\varphi(L_{a+b}, I_c) - (c-\lambda b)\varphi(L_a, I_{b+c}) + (c-\lambda a)\varphi(L_b, I_{a+c}) = 0.$$
 (2.5)

Take a=0,

$$(b+c)\varphi(L_b, I_c) - (c-\lambda b)\varphi(L_0, I_{b+c}) = 0,$$

which implies

$$\varphi(L_b, I_c) = 0 \quad \text{if} \quad b + c \neq 0. \tag{2.6}$$

Moreover, let b+c=0 and $c-\lambda b\neq 0$ (such b,c exist since $\lambda\neq -1$), we get $\varphi(L_0,I_0)=0$. Set c=-a-b in (2.5) and we obtain

$$(b-a)\varphi(L_{a+b},I_{-a-b}) + (a+b(\lambda+1))\varphi(L_a,I_{-a}) - (a(\lambda+1)+b)\varphi(L_b,I_{-b}) = 0.$$
 (2.7)

Let a + b = 0 in (2.7) and we get

$$\varphi(L_{-a}, I_a) = -\varphi(L_a, I_{-a}) \text{ if } \lambda \neq 0.$$
(2.8)

Let b = 1 in (2.7), we have

$$(1-a)\varphi(L_{a+1},I_{-a-1}) + (a+\lambda+1)\varphi(L_a,I_{-a}) - (a(\lambda+1)+1)\varphi(L_1,I_{-1}) = 0.$$
 (2.9)

Moreover, setting b = -1 and replacing a by a + 1 in (2.7), it gives

$$(a - \lambda)\varphi(L_{a+1}, I_{-a-1}) - (a+2)\varphi(L_a, I_{-a}) = 0.$$
(2.10)

Now we continue the proof in the following four cases.

Case I: $\lambda = 0$. Combining (2.9) and (2.10), we get

$$\varphi(L_a, I_{-a}) = \frac{1}{2}(a^2 + a)\varphi(L_1, I_{-1}).$$

Set $C_{LI}^{(0)} = \frac{1}{2}\varphi(L_1, I_{-1})$, and we get the lemma by (2.6).

We emphasize that in the remaining three cases we have $\varphi(L_{-a}, I_a) = -\varphi(L_a, I_{-a})$ for any $a \in G$. Especially, $\varphi(L_1, I_{-1}) = -\varphi(L_{-1}, I_1) = 0$.

Case II: $\lambda \neq 0, \pm 1, -2$. Notice that (2.9) and (2.10) form a system of homogeneous liner equations in variables $\varphi(L_a, I_{-a})$ and $\varphi(L_{a+1}, I_{-a-1})$ with a coefficient matrix whose determinant is

$$\begin{vmatrix} 1-a & a+\lambda+1 \\ a-\lambda & -a-2 \end{vmatrix} = (\lambda-1)(\lambda+2) \neq 0.$$

So $\varphi(L_a, I_{-a}) = 0$ and hence by (2.6) $\varphi(L_a, I_b) = 0$ for any $a, b \in G$.

Case III: $\lambda = 1$. Set $C_{LI}^{(1)} = 2\varphi(L_2, I_{-2})$. A same calculation as in Lemma 2.1 shows

$$\varphi(L_a, I_b) = \frac{1}{12} (a^3 - a) C_{LI}^{(1)} \delta_{a+b,0}.$$

Case IV: $\lambda = -2$. In this case, the equations (2.9) and (2.10) turn to

$$(a-1)\varphi(L_a, I_{-a}) = (a-1)\varphi(L_{a+1}, I_{-a-1}),$$

$$(a+2)\varphi(L_a, I_{-a}) = (a+2)\varphi(L_{a+1}, I_{-a-1}),$$

which implies that $\varphi(L_{a+1}, I_{-a-1}) = \varphi(L_a, I_{-a})$ for any $a \in G$. So

$$\varphi(L_k, I_{-k}) = 0, \quad \varphi(L_{a+k}, I_{-a-k}) = \varphi(L_a, I_{-a}) \text{ for any } k \in \mathbb{Z}, a \in G \setminus \mathbb{Z}.$$

Then (2.5) gives

$$\varphi(L_{a+b}, I_{-a-b}) = \varphi(L_a, I_{-a}) + \varphi(L_b, I_{-b})$$
 if $a \neq b$.

For $a \in G \setminus \mathbb{Z}$, we still have

$$\varphi(L_{2a}, I_{-2a}) = \varphi(L_{a+1}, I_{-a-1}) + \varphi(L_{a-1}, I_{-a+1}) = 2\varphi(L_a, I_{-a}).$$

So we have proved $\varphi(L_{a+b}, I_{-a-b}) = \varphi(L_a, I_{-a}) + \varphi(L_b, I_{-b})$ for any $a, b \in G$. Therefore $\varphi(L_a, I_{-a}) = \sum_{i=2}^n a_i \varphi(L_{\epsilon_i}, I_{-\epsilon_i})$ if $a = \sum_{i=1}^n a_i \epsilon_i$. Put $C_{LI}^{(i)} = \varphi(L_{\epsilon_i}, I_{-\epsilon_i})$ for $2 \le i \le n$, and we see that $\varphi(L_a, I_b) = \sum_{i=2}^n a_i C_{LI}^{(i)} \delta_{a+b,0}$. This finishes the proof.

Now from Lemma 2.1, 2.2 and 2.3 we get the main theorem of this section.

Theorem 2.4. Denote by $\overline{\mathfrak{g}}$ the universal central extension of $\mathfrak{g}(G,\lambda)$ for $\lambda \neq -1$ and G be a free subgroup of \mathbb{C} of rank n. Then $\overline{\mathfrak{g}}$ satisfies the following Lie brackets

$$[L_a, L_b] = (b - a)L_{a+b} + \frac{1}{12}(a^3 - a)C_L\delta_{a+b,0}; [I_a, I_b] = aC_I\delta_{a+b,0}\delta_{\lambda,0};$$

$$[L_a, I_b] = (b - \lambda a)I_{a+b} + \delta_{a+b,0}\left(C_{LI}^{(0)}(a^2 + a)\delta_{\lambda,0} + \frac{1}{12}(a^3 - a)C_{LI}^{(1)}\delta_{\lambda,1} + \sum_{i=2}^n a_i C_{LI}^{(i)}\delta_{\lambda,-2}\right),$$

where $C_L, C_I, C_{LI}^{(i)}, 0 \le i \le n$ are central elements and $a = \sum_{i=1}^n a_i \epsilon_i$.

Remark: (1) If $\lambda \neq -1, -2$, Theorem 2.4 stands for arbitrary additive subgroup G of \mathbb{C} . Especially, when $\lambda = 0$, we get the generalized Heisenberg-Virasoro algebras, which was originally given in [8]. When $G = \mathbb{Z}$, we get the deformed Heisenberg-Virasoro algebra, which was studied in [7].

(2) The Lie algebra $\mathfrak{g} = \mathfrak{g}(G, -1)$ is not perfect $(I_0 \notin [\mathfrak{g}, \mathfrak{g}])$ and there is no universal central extension of \mathfrak{g} . But one can still consider the universal central extension of the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \operatorname{span}_{\mathbb{C}}\{L_a, I_b \mid a \in G, b \in G \setminus \{0\}\}$. One can prove that the universal central extension of \mathfrak{g}' is governed by four nontrivial 2-cocycles

$$C_L(L_a, L_b) = \frac{1}{12}(a^3 - a)\delta_{a+b,0}; \quad C_I(I_a, I_b) = \frac{1}{a}\delta_{a+b,0};$$

$$C_{LI}(L_a, I_b) = a\delta_{a+b,0}; \quad C'_{LI}(L_a, I_b) = \delta_{a+b,0}.$$

3 Derivations of \mathfrak{g} and $\overline{\mathfrak{g}}$

In this section we compute derivations of \mathfrak{g} for arbitrary G and λ , and consider the lifts of these derivations of \mathfrak{g} to derivations of $\overline{\mathfrak{g}}$.

The algebra \mathfrak{g} has a natural G-grading $\mathfrak{g} = \bigoplus_{a \in G} \mathfrak{g}_a$, where $\mathfrak{g}_a = \operatorname{span}_{\mathbb{C}} \{L_a, I_a\}$. From a well known result about derivations of graded Lie algebras in [4] we know that the algebra $\operatorname{Der}\mathfrak{g}$ of derivations of \mathfrak{g} is also graded by G, i.e, $\operatorname{Der}\mathfrak{g} = \bigoplus_{a \in G} (\operatorname{Der}\mathfrak{g})_a$, where

$$(\operatorname{Der}\mathfrak{g})_a = \operatorname{span}_{\mathbb{C}} \{ \sigma \in \operatorname{Der}\mathfrak{g} \mid \sigma(\mathfrak{g}_b) \subseteq \mathfrak{g}_{a+b}, \ \forall \ b \in G \},$$

and all outer derivations of \mathfrak{g} lie in $(\operatorname{Der}\mathfrak{g})_0$. Now we construct some derivations of \mathfrak{g} which are all of degree 0. Define linear maps $\varphi, \psi : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\varphi(L_a) = aI_a, \ \varphi(I_a) = 0;$$
 $\psi(L_a) = 0, \ \psi(I_a) = I_a.$

For $\lambda = 0$, we define a linear map $\sigma_{(0)} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\sigma_{(0)}(L_a) = I_a \delta_{\lambda,0}, \ \sigma_{(0)}(I_a) = 0.$$

For $\lambda = -1$, we define a linear map $\sigma_{(-1)} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\sigma_{(-1)}(L_a) = a^2 I_a \delta_{\lambda,-1}, \ \sigma_{(-1)}(I_a) = 0.$$

For $\lambda = -2$, we define a linear map $\sigma_{(-2)} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\sigma_{(-2)}(L_a) = a^3 I_a \delta_{\lambda,-2}, \ \sigma_{(-2)}(I_a) = 0.$$

Let $A \in \text{Hom}(G, \mathbb{C})$. We define a linear map $\xi_A : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\xi_{\mathcal{A}}(L_a) = \mathcal{A}(a)L_a, \ \xi_{\mathcal{A}}(I_a) = \mathcal{A}(a)I_a,$$

and for $\lambda = 1$, a linear map $\eta_{A,1} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\eta_{\mathcal{A},1}(L_a) = \mathcal{A}(a)I_a\delta_{\lambda,1}, \ \eta_{\mathcal{A},1}(I_a) = 0.$$

It is easy to check that the linear maps $\varphi, \psi, \sigma_{(0)}, \sigma_{(-1)}, \sigma_{(-2)}, \xi_{\mathcal{A}}, \eta_{\mathcal{A},1}$ are all derivations of degree 0. The main result of this section is the following

Theorem 3.1. For $a \neq 0$, $(Der\mathfrak{g})_a = span_{\mathbb{C}}\{adL_a, adI_a\}$, and

$$(Der\mathfrak{g})_0 = span_{\mathbb{C}}\{\varphi, \psi, \sigma_{(0)}, \sigma_{(-1)}, \sigma_{(-2)}, \xi_{\mathcal{A}}, \eta_{\mathcal{A}, 1} \mid \mathcal{A} \in Hom(G, \mathbb{C})\}.$$

Proof. Let $\sigma \in (\text{Der}\mathfrak{g})_0$ and suppose

$$\sigma(L_a) = \mu_L(a)L_a + \tau_L(a)I_a, \quad \sigma(I_a) = \mu_I(a)L_a + \tau_I(a)I_a,$$

for some functions $\mu_L, \tau_L, \mu_I, \tau_I$ on G. Apply σ to $[L_a, L_b], [L_a, I_b]$ and $[I_a, I_b]$, we get the following equations

$$\mu_I(a)(b - \lambda a) = \mu_I(b)(a - \lambda b), \tag{3.1}$$

$$\mu_I(a+b)(b-\lambda a) = \mu_I(b)(b-a),$$
(3.2)

$$(b - \lambda a)\tau_I(a+b) = (b - \lambda a)\mu_L(a) + (b - \lambda a)\tau_I(b), \tag{3.3}$$

$$(b-a)\tau_L(a+b) = (b-\lambda a)\tau_L(b) - (a-\lambda b)\tau_L(a), \tag{3.4}$$

$$(b-a)\mu_L(a+b) = (b-a)(\mu_L(a) + \mu_L(b)). \tag{3.5}$$

Claim 1: $\mu_L \in \text{Hom}(G,\mathbb{C})$. From (3.5) we see that $\mu_L(a+b) = \mu_L(a) + \mu_L(b)$ if $a \neq b$. Moreover, let $a = 0, b \neq 0$ in (3.5) we get $\mu_L(0) = 0$. Hence $\mu_L(-a) = -\mu_L(a)$ for any $a \in G$. Then choose $b \neq 0, \pm a$ and we have

$$\mu_L(2a) = \mu_L(a+b) + \mu_L(a-b) = 2\mu_L(a) + \mu_L(-b) + \mu_L(b) = 2\mu_L(a).$$

So we have proved $\mu_L(a+b) = \mu_L(a) + \mu_L(b)$ for any $a, b \in G$. Claim 1 stands.

Claim 2: $\mu_I = 0$. Let b = 0 in (3.2) we get

$$\lambda \mu_I(a) = \mu_I(0) \text{ for any } a \neq 0. \tag{3.6}$$

If $\lambda = 0$, then $\mu_I(0) = 0$. Take b = -a in (3.2) and we have $\mu_I(a) = 0$ for any $a \in G$. Suppose $\lambda \neq 0$, then $\mu_I(a) = \frac{1}{\lambda}\mu_I(0)$ for $a \neq 0$. Then (3.1) turns to

$$\mu_I(0)(b-a)(\lambda+1) = 0,$$

which implies $\mu_I(0) = 0$ and hence $\mu_I = 0$ if $\lambda \neq -1$. If $\lambda = -1$, then take $a = -b \neq 0$ in (3.2) and we have $\mu_I(b) = 0$ for any $b \neq 0$. So $\mu_I(0) = 0$ by (3.6). This proves Claim 2.

Claim 3: $\tau_I(a) = \mu_L(a) + \tau_I(0)$. If $\lambda = 0$, then (3.3) shows that

$$\tau_I(a+b) = \mu_L(a) + \tau_I(b)$$
 for $b \neq 0$.

Choose $a = -b \neq 0$ then we have

$$\tau_I(0) = \mu_L(-b) + \tau_I(b) = -\mu_L(b) + \tau_I(b),$$

that is, $\tau_I(b) = \mu_L(b) + \tau_I(0)$. This proves the claim for $\lambda = 0$ since $\mu_L(0) = 0$. Suppose $\lambda \neq 0$. Let $b = 0, a \neq 0$ in (3.3), and Claim 3 follows.

Claim 4: $\tau_L(a) = l_0 \delta_{\lambda,0} + l_1 a + l_2 a^2 \delta_{\lambda,-1} + l_3 a^3 \delta_{\lambda,-2} + \mathcal{B}(a) \delta_{\lambda,1}$ for some $l_0, l_1, l_2, l_3 \in \mathbb{C}$ and $\mathcal{B} \in \text{Hom}(G, \mathbb{C})$. Take a = 0 in (3.4) and we get

$$\lambda \tau_L(0) = 0. (3.7)$$

Replacing b by -b and a by a + b in (3.4) we have

$$(a + b(\lambda + 1))\tau_L(a + b) - (a + 2b)\tau_L(a) = -(\lambda a + b(\lambda + 1))\tau_L(-b).$$
 (3.8)

Combining (3.4) and (3.8), and letting b = 1, we obtain

$$(\lambda - 1)(\lambda + 2)\tau_L(a) = (\lambda a - 1)(a + \lambda + 1)\tau_L(1) + (a - 1)(\lambda a + \lambda + 1)\tau_L(-1).$$
 (3.9)

If $\lambda = 0$, then we get

$$\tau_L(a) = l_1 a + l_0$$
 with $l_1 = \frac{1}{2}(\tau_L(1) - \tau_L(-1)), l_0 = \frac{1}{2}(\tau_L(1) + \tau_L(-1)).$

If $\lambda \neq 0$ then $\tau_L(0) = 0$ by (3.7). Let a + b = 0 in (3.4) and we get

$$\tau_L(-a) = -\tau_L(a)$$
 if $\lambda \neq 0, -1$.

So by (3.9) we see $\tau_L(a) = a\tau_L(1)$ if $\lambda \neq 0, \pm 1, -2$.

If $\lambda = -1$, then from (3.9) it follows

$$\tau_L(a) = l_2 a^2 + l_1 a$$
 with $l_2 = \frac{1}{2} (\tau_L(1) + \tau_L(-1)), l_1 = \frac{1}{2} (\tau_L(1) - \tau_L(-1)).$

If $\lambda = 1$, then (3.4) and (3.8) imply that

$$\tau_L(a+b) = \tau_L(a) + \tau_L(b)$$
 for any $a, b \in G$,

that is, $\tau_L \in \text{Hom}(G, \mathbb{C})$. We may write $\tau_L(a) = \mathcal{B}(a) + l_1 a$ for some $l_1 \in \mathbb{C}$ and $\mathcal{B} \in \text{Hom}(G, \mathbb{C})$.

Now suppose $\lambda = -2$. Let b = 1 in (3.4),

$$(1-a)\tau_L(a+1) + (a+2)\tau_L(a) = (2a+1)\tau_L(1).$$

Furthermore, replace a by a + 1 and we have

$$-a\tau_L(a+2) + (a+3)\tau_L(a+1) = (2a+3)\tau_L(1).$$

Let b = 2 in (3.4),

$$(a-2)\tau_L(a+2) - (a+4)\tau_L(a) = -2(a+1)\tau_L(2).$$

Then the above three equations imply that

$$\tau_L(a) = l_3 a^3 + l_1 a$$
 with $l_3 = \frac{1}{6} (\tau_L(2) - 2\tau_L(1)), l_1 = \frac{1}{6} (8\tau_L(1) - \tau_L(2)).$

Combing the above cases we see that Claim 4 is valid.

In conclusion, we have

$$\sigma(L_a) = \mathcal{A}(a)L_a + \left(l_0\delta_{\lambda,0} + l_1a + l_2a^2\delta_{\lambda,-1} + l_3a^3\delta_{\lambda,-2} + \mathcal{B}(a)\delta_{\lambda,1}\right)I_a;$$

$$\sigma(I_a) = \left(\mathcal{A}(a) + l\right)I_a,$$

for some $l, l_0, l_1, l_2, l_3 \in \mathbb{C}$ and $\mathcal{A}, \mathcal{B} \in \text{Hom}(G, \mathbb{C})$. This proves Theorem 3.1.

Remark: (1) Denote by id: $G \longrightarrow \mathbb{C}$ be the identity map. The corresponding derivation ξ_{id} is exactly the inner derivation adL_0 . The inner derivation $adI_0 = 0$ if $\lambda = 0$, and $adI_0 = \frac{1}{\lambda}\varphi$ if $\lambda \neq 0$.

(2) In [8], derivations for higher rank Heisenberg-Virasoro algebras were computed. However, there is one derivation missed, the one we denoted by $\sigma_{(0)}$. The reason why they missed this derivation is that they assumed $\tau_L(0) = 0$ out of nowhere (or in the terminology of [8] $\beta_0 = 0$, see Line 3 Page 9 in [8]).

In the following we consider lifts of the derivations of \mathfrak{g} we obtained above to derivations of $\overline{\mathfrak{g}}$ with $\lambda \neq -1$ and G being free of rank $n \geq 1$. We shall recall a result about derivations of the universal central extension of a perfect Lie algebra from [2].

Proposition 3.2 ([2]). Suppose the Lie algebra \mathcal{G} is perfect and denote by $\overline{\mathcal{G}}$ the universal central extension of \mathcal{G} . Then every derivation of \mathcal{G} lifts to a derivation of $\overline{\mathcal{G}}$. Moreover, if \mathcal{G} is centerless, then this lift is unique and $Der\overline{\mathcal{G}} \cong Der\mathcal{G}$.

Denote $\mathfrak{c} = \operatorname{span}_{\mathbb{C}}\{C_L, C_I\delta_{\lambda,0}, C_{LI}^{(0)}\delta_{\lambda,0}, C_{LI}^{(1)}\delta_{\lambda,1}, C_{LI}^{(i)}\delta_{\lambda,-2} \mid 2 \leq i \leq n\}$. From Proposition 3.2 we know that if $\lambda \neq 0, -1$, every derivation of \mathfrak{g} lifts uniquely to a derivation of $\overline{\mathfrak{g}}$. For any $\sigma \in \operatorname{Der}\mathfrak{g}$, define a linear map $\overline{\sigma} : \overline{\mathfrak{g}} \longrightarrow \overline{\mathfrak{g}}$ by

$$\overline{\sigma}(L_a) = \sigma(L_a); \ \overline{\sigma}(I_a) = \sigma(I_a); \ \overline{\sigma}(\mathfrak{c}) = 0 \text{ for any } a \in G.$$
 (3.10)

Clearly, $\overline{\sigma}$ is a derivation of $\overline{\mathfrak{g}}$ and it lifts σ . Therefore we get

Theorem 3.3. If $\lambda \neq 0, -1$, then

$$Der\overline{\mathfrak{g}} = span_{\mathbb{C}}\{\overline{\varphi}, \overline{\psi}, \overline{\sigma_{(-2)}}, \overline{\xi_{\mathcal{A}}}, \overline{\eta_{\mathcal{A},1}}, adL_a, adI_a \mid \mathcal{A} \in Hom(G, \mathbb{C}), a \in G\},$$

where $\varphi, \psi, \sigma_{(-2)}, \xi_{\mathcal{A}}, \eta_{\mathcal{A},1}$ are as in Theorem 3.1 and $\overline{\varphi}, \overline{\psi}, \overline{\sigma_{(-2)}}, \overline{\xi_{\mathcal{A}}}, \overline{\eta_{\mathcal{A},1}}$ defined in (3.10).

Now we consider the $\lambda=0$ case. We emphasize again that $\overline{\mathfrak{g}}$ is the higher rank Heisenberg-Virasoro algebra in this case. There are four kinds of derivations $\varphi, \psi, \sigma_{(0)}, \xi_{\mathcal{A}}$, which are of degree 0.

Theorem 3.4. Let $\lambda = 0$ and $A \in Hom(G, \mathbb{C})$ (here we denote $C_{LI} = C_{LI}^{(0)}$).

(1) The derivation φ lifts uniquely to a derivation $\overline{\varphi}$ of $\overline{\mathfrak{g}}$ defined by

$$L_a \mapsto aI_a + \delta_{a,0}C_{LI}, \ I_a \mapsto C_I\delta_{a,0}, \ C_L \mapsto -24C_{LI}, \ C_{LI} \mapsto C_I, \ C_I \mapsto 0.$$

(2) The derivation $\sigma_{(0)}$ lifts uniquely to a derivation $\overline{\sigma_{(0)}}$ of $\overline{\mathfrak{g}}$ defined by

$$L_a \mapsto I_a - \delta_{a,0}C_{LI}, \ I_a \mapsto -C_I\delta_{a,0}, \ C_L \mapsto 0, \ C_{LI} \mapsto 0, \ C_I \mapsto 0.$$

(3) The derivation $\xi_{\mathcal{A}}$ lifts to a family of derivations $\{\overline{\xi_{\mathcal{A},l,k}} \mid l,k \in \mathbb{C}\}$ of $\overline{\mathfrak{g}}$ defined by

$$L_a \mapsto \mathcal{A}(a)L_a + (l+ka)I_a + \delta_{a,0}(k-l)C_{LI},$$

$$I_a \mapsto \mathcal{A}(a)I_a - \delta_{a,0}C_I(l+k),$$

$$C_L \mapsto -24kC_{LI}, C_{LI} \mapsto -kC_I, C_I \mapsto 0.$$

(4) The derivation ψ lifts to a family of derivations $\{\overline{\psi_{l,k}} \mid l,k \in \mathbb{C}\}$ of $\overline{\mathfrak{g}}$ defined by

$$L_a \mapsto (l+ka)I_a + \delta_{a,0}(k-l)C_{LI},$$

$$I_a \mapsto I_a + \delta_{a,0}C_I(k-l),$$

$$C_L \mapsto -24kC_{LI}, C_{LI} \mapsto C_{LI} + kC_I, C_I \mapsto 2C_I.$$

Proof. We only prove (3), the other three are similar and we omit it. Let ϕ be a lift of ξ_A . Notice that ϕ is homogeneous of degree 0. We may write

$$\phi(L_a) = \mathcal{A}(a)L_a + f_L(a)I_a + \delta_{a,0}C_1; \quad \phi(I_a) = f_I(a)L_a + \mathcal{A}(a)I_a + \delta_{a,0}C_2,$$

for some $C_1, C_2 \in \mathfrak{c}$ and functions $f_L, f_I : G \longrightarrow \mathbb{C}$. Expanding the equation $\phi([L_a, L_b]) = [\phi(L_a), L_b] + [L_a, \phi(L_b)]$ and comparing coefficients we get $(b-a)f_L(a+b) = bf_L(b) - af_L(a)$, which implies

$$f_L(a) = l + ka$$
 for some $l, k \in \mathbb{C}$,

and moreover we get

$$\frac{1}{12}(a^3 - a)\phi(C_L) - 2aC_1 = \left(f_L(-a)(a^2 + a) - f_L(a)(a^2 - a)\right)C_{LI} = 2a(l - ka^2)C_{LI}.$$

Let a=1, we see $C_1=(k-l)C_{LI}$. Hence $\phi(C_L)=-24kC_{LI}$.

Expanding $\phi([L_a, I_b]) = [\phi(L_a), I_b] + [L_a, \phi(I_b)]$ we get $bf_I(a+b) = f_I(b)(b-a)$, which implies $f_I(a) = 0$ for any $a \in G$, and we get

$$(a^{2} + a)\phi(C_{LI}) - aC_{2} = a(l - k)C_{I}.$$

Let a = -1 and we see that $C_2 = -(l+k)C_I$, $\phi(C_{LI}) = -kC_I$.

At last expand $\phi([I_a, I_b])$ and we have $\phi(C_I) = 0$. This proves (3).

4 Isomorphism classes and automorphisms

In this section we determine the isomorphism classes and automorphism groups of the Lie algebras $\mathfrak{g}(G,\lambda)$ for arbitrary G and λ , and then consider the lifts of these automorphisms to automorphisms of $\overline{\mathfrak{g}}$ for $\lambda \neq -1$ and free G.

Theorem 4.1. The Lie algebras $\mathfrak{g}(G,\lambda)$ and $\mathfrak{g}(G',\lambda')$ are isomorphic if and only if $\lambda'=\lambda$ and $G'=\xi G$ for some nonzero $\xi\in\mathbb{C}$. Moreover, any Lie algebra isomorphism $\pi:\mathfrak{g}(G,\lambda)\longrightarrow\mathfrak{g}(\xi G,\lambda)$, aside from an inner automorphism of $\mathfrak{g}(\xi G,\lambda)$, has the form

$$\pi(L_a) = \xi^{-1} \chi(a) L'_{\xi a} + \chi(a) I'_{\xi a} \left(l_0 \delta_{\lambda,0} + l_1 a + l_2 a^2 \delta_{\lambda,-1} + l_3 a^3 \delta_{\lambda,-2} + f(a) \delta_{\lambda,1} \right);$$

$$\pi(I_a) = l \chi(a) I'_{\xi a},$$
(4.1)

where $l \in \mathbb{C}^{\times}$, $l_0, l_1, l_2, l_3 \in \mathbb{C}$, $\chi \in G^*$ is a character of G, and $f \in Hom(G, \mathbb{C})$.

Proof. Denote $\mathfrak{g} = \mathfrak{g}(G,\lambda), \mathfrak{g}' = \mathfrak{g}(G',\lambda')$, and we will use an extra dash to denote elements in \mathfrak{g}' in the following. If $\lambda' = \lambda$, $G' = \xi G$ for some nonzero $\xi \in \mathbb{C}$, it is easy to check that the linear map defined by

$$L_a \mapsto \xi^{-1} L'_{\xi a}; \quad I_a \mapsto \xi^{-1} I'_{\xi a}$$

is a Lie algebra isomorphism from \mathfrak{g} to \mathfrak{g}' .

On the other hand, suppose $\mathfrak{g} \cong \mathfrak{g}'$ and let $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ be a Lie algebra isomorphism.

Claim 1: There exists some $\xi \in \mathbb{C}^{\times}$ and an inner automorphism θ of \mathfrak{g}' such that $G' = \xi G$ and $\theta \pi(\mathfrak{g}_a) = \mathfrak{g}'_{\xi a}$ for any $a \in G$.

Notice that the set of locally finite elements in \mathfrak{g} is $\operatorname{span}_{\mathbb{C}}\{L_0, I_a \mid a \in G\}$. Since π maps a locally finite element in \mathfrak{g} to a locally finite element in \mathfrak{g}' , we may assume that

$$\pi(L_0) = \xi^{-1} L_0' + \sum_{b \in G'(L_0)} \gamma_b I_b',$$

for some $\xi \in \mathbb{C}^{\times}$, finite subset $G'(L_0)$ of G' and $\gamma_b \in \mathbb{C}$. Define

$$\eta = \prod_{0 \neq b \in G'(L_0)} \exp\left\{-\frac{\xi \gamma_b}{b} \operatorname{ad} I_b'\right\},\,$$

which is an inner automorphism of \mathfrak{g}' . Clearly, $\eta \pi(L_0) = \xi^{-1} L'_0 + \gamma_0 I'_0$. So replacing π by $\eta^{-1} \pi$ we may assume

$$\pi(L_0) = \xi^{-1}L_0' + \gamma_0 I_0'.$$

Since span_{\mathbb{C}}{ $I_a \mid a \in G$ } is the unique maximal abelian ideal of \mathfrak{g} , it forces

$$\pi(I_a) = \sum_{b \in G'(I_a)} \nu_a(b) I_b'$$

for some finite subset $G'(I_a)$ of G' and some function $\nu_a: G' \longrightarrow \mathbb{C}$. For any $a \neq 0$, from

$$\pi[L_0, I_a] = a \sum_{b \in G'(I_a)} \nu_a(b) I_b' = [\pi(L_0), \pi(I_a)] = \xi^{-1} \sum_{b \in G'(I_a)} b \nu_a(b) I_b'$$

we see that $G'(I_a) = \{\xi a\} \subset G'$ for any $a \neq 0$. This implies $G' = \xi G$ and

$$\pi(I_a) = \nu_a(\xi a) I'_{\xi a}$$
 for $a \neq 0$.

Since $\pi[L_a, I_b] = (b - \lambda a)\pi(I_{a+b}) = (b - \lambda a)\nu_{a+b}(\xi(a+b))I'_{\xi(a+b)} = [\pi(L_a), \nu_b(\xi b)I'_{\xi b}]$ for $a, b \neq 0, a+b \neq 0$, it follows that $\pi(L_a), a \neq 0$, must have the form

$$\pi(L_a) = \rho(a)L'_{\xi a} + \sum_{c \in G'(L_a)} \mu(c)I'_c$$

for some finite subset $G'(L_a)$ of G', functions $\rho: G \longrightarrow \mathbb{C}^{\times}$ and $\mu: G' \longrightarrow \mathbb{C}$. Then

$$(b - \lambda a)\nu_{a+b}(\xi(a+b))I'_{\xi(a+b)} = [\rho(a)L'_{\xi a}, \nu_b(\xi b)I'_{\xi b}] = \xi \rho(a)\nu_b(\xi b)(b - \lambda' a)I'_{\xi(a+b)}.$$
(4.2)

Moreover, from

$$\pi[L_a, I_0] = -\lambda a \pi(I_a) = -\lambda a \nu_a(\xi a) I'_{\xi a} = [\rho(a) L'_{\xi a}, \sum_{b \in G'(I_0)} \nu_0(b) I'_b]$$

we see that $G'(I_0) = \{0\} \subset G'$, and hence $\pi(I_0) = \nu_0(0)I'_0$. Define

$$\nu(\xi b) = \nu_b(\xi b)$$
 for any $b \in G$.

We get a function $\nu: G' \longrightarrow \mathbb{C}^{\times}$ such that

$$\pi(I_a) = \nu(\xi a) I'_{\xi a} \in \mathfrak{g}'_{\xi a}$$
 for any $a \in G$.

Then (4.2) turns to

$$(b - \lambda a)\nu(\xi(a+b)) = \xi\rho(a)\nu(\xi b)(b - \lambda' a). \tag{4.3}$$

Consider $\pi[L_0, L_a]$ for $a \neq 0$, we get

$$\sum_{b \in G'(L_a)} \mu(b)(a - \xi^{-1}b)I'_b - \gamma_0 \lambda' \xi a \rho(a)I'_{\xi a} = 0,$$

which implies $G'(L_a) = \{\xi a\}$ and $\lambda' \gamma_0 = 0$. Write $\mu(0) = \gamma_0$ and $\rho(0) = \xi^{-1}$. Then we have

$$\pi(L_a) = \rho(a)L'_{\xi a} + \mu(\xi a)I'_{\xi a} \in \mathfrak{g}'_{\xi a}$$
 for any $a \in G$.

This proves Claim 1.

In the following we determine the functions ρ, μ, ν .

Claim 2: $\chi = \xi \rho$ is a character of G, $\lambda' = \lambda$ and $\nu(\xi a) = \nu(0)\chi(a)$ for any $a \in G$. Since

$$\pi[L_a, L_b] = (b - a)\pi(L_{a+b}) = (b - a)\rho(a + b)L'_{\xi(a+b)} + (b - a)\mu(\xi(a + b))I'_{\xi(a+b)}$$

$$= [\rho(a)L'_{\xi a} + \mu(\xi a)I'_{\xi a}, \rho(b)L'_{\xi b} + \mu(\xi b)I'_{\xi b}]$$

$$= \xi(b - a)\rho(a)\rho(b)L'_{\xi(a+b)} + \rho(a)\mu(\xi b)\xi(b - \lambda'a)I'_{\xi(a+b)} - \rho(b)\mu(\xi a)\xi(a - \lambda'b)I'_{\xi(a+b)},$$

we get

$$(b-a)(\rho(a+b) - \xi \rho(a)\rho(b)) = 0, \tag{4.4}$$

$$(b-a)\mu(\xi(a+b)) = \xi \rho(a)\mu(\xi b)(b-\lambda' a) - \xi \rho(b)\mu(\xi a)(a-\lambda' b). \tag{4.5}$$

From (4.4) we get that if $a \neq b$ then

$$\rho(a+b) = \xi \rho(a)\rho(b). \tag{4.6}$$

Let b=0 in (4.6), we have $\rho(0)=\xi^{-1}$. Choose $b\neq 0,\pm a$. Using (4.6) we get

$$\rho(2a) = \xi \rho(a+b)\rho(a-b) = \xi^3 \rho(a)^2 \rho(b)\rho(-b) = \xi^2 \rho(a)^2 \rho(0) = \xi \rho(a)^2 \text{ for any } a \in G.$$

This proves that (4.6) stands for any $a, b \in G$. So $\chi = \xi \rho$ is a character of G.

Let b = 0 in (4.3) we get

$$\lambda \nu(\xi a) = \lambda' \xi \nu(0) \rho(a) \text{ for any } a \neq 0, \tag{4.7}$$

which implies that

$$\lambda' = 0$$
 if and only if $\lambda = 0$.

If $\lambda \neq 0$, then (4.7) says $\nu(\xi a) = \frac{\lambda' \xi}{\lambda} \nu(0) \rho(a)$ for any $a \neq 0$. Putting it into (4.3) and using (4.6) we see $\lambda' = \lambda$. Hence $\nu(\xi a) = \xi \nu(0) \rho(a) = \nu(0) \chi(a)$ if $\lambda \neq 0$.

If $\lambda = \lambda' = 0$, let $a = -b \neq 0$ in (4.3) and we have

$$\nu(0) = \xi \rho(-b)\nu(\xi b).$$

Then $\nu(0)\chi(b) = \xi^2 \rho(b)\rho(-b)\nu(\xi b) = \xi \rho(0)\nu(\xi b) = \nu(\xi b)$. This proves Claim 2.

At last we determine the function μ . Set $\varphi(a) = \frac{\mu(\xi a)}{\chi(a)}$. Divide $\chi(a+b)$ to (4.5) and we obtain

$$(b-a)\varphi(a+b) + (a-\lambda b)\varphi(a) = (b-\lambda a)\varphi(b),$$

which is the same equation as (3.4). So from the computation in Section 3 we have

$$\varphi(a) = l_0 \delta_{\lambda,0} + l_1 a + l_2 a^2 \delta_{\lambda,-1} + l_3 a^3 \delta_{\lambda,-2} + f(a) \delta_{\lambda,1},$$

where $l_0, l_1, l_2, l_3 \in \mathbb{C}$ and $f \in \text{Hom}(G, \mathbb{C})$. Therefore

$$\mu(\xi a) = \chi(a) \left(l_0 \delta_{\lambda,0} + l_1 a + l_2 a^2 \delta_{\lambda,-1} + l_3 a^3 \delta_{\lambda,-2} + f(a) \delta_{\lambda,1} \right).$$

This proves the theorem.

Now using Theorem 4.1 we may determine the automorphism group of \mathfrak{g} . Denote $\mathcal{E} = \{\epsilon \in \mathbb{C}^{\times} \mid \epsilon G = G\}$, which is a subgroup of \mathbb{C}^{\times} . Let $\xi \in \mathcal{E}, \chi \in G^*, f \in \text{Hom}(G, \mathbb{C}), l \in \mathbb{C}^{\times}, l_0, l_1, l_2, l_3 \in \mathbb{C}$, denote by $\theta_{\lambda}(\xi, \chi, f, l, l_0, l_1, l_2, l_3)$ the linear map given by

$$L_{a} \mapsto \xi^{-1} \chi(a) L_{\xi a} + \chi(a) I_{\xi a} \left(l_{0} \delta_{\lambda,0} + l_{1} a + l_{2} a^{2} \delta_{\lambda,-1} + l_{3} a^{3} \delta_{\lambda,-2} + f(a) \delta_{\lambda,1} \right);$$

$$I_{a} \mapsto l \chi(a) I_{\xi a}.$$

Here we shall point out that if $\lambda = 1$ we may assume $l_1 = 0$ since the map $a \mapsto f(a) + l_1 a$ still lies in $\text{Hom}(G, \mathbb{C})$. Then by Theorem 4.1 we have

Theorem 4.2. The outer automorphism group of \mathfrak{g} is

$$Out\mathfrak{g} = \{\theta_{\lambda}(\xi, \chi, f, l, l_0, l_1, l_2, l_3) \mid \xi \in \mathcal{E}, \chi \in G^*, f \in Hom(G, \mathbb{C}), l \in \mathbb{C}^{\times}, l_0, l_1, l_2, l_3 \in \mathbb{C}\},\$$

and $Aut\mathfrak{g} = Out\mathfrak{g} \ltimes Inn\mathfrak{g}$, where $Inn\mathfrak{g}$ is the inner automorphism group of \mathfrak{g} , generated by $\{\exp adI_a \mid a \in G\}$.

Notice that

$$\theta_{\lambda}(\xi', \chi', f', l', l'_0, l'_1, l'_2, l'_3) \cdot \theta_{\lambda}(\xi, \chi, f, l, l_0, l_1, l_2, l_3)$$

$$= \theta_{\lambda} \left(\xi' \xi, (\chi' \cdot \xi) \chi, f' + l' f, l' l, \xi^{-1} l'_0 + l' l_0, l'_1 + l' l_1, \xi l'_2 + l' l_2, \xi^2 l'_3 + l' l_3 \right)$$

and

$$\theta_{\lambda}(\xi, \chi, f, l, l_0, l_1, l_2, l_3)^{-1} = \theta_{\lambda}(\xi^{-1}, \chi^{-1} \cdot \xi^{-1}, -l^{-1}f, l^{-1}, -l^{-1}\xi l_0, -l^{-1}l_1, -l^{-1}\xi^{-1}l_2, -l^{-1}\xi^{-2}l_3).$$

Clearly, the map $G^* \longrightarrow \text{Out}\mathfrak{g}$ defined by $\chi \mapsto \theta_{\lambda}(1, \chi, 0, 1, 0, 0, 0, 0)$ is a group monomorphism and denote by N its image. Set $S = \{\theta_{\lambda}(1, 1, 0, l, 0, 0, 0, 0) \mid l \in \mathbb{C}^{\times}\}$, and

$$K = \{ \theta_{\lambda}(1, 1, f, 1, l_0, l_1, l_2, l_3) \mid l_0, l_1, l_2, l_3 \in \mathbb{C}, f \in \text{Hom}(G, \mathbb{C}) \}.$$

Here $\mathbb{1}$ denotes the identity in G^* . Clearly, N, S, K are normal subgroups of $Out\mathfrak{g}$, and

$$N \cong G^*, \ S \cong \mathbb{C}^{\times}, \ \mathrm{and} \ K \cong \begin{cases} \mathrm{Hom}(G,\mathbb{C}) \cong \mathbb{C}^n & \mathrm{if} \ \lambda = 1; \\ \mathbb{C}^2 & \mathrm{if} \ \lambda = 0, -1, -2; \\ \mathbb{C} & \mathrm{otherwise}. \end{cases}$$

Moreover, let $T = \{\theta_{\lambda}(\xi, \mathbb{1}, 0, 1, 0, 0, 0, 0) \mid \xi \in \mathcal{E}\}$, which is a subgroup of Out \mathfrak{g} and isomorphic to \mathcal{E} . Then we have a projection Out $\mathfrak{g} \longrightarrow T$, whose kernel is the normal subgroup NSK. So we get

Theorem 4.3. $Out\mathfrak{g} = T \ltimes (NSK) \cong \mathcal{E} \ltimes (G^*\mathbb{C}^{\times}K).$

Using a result from [10] one may extend automorphisms of \mathfrak{g} to automorphisms of $\overline{\mathfrak{g}}$.

Proposition 4.4 ([10]). Let \mathcal{G} be a perfect Lie algebra and $\overline{\mathcal{G}}$ the universal central extension of \mathcal{G} . Then every automorphism θ of \mathfrak{g} admits a unique extension to an automorphism $\overline{\theta}$ of $\overline{\mathcal{G}}$, and the map $\theta \mapsto \overline{\theta}$ is a group monomorphism. Moreover, if \mathcal{G} is centerless, then the map $\theta \mapsto \overline{\theta}$ is an isomorphism.

Explicitly we have the form of extended automorphisms.

Theorem 4.5. Suppose $\lambda \neq -1$. Let $\xi \in \mathcal{E}, \chi \in G^*, f \in Hom(G, \mathbb{C}), l \in \mathbb{C}^{\times}, l_0, l_1, l_3 \in \mathbb{C}$, then the unique automorphism $\overline{\theta} = \overline{\theta_{\lambda}(\xi, \chi, f, l, l_0, l_1, 0, l_3)}$ of $\overline{\mathfrak{g}}$ obtained by extension from $\theta = \theta_{\lambda}(\xi, \chi, f, l, l_0, l_1, 0, l_3)$ is such that

$$\overline{\theta}(L_a) = \xi^{-1}\chi(a)L_{\xi a} + \chi(a)I_{\xi a} \left(l_0\delta_{\lambda,0} + l_1a + l_3a^3\delta_{\lambda,-2} + f(a)\delta_{\lambda,1}\right)$$

$$+ \delta_{a,0} \left(\frac{\xi^{-1} - \xi}{24}C_L + (l_1\xi - l_0)C_{LI}^{(0)}\delta_{\lambda,0} + \frac{1}{2}\xi(l_1^2 - l_0^2)C_I\delta_{\lambda,0}\right);$$

$$(4.8)$$

$$\overline{\theta}(I_a) = l\chi(a)I_{\xi a} + l\xi \delta_{a,0} \left(\frac{\xi^{-1} - \xi}{24} C_{LI}^{(1)} \delta_{\lambda,1} + \left((1 - \xi^{-1}) C_{LI}^{(0)} + (l_1 - l_0) C_I \right) \delta_{\lambda,0} \right)$$
(4.9)

+
$$l(\xi \epsilon_1)^{-1} \delta_{a,0} \sum_{i=2}^{n} (\xi \epsilon_1)_i C_{LI}^{(i)} \delta_{\lambda,-2};$$

$$\overline{\theta}(C_L) = \xi C_L - 12l_1 \xi \left(2C_{LI}^{(0)} + l_1 C_I \right) \delta_{\lambda,0}; \tag{4.10}$$

$$\overline{\theta}(C_{LI}^{(0)}) = l\xi \left(C_{LI}^{(0)} + l_1 C_I\right); \qquad \overline{\theta}(C_{LI}^{(1)}) = l\xi^2 C_{LI}^{(1)};$$

$$\overline{\theta}(C_{LI}^{(i)}) = l(\xi \epsilon_1)^{-1} \sum_{j=2}^{n} \left\{ \epsilon_1(\xi \epsilon_i)_j - \epsilon_i(\xi \epsilon_1)_j \right\} C_{LI}^{(j)}, \ i \ge 2; \tag{4.11}$$

$$\overline{\theta}(C_I) = l^2 \xi C_I. \tag{4.12}$$

Here for $a \in G$ we use a_i to denote the coefficients of a with respect to the basis $\epsilon_1, \ldots, \epsilon_n$.

Proof. Recall $\mathfrak{c} = \operatorname{span}_{\mathbb{C}}\{C_L, C_I\delta_{\lambda,0}, C_{LI}^{(0)}\delta_{\lambda,0}, C_{LI}^{(1)}\delta_{\lambda,1}, C_{LI}^{(i)}\delta_{\lambda,-2} \mid 2 \leq i \leq n\}$. For later convenience, we denote

$$\tau(a) = l_0 \delta_{\lambda,0} + l_1 a + l_3 a^3 \delta_{\lambda,-2} + f(a) \delta_{\lambda,1},$$

$$\nu_C(a) = C_{LI}^{(0)}(a^2 + a) \delta_{\lambda,0} + \frac{1}{12}(a^3 - a) C_{LI}^{(1)} \delta_{\lambda,1} + \sum_{i=2}^n a_i C_{LI}^{(i)} \delta_{\lambda,-2} \in \mathfrak{c}.$$

Clearly, $\overline{\theta}(L_a)$ and $\overline{\theta}(I_a)$ have the form

$$\overline{\theta}(L_a) = \xi^{-1} \chi(a) L_{\xi a} + \chi(a) \tau(a) I_{\xi a} + K_L(a);$$

$$\overline{\theta}(I_a) = l \chi(a) I_{\xi a} + K_I(a),$$

for some maps $K_L, K_I : G \longrightarrow \mathfrak{c}$. Expanding

$$\overline{\theta}([L_a, L_b]) = [\xi^{-1}\chi(a)L_{\xi a} + \chi(a)\tau(a)I_{\xi a}, \xi^{-1}\chi(b)L_{\xi b} + \chi(b)\tau(b)I_{\xi b}]$$

we see that $K_L(a) = 0$ if $a \neq 0$, and

$$-2K_L(0) + \frac{1}{12}(a^2 - 1)\overline{\theta}(C_L) = \frac{\xi a^2 - \xi^{-1}}{12}C_L + 2(l_0 - l_1\xi a^2)C_{LI}^{(0)}\delta_{\lambda,0} + (l_0^2 - l_1^2a^2)\xi C_I\delta_{\lambda,0},$$

which implies

$$K_L(0) = \frac{\xi^{-1} - \xi}{24} C_L + (l_1 \xi - l_0) C_{LI}^{(0)} \delta_{\lambda,0} + \frac{1}{2} \xi (l_1^2 - l_0^2) C_I \delta_{\lambda,0}$$

and

$$\overline{\theta}(C_L) = \xi C_L - 12l_1 \xi \left(2C_{LI}^{(0)} + l_1 C_I \right) \delta_{\lambda,0}.$$

This proves (4.8) and (4.10).

Expanding $\overline{\theta}([L_a, I_b]) = [\overline{\theta}(L_a), \overline{\theta}(I_b)]$ we get that $K_I(a) = 0$ for $a \neq 0$ and

$$-(1+\lambda)aK_{I}(0) + (a^{2}+a)\overline{\theta}(C_{LI}^{(0)})\delta_{\lambda,0} + \frac{a^{3}-a}{12}\overline{\theta}(C_{LI}^{(1)})\delta_{\lambda,1} + \sum_{i=2}^{n}a_{i}\overline{\theta}(C_{LI}^{(i)})\delta_{\lambda,-2}$$

$$= l\xi^{-1}\left(\left((\xi a)^{2} + \xi a\right)C_{LI}^{(0)}\delta_{\lambda,0} + \frac{(\xi a)^{3} - \xi a}{12}C_{LI}^{(1)}\delta_{\lambda,1} + \xi\sum_{i=2}^{n}a_{i}C_{LI}^{(i)}\delta_{\lambda,-2}\right) + l\tau(a)\xi aC_{I}\delta_{\lambda,0}.$$

$$(4.13)$$

If $\lambda = 0$, let a = -1 in (4.13), then we get

$$K_I(0)\delta_{\lambda,0} = l\xi\left((1-\xi^{-1})C_{LI}^{(0)} + (l_1-l_0)C_I\right)\delta_{\lambda,0} \text{ and } \overline{\theta}(C_{LI}^{(0)}) = l\xi\left(C_{LI}^{(0)} + l_1C_I\right).$$

If $\lambda = 1$, let a = 1 in (4.13), then we get

$$K_I(0)\delta_{\lambda,1} = -\frac{l(\xi^2 - 1)}{24}C_{LI}^{(1)}\delta_{\lambda,1}$$
 and $\overline{\theta}(C_{LI}^{(1)}) = l\xi^2C_{LI}^{(1)}$.

If $\lambda = -2$, let $0 \neq a \in \mathbb{Z}\epsilon_1$ in (4.13), then we get

$$K_I(0)\delta_{\lambda,-2} = l(\xi\epsilon_1)^{-1} \sum_{i=2}^n (\xi\epsilon_1)_i C_{LI}^{(i)} \delta_{\lambda,-2};$$

$$\overline{\theta}(C_{LI}^{(i)}) = l(\xi \epsilon_1)^{-1} \sum_{i=2}^{n} \left(\epsilon_1(\xi \epsilon_i)_j - \epsilon_i(\xi \epsilon_1)_j \right) C_{LI}^{(j)}.$$

This proves (4.9) and (4.11). Similarly, (4.12) follows from $\overline{\theta}([I_a, I_b]) = [\overline{\theta}(I_a), \overline{\theta}(I_b)].$

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