

On the Frobenius number of certain numerical semigroups

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Abstract

Let $0 < \lambda \leq 1$, $\lambda \notin \{\frac{2}{4}, \frac{2}{7}, \frac{2}{10}, \frac{2}{13}, \dots\}$, be a real and p a prime number, with $[p, p + \lambda p]$ containing at least two primes. Denote by $f_\lambda(p)$ the largest integer which cannot be written as a sum of primes from $[p, p + \lambda p]$. Then

$$f_\lambda(p) \sim \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor \cdot p, \text{ as } p \text{ goes to infinity.}$$

Further a question of Wilf about the 'Money-Changing Problem' has a positive answer for all semigroups of multiplicity p containing the primes from $[p, 2p]$. In particular, this holds for the semigroup generated by all primes not less than p . The latter special case was already shown in a previous paper.

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1 Introduction

A *numerical semigroup* is an additively closed subset S of \mathbb{N} with $0 \in S$ and only finitely many positive integers outside from S , the so-called *gaps* of S . The *genus* g of S is the number of its gaps. The set $E = S^* \setminus (S^* + S^*)$, where $S^* = S \setminus \{0\}$, is the (unique) minimal system of generators of S . Its elements are called the *atoms* of S ; their number e is the *embedding dimension* of S . The *multiplicity* of S is the smallest element p of S^* .

From now on we assume that $S \neq \mathbb{N}$. Then the greatest gap f is called the *Frobenius number* of S .

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For a certain class of numerical semigroups we shall study the relationship between the various invariants mentioned above.

In particular for some of these semigroups we will give an affirmative answer to

Wilf's question [15]: Is it true that

$$(1) \quad \frac{g}{1+f} \leq \frac{e-1}{e} ?$$

We shall consider the following semigroups: Let p be a prime, λ a positive real number, $I_\lambda(p)$ the interval $[p, p + \lambda p]$ and D_λ the set of all primes p such that $I_\lambda(p)$ contains at least two primes. For such a p we denote by $S_\lambda(p)$ the numerical semigroup generated by all primes from $I_\lambda(p)$ and by $f_\lambda(p)$ its Frobenius number. According to Bertrand's postulate, D_1 is the set \mathbb{P} of all primes, further $\mathbb{P} \setminus D_\lambda$ is finite for all $\lambda > 0$ by the prime number theorem.

Let $p_1 = 2, p_2 = 3, p_3 = 5, \dots$ be the sequence of prime numbers in natural order and let S_n be the semigroup generated by all primes not less than p_n . Proposition 1.1 b) below generalizes the corresponding assertion [8, Proposition 5] about S_n .

Proposition 1.1. *Let $\lambda > 0$, $p \in D_\lambda$ and S any numerical semigroup of multiplicity p containing $S_\lambda(p)$.*

- a) *There is an integer $C(\lambda) > 0$ such that for $p > C(\lambda)$, the semigroups S from above satisfy Wilf's inequality (1).*
- b) *In case $\lambda = 1$ formula (1) holds for all p . In particular (1) is true for $S = S_n$. \square*

This will be seen in section 3.

In section 2, we shall show the following result.

Theorem 1.2. *Let $0 < \lambda \leq 1$, $\lambda \notin \{\frac{2}{4}, \frac{2}{7}, \frac{2}{10}, \frac{2}{13}, \dots\}$, be a real. Then*

$$\lim_{\substack{p \in D_\lambda \\ p \rightarrow \infty}} \frac{f_\lambda(p)}{p} = \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor.$$

For $\lambda = \frac{2}{m}$ with an integer $m \geq 2$ and $m \equiv 1 \pmod{3}$, at least

$$\limsup_{p \rightarrow \infty} \frac{f_\lambda(p)}{p} = \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor. \quad \square$$

In particular, since $f_\lambda(p)$ is decreasing as a function of λ , for each $\lambda > 0$ there is a constant c_λ , such that $f_\lambda(p) \leq c_\lambda \cdot p$ for all primes $p \in D_\lambda$, cf. [8, Remark 2 c)].

According to our table `t4_quotient_not_always_6.pdf` from [19] possibly the series $\frac{f_{\frac{1}{2}}(p)}{p}$ may not converge as p goes to infinity. Hence we do not expect

that $\frac{f_{\frac{2}{m}}(p)}{p} \sim 2 + m$ holds in the exceptional cases $m \geq 4$ and $m \equiv 1 \pmod{3}$ from Theorem 1.2 as well. See figure 1.

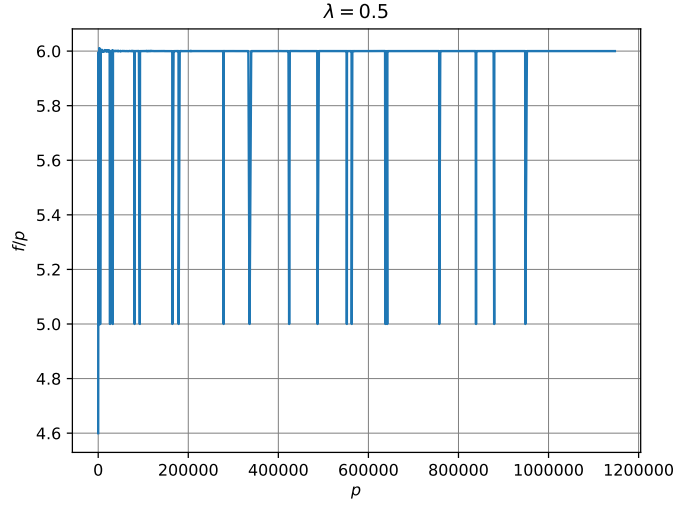


Figure 1: Plot of $f_{1/2}(p)/p$

By computational evidence (see [19]), we suspect that $\lim_{p \rightarrow \infty} \frac{f_{\lambda}(p)}{p} = 3$ for $\lambda > 1$. See the algorithm in [19] and figure 2 below.

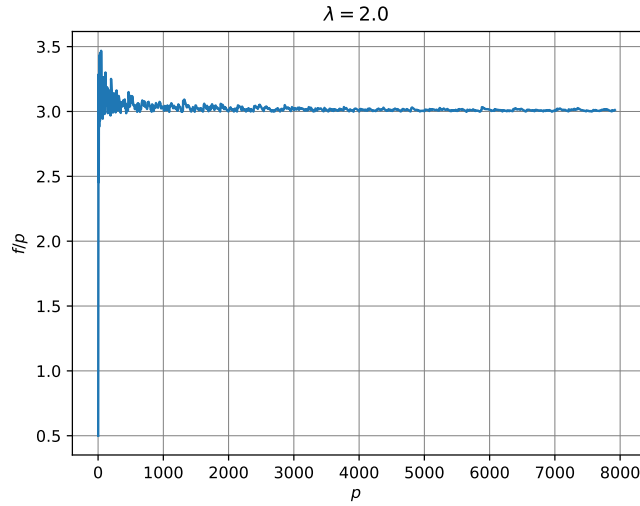


Figure 2: Plot of $f_2(p)/p$

In particular for the Frobenius number f_n of S_n we would have $\lim_{n \rightarrow \infty} \frac{f_n}{p_n} =$

3, hence large even numbers would be the sum of two primes (cf. [8, Proposition 2]).

2 Bounds for the Frobenius number of certain numerical semigroups generated by primes

In order to verify Theorem 1.2, for integers $m \geq 2$ consider the statement

A(m) For every $\delta > 0$ there is an $N(\delta, m) > 0$ such that all integers $N \geq N(\delta, m)$ of the same parity as m can be written as a sum of primes

$$(2) \quad N = q_1 + \cdots + q_m \text{ with the restriction } \left| \frac{N}{m} - q_i \right| < \delta \cdot N \text{ for } i = 1, \dots, m.$$

For short: “Large N of the same parity as m are sums of m almost equal primes.”

Proposition 2.1. *Suppose $A(m)$ holds for some $m \geq 2$. Then for each pair (ε, λ) of reals $\varepsilon > 0$ and $\lambda > \frac{2}{m}$,*

$$(3) \quad f_\lambda(p) < (m + 1 + \varepsilon)p \text{ for large } p \in D_\lambda.$$

Proof: W.l.o.g. we may assume that $\varepsilon < \lambda - \frac{2}{m}$. Let $p \in D_\lambda$. Set $\delta := \frac{\varepsilon}{m(m+2+\varepsilon)}$. From $A(m)$ we get for large p :

Every integer $N \in [(m + \varepsilon)p, (m + 2 + \varepsilon)p]$ of the same parity as m can be written as $N = q_1 + \cdots + q_m$ with primes q_i such that

$$(4) \quad \left| \frac{N}{m} - q_i \right| < \delta \cdot N \text{ for } i = 1, \dots, m.$$

We will see in a moment, that $q_i \in S_\lambda(p)$ for $i = 1, \dots, m$, hence $N \in S_\lambda(p)$: Inequality (4) implies

$$q_i > \frac{N}{m} - \delta \cdot N = \frac{m+2}{m(m+2+\varepsilon)}N \geq \frac{(m+2)(m+\varepsilon)}{m(m+2+\varepsilon)}p > p$$

and

$$q_i < \frac{N}{m} + \delta \cdot N = \frac{m+2+2\varepsilon}{m(m+2+\varepsilon)}N \leq \frac{m+2+2\varepsilon}{m}p \leq p + \left(\frac{2}{m} + \varepsilon \right)p < p + \lambda p,$$

since $m \geq 2$ and $\varepsilon < \lambda - \frac{2}{m}$. Hence $N \in S_\lambda(p)$.

Considering N and $N + p$ we see, that $\mathbb{Z} \cap [(m + 1 + \varepsilon)p, (m + 2 + \varepsilon)p]$ is contained in $S_\lambda(p)$ and is a set of at least p consecutive integers. Hence $f_\lambda(p) < (m + 1 + \varepsilon)p$. \square

It will be immediate from the following improved version of [11, Theorem 1.1], that $A(m)$ holds for all integers $m \geq 3$.

Proposition 2.2. *Let $\theta = \frac{11}{20} + \varepsilon$, $\varepsilon > 0$, and $m \geq 3$ an integer. Then every sufficiently large integer N of the same parity as m can be written as the sum $N = q_1 + \cdots + q_m$ of m primes with*

$$\left| \frac{N}{m} - q_i \right| \leq N^\theta \text{ for } i = 1, \dots, m.$$

Proof, due to Kaisa Matomäki (private communication [10]).

Given $\varepsilon > 0$ and $m \geq 3$ as above, let N be a sufficiently large integer with $N - m$ even. By the existence of primes in short intervals, cf. [1, Theorem 1], there is a prime $p \in \left[\frac{N}{m} - \left(\frac{N}{m} \right)^{\frac{21}{40}}, \frac{N}{m} \right]$, hence

$$(5) \quad 0 \leq \frac{N}{m} - p \leq \left(\frac{N}{m} \right)^{\frac{21}{40}} < N^{\frac{11}{20} + \frac{\varepsilon}{2}} < N^{\frac{11}{20} + \varepsilon}.$$

Now use [11, Theorem 1.1] for the odd integer $n := N - (m-3)p = 3\frac{N}{m} + (m-3)\left(\frac{N}{m} - p\right)$ and $\theta = \frac{11}{20} + \frac{\varepsilon}{2}$. You get primes q_1, q_2, q_3 with $n = q_1 + q_2 + q_3$ and

$$(6) \quad \left| \frac{n}{3} - q_i \right| \leq n^{\frac{11}{20} + \frac{\varepsilon}{2}} \leq N^{\frac{11}{20} + \frac{\varepsilon}{2}} \text{ for } i = 1, 2, 3.$$

For $N \gg 0$ we also have

$$(7) \quad \frac{m}{3} \leq N^{\frac{\varepsilon}{2}}.$$

Finally by (5), (6) and (7)

$$\begin{aligned} \left| \frac{N}{m} - q_i \right| &= \left| \left(\frac{n}{3} - q_i \right) - (m-3) \frac{\frac{N}{m} - p}{3} \right| \\ &\leq N^{\frac{11}{20} + \frac{\varepsilon}{2}} + \frac{m-3}{3} N^{\frac{11}{20} + \frac{\varepsilon}{2}} \\ &= \frac{m}{3} N^{\frac{11}{20} + \frac{\varepsilon}{2}} \\ &\leq N^{\frac{11}{20} + \varepsilon} \text{ for } i = 1, 2, 3. \end{aligned}$$

Hence $n = q_1 + q_2 + q_3 + (m-3)p$ is the sum of m primes of size as desired. \square

Corollary 2.3. *$A(m)$ is true for all integers $m \geq 3$.*

Proof. Let $m \geq 3$ and $\delta > 0$. Apply 2.2 with $\varepsilon = \frac{1}{20}$. Then $1 > \theta = \frac{3}{5} > \frac{11}{20}$ and $N^\theta < \delta \cdot N$ for large N . Hence $A(m)$ is true by 2.2. \square

Proposition 2.1 and Corollary 2.3 together imply the following result.

Corollary 2.4. *Let $\varepsilon > 0$, $m \geq 3$ and $\lambda > \frac{2}{m}$. Then*

$$f_\lambda(p) < (m+1+\varepsilon)p \text{ for large } p \in D_\lambda. \quad \square$$

Remark. Let $p = p_n$ the n th prime in the natural order, S_n the semigroup generated by all primes not less than p and f_n its Frobenius number. $S_{\lambda=1}(p_n)$ is contained in S_n , hence f_n is at most $f_{\lambda=1}(p_n)$. Therefore, an application of 2.4 with $m = 3$ gives $\limsup_{n \rightarrow \infty} \frac{f_n}{p_n} \leq 4$. This has been shown by a somewhat different way in our former paper [8, Remark 2. a)].

On our way to Theorem 1.2, Proposition 2.6 below will give us lower bounds for $\frac{f_\lambda(p)}{p}$, $p \in D_\lambda$. Let $p(\lambda) := \max(I_\lambda(p) \cap \mathbb{P})$. From now on let $m \geq 2$. Set

$$T(m) := \{t \in \mathbb{N} \mid 1 + tm, 3 + tm, 1 + t(m+2) \in \mathbb{P}\}.$$

Lemma 2.5. a) Let $p \in D_{\frac{2}{m}}$ and $p > m$. If $f_{\frac{2}{m}}(p) < (m+2)p - 2$, then there is a $t \in T(m)$ such that $p = 1 + tm$, in particular $p + 2$ is a prime as well.

b) If $T(m)$ is finite, then

$$\frac{f_{\frac{2}{m}}(p)}{p} \geq m + 2 - \frac{2}{p} \text{ for large } p \in D_{\frac{2}{m}}.$$

c) $T(2) = \{1\}$, and $T(m)$ is empty if $m > 2$ and m is incongruent to 1 modulo 3.

Proof. Let $p \in D_\lambda$.

a) By definition of $p(\frac{2}{m})$, $mp(\frac{2}{m}) \leq (m+2)p$, hence $mp(\frac{2}{m}) < (m+2)p$ since $p > m$. For reasons of parity we even have $mp(\frac{2}{m}) \leq (m+2)p - 2$.

$z := (m+2)p + 2 > f_{\frac{2}{m}}(p)$, hence $z \in S_{\frac{2}{m}}(p)$. Since $mp(\frac{2}{m}) < z < (m+3)p$, because of parity z is the sum of exactly $m+2$ atoms from $S_{\frac{2}{m}}(p)$, hence $p+2$ must be a prime. Similarly $w := (m+2)p - 2 > f_{\frac{2}{m}}(p)$ is in $S_{\frac{2}{m}}(p)$; hence $w = mp(\frac{2}{m})$ because of its parity and since $mp(\frac{2}{m}) \leq w < (m+2)p$. For $t := \frac{p-1}{m}$ we have primes $p = 1 + tm$, $p + 2 = 3 + tm$ and $p(\frac{2}{m}) = 1 + t(m+2)$. Further $2t = p(\frac{2}{m}) - p$ is an even integer; hence $t \in T(m)$.

b) is immediate from a).

c) Let $m \geq 2$ and $t > 0$ be integers. Elementary calculations modulo 3 show: If m is incongruent to 1 modulo 3, then 3 divides $(1 + tm)(3 + tm)(1 + t(m+2))$. Hence $1 + tm$, $3 + tm$ and $1 + t(m+2)$ are primes if and only if $m = 2$ and $t = 1$. \square

Proposition 2.6. a) $\frac{f_\lambda(p)}{p} \geq 3 - \frac{6}{p}$ for all $\lambda > 0$ and $p \in D_\lambda$ (cf. [8, Proposition 1]).

b) If m is incongruent to 1 modulo 3, then

$$(8) \quad \frac{f_{\frac{2}{m}}(p)}{p} \geq m + 2 - \frac{2}{p} \text{ for large } p \in D_{\frac{2}{m}}.$$

In case $m \equiv 1 \pmod{3}$, (8) at least holds for large isolated primes.

c) Let $m \geq 2$ be arbitrary and $0 < \lambda < \frac{2}{m}$. Then

$$\frac{f_\lambda(p)}{p} \geq m + 2 - \frac{2}{p} \text{ for } p > \frac{2}{2 - \lambda \cdot m}, p \in D_\lambda.$$

Proof:

b) immediately follows from Lemma 2.5.

c) Elementary calculation shows that, since $p > \frac{2}{2 - \lambda \cdot m}$,

$$m(1 + \lambda)p < (m + 2)p - 2,$$

consequently

$$m \cdot p(\lambda) \leq m(1 + \lambda) \cdot p < (m + 2)p - 2 < (m + 2)p.$$

Hence $(m + 2)p - 2$ is a gap of $S_\lambda(p)$, for reasons of parity and magnitude. \square

Now we are ready to restate and prove:

Theorem 1.2. Let $0 < \lambda \leq 1$, $\lambda \notin \{\frac{2}{4}, \frac{2}{7}, \frac{2}{10}, \frac{2}{13}, \dots\}$, be a real. Let $D_\lambda := \{p \in \mathbb{P} \mid [p, p + \lambda p] \text{ contains at least two primes}\}$ and let $f_\lambda(p)$ be the Frobenius number of the numerical semigroup generated by all primes from $[p, p + \lambda p]$. Then

$$\lim_{\substack{p \in D_\lambda \\ p \rightarrow \infty}} \frac{f_\lambda(p)}{p} = \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor.$$

For $\lambda = \frac{2}{m}$ with an integer $m \geq 2$ and $m \equiv 1 \pmod{3}$, at least

$$\limsup_{p \rightarrow \infty} \frac{f_\lambda(p)}{p} = \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor.$$

Proof. Immediate from Corollary 2.4 and Proposition 2.6. \square

Notice, that in case $m \equiv 1 \pmod{3}$ Dickson's conjecture [5] implies, that $T(m)$ is infinite, so the above proof probably will not work. So in this case, Lemma 2.5 does not include formula (8) for large p .

3 The question of Wilf for certain numerical semigroups

This section is devoted to the proof of Proposition 1.1, restated below for the reader's comfort.

Since Wilf's inequality (1) holds by [2] and [6] if $p < 19$ or $f < 3p$, in what follows we may assume that $p \geq 19$ and $f > 3p$. Proposition 1.1 a) will be an easy consequence of the following result.

Theorem 3.1. *There is a constant $c > 0$ such that every numerical semigroup S of multiplicity $p \geq c$ (p not necessarily a prime number) and containing the primes from $J(p) := [p, p + p^{0.525}]$, satisfies Wilf's inequality*

$$(9) \quad \frac{g}{1+f} \leq 1 - \frac{1}{e}, \text{ equivalently } e(1+f-g) \geq 1+f.$$

Proof. Let $\pi(x)$ be the number of primes less than or equal to x . For sufficiently large integers p , by [1, p. 562] we have

$$(10) \quad e \geq |J(p) \cap \mathbb{P}| \geq 0.09 \cdot \frac{p^{0.525}}{\log p} =: e^*(p) \geq 2p^{0.5},$$

the latter since $\lim_{x \rightarrow \infty} \frac{x^{0.025}}{\log x} = \infty$. Notice, that $1+f-g$ is the number of elements of S lying below f , sometimes called *sporadic* for S . Let m be an integer such that $mp < f < (m+1)p$. We have $m \geq 3$ since by assumption $f > 3p$. Hence the $(m-1) \cdot |J(p) \cap \mathbb{P}|$ many elements

$$s = ip + q, 0 \leq i \leq m-2 \text{ and } q \in J(p) \text{ a prime,}$$

are sporadic for S , since $s \leq (m-2)p + q < mp < f$. Finally we get by (10) and since $m \geq 3$

$$e(1+f-g) \geq e^*(p) \cdot (m-1) \cdot e^*(p) \geq (m-1)4p > (m+1)p > f. \quad \square$$

Lemma 3.2. *Let $S(p)$ be the semigroup generated by the primes from $I(p) = [p, 2p]$ and $f(p)$ its Frobenius number. Then*

$$(11) \quad f(p) < 2(\pi(2p) - \pi(p))^2, \text{ if } n = \pi(p) > 674.$$

Proof. Fundamental for this are the approximate formulas for the functions p_n and $\pi(x)$ from the papers [12] and [13] by Rosser and Schoenfeld. According to [13] we have

$$(12) \quad \pi(2x) < 2\pi(x) \text{ for } x \geq 11.$$

In [12] it is shown, that

$$(13) \quad p_n < n(\log n + \log \log n) \text{ for } n > 5,$$

$$(14) \quad \pi(x) < \frac{x}{\log x - \frac{3}{2}} \text{ if } \log x > \frac{3}{2} \text{ and}$$

$$(15) \quad \pi(x) > \frac{x}{\log x - \frac{1}{2}} \text{ for } x \geq 67.$$

Since for the embedding dimension $e(p_n)$ of $S(p_n)$ we have

$$e(p_n) = \pi(2p_n) - n + 1 < n + 1 < p_n \text{ for } n > 674 \text{ by (12),}$$

the approximation of the Frobenius number by [14] page 2, last line can be applied to $S(p)$ if $\pi(p) = n > 674$. We get:

$$(16) \quad \begin{aligned} f(p) &\leq 2p_{\pi(2p)} \cdot \lfloor p/(\pi(2p) - n + 1) \rfloor - p \\ &< 2pp_{\pi(2p)}/(\pi(2p) - n + 1) \\ &< 2p_n \cdot p_{2n}/(\pi(2p) - n). \end{aligned}$$

From (14) and (15) we get

$$(17) \quad \pi(2x) > 2 \cdot \frac{x}{\log(2x) - \frac{1}{2}} > 2 \cdot \frac{\log x - \frac{3}{2}}{\log(2x) - \frac{1}{2}} \cdot \pi(x) =: l(x) \cdot \pi(x), \quad x \geq 67.$$

It is easily seen that the function $l(x)$, $x \geq 67$, is strictly increasing. Together with (12) and (17) we get at the places $x = p_n$, $n \geq 675$, i.e. $p_n \geq 5039$

$$(18) \quad 2n > \pi(2p_n) > l(p_n) \cdot \pi(p_n) \geq l(5039) \cdot n,$$

where $l(5039)$ is approximately 1.61158.

The function

$$l_2(x) := \frac{2 \cdot (\log x + \log \log x)(\log(2x) + \log \log(2x))}{x}, \quad x \geq 675$$

is strictly decreasing. As one can check,

$$(19) \quad l_2(675) < (l(5039) - 1)^3.$$

Applying (13) to the right hand side of formula (16), we get for $n \geq 675$

$$\begin{aligned} f(p_n) &< 2p_n p_{2n}/(\pi(2p_n) - n) \\ &\stackrel{(13)}{<} 2 \cdot l_2(n) \frac{n^3}{\pi(2p_n) - n} \\ &\leq 2 \cdot l_2(675) \cdot \frac{n^3}{\pi(2p_n) - n} \\ &\stackrel{(19)}{<} 2 \cdot (l(5039) - 1)^3 \cdot \frac{n^3}{\pi(2p_n) - n} \\ &\leq 2 \cdot \frac{((l(p_n) - 1)n)^3}{\pi(2p_n) - n} \\ &\stackrel{(17)}{<} 2 \cdot (\pi(2p_n) - n)^2. \end{aligned} \quad \square$$

For $\lambda > 0$, let $D_\lambda := \{p \in \mathbb{P} \mid [p, p + \lambda p] \text{ contains at least two primes}\}$.

Proposition 1.1. Let $\lambda > 0$, $p \in D_\lambda$ and S any numerical semigroup of multiplicity p containing $[p, p + \lambda p] \cap \mathbb{P}$.

- a) There is an integer $C(\lambda) > 0$ such that for $p > C(\lambda)$, the semigroups S from above satisfy Wilf's inequality (1).

- b) In case $\lambda = 1$ formula (1) holds for all p . In particular (1) is true for $S = S_n$:= numerical semigroup generated by all primes not less than p_n .

Proof. a) Let $\lambda > 0$, $p \in D_\lambda$ and S as in the statement. Let c be the constant from Theorem 3.1, choose $C(\lambda) \geq c$ such that $C(\lambda)^{0.525} < \lambda \cdot C(\lambda)$. Then for every prime $p \geq C(\lambda)$ we have $p^{0.525} < \lambda p$ as well. Hence $p \geq c$ and $J(p) \cap \mathbb{P} \subseteq [p, p + \lambda p] \cap \mathbb{P} \subseteq S$, as requested in Theorem 3.1, and S satisfies (9).

b) Let S be as in 1.1 b), $S(p)$ the semigroup generated by the primes from $I(p) := [p, 2p]$ and $f(p)$ its Frobenius number. Since $3p < f$, the primes from $I(p) \subseteq S$ are atoms as well as sporadic elements for S . The latter also holds for the even numbers $p + q$, q a prime from $I(p)$ as well as for $3p$. Hence $1 + f - g \geq 2(\pi(2p) - \pi(p) + 1) + 1$, and all together

$$(20) \quad e(1 + f - g) \geq 2(\pi(2p) - \pi(p) + 1)^2 + \pi(2p) - \pi(p) + 1.$$

Since $f \leq f(p)$, (20) together with (11) from Lemma 3.2 imply (9) for $n > 674$. Therefore, it remains only to prove Proposition 1.1 b) in case $7 < n < 675$.

According to the last column of table `wilf_for_p_to_2p.pdf` from [19] inequality (9) holds for $S(p)$, if $8 \leq \pi(p) \leq 675$.

Hence we may assume that S is different from $S(p)$, $p = p_n$. Then $e \geq \pi(2p_n) - n + 2$, and (20) can be improved to

$$e \cdot (1 + f - g) \geq (2 \cdot (\pi(2p_n) - n + 1) + 1)(\pi(2p_n) - n + 2).$$

The second last column of table `wilf_for_p_to_2p.pdf` from [19] mentioned above shows, that $f(p)$ (and, a fortiori, f) is less than the right hand side of this inequality, if $10 \leq \pi(p) < 675$. The remaining cases are $p = 19$ and $p = 23$.

For $p = 23$, by assumption we have $f > 69 = 3 \cdot p$, and $S(23)$ contains 17 elements less than 70, which then are sporadic for S . Since S is different from $S(23)$, we have $e \geq e(23) + 1 = 7$; finally $e(1 + f - g) \geq 7 \cdot 17 > 102 = f(23) \geq f$.

Analogously for $p = 19$ we have $f > 57 = 3 \cdot p$ and $e \geq 6$. Further $58 = 29 + 29$ is in S , hence $f \geq 59$. Since 60, 61 and 62 are also in S , either $f = 59$ or $f \geq 63$.

- a) Case $f \geq 63$: $S(19)$ contains 19 elements < 63 . As above $e(1 + f - g) \geq 6 \cdot 19 > 101 = f(19) \geq f$.

- b) Case $f = 59$: $S(19)$ contains 16 elements < 59 . It follows $e(1 + f - g) \geq 6 \cdot 16 > 59 = f$. \square

The fraction $d = \frac{1+f-g}{1+f}$ describes the density of the sporadic elements of S in $[0, f] \cap \mathbb{Z}$. In terms of this density, Wilf's conjecture says that d is at least $\frac{1}{e}$.

We will see in a moment that for the semigroups $S(p)$ generated by the primes from $[p, 2p]$ this bound becomes extremely weak, as p goes to infinity.

As above let $2 \geq \lambda > 0$ be a real parameter, $S_\lambda(p)$ the semigroup generated by the primes from $I_\lambda(p)$ and $e_\lambda(p)$ its embedding dimension. Here $e_\lambda(p) \sim \lambda \cdot \pi(p)$, hence $\frac{1}{e_\lambda(p)}$ is a null sequence. In contrast, the results of [3] and [11] imply the following result.

Proposition 3.3. *If $S = S(p)$ is the semigroup generated by the primes from $[p, 2p]$ then*

$$d \sim \frac{3}{8}.$$

Proof Let $\frac{5}{8} < t < 1$. By [3, Corollary], for all $2N \in [2p, 4p]$ but $O\left(\frac{2p}{\log(2p)}\right)$ exceptions, we have $2N = q_1 + q_2$ with

$$N - N^t \leq q_i \leq N + N^t, q_i \text{ a prime for } i = 1, 2.$$

If even

$$p + (2p)^t \leq N \leq 2p - (2p)^t \text{ and } 2N \text{ is not an exception,}$$

then it easily follows that $p < q_i < 2p$, hence $2N \in S(p)$. Since $\frac{p^t}{p}$ and $\frac{1}{\log(2p)}$ are null sequences, this shows that for large p , almost all even elements from $[2p, 4p]$ are in $S(p)$.

By similar arguments we see from [11, Theorem 1.1], that for large primes p , every odd integer N with

$$p + (6p)^t \leq \frac{N}{3} \leq 2p - (6p)^t$$

is contained in $S(p)$. Hence for large p , almost all odd elements from $[3p, 6p]$ are in $S(p)$. Further by Theorem 1.2, $f(p) \sim 4p$. \square

4 Binary Goldbach for large numbers: Sufficient conditions

The *Binary Goldbach conjecture* for large numbers, which seems to be open, states that each large enough even integer can be written as a sum of two primes.

In this section we present some consequences which would follow if “Binary Goldbach for large numbers” should be false. We cannot disprove any of these consequences and we do not believe that our results mean some practical progress on a way to prove Binary Goldbach for large numbers.

Recall that by S_n we denote the semigroup generated by all primes not less than p_n , and by f_n we denote the Frobenius number of S_n .

Obviously $3p_n \notin S_{n+1}$, hence $3p_n \leq f_{n+1}$ for all $n \geq 2$.

On the other hand, the table `full_numerical_semigroups_created_by_primes_greater_nth.pdf` from [19] shows that, for $2 \leq n \leq 10\,000$,

- (i) $f_{n+1} < 3p_n + 2n$.
- (ii) f_{n+1} is odd, with the exception $f_4 = 16$.
- (iii) $f_{n+1} = 3p_n + 2n - 2$ for $n = 4, 6, 7, 9$ or 15 .

Analogously, improving [8, Lemma 3] we get the following result.

Lemma 4.1. *For large n , each odd integer $N \geq 3p_n + 2n$ is contained in S_{n+1} . In particular,*

$$\begin{aligned} f_{n+1} &< 3p_n + 2n \text{ if } f_{n+1} \text{ is odd, and} \\ f_{n+1} &< 3p_n + p_{n+1} + 2n \text{ if } f_{n+1} \text{ is even.} \end{aligned}$$

Proof: By [11, Theorem 1.1] each odd number $N \geq U_n := 3p_n + 2n \gg 0$ is the sum

$$N = q_1 + q_2 + q_3 \text{ of primes } q_i \text{ with } \left| \frac{N}{3} - q_i \right| \leq N^{\frac{3}{5}}, i = 1, 2, 3.$$

Since $F(x) := \frac{x}{3} - x^{\frac{3}{5}}$ is increasing for $x \geq 5$, and $p_n \sim n \log n$,

$$\begin{aligned} q_i &\geq \frac{N}{3} - N^{\frac{3}{5}} \\ &\geq \frac{U_n}{3} - U_n^{\frac{3}{5}} \\ &> p_n + \frac{2n}{3} - (6n \log n + 2n)^{\frac{3}{5}} \\ &> p_n \end{aligned}$$

for large n . Hence $N = q_1 + q_2 + q_3$ is contained in S_{n+1} . \square

This implies for each $1 \geq \varepsilon > 0$: Since $\frac{2n}{p_n}$ is a null sequence and $\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = 1$, there is an $n(\varepsilon) > 0$ such that for $n \geq n(\varepsilon)$

$$(21) \quad \frac{f_{n+1}}{p_n} < 3 + \varepsilon \text{ if } f_{n+1} \text{ is odd, } \frac{f_{n+1}}{p_{n+1}} < 4 + \varepsilon \text{ if } f_{n+1} \text{ is even and}$$

$$(22) \quad \frac{p_{n+1}}{p_n} < 1 + \frac{\varepsilon}{5}.$$

Hence for all $n \geq n(\varepsilon)$

$$\begin{aligned} (23) \quad \frac{f_{n+1}}{p_n} - \frac{f_{n+2}}{p_{n+1}} &\leq \frac{f_{n+1}}{p_{n+1}} \cdot \frac{p_{n+1}}{p_n} - \frac{f_{n+1}}{p_{n+1}} \\ &= \frac{f_{n+1}}{p_{n+1}} \left(\frac{p_{n+1}}{p_n} - 1 \right) \\ &< 5 \cdot \frac{\varepsilon}{5} \\ &= \varepsilon \end{aligned}$$

by (21) and (22).

Assumption The Binary Goldbach conjecture for large numbers is false.

Conclusions for the sequence (f_n)

- a) For infinitely many $n > 0$, $\frac{f_{n+1}}{p_n} \geq 4$. See [8, Lemma 1].
- b) For each integer $k > 0$ there is an integer $n \geq n(\frac{1}{k})$ such that the k Frobenius numbers f_{n+1}, \dots, f_{n+k} are even.
- Proof:* By a) we can find an $n \geq n(\frac{1}{k})$ such that $\frac{f_{n+1}}{p_n} \geq 4$. By (23), for $1 \leq m \leq k$ we have

$$\begin{aligned} \frac{f_{n+m}}{p_{n+m-1}} &= \frac{f_{n+1}}{p_n} - \left(\frac{f_{n+1}}{p_n} - \frac{f_{n+2}}{p_{n+1}} \right) - \dots - \left(\frac{f_{n+m-1}}{p_{n+m-2}} - \frac{f_{n+m}}{p_{n+m-1}} \right) \\ &\geq 4 - (m-1) \cdot \frac{1}{k} \\ &\geq 3 + \frac{1}{k}, \end{aligned}$$

hence the integers f_{n+1}, \dots, f_{n+k} are even by (21). \square

- c) Either f_{n+1} is even for almost all $n > 0$, or $[3, 4]$ is contained in the closure of $\left\{ \frac{f_{n+1}}{p_n} \mid n > 0 \right\}$.

Proof: Under the **additional assumption**, that f_{n+1} is odd for infinitely many n we have to show:

Let $1 \geq \varepsilon > 0$ and $x \in [3, 4]$ be arbitrary. Then

$$\left| x - \frac{f_{m+1}}{p_m} \right| < \varepsilon \text{ for some integer } m > 0.$$

Proof: By a) and since f_{n+1} is odd infinitely often, there are integers $n \geq n(\varepsilon)$ and $k > 0$ such that

$$\frac{f_{n+1}}{p_n} \geq 4 \text{ and } \frac{f_{n+k+1}}{p_{n+k}} < 3 + \varepsilon.$$

In case $x \leq \frac{f_{n+k+1}}{p_{n+k}}$ we take $m = n + k$. Otherwise

$$\frac{f_{n+k+1}}{p_{n+k}} < x \leq 4 \leq \frac{f_{n+1}}{p_n},$$

and we can find an m with $n+1 \leq m \leq n+k$ such that $\frac{f_{m+1}}{p_m} < x \leq \frac{f_m}{p_{m-1}}$.

Hence by (23)

$$0 \leq \frac{f_m}{p_{m-1}} - x < \frac{f_m}{p_{m-1}} - \frac{f_{m+1}}{p_m} < \varepsilon. \quad \square$$

Note on the even f_n :

1. We apply [3, Corollary] to any $0 < \varepsilon < \frac{3}{8}$ like e.g. $\varepsilon = \frac{1}{8}$: $U = m^{\frac{5}{8}+\varepsilon}$ is, because of $\varepsilon < \frac{3}{8}$, smaller than $\frac{m}{10}$ for m large.

Hence apart from at most $O\left(\frac{N}{(\log N)^A}\right)$ exceptions (A arbitrary) each even integer

$$2m \in [N, 2N]$$

is a sum

$$2m = q_1 + q_2$$

of two primes numbers

$$\frac{9}{10}m \leq q_1, q_2 \leq \frac{11}{10}m.$$

2. Let n be large enough and suppose f_n is even. By [8, Proposition 1 and Lemma 3], $N := 3p_n - 6 \leq f_n \leq 2N$; in particular, $p_n \leq \frac{9}{10} \cdot \frac{f_n}{2}$. Hence the gap f_n of S_n is always an exception in the sense of (1).

Note on coding Our numerical experiments may be reproduced by using the corresponding codes from the repository `Bleiglanz/On_The_FrobeniusNumber` in [19].

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