

# SYMBOLIC POWERS IN WEIGHTED ORIENTED GRAPHS

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ABSTRACT. Let  $D$  be a weighted oriented graph with the underlying graph  $G$  when vertices with non-trivial weights are sinks and  $I(D), I(G)$  be the edge ideals corresponding to  $D$  and  $G$ , respectively. We give an explicit description of the symbolic powers of  $I(D)$  using the concept of strong vertex covers. We show that the ordinary and symbolic powers of  $I(D)$  and  $I(G)$  behave in a similar way. We provide a description for symbolic powers and Waldschmidt constant of  $I(D)$  for certain classes of weighted oriented graphs. When  $D$  is a weighted oriented odd cycle, we compute  $\text{reg}(I(D)^{(s)}/I(D)^s)$  and prove  $\text{reg} I(D)^{(s)} \leq \text{reg} I(D)^s$  and show that equality holds when there is only one vertex with non-trivial weight.

Keywords: Weighted oriented graph, edge ideal, symbolic power, Waldschmidt constant, Castelnuovo-Mumford regularity.

## 1. INTRODUCTION

A *directed graph* or *digraph*  $D = (V(D), E(D))$  consists of a finite set  $V(D)$  of vertices together with a prescribed collection  $E(D)$  of ordered pairs of distinct vertices called edges or arrows. If  $(u, v) \in E(D)$ , then we call it a directed edge where the direction is from  $u$  to  $v$  and  $u$  (respectively  $v$ ) is called the initial vertex (respectively the terminal vertex). An oriented graph is a directed graph without multiple edges or loops. In other words, an oriented graph  $D$  is a simple graph  $G$  together with an orientation of its edges. An oriented graph  $D$  is called vertex weighted oriented if it is equipped with a weight function  $w : V(D) \rightarrow \mathbb{N}$ . In fact, a vertex weighted oriented graph  $D$  is a triplet  $D = (V(D), E(D), w)$  where  $V(D)$  and  $E(D)$  are the vertex set and edge set, respectively, and the weight of a vertex  $x_i \in V(D)$  is  $w(x_i)$  denoted by  $w_i$  or  $w_{x_i}$ . We set  $V^+(D) = \{x \in V(D) \mid w(x) \neq 1\}$  and it is denoted by  $V^+$ . If  $V(D) = \{x_1, \dots, x_n\}$ , we can regard each vertex  $x_i$  as a variable and consider the polynomial ring  $R = k[x_1, \dots, x_n]$  over a field  $k$ . Then the edge ideal of  $D$  is defined as

$$I(D) = (x_i x_j^{w_j} \mid (x_i, x_j) \in E(D)).$$

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The underlying graph of  $D$  is the simple graph  $G$  whose vertex set is  $V(G) = V(D)$  and whose edge set is  $E(G) = \{\{u, v\} \mid (u, v) \in E(D)\}$ . The edge ideal of  $G$  is  $I(G) = (uv \mid \{u, v\} \in E(G)) \subseteq R = k[x_1, \dots, x_n]$ . If a vertex  $x_i$  of  $D$  is a source (i.e., has only arrows leaving  $x_i$ ), we shall always assume  $w_i = 1$  because in this case, the definition of  $I(D)$  does not depend on the weight of  $x_i$ . If  $w(x) = 1$  for all  $x \in V$ , then  $I(D)$  recovers the usual edge ideal of the underlying graph  $G$ . Here, we shall always mean that a weighted oriented graph is the same as a vertex weighted oriented graph. The interest in edge ideals of weighted digraphs comes from coding theory, especially, in the study of Reed-Muller types codes. The edge ideal of a vertex weighted digraph appears as initial ideals of vanishing ideals of certain projective spaces over finite fields [15].

Algebraic invariants and properties like Cohen-Macaulayness and unmixedness of edge ideals of weighted oriented graphs have been studied in [13] and [16]. Recently in [1] and [20], regularity and projective dimension of the edge ideals of some classes of weighted oriented graphs have been studied. In [16], Y. Pitones et al. studied some properties of weighted oriented graphs when  $V(D)$  is a strong vertex cover and the elements of  $V^+$  are sinks. They described the irreducible decomposition of  $I(D)$  for any weighted oriented graph  $D$  using the concept of strong vertex cover. In the geometrical context, symbolic powers are important since they capture all the polynomials that vanish with a given multiplicity. For any homogeneous ideal  $I \subset R$ , its  $m$ -th symbolic power is defined as  $I^{(m)} = \bigcap_{p \in \text{Ass } I} (I^m R_p \cap R)$ . In [19], a classical result of Zariski and Samuel states that  $I^{(m)} = I^m$  for all  $m \geq 1$  if  $I$  is generated by a regular sequence, or equivalently, a complete intersection. Ideals that have the property  $I^{(m)} = I^m$  for all  $m \geq 1$  are called normally torsion free because their Rees algebra is normal. In particular, in [7] Gitler et al. showed that a squarefree monomial ideal is normally torsion free if and only if the corresponding hypergraph satisfies the max-flow min-cut property. Symbolic powers of codimension 2 Cohen-Macaulay ideals have been studied recently in [6]. In the last decade the comparison between symbolic and regular powers of ideals is studied not only for graphs but also for ideal defining a set of points both in projective and multiprojective spaces [2, 8, 9]. Recently symbolic powers and invariants of the edge ideals of simple graphs have been studied in [4, 11, 14]. But nothing is known about the symbolic powers of edge ideals of weighted oriented graphs.

In general, computing the symbolic powers of ideals is a difficult problem. Symbolic powers of square free monomial ideals have been studied via the concept of minimal vertex cover for edge ideals of graphs. Here we study the symbolic powers of some classes of non square free monomial ideals through the edge ideals of weighted oriented graphs using the concept of strong vertex cover. We also compare the regularity of the

symbolic and ordinary powers of the edge ideals of weighted oriented odd cycles when the elements of  $V^+$  are sinks.

In Section 2, we recall all the definitions and results which will be required for the rest of the paper. In Section 3, we provide a relation between ordinary and symbolic powers of edge ideals of some classes of weighted naturally oriented graphs in Corollary 3.2. In Theorem 3.4, we show that the  $m$ -th symbolic power of edge ideal of a weighted oriented graph, when the elements of  $V^+$  are sinks, is the intersection of the  $m$ -th powers of irreducible ideals associated to the strong vertex covers. In Theorem 3.7, we prove that the ordinary and symbolic powers of edge ideal of a weighted oriented graph behaves in the similar way as in the edge ideal of its underlying graph in case of elements of  $V^+$  are sinks.

We also give an explicit description of symbolic powers and Waldschmidt constants of edge ideals of weighted oriented graphs when the elements of  $V^+$  are sinks and underlying graphs are unicyclic graph with a unique odd cycle, complete graph and clique sum of two odd cycles of same length joining at a common vertex (see Proposition 3.9, Proposition 3.10 and Proposition 3.12, respectively). In Section 4, in Theorem 4.3, we provide an inequality between the regularity of ordinary and symbolic powers of edge ideal of a weighted oriented odd cycle when the elements of  $V^+$  are sinks and in Corollary 4.5, we prove the equality when  $V^+$  contains a single element.

## 2. PRELIMINARIES

In this section, we recall some definitions and results regarding a weighted oriented graph  $D$  and its underlying graph  $G$ . A vertex cover of  $G$  is a subset  $V' \subseteq V(G)$  such that for all  $e \in E(G)$ ,  $e \cap V' \neq \emptyset$ . A minimal vertex cover is a vertex cover which is minimal with respect to inclusion.

The next two lemmas describe the symbolic powers of the edge ideals in terms of minimal vertex covers.

**Lemma 2.1.** [2, Lemma 2.5] *Let  $G$  be a graph on vertices  $\{x_1, \dots, x_n\}$ ,  $I = I(G) \subseteq k[x_1, \dots, x_n]$  be the edge ideal of  $G$  and  $V_1, \dots, V_r$  be the minimal vertex covers of  $G$ . Let  $P_j$  be the monomial prime ideal generated by the variables in  $V_j$ . Then*

$$I = P_1 \cap \dots \cap P_r$$

and

$$I^{(m)} = P_1^m \cap \dots \cap P_r^m.$$

**Lemma 2.2.** [2, Lemma 2.6] *Let  $I \subseteq S$  be a squarefree monomial ideal with minimal primary decomposition  $I = P_1 \cap \dots \cap P_r$  with  $P_j = (x_{j_1}, \dots, x_{j_{s_j}})$  for  $j = 1, \dots, r$ . Then  $x_1^{a_1} \dots x_n^{a_n} \in I^{(m)}$  if and only if  $a_{j_1} + \dots + a_{j_{s_j}} \geq m$  for  $j = 1, \dots, r$ .*

Below we recall some definitions and results for the weighted oriented graph  $D$ .

**Definition 2.3.** A vertex cover  $C$  of  $D$  is a subset of  $V(D)$  such that if  $(x, y) \in E(D)$ , then  $x \in C$  or  $y \in C$ . A vertex cover  $C$  of  $D$  is minimal if each proper subset of  $C$  is not a vertex cover of  $D$ . We set  $\langle C \rangle$  to be the ideal generated by the variables in  $C$ .

**Remark 2.4.** [16, Remark 2]  $C$  is a minimal vertex cover of  $D$  if and only if  $C$  is a minimal vertex cover of  $G$ .

**Definition 2.5.** Let  $x$  be a vertex of a weighted oriented graph  $D$ , then the sets  $N_D^+(x) = \{y : (x, y) \in E(D)\}$  and  $N_D^-(x) = \{y : (y, x) \in E(D)\}$  are called the out-neighbourhood and the in-neighbourhood of  $x$ , respectively. Moreover, the neighbourhood of  $x$  is the set  $N_D(x) = N_D^+(x) \cup N_D^-(x)$ . Define  $\deg_D(x) = |N_D(x)|$  for  $x \in V(D)$ . A vertex  $x \in V(D)$  is called a source vertex if  $N_D(x) = N_D^+(x)$ . A vertex  $x \in V(D)$  is called a sink vertex if  $N_D(x) = N_D^-(x)$ .

**Definition 2.6.** [16, Definition 4] Let  $C$  be a vertex cover of a weighted oriented graph  $D$ , we define

$$\begin{aligned} L_1(C) &= \{x \in C \mid N_D^+(x) \cap C^c \neq \phi\}, \\ L_2(C) &= \{x \in C \mid x \notin L_1(C) \text{ and } N_D^-(x) \cap C^c \neq \phi\} \text{ and} \\ L_3(C) &= C \setminus (L_1(C) \cup L_2(C)) \end{aligned}$$

where  $C^c$  is the complement of  $C$ , i.e.,  $C^c = V(D) \setminus C$ .

**Lemma 2.7.** [16, Proposition 6] *Let  $C$  be a vertex cover of  $D$ . Then  $L_3(C) = \phi$  if and only if  $C$  is a minimal vertex cover of  $D$ .*

**Definition 2.8.** [16, Definition 7] A vertex cover  $C$  of  $D$  is strong if for each  $x \in L_3(C)$  there is  $(y, x) \in E(D)$  such that  $y \in L_2(C) \cup L_3(C)$  with  $y \in V^+$  (i.e.,  $w(y) \neq 1$ ).

**Definition 2.9.** [16, Definition 32] A weighted oriented graph  $D$  has the minimal-strong property if each strong vertex cover is a minimal vertex cover.

The following lemma gives a class of weighted oriented graphs which satisfy the minimal-strong property.

**Lemma 2.10.** [16, Lemma 47] *If the elements of  $V^+$  are sinks, then  $D$  has the minimal-strong property.*

The next lemma describes the irreducible decomposition of the edge ideal of weighted oriented graph  $D$  in terms of irreducible ideals associated with the strong vertex covers of  $D$ .

**Lemma 2.11.** [16, Theorem 25, Remark 26] *Let  $D$  be a weighted oriented graph and  $C_1, \dots, C_s$  are the strong vertex covers of  $D$ , then the irredundant irreducible decomposition of  $I(D)$  is*

$$I(D) = I_{C_1} \cap \dots \cap I_{C_s}$$

where each  $I_{C_i} = (L_1(C_i) \cup \{x_j^{w(x_j)} \mid x_j \in L_2(C_i) \cup L_3(C_i)\})$ ,  $\text{rad}(I_{C_i}) = P_i = (C_i)$ .

**Corollary 2.12.** [16, Remark 26] *Let  $D$  be a weighted oriented graph. Then  $P$  is an associated prime of  $I(D)$  if and only if  $P = (C)$  for some strong vertex cover  $C$  of  $D$ .*

**Notation 2.13.** *The degree of a monomial  $m \in k[x_1, \dots, x_n]$  is denoted by  $\deg(m)$ .*

Now we recall the definition of Castelnuovo-Mumford regularity using local cohomology. Let  $M$  be a non zero module over the polynomial ring  $R$ . If  $H_{\mathfrak{m}}^i(M)$  denotes the  $i$ -th local cohomology module of the  $R$ -module  $M$  with support on the maximal ideal  $\mathfrak{m}$ , we set

$$a_i(M) = \max\{j \in \mathbb{Z} : [H_{\mathfrak{m}}^i(M)]_j \neq 0\}$$

(or  $a_i(M) = -\infty$  if  $H_{\mathfrak{m}}^i(M) = 0$ ) where  $[H_{\mathfrak{m}}^i(M)]_j$  denotes the  $j$ -th graded component of the  $i$ -th local cohomology module of  $M$ . Then the Castelnuovo-Mumford regularity of  $M$  is defined as

$$\text{reg } M = \max_{0 \leq i \leq \dim M} \{a_i(M) + i\}.$$

One of the important asymptotic invariants for symbolic powers is resurgence which was introduced by Bocci and Harbourne [3] to answer the ideal containment problem. In general, computation of resurgence of an ideal is difficult. In [3], one lower bound of resurgence of an ideal is given in terms of another invariant known as Waldschmidt constant of  $I$ . Let  $\alpha(I) = \min\{d \mid I_d \neq 0\}$ , i.e.,  $\alpha(I)$  is the smallest degree of a nonzero element in  $I$ . Then the Waldschmidt constant of  $I$  is defined as

$$\widehat{\alpha}(I) = \lim_{s \rightarrow \infty} \frac{\alpha(I^{(s)})}{s}.$$

### 3. SYMBOLIC POWERS IN ORIENTED GRAPHS

We provide an explicit description of the symbolic powers for edge ideals of weighted oriented graphs when the elements of  $V^+$  are sinks. In the next lemma, we give a sufficient condition for the equality of the symbolic and ordinary powers of the edge ideals for some classes of weighted oriented graphs.

**Lemma 3.1.** *Let  $I = I(D)$  be the edge ideal of a weighted oriented graph  $D$ . If  $V(D)$  is a strong vertex cover of  $D$ , then  $I^{(k)} = I^k$  for all  $k$ .*

*Proof.* Since  $V(D)$  is the strong vertex cover of  $D$ , by Corollary 2.12,  $P = (V(D))$  is an associated prime of  $I(D)$ . Here  $P$  is the unique maximal associated prime in  $\text{Ass}(I)$ . Thus by [5, Lemma 3.3], we have  $I^{(k)} = I^k$  for all  $k$ .  $\square$

A cycle is naturally oriented if all edges of cycle are oriented in clockwise direction. In a naturally oriented unicyclic graph, the cycle is naturally oriented and each edge of the tree connected with the cycle is oriented away from the cycle.

**Corollary 3.2.** *Let  $I = I(D)$  be the edge ideal of a weighted oriented graph  $D$ . If there are weighted naturally oriented unicyclic graphs  $D_1, \dots, D_s$  of  $D$  such that  $V(D_1), \dots, V(D_s)$  is a partition of  $V(D)$  and  $w(x) \neq 1$  if  $\deg_{D_i}(x) > 1$  for each  $i$ , then  $I^{(k)} = I^k$  for all  $k$ .*

*Proof.* Since there are weighted naturally oriented unicyclic graphs  $D_1, \dots, D_s$  of  $D$  such that  $V(D_1), \dots, V(D_s)$  is a partition of  $V(D)$  and  $w(x) \neq 1$  if  $\deg_{D_i}(x) > 1$  for each  $i$ , by [16, Proposition 15],  $V(D)$  is a strong vertex cover of  $D$ . Therefore by Lemma 3.1, we have  $I^{(k)} = I^k$  for all  $k$ .  $\square$

**Corollary 3.3.** *Let  $I = I(D)$  be the edge ideal of a weighted oriented complete graph  $D$  on the vertex set  $\{x_1, \dots, x_n\}$  where the cycle  $C_n = (x_1, \dots, x_n)$  is naturally oriented and the diagonals are oriented in any direction such that  $w(x) \neq 1$  for any vertex  $x$ . Then  $I^{(k)} = I^k$  for all  $k$ .*

*Proof.* Let  $D_1$  be the weighted naturally oriented cycle whose underlying graph is the cycle  $C_n = (x_1, \dots, x_n)$ . Here  $V(D_1) = V(D)$  and  $w(x) \neq 1$  for all  $x \in V(D_1)$ . Then by Corollary 3.2, we have  $I^{(k)} = I^k$  for all  $k$ .  $\square$

In the next theorem, we describe the symbolic powers for edge ideals of weighted oriented graphs when the elements of  $V^+$  are sinks, in terms of irreducible ideals associated to the strong vertex covers. For the remainder of this section, we set  $V^+ \neq \emptyset$ .

**Theorem 3.4.** *Let  $I = I(D)$  be the edge ideal of a weighted oriented graph  $D$  where the elements of  $V^+$  are sinks and  $C_1, \dots, C_s$  be the strong vertex covers of  $D$ . Let  $w_j = w(x_j)$  if  $x_j \in V^+$ . Then the irredundant irreducible decomposition of  $I$  is*

$$I = I_{C_1} \cap \dots \cap I_{C_s}$$

and

$$I^{(m)} = I_{C_1}^m \cap \dots \cap I_{C_s}^m$$

where each  $I_{C_i} = (\{x_j \mid x_j \in C_i \setminus V^+\} \cup \{x_j^{w_j} \mid x_j \in C_i \cap V^+\})$ ,  $\text{rad}(I_{C_i}) = P_i = (C_i)$  and  $P_i$ 's are minimal primes of  $I$ .

*Proof.* Here the elements of  $V^+$  are sinks. By Lemma 2.11,  $I = I_{C_1} \cap \dots \cap I_{C_s}$  be the irredundant irreducible decomposition of  $I$  where each  $I_{C_i} = (L_1(C_i) \cup \{x_j^{w_j} \mid x_j \in L_2(C_i) \cup$

$L_3(C_i)\}$ ),  $\text{rad}(I_{C_i}) = P_i = (C_i)$  and  $P_i$ 's are associated primes of  $I$ . By Lemma 2.10,  $C_1, \dots, C_s$  are minimal vertex covers of  $D$ . Thus by Lemma 2.7, we have  $L_3(C_i) = \phi$  for  $i = 1, \dots, s$ . For a fixed  $i$ , if  $x \in C_i$  is a sink vertex, then  $N_D^+(x) \cap C_i^c = \phi$ , which implies  $x \notin L_1(C_i)$ . Thus  $L_2(C_i) = C_i \cap V^+$  and  $L_1(C_i) = C_i \setminus V^+$ . Therefore  $I_{C_i} = (\{x_j \mid x_j \in C_i \setminus V^+\} \cup \{x_j^{w_j} \mid x_j \in C_i \cap V^+\})$ . Since  $C_i$ 's are minimal vertex covers of  $D$ , the  $P_i$ 's are minimal primes of  $I$ . Hence by [5, Lemma 3.7], we have  $I^{(m)} = I_{C_1}^m \cap \dots \cap I_{C_s}^m$ .  $\square$

**Proposition 3.5.** *Let  $I$  be the edge ideal of a weighted oriented graph  $D$  where the elements of  $V^+$  are sinks with irredundant irreducible decomposition  $I = I_{C_1} \cap \dots \cap I_{C_s}$  where  $\text{rad}(I_{C_i}) = P_i = (C_i) = (x_{i_1}, \dots, x_{i_{r_i}})$  for  $i = 1, \dots, s$  and  $C_i$ 's are the strong vertex covers of  $D$ . Let  $w_j = w(x_j)$  if  $x_j \in V^+$ . Then every minimal generator of  $I^{(m)}$  is of the form  $(\prod x_j^{a_j} \mid x_j \notin V^+)(\prod x_j^{w_j a_j} \mid x_j \in V^+)$  for some  $a_j$ 's where  $a_{i_1} + \dots + a_{i_{r_i}} \geq m$  for  $i = 1, \dots, s$ .*

*Proof.* Here each  $I_{C_i}$  is of the form  $I_{C_i} = (\{x_j \mid x_j \in C_i \setminus V^+\} \cup \{x_j^{w_j} \mid x_j \in C_i \cap V^+\})$  and  $\text{rad}(I_{C_i}) = P_i = (C_i) = (x_{i_1}, \dots, x_{i_{r_i}})$ . By Theorem 3.4, we have  $I^{(m)} = I_{C_1}^m \cap \dots \cap I_{C_s}^m$ . Let  $t = (\prod x_j^{a_j} \mid x_j \notin V^+)(\prod x_j^{w_j a_j} \mid x_j \in V^+)(\prod x_j^{b_j} \mid x_j \in V^+, b_j < w_j)$  for some  $a_j$ 's and  $b_j$ 's. Then  $t \in I^{(m)}$  if and only if  $t \in I_{C_i}^m$  for  $i = 1, \dots, s$ . This is possible if and only if there exists at least one generator  $f_i \in I_{C_i}^m$  such that  $f_i$  divides  $t$  for  $i = 1, \dots, s$  which is equivalent to the condition  $a_{i_1} + \dots + a_{i_{r_i}} \geq m$  for  $i = 1, \dots, s$ .

From the structure of  $I^{(m)} = I_{C_1}^m \cap \dots \cap I_{C_s}^m$ , it is clear that every minimal generator of  $I^{(m)}$  is of the form  $(\prod x_j^{a_j} \mid x_j \notin V^+)(\prod x_j^{w_j a_j} \mid x_j \in V^+)$  for some  $a_j$ 's where  $a_{i_1} + \dots + a_{i_{r_i}} \geq m$  for  $i = 1, \dots, s$ .  $\square$

Now we relate the symbolic powers of edge ideals of weighted oriented graphs with the edge ideals of underlying graphs and show that both of them behave in a similar way when the elements of  $V^+$  are sinks.

**Notation 3.6.** *Let  $D$  be a weighted oriented graph where the elements of  $V^+$  are sinks and  $w_j = w(x_j)$  if  $x_j \in V^+$ . Here  $R = k[x_1, \dots, x_n] = \bigoplus_{d=0}^{\infty} R_d$  is the standard graded polynomial ring. Consider the map*

$$\Phi : R \longrightarrow R \text{ where } x_j \longrightarrow x_j \text{ if } x_j \notin V^+ \text{ and } x_j \longrightarrow x_j^{w_j} \text{ if } x_j \in V^+.$$

Here  $\Phi$  is an injective homomorphism of  $k$ -algebras. We use this definition of  $\Phi$  for the following theorem.

**Theorem 3.7.** *Let  $D$  be a weighted oriented graph where the elements of  $V^+$  are sinks and  $G$  be the underlying graph of  $D$ . Let  $I$  and  $\tilde{I}$  be the edge ideals of  $G$  and  $D$ , respectively. Then  $\Phi(I^k) = \tilde{I}^k$  and  $\Phi(I^{(k)}) = \tilde{I}^{(k)}$  for all  $k \geq 1$ .*

*Proof.* Let  $V(G) = V(D) = \{x_1, \dots, x_n\}$ . Let  $C_1, \dots, C_s$  are the strong vertex covers of  $D$  and by Lemma 2.10, they are minimal vertex covers of  $D$ . By Remark 2.4,  $C_1, \dots, C_s$  are also minimal vertex covers of  $G$ . Thus by Lemma 2.1, the minimal primary decomposition of  $I$  is  $I = P_1 \cap \dots \cap P_s$  where  $P_i = (C_i) = (x_{i_1}, \dots, x_{i_{r_i}})$  for  $i = 1, \dots, s$ . By Lemma 2.11,  $\tilde{I} = \tilde{I}_{C_1} \cap \dots \cap \tilde{I}_{C_s}$  be the irredundant irreducible decomposition of  $I$  where each  $\tilde{I}_{C_i} = (\{x_j \mid x_j \in C_i \setminus V^+\} \cup \{x_j^{w_j} \mid x_j \in C_i \cap V^+\})$ ,  $w_j = w(x_j)$  if  $x_j \in V^+$ ,  $\text{rad}(\tilde{I}_{C_i}) = P_i = (C_i)$  and  $P_i$ 's are minimal primes of  $\tilde{I}$ . Here  $\Phi(I) = \tilde{I}$ . Then  $\Phi(I^k) = [\Phi(I)]^k = \tilde{I}^k$  for all  $k \geq 1$ . Next we want to prove that  $\Phi(I^{(k)}) = \tilde{I}^{(k)}$ . By Lemma 2.1,  $I^{(k)} = P_1^k \cap \dots \cap P_s^k$  and by Theorem 3.4, we have  $\tilde{I}^{(k)} = \tilde{I}_{C_1}^k \cap \dots \cap \tilde{I}_{C_s}^k$ . Here  $\Phi(P_i) = \tilde{I}_{C_i}$  for  $i = 1, \dots, s$ .

Let  $q$  be a minimal generator of  $\Phi(I^{(k)})$ . Then there exists  $p \in I^{(k)}$  such that  $q = \Phi(p)$ . Since  $p \in I^{(k)} = P_1^k \cap \dots \cap P_s^k$ , then  $p \in P_i^k$  for each  $i = 1, \dots, s$  which implies that  $\Phi(p) \in \Phi(P_i^k) = [\Phi(P_i)]^k = \tilde{I}_{C_i}^k$ . Thus  $q = \Phi(p) \in \tilde{I}_{C_1}^k \cap \dots \cap \tilde{I}_{C_s}^k = \tilde{I}^{(k)}$ . Therefore  $\Phi(I^{(k)}) \subseteq \tilde{I}^{(k)}$ .

Let  $q$  be a minimal generator of  $\tilde{I}^{(k)}$ . Then by Proposition 3.5, we have  $q = (\prod x_j^{a_j} \mid x_j \notin V^+) (\prod x_j^{w_j a_j} \mid x_j \in V^+)$  for some  $n$ -tuple  $(a_1, \dots, a_n)$  where  $a_{i_1} + \dots + a_{i_{r_i}} \geq k$  for  $i = 1, \dots, s$ . By letting  $p := x_1^{a_1} \dots x_n^{a_n}$ , we see that  $\Phi(p) = q$  and by Lemma 2.2,  $p \in I^{(k)}$  since  $a_{i_1} + \dots + a_{i_{r_i}} \geq k$  for  $i = 1, \dots, s$ . Thus  $q = \Phi(p) \in \Phi(I^{(k)})$ . Therefore  $\tilde{I}^{(k)} \subseteq \Phi(I^{(k)})$ . Hence  $\Phi(I^{(k)}) = \tilde{I}^{(k)}$  for all  $k \geq 1$ .  $\square$

Next, we apply the above theorem to some specific classes of weighted oriented graphs and give an explicit description of the symbolic powers of their edge ideals. Then we compute the Waldschmidt constant.

**Corollary 3.8.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented bipartite graph  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  which is a bipartite graph. Then  $\tilde{I}^{(k)} = \tilde{I}^k$  for all  $k \geq 1$ .*

*Proof.* Let  $I$  be the edge ideal of  $G$  and  $\Phi$  is same as defined in Notation 3.6. Since  $G$  is a bipartite graph, we have  $I^{(k)} = I^k$  for all  $k \geq 1$  by [17, Theorem 5.9]. Thus by Theorem 3.7, we have  $\Phi(I^{(k)}) = \Phi(I^k)$ , i.e.,  $\tilde{I}^{(k)} = \tilde{I}^k$  for all  $k \geq 1$ .  $\square$

In [2, Theorem 6.7], C. Bocci et al. found that the Waldschmidt constant of edge ideal of an odd cycle  $C_{2n+1}$  is  $\frac{2n+1}{n+1}$ . In the next Proposition, we observe that the Waldschmidt constant of edge ideal of a weighted oriented unicyclic graph depends on the position of the elements of  $V^+$ .

**Proposition 3.9.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented unicyclic graph  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  with a unicycle  $C_{2n+1} = (x_1, \dots, x_{2n+1})$ . Let  $c = (\prod x_j \mid x_j \in V(C_{2n+1}) \setminus V^+) (\prod x_j^{w(x_j)} \mid x_j \in V(C_{2n+1}) \cap V^+)$ .*

(1) *Then  $\tilde{I}^{(k)} = \tilde{I}^k$  for  $1 \leq k \leq n$  and  $\tilde{I}^{(n+1)} = \tilde{I}^{n+1} + (c)$ .*

(2) Let  $s \in \mathbb{N}$  and write  $s = k(n+1) + r$  for some  $k \in \mathbb{Z}$  and  $0 \leq r \leq n$ . Then

$$\tilde{I}^{(s)} = \sum_{t=0}^k \tilde{I}^{s-t(n+1)}(c)^t.$$

(3) If none of the vertices of  $V(C_{2n+1})$  is in  $V^+$  and some vertex of  $V(G) \setminus V(C_{2n+1})$  is in  $V^+$ , then  $\alpha(\tilde{I}^{(s)}) = 2s - \left\lfloor \frac{s}{n+1} \right\rfloor$  for  $s \in \mathbb{N}$ .

In particular, the Waldschmidt constant of  $\tilde{I}$  is given by

$$\widehat{\alpha}(\tilde{I}) = \frac{2n+1}{n+1}.$$

(4) If at least one of the vertices of  $V(C_{2n+1})$  is in  $V^+$ , then  $\alpha(\tilde{I}^{(s)}) = 2s$  for  $s \in \mathbb{N}$ . In particular, the Waldschmidt constant of  $\tilde{I}$  is given by

$$\widehat{\alpha}(\tilde{I}) = 2.$$

*Proof.* Let  $I$  be the edge ideal of  $G$  and  $\Phi$  is same as defined in Notation 3.6.

(1) Since  $G$  is a unicyclic graph with a unique odd cycle  $C_{2n+1} = (x_1, \dots, x_{2n+1})$ , by [11, Lemma 3.3], we have  $I^{(k)} = I^k$  for  $1 \leq k \leq n$  and  $I^{(n+1)} = I^{n+1} + (x_1 \cdots x_{2n+1})$ . Thus, by Theorem 3.7,  $\Phi(I^{(k)}) = \Phi(I^k)$ , i.e.,  $\tilde{I}^{(k)} = \tilde{I}^k$  for  $1 \leq k \leq n$  and  $\Phi(I^{(n+1)}) = \Phi(I^{n+1}) + \Phi(x_1 \cdots x_{2n+1})$ , i.e.,  $\tilde{I}^{(n+1)} = \tilde{I}^{n+1} + (c)$ .

(2) By [11, Lemma 3.4], we have  $I^{(s)} = \sum_{t=0}^k I^{s-t(n+1)}(x_1 \cdots x_{2n+1})^t$ . Then by Theorem 3.7, applying  $\Phi$  on both sides, we get  $\tilde{I}^{(s)} = \sum_{t=0}^k \tilde{I}^{s-t(n+1)}(c)^t$ .

(3) Assume that none of the vertices of  $V(C_{2n+1})$  is in  $V^+$ . Thus by (1),  $\tilde{I}^{(n+1)} = \tilde{I}^{n+1} + (c) = \tilde{I}^{n+1} + (x_1 \cdots x_{2n+1})$ . Since all the edges of cycle are without any element of  $V^+$ ,  $\alpha(\tilde{I}^s) = \alpha(I^s) = 2s$  for all  $s$ . Here  $\alpha(c) = 2n+1$ , thus we have  $\alpha(\tilde{I}^{(n+1)}) = \alpha(I^{(n+1)}) = 2n+1$ . Since  $I^{(s)}$  is generated by  $I$  and  $I^{(n+1)}$ ,  $\tilde{I}^{(s)}$  is generated by  $\tilde{I}$  and  $\tilde{I}^{(n+1)}$  which implies  $\alpha(\tilde{I}^{(s)}) = \alpha(I^{(s)})$ . Therefore by [11, Theorem 3.6], we have  $\alpha(\tilde{I}^{(s)}) = 2s - \left\lfloor \frac{s}{n+1} \right\rfloor$ . Hence  $\widehat{\alpha}(\tilde{I}) = 2 - \frac{1}{n+1} = \frac{2n+1}{n+1}$ .

(4) Assume that at least one of the vertices of  $V(C_{2n+1})$  is in  $V^+$ . By (1), we have  $\tilde{I}^{(s)} = \tilde{I}^s + \sum_{t=1}^k \tilde{I}^{s-t(n+1)}(c)^t$ . Since  $G$  contains a unique odd cycle, there is an edge of odd cycle without any element of  $V^+$ . Thus  $\alpha(\tilde{I}^s) = \alpha(I^s) = 2s$  for all  $s$ . Since the cycle  $C_{2n+1}$  contains at least one element of  $V^+$ ,  $\alpha(c) \geq 2(n+1)$  and  $\alpha(c^t) \geq 2t(n+1)$ . Also  $\alpha(\tilde{I}^{s-t(n+1)}) = 2s - 2t(n+1)$  and  $\alpha(\tilde{I}^{s-t(n+1)}(c)^t) \geq 2s - 2t(n+1) + 2t(n+1) = 2s$  for  $1 \leq t \leq k$ . Since  $\alpha(\tilde{I}^s) = 2s$ ,  $\alpha(\tilde{I}^{(s)}) = \alpha(\tilde{I}^s) = 2s$ . Hence  $\widehat{\alpha}(\tilde{I}) = 2$ .

□

**Proposition 3.10.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented complete graph  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  on the vertices  $\{x_1, \dots, x_n\}$ . Then*

$$(1) \text{ For any } s \geq n, \text{ we have } \tilde{I}^{(s)} = \sum_{\substack{(r_1, \dots, r_{n-1}) \text{ and} \\ s=r_1+2r_2+\dots+(n-1)r_{n-1}}} \tilde{I}^{r_1} \tilde{I}^{(2)^{r_2}} \dots \tilde{I}^{(n-1)^{r_{n-1}}}.$$

$$(2) \text{ For any } s \in \mathbb{N}, \alpha(\tilde{I}^{(s)}) = s + \left\lceil \frac{s}{n-2} \right\rceil.$$

*In particular, the Waldschmidt constant of  $\tilde{I}$  is given by*

$$\widehat{\alpha}(\tilde{I}) = \frac{n-1}{n-2}.$$

*Proof.* Let  $I$  be the edge ideal of  $G$  and  $\Phi$  is same as defined in Notation 3.6.

(1) By [4, Theorem 4.2] for any  $s \geq n$ , we have

$$I^{(s)} = \sum_{\substack{(r_1, \dots, r_{n-1}) \text{ and} \\ s=r_1+2r_2+\dots+(n-1)r_{n-1}}} I^{r_1} I^{(2)^{r_2}} \dots I^{(n-1)^{r_{n-1}}}.$$

By Theorem 3.7, applying  $\Phi$  on both sides we get

$$\tilde{I}^{(s)} = \sum_{\substack{(r_1, \dots, r_{n-1}) \text{ and} \\ s=r_1+2r_2+\dots+(n-1)r_{n-1}}} \tilde{I}^{r_1} \tilde{I}^{(2)^{r_2}} \dots \tilde{I}^{(n-1)^{r_{n-1}}}.$$

(2) Since  $D$  is a weighted oriented complete graph and all the elements of  $V^+$  are sinks, there is only one element in  $V^+$ . Without loss of generality, we may assume that the only element in  $V^+$  is  $x_n$  where  $w_n := w(x_n)$ . For  $1 \leq s \leq n-1$  by the definition of  $I^{(s)}$ , no monomial of degree  $s$  is in  $I^{(s)}$ . Note that for  $1 \leq s \leq n-1$ ,  $x_1 x_2 \dots x_{s+1} \in I^{(s)}$ . Then by Theorem 3.7,  $\Phi(x_1 x_2 \dots x_{s+1}) = x_1 x_2 \dots x_{s+1} \in \tilde{I}^{(s)}$  for  $1 \leq s \leq n-2$  and  $\Phi(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n^{w_n} \in \tilde{I}^{(n-1)}$ . Since degree of any monomial can not be reduced under the map  $\Phi$ ,  $x_1 x_2 \dots x_{s+1}$  is one of the least degree generator in both  $I^{(s)}$  and  $\tilde{I}^{(s)}$  for  $1 \leq s \leq n-2$ . Hence  $\alpha(\tilde{I}^{(s)}) = \alpha(I^{(s)}) = s+1$  for  $1 \leq s \leq n-2$ . Since  $x_1 \dots x_n$  is the only minimal generator of degree  $n$  in  $I^{(n-1)}$ , there is no minimal generator of degree  $n$  in  $\tilde{I}^{(n-1)}$  as  $\deg(x_1 x_2 \dots x_n^{w_n}) \geq n+1$ . By the definition of  $I^{(s)}$ ,  $x_1^2 x_2^2 \dots x_{n-1} \in I^{(n-1)}$ . Then by Theorem 3.7,  $\Phi(x_1^2 x_2^2 \dots x_{n-1}) = x_1^2 x_2^2 \dots x_{n-1} \in \tilde{I}^{(n-1)}$ . Since  $\deg(x_1^2 x_2^2 \dots x_{n-1}) = n+1$ ,  $x_1^2 x_2^2 \dots x_{n-1}$  is one of the least degree generator of  $\tilde{I}^{(n-1)}$  and hence  $\alpha(\tilde{I}^{(n-1)}) = n+1$ . Here  $x_1^2 x_2^2 \dots x_{n-1} = (x_1 x_2 \dots x_{n-1})(x_1 x_2) \in \tilde{I}^{(n-1)}$  where  $x_1 x_2 \dots x_{n-1}$  and  $x_1 x_2$  are one of the least degree generators of  $\tilde{I}^{(n-2)}$  and  $\tilde{I}$  respectively. Therefore one of the least degree generator of  $\tilde{I}^{(n-1)}$  can be generated by the least degree generators of  $\tilde{I}^{(n-2)}$  and  $\tilde{I}$ .

By the above argument, one of the least degree generators of  $\tilde{I}^{(s)}$  can be generated

by the least degree generators of  $\tilde{I}$ ,  $\tilde{I}^{(2)}$ ,  $\dots$ ,  $\tilde{I}^{(n-2)}$ . Now from (1), by assuming  $r_{n-1} = 0$ , it follows that for  $s \geq n$ ,  $\alpha(\tilde{I}^{(s)}) = \min\{2r_1 + 3r_2 + \dots + (n-1)r_{n-2} \mid s = r_1 + 2r_2 + \dots + (n-2)r_{n-2}\}$ . Now  $2r_1 + 3r_2 + \dots + (n-1)r_{n-2} = s + r_1 + r_2 + \dots + r_{n-2}$ . Then it is equivalent to find the minimum of  $r = r_1 + r_2 + \dots + r_{n-2}$  with the condition  $s = r_1 + 2r_2 + \dots + (n-2)r_{n-2}$ . Write  $s = k(n-2) + p$  for some  $k \in \mathbb{Z}$  and  $0 \leq p \leq n-3$ . Here the minimum value of  $r$  will occur for maximum value of  $r_{n-2}$  and the maximum value of  $r_{n-2}$  is  $k$ . So the minimal generating degree term will come from  $\tilde{I}^{(n-2)^k} \tilde{I}^{(p)}$ . Thus

$$\alpha(\tilde{I}^{(s)}) = \begin{cases} k(n-1) + (p+1) & \text{if } p \neq 0, \\ k(n-1) & \text{if } p = 0 \end{cases}$$

$$\alpha(\tilde{I}^{(s)}) = \begin{cases} s + 1 + \left\lfloor \frac{s}{n-2} \right\rfloor & \text{if } p \neq 0, \\ s + \left\lfloor \frac{s}{n-2} \right\rfloor & \text{if } p = 0 \end{cases}$$

Therefore  $\alpha(\tilde{I}^{(s)}) = s + \left\lfloor \frac{s}{n-2} \right\rfloor$ . Also  $\frac{s}{n-2} \leq \left\lfloor \frac{s}{n-2} \right\rfloor \leq \frac{s}{n-2} + 1$ . Hence

$$\widehat{\alpha}(\tilde{I}) = \lim_{s \rightarrow \infty} \frac{\alpha(\tilde{I}^{(s)})}{s} = 1 + \frac{1}{n-2} = \frac{n-1}{n-2}.$$

□

**Remark 3.11.** In Corollaries 3.2 and 3.3, we proved the equality of ordinary and symbolic powers for weighted oriented unicyclic graphs and complete graphs under certain orientations and weights. If we change the orientation and weights, we may lose the equality of ordinary and symbolic powers as it was shown in Propositions 3.9 and 3.10.

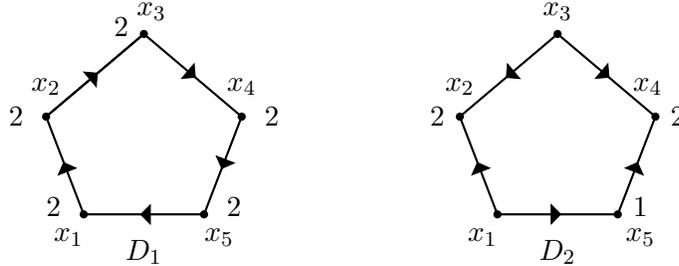


FIGURE 1. Two weighted oriented odd cycle's of length 5.

For example consider  $D_1$  and  $D_2$  be the weighted oriented odd cycles as in Figure 1. Then  $I(D_1) = (x_1x_2^2, x_2x_3^2, x_3x_4^2, x_4x_5^2, x_5x_1^2)$  and  $I(D_2) = (x_1x_2^2, x_2^2x_3, x_3x_4^2, x_4^2x_5, x_1x_5)$ . In  $D_1$ , edges are naturally oriented and  $w(x_i) = 2$  for each vertex  $x_i$ . In  $D_2$ , vertices of

$V^+(D_2) = \{x_2, x_4\}$  are sinks. By Corollary 3.2,  $I(D_1)^{(s)} = I(D_1)^s$  for all  $s \geq 2$  where as by Proposition 3.9,  $I(D_2)^{(s)} \neq I(D_2)^s$  for all  $s \geq 3$ .

**Proposition 3.12.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented graph  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  which is a clique sum of two odd cycles  $C_{2n+1} = (x_1, \dots, x_{2n+1})$  and  $C'_{2n+1} = (x_1, y_2, \dots, y_{2n+1})$  of same length with a common vertex  $x_1$ . Let  $c = (\prod v|v \in V(C_{2n+1}) \setminus V^+)(\prod v^{w(v)}|v \in V(C_{2n+1}) \cap V^+)$  and  $c' = (\prod v|v \in V(C'_{2n+1}) \setminus V^+)(\prod v^{w(v)}|v \in V(C'_{2n+1}) \cap V^+)$ .*

- (1) *Then  $\tilde{I}^{(s)} = \tilde{I}^s$  for  $1 \leq s \leq n$  and  $\tilde{I}^{(n+1)} = \tilde{I}^{n+1} + (c) + (c')$ .*
- (2) *Let  $s \in \mathbb{N}$  and write  $s = k(n+1) + r$  for some  $k \in \mathbb{Z}$  and  $0 \leq r \leq n$ . Then*

$$\tilde{I}^{(s)} = \sum_{p+q=t=0}^k \tilde{I}^{s-t(n+1)}(c)^p(c')^q.$$

- (3) *If only one of the two cycles does not contain any element of  $V^+$ , then for  $s \in \mathbb{N}$ ,  $\alpha(\tilde{I}^{(s)}) = 2s - \left\lfloor \frac{s}{n+1} \right\rfloor$ . In particular, the Waldschmidt constant of  $\tilde{I}$  is given by*

$$\widehat{\alpha}(\tilde{I}) = \frac{2n+1}{n+1}.$$

- (4) *If both cycles contain some element of  $V^+$ , then for  $s \in \mathbb{N}$ ,  $\alpha(\tilde{I}^{(s)}) = 2s$ . In particular, the Waldschmidt constant of  $\tilde{I}$  is given by  $\widehat{\alpha}(\tilde{I}) = 2$ .*

*Proof.* Let  $I$  be the edge ideal of  $G$  and  $\Phi$  is same as defined in Notation 3.6.

- (1) Since  $G$  is the clique sum of two odd cycles of same length with a common vertex, by [4, Corollary 3.15], we have  $I^{(k)} = I^k$  for  $1 \leq k \leq n$  and  $I^{(n+1)} = I^{n+1} + (x_1 \cdots x_{2n+1}) + (x_1 y_2 \cdots y_{2n+1})$ . Thus by Theorem 3.7,  $\Phi(I^{(k)}) = \Phi(I^k)$ , i.e.,  $\tilde{I}^{(k)} = \tilde{I}^k$  for  $1 \leq k \leq n$  and  $\Phi(I^{(n+1)}) = \Phi(I^{n+1}) + \Phi(x_1 \cdots x_{2n+1}) + \Phi(x_1 y_2 \cdots y_{2n+1})$ , i.e.,  $\tilde{I}^{(n+1)} = \tilde{I}^{n+1} + (c) + (c')$ .

- (2) Let  $s \in \mathbb{N}$  and write  $s = k(n+1) + r$  for some  $k \in \mathbb{Z}$  and  $0 \leq r \leq n$ . By [4, Theorem 3.16], we have  $I^{(s)} = \sum_{p+q=t=0}^k I^{s-t(n+1)}(x_1 \cdots x_{2n+1})^p(x_1 y_2 \cdots y_{2n+1})^q$ . Thus

$$\text{by Theorem 3.7, applying } \Phi \text{ on both sides we get } \tilde{I}^{(s)} = \sum_{p+q=t=0}^k \tilde{I}^{s-t(n+1)}(c)^p(c')^q.$$

- (3) Assume only one of the both cycles does not contain any element of  $V^+$ . Without loss of generality let  $C_{2n+1}$  be the one without any element of  $V^+$ . Thus by (1),  $\tilde{I}^{(n+1)} = \tilde{I}^{n+1} + (c) + (c') = \tilde{I}^{n+1} + (x_1 \cdots x_{2n+1}) + (c')$ . Since all the edges of cycle  $C_{2n+1}$  are without any element of  $V^+$ ,  $\alpha(\tilde{I}^s) = \alpha(I^s) = 2s$  for all  $s$ . Here  $\alpha(c) = 2n+1$  and  $\alpha(c') \geq 2n+2$ . Thus we have  $\alpha(\tilde{I}^{(n+1)}) = \alpha(I^{(n+1)}) = 2n+1$ . Since  $I^{(s)}$  is generated by  $I$  and  $I^{(n+1)}$ ,  $\tilde{I}^{(s)}$  is generated by  $\tilde{I}$  and  $\tilde{I}^{(n+1)}$  which implies

$\alpha(\tilde{I}^{(s)}) = \alpha(I^{(s)})$ . Therefore by [4, Theorem 3.12], we have  $\alpha(\tilde{I}^{(s)}) = 2s - \left\lfloor \frac{s}{n+1} \right\rfloor$ .

Hence  $\widehat{\alpha}(\tilde{I}) = 2 - \frac{1}{n+1} = \frac{2n+1}{n+1}$ .

- (4) Assume both cycles contain some element of  $V^+$ . By (1), we have  $\tilde{I}^{(s)} = \tilde{I}^s + \sum_{p+q=t=1}^k \tilde{I}^{s-t(n+1)}(c)^p(c')^q$ . Since  $G$  contains odd cycles, there exists an edge in each odd cycle without any element of  $V^+$ . Thus  $\alpha(\tilde{I}^s) = \alpha(I^s) = 2s$  for all  $s$ . Since each of the both cycles contain some element of  $V^+$ ,  $\alpha(c) \geq 2(n+1)$ ,  $\alpha(c') \geq 2(n+1)$ ,  $\alpha(c^p) \geq 2p(n+1)$  and  $\alpha(c'^q) \geq 2q(n+1)$ . Also  $\alpha(\tilde{I}^{s-t(n+1)}) = 2s - 2t(n+1)$  and  $\alpha(\tilde{I}^{s-t(n+1)}(c)^p(c')^q) \geq 2s - 2t(n+1) + 2p(n+1) + 2q(n+1) = 2s$  for  $1 \leq t \leq k$ . Since  $\alpha(\tilde{I}^s) = 2s$ ,  $\alpha(\tilde{I}^{(s)}) = \alpha(\tilde{I}^s) = 2s$ . Hence  $\widehat{\alpha}(\tilde{I}) = 2$ . □

#### 4. REGULARITY IN WEIGHTED ORIENTED ODD CYCLE

In this section, we focus on weighted oriented odd cycles where the elements of  $V^+$  are sinks and we set  $V^+ \neq \phi$ . We first show that  $\text{reg } \tilde{I}^{(s)} \leq \text{reg } \tilde{I}^s$  and then we explicitly compute  $\text{reg}(\tilde{I}^{(s)}/\tilde{I}^s)$ . As a Corollary, we conclude that the equality  $\text{reg } \tilde{I}^{(s)} = \text{reg } \tilde{I}^s$  occurs if  $V^+$  has only one element.

**Lemma 4.1.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented cycle  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  which is the cycle  $C_{2n+1} = (x_1, \dots, x_{2n+1})$ . Let  $w_i = w(x_i)$  where  $x_i \in V^+$ ,  $w_v := \max\{w_i \mid x_i \in V^+\}$  and  $w = \sum_{x_i \in V^+} (w_i - 1)$ . For  $s \in \mathbb{N}$ , let  $s = k(n+1) + r$  for some  $k \in \mathbb{Z}$  and  $0 \leq r \leq n$ . Then  $\mathfrak{m}^{(t-1)w_v+w}(c)^t \notin \tilde{I}^{t(n+1)}$  for  $1 \leq t \leq k$  where  $c = (\prod x_j \mid x_j \notin V^+)(\prod x_j^{w_j} \mid x_j \in V^+)$  and  $\mathfrak{m} = (x_1, \dots, x_{2n+1})$  is the maximal homogeneous ideal.*

*Proof.* Here  $c = (\prod x_j \mid x_j \notin V^+)(\prod x_j^{w_j} \mid x_j \in V^+)$  and  $x_v$  be an element of  $V^+$  with maximum weight. Here

$$\tilde{I} = (x_i x_j^{w_j} \text{ if } x_j \in V^+ \text{ or } x_i x_j \text{ if } x_i, x_j \notin V^+ \mid \{x_i, x_j\} \in E(C_{2n+1})).$$

Let  $f = \prod_{x_j \in V^+} x_j^{(w_j-1)}$ . Then  $\deg(f) = w$  and  $f$  is a minimal generator of  $\mathfrak{m}^w$ . Let  $g = x_v^{(t-1)w_v} f c^t$  for some  $t$  where  $1 \leq t \leq k$ . Then  $g \in \mathfrak{m}^{(t-1)w_v} \mathfrak{m}^w(c)^t = \mathfrak{m}^{(t-1)w_v+w}(c)^t$ . We want to prove that  $g \notin \tilde{I}^{t(n+1)}$ .

Note that  $c$  can be expressed as the product of  $n$  minimal generators of  $\tilde{I}$  and any  $x_i$  where  $x_i \notin V^+$  or the product of  $n$  minimal generators of  $\tilde{I}$  and any  $x_i^{w_i}$  where  $x_i \in V^+$ . So the product of  $c$  only with any  $x_j^{w_j}$  for some  $x_j \in V^+$  or  $x_j$  for some  $x_j \notin V^+$  is exactly the product of  $n+1$  minimal generators of  $\tilde{I}$ . Hence the product of  $c^t$  only with  $t$  number

of  $x_j^{w_j}$  for some  $x_j \in V^+$  or  $x_j$  for some  $x_j \notin V^+$  is exactly the product of  $t(n+1)$  minimal generators of  $\tilde{I}$ . Since  $f = \prod_{x_j \in V^+} x_j^{(w_j-1)}$  is neither a multiple of  $x_j^{w_j}$  for some  $x_j \in V^+$  nor  $x_j$  for some  $x_j \notin V^+$ ,  $x_v^{(t-1)w_v} f$  can not be a multiple of product of  $t$  number of  $x_j^{w_j}$  for some  $x_j \in V^+$  or  $x_j$  for some  $x_j \notin V^+$ . Note that  $x_v^{(t-1)w_v} f$  is a multiple of product of only  $t-1$  number of  $x_j^{w_j}$  for  $x_j = x_v \in V^+$ . Therefore  $x_v^{(t-1)w_v} f c^t$  is not a multiple of product of any  $t(n+1)$  minimal generators of  $\tilde{I}$ . So  $g = x_v^{(t-1)w_v} f c^t \notin \tilde{I}^{t(n+1)}$ . Hence  $\mathfrak{m}^{(t-1)w_v+w}(c)^t \not\subseteq \tilde{I}^{t(n+1)}$  for  $1 \leq t \leq k$ .  $\square$

**Lemma 4.2.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented cycle  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  which is the cycle  $C_{2n+1} = (x_1, \dots, x_{2n+1})$ . Let  $w_i = w(x_i)$  where  $x_i \in V^+$ ,  $w_v := \max\{w_i \mid x_i \in V^+\}$  and  $w = \sum_{x_i \in V^+} (w_i - 1)$ . For  $s \in \mathbb{N}$ , let  $s = k(n+1) + r$  for some  $k \in \mathbb{Z}$  and  $0 \leq r \leq n$ . Then  $\mathfrak{m}^{(k-1)w_v+w+1} \tilde{I}^s \subseteq \tilde{I}^s$  where  $\mathfrak{m} = (x_1, \dots, x_{2n+1})$  is the maximal homogeneous ideal.*

*Proof.* Let  $c = (\prod x_j \mid x_j \notin V^+) (\prod x_j^{w_j} \mid x_j \in V^+)$ . Let  $x_v$  be a vertex of  $V^+$  with maximum weight. Here

$$\tilde{I} = (x_i x_j^{w_j} \mid x_j \in V^+ \text{ or } x_i x_j \mid x_i, x_j \notin V^+ \mid \{x_i, x_j\} \in E(C_{2n+1})).$$

By Proposition 3.9, we have  $\tilde{I}^s = \tilde{I}^s + \sum_{t=1}^k \tilde{I}^{s-t(n+1)}(c)^t$ . Since  $w+1 = \sum_{x_i \in V^+} (w_i - 1) + 1$ , any element of  $\mathfrak{m}^{w+1}$  can be viewed as a multiple of  $x_j^{w_j}$  for some  $x_j \in V^+$  or a multiple of  $x_j$  for some  $x_j \notin V^+$ . Note that the product of  $c$  only with any  $x_j^{w_j}$  for some  $x_j \in V^+$  or  $x_j$  for some  $x_j \notin V^+$  is exactly the product of  $n+1$  minimal generators of  $\tilde{I}$ . Thus any element of  $\mathfrak{m}^{w+1}(c)$  can be represented as a multiple of product of  $n+1$  minimal generators of  $\tilde{I}$  and hence  $\mathfrak{m}^{w+1}(c) \subseteq \tilde{I}^{n+1}$ . First we want to show that  $\mathfrak{m}^{w_v} \mathfrak{m}^{w+1}(c) \subseteq \mathfrak{m}^{w+1} \tilde{I}^{n+1}$ .

Let  $t_1 \in \mathfrak{m}^{w_v} \mathfrak{m}^{w+1}(c)$  such that  $t_1 = t_2 t_3$  where  $t_2 \in \mathfrak{m}^{w_v}$  and  $t_3 \in \mathfrak{m}^{w+1}(c)$ . Since  $\mathfrak{m}^{w+1}(c) \subseteq \tilde{I}^{n+1}$ ,  $t_3$  is a multiple of product of  $n+1$  minimal generators of  $\tilde{I}$ . Let the product of those  $n+1$  minimal generators of  $\tilde{I}$  is  $t_4$ , i.e.,  $t_4 \in \tilde{I}^{n+1}$ . Then  $t_5 = t_3/t_4$  is a monomial of degree at least  $\sum_{x_i \in V^+ \setminus \{x_v\}} (w_i - 1)$ . Here  $t_2 t_5 \in \mathfrak{m}^{w+1}$  because  $w_v + \sum_{x_i \in V^+ \setminus \{x_v\}} (w_i - 1) = w+1$ . So  $t_1 = t_2 t_3 = t_2 t_4 t_5 = (t_2 t_5) t_4 \in \mathfrak{m}^{w+1} \tilde{I}^{n+1}$ . Thus  $\mathfrak{m}^{w_v} \mathfrak{m}^{w+1}(c) \subseteq \mathfrak{m}^{w+1} \tilde{I}^{n+1}$ . Hence

$$\begin{aligned} \mathfrak{m}^{(t-1)w_v+w+1}(c)^t &= \mathfrak{m}^{(t-1)w_v}(c)^{t-1} \mathfrak{m}^{w+1}(c) \\ &= \mathfrak{m}^{(t-2)w_v}(c)^{t-1} \mathfrak{m}^{w_v} \mathfrak{m}^{w+1}(c) \\ &\subseteq \mathfrak{m}^{(t-2)w_v}(c)^{t-1} \mathfrak{m}^{w+1} \tilde{I}^{n+1} \\ &= \mathfrak{m}^{(t-3)w_v}(c)^{t-2} \mathfrak{m}^{w_v} \mathfrak{m}^{w+1}(c) \tilde{I}^{n+1} \\ &\subseteq \mathfrak{m}^{(t-3)w_v}(c)^{t-2} \mathfrak{m}^{w+1} \tilde{I}^{2(n+1)} \end{aligned}$$

$$\begin{aligned}
 & \vdots \\
 & \subseteq \mathfrak{m}^{w+1}(c)\tilde{I}^{(t-1)(n+1)} \\
 & \subseteq \tilde{I}^{n+1}\tilde{I}^{(t-1)(n+1)} \\
 & = \tilde{I}^{t(n+1)}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \mathfrak{m}^{(k-1)w_v+w+1}\tilde{I}^{s-t(n+1)}(c)^t &= \mathfrak{m}^{(k-t)w_v}\tilde{I}^{s-t(n+1)}\mathfrak{m}^{(t-1)w_v+w+1}(c)^t \\
 &\subseteq \mathfrak{m}^{(k-t)w_v}\tilde{I}^{s-t(n+1)}\tilde{I}^{t(n+1)} \\
 &\subseteq \mathfrak{m}^{(k-t)w_v}\tilde{I}^s \\
 &\subseteq \tilde{I}^s \text{ for } 1 \leq t \leq k.
 \end{aligned}$$

Hence  $\mathfrak{m}^{(k-1)w_v+w+1}\tilde{I}^{(s)} = \mathfrak{m}^{(k-1)w_v+w+1}(\tilde{I}^s + \sum_{t=1}^k \tilde{I}^{s-t(n+1)}(c)^t) \subseteq \tilde{I}^s$ .  $\square$

**Theorem 4.3.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented cycle  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  which is the cycle  $C_{2n+1} = (x_1, \dots, x_{2n+1})$ . Let  $w_i = w(x_i)$  where  $x_i \in V^+$ ,  $w_v := \max\{w_i \mid x_i \in V^+\}$  and  $w = \sum_{x_i \in V^+} (w_i - 1)$ . For  $s \in \mathbb{N}$ , let  $s = k(n+1) + r$  for some  $k \in \mathbb{Z}$  and  $0 \leq r \leq n$ . Then  $\text{reg } \tilde{I}^{(s)} \leq \text{reg } \tilde{I}^s$ .*

*Proof.* Let  $\mathfrak{m} = (x_1, \dots, x_{2n+1})$  is the maximal homogeneous ideal. By Lemma 4.2, we have  $\mathfrak{m}^{(k-1)w_v+w+1}\tilde{I}^{(s)} \subseteq \tilde{I}^s$ . Thus  $\tilde{I}^{(s)}/\tilde{I}^s$  is an Artinian module. Therefore  $\dim(\tilde{I}^{(s)}/\tilde{I}^s) = 0$  and hence  $H_{\mathfrak{m}}^i(\tilde{I}^{(s)}/\tilde{I}^s) = 0$  for  $i > 0$ . Consider the following short exact sequence

$$0 \rightarrow \tilde{I}^{(s)}/\tilde{I}^s \rightarrow R/\tilde{I}^s \rightarrow R/\tilde{I}^{(s)} \rightarrow 0.$$

Applying local cohomology functor we get  $H_{\mathfrak{m}}^i(R/\tilde{I}^{(s)}) \cong H_{\mathfrak{m}}^i(R/\tilde{I}^s)$  for  $i \geq 1$  and the following short exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(\tilde{I}^{(s)}/\tilde{I}^s) \rightarrow H_{\mathfrak{m}}^0(R/\tilde{I}^s) \rightarrow H_{\mathfrak{m}}^0(R/\tilde{I}^{(s)}) \rightarrow 0. \quad (1)$$

Now by the exact sequence (1), we have  $a_0(R/\tilde{I}^{(s)}) \leq a_0(R/\tilde{I}^s)$ . Thus we can conclude that  $\text{reg}(R/\tilde{I}^{(s)}) = \max\{a_i(R/\tilde{I}^{(s)}) + i \mid i \geq 0\} \leq \max\{a_i(R/\tilde{I}^s) + i \mid i \geq 0\} = \text{reg}(R/\tilde{I}^s)$ . Hence  $\text{reg } \tilde{I}^{(s)} \leq \text{reg } \tilde{I}^s$ .  $\square$

**Theorem 4.4.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented cycle  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  which is the cycle  $C_{2n+1} = (x_1, \dots, x_{2n+1})$ . Let  $w_i = w(x_i)$  where  $x_i \in V^+$ ,  $w_v := \max\{w_i \mid x_i \in V^+\}$  and  $w = \sum_{x_i \in V^+} (w_i - 1)$ . For  $s \in \mathbb{N}$ , let  $s = k(n+1) + r$  for some  $k \in \mathbb{Z}$  and  $0 \leq r \leq n$ . Then  $\text{reg}(\tilde{I}^{(s)}/\tilde{I}^s) = (s - n - 1)(1 + w_v) + 2w + 2n + 1$  for  $s \geq n + 1$ .*

*Proof.* We fix  $s \geq n + 1$ . Let  $c = (\prod x_j | x_j \notin V^+) (\prod x_j^{w_j} | x_j \in V^+)$  and  $\mathfrak{m} = (x_1, \dots, x_{2n+1})$  be the maximal homogeneous ideal. Let  $x_v$  be an element of  $V^+$  with maximum weight. By Lemma 4.2, we have  $\mathfrak{m}^{(k-1)w_v+w+1} \tilde{I}^{(s)} \subseteq \tilde{I}^s$ . Thus  $\tilde{I}^{(s)}/\tilde{I}^s$  is an Artinian module. Therefore  $\dim(\tilde{I}^{(s)}/\tilde{I}^s) = 0$ ,  $H_{\mathfrak{m}}^0(\tilde{I}^{(s)}/\tilde{I}^s) = \tilde{I}^{(s)}/\tilde{I}^s$  and  $H_{\mathfrak{m}}^i(\tilde{I}^{(s)}/\tilde{I}^s) = 0$  for  $i > 0$ . So  $\text{reg}(\tilde{I}^{(s)}/\tilde{I}^s) = a_0(\tilde{I}^{(s)}/\tilde{I}^s)$ . We want to prove that one of the maximum degree element of  $I^{(s)}$  which does not lie in  $\tilde{I}^s$  will come from the ideal  $\tilde{I}^{s-(n+1)}(c)^1$ .

By Proposition 3.9, we have  $\tilde{I}^{(s)} = \tilde{I}^s + \sum_{t=1}^k \tilde{I}^{s-t(n+1)}(c)^t$ . Here  $\deg(c) = 2n + 1 + w$  and  $\deg(c^t) = t(2n + 1 + w)$ . Without loss of generality we can assume that  $x_v \neq x_{2n+1}$ . Then  $x_v^{w_v} x_{v+1}$  is a maximum degree minimal generator of  $\tilde{I}$  which implies that one maximum degree minimal generator of  $\tilde{I}^{s-t(n+1)}$  is  $(x_v^{w_v} x_{v+1})^{s-t(n+1)}$  and its degree is  $(s - t(n + 1))(w_v + 1)$ . From the proof of Lemma 4.2, we observe that  $\mathfrak{m}^{(t-1)w_v+w+1}(c)^t \subseteq \tilde{I}^{t(n+1)}$  for  $1 \leq t \leq k$ . Thus by Lemma 4.1, the maximum value of  $u$  such that  $\mathfrak{m}^u(c)^t \not\subseteq \tilde{I}^{t(n+1)}$  is  $(t - 1)w_v + w$  for  $1 \leq t \leq k$ . So the maximum degree of an element of the ideal  $(c)^t$  which does not lie in  $\tilde{I}^{t(n+1)}$  is  $(t - 1)w_v + w + t(2n + 1 + w)$  for  $1 \leq t \leq k$ . Hence the maximum degree of an element of  $\tilde{I}^{s-t(n+1)}(c)^t$  which does not lie in  $\tilde{I}^{s-t(n+1)}\tilde{I}^{t(n+1)} = \tilde{I}^s$  is  $(s - t(n + 1))(w_v + 1) + (t - 1)w_v + w + t(2n + 1 + w)$  for  $1 \leq t \leq k$ . Let  $d(t) = (s - t(n + 1))(w_v + 1) + (t - 1)w_v + w + t(2n + 1 + w)$  for  $1 \leq t \leq k$ . If  $t_1 < t_2$ , then

$$\begin{aligned} d(t_1) - d(t_2) &= (s - t_1(n + 1))(w_v + 1) + (t_1 - 1)w_v + w + t_1(2n + 1 + w) \\ &\quad - (s - t_2(n + 1))(w_v + 1) - (t_2 - 1)w_v - w - t_2(2n + 1 + w) \\ &= (t_2 - t_1)(n + 1)(w_v + 1) - (t_2 - t_1)w_v - (t_2 - t_1)(2n + 1 + w) \\ &= (t_2 - t_1)((n + 1)(w_v + 1) - w_v - (2n + 1 + w)) \\ &= (t_2 - t_1)(n(w_v + 1) - (2n + w)) \\ &= (t_2 - t_1)(n(w_v - 1) + 2n - 2n - w) \\ &= (t_2 - t_1)(n(w_v - 1) - w). \end{aligned}$$

Since  $|E(D)| = 2n + 1$ , there are at most  $n$  elements of  $V^+$ . So  $n(w_v - 1) \geq \sum_{x_i \in V^+} (w_i - 1) = w$  which implies that  $d(t_1) - d(t_2) \geq 0$ . Thus  $d(1) \geq d(2) \geq \dots \geq d(k)$ . Therefore one of the maximum degree elements of  $\tilde{I}^{(s)}$  which does not lie in  $\tilde{I}^s$  will come from  $\tilde{I}^{s-(n+1)}(c)^1$  for  $t = 1$  and its degree is  $d(1) = (s - n - 1)(w_v + 1) + w + (2n + 1 + w) = (s - n - 1)(w_v + 1) + 2w + 2n + 1$ . Hence  $\text{reg}(\tilde{I}^{(s)}/\tilde{I}^s) = a_0(\tilde{I}^{(s)}/\tilde{I}^s) = (s - n - 1)(w_v + 1) + 2w + 2n + 1$  for  $s \geq n + 1$ .  $\square$

**Corollary 4.5.** *Let  $\tilde{I}$  be the edge ideal of a weighted oriented cycle  $D$  where the elements of  $V^+$  are sinks and let  $G$  be the underlying graph of  $D$  which is the cycle  $C_{2n+1} = (x_1, \dots, x_{2n+1})$ . Let  $V^+ = \{x_v\}$  and  $w_v = w(x_v)$  for some  $x_v \in V(C_{2n+1})$ . For  $s \in \mathbb{N}$ , let  $s = k(n + 1) + r$  for some  $k \in \mathbb{Z}$  and  $0 \leq r \leq n$ . Then  $\text{reg} \tilde{I}^{(s)} = \text{reg} \tilde{I}^s$  for all  $s$ .*

*Proof.* Let  $I$  be the edge ideal of  $G$  and  $\Phi$  is same as defined in Notation 3.6. First we fix  $s \geq n + 1$ . Consider the following short exact sequence

$$0 \rightarrow \tilde{I}^{(s)}/\tilde{I}^s \rightarrow R/\tilde{I}^s \rightarrow R/\tilde{I}^{(s)} \rightarrow 0. \quad (2)$$

Without loss of generality we can assume that  $V^+ = \{x_1\}$ . Thus by Theorem 4.4, we have  $\text{reg}(\tilde{I}^{(s)}/\tilde{I}^s) = (s - n - 1)(w_1 + 1) + 2(w_1 - 1) + 2n + 1$ . Here  $x_2x_1 \in I$  and by definition of  $I^{(s)}$ , we have  $(x_2x_1)^s \in I^{(s)}$ . Observe that  $(x_2x_1)^s$  is a minimal generator of  $I^{(s)}$ . Then by Theorem 3.7,  $\Phi((x_2x_1)^s) = (x_2x_1^{w_1})^s \in \tilde{I}^{(s)}$  and  $(x_2x_1^{w_1})^s$  is a minimal generator of degree  $s(w_1 + 1)$  in  $\tilde{I}^{(s)}$ . Since there exist a minimal generator of degree  $s(w_1 + 1)$  in  $\tilde{I}^{(s)}$ , we have

$$\begin{aligned} \text{reg}(\tilde{I}^{(s)}) &\geq s(w_1 + 1) \\ &= (s - n - 1)(w_1 + 1) + (n + 1)(w_1 - 1 + 2) \\ &= (s - n - 1)(w_1 + 1) + (n + 1)(w_1 - 1) + 2(n + 1) \\ &> \text{reg}(\tilde{I}^{(s)}/\tilde{I}^s) \end{aligned}$$

as  $n \geq 1$ . So  $\text{reg}(R/\tilde{I}^{(s)}) \geq \text{reg}(\tilde{I}^{(s)}/\tilde{I}^s)$ . Thus by the exact sequence (2) and [12, Lemma 1.2], we have  $\text{reg}(R/\tilde{I}^s) \leq \text{reg}(R/\tilde{I}^{(s)})$ . Also by Theorem 4.3, we have  $\text{reg}(R/\tilde{I}^{(s)}) \leq \text{reg}(R/\tilde{I}^s)$ . Therefore  $\text{reg}(R/\tilde{I}^{(s)}) = \text{reg}(R/\tilde{I}^s)$ , i.e.,  $\text{reg} \tilde{I}^{(s)} = \text{reg} \tilde{I}^s$  for  $s \geq n + 1$ . By Proposition 3.9, we have  $\text{reg} \tilde{I}^{(s)} = \text{reg} \tilde{I}^s$  for  $1 \leq s \leq n$ . Hence  $\text{reg} \tilde{I}^{(s)} = \text{reg} \tilde{I}^s$  for all  $s$ .  $\square$

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## REFERENCES

- [1] S. K. Beyarslan, J. Biermann, K.N. Lin and A. O’Keefe, Algebraic invariants of weighted oriented graphs, arXiv:1910.11773.
- [2] C. Bocci, S. Cooper, E. Guardo, B. Harbourne, M. Janssen, U. Nagel, A. Seceleanu, A. Van Tuyl and T. Vu, The Waldschmidt constant for squarefree monomial ideals, *J. Algebraic Combin.* **44** (2016), no. 4, 875-904.
- [3] C. Bocci and B. Harbourne, Comparing powers and symbolic powers of ideals, *J. Algebraic Geom.* **19** (2010), no. 3, 399-417.
- [4] B. Chakraborty and M. Mandal, Invariants of the symbolic powers of edge ideals, *J. Alg. Appl.* **19** (2020), no. 10, 2050184.
- [5] S. Cooper, R. Embree, H. T. Hà and A. H. Hoefel, Symbolic powers of monomial ideals, *Proc. Edinb. Math. Soc.* (2) **60** (2017), no. 1, 39-55.

- [6] S. Cooper, G. Fatabbi, E. Guardo, A. Lorenzini, J. Migliore, U. Nagel, A. Seceleanu, J. Szpond and A. Van Tuyl, Symbolic Powers of Codimension two Cohen-Macaulay Ideals, *Comm. In Alg.* **48** (2020), no. 11, 4663–4680.
- [7] I. Gitler, C. Valencia and R. H. Villarreal, A note on Rees algebras and the *MFMC* property, *Beiträge Algebra Geom.* **48** (2007), no. 1, 141–150.
- [8] E. Guardo, B. Harbourne and A. Van Tuyl, Symbolic powers versus regular powers of ideals of general points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , *Canad. J. Math.* **65** (2013), no. 4, 823–842, <http://dx.doi.org/10.4153/CJM-2012-045-3>.
- [9] E. Guardo, B. Harbourne and A. Van Tuyl, Fat lines in  $\mathbb{P}^3$  : powers versus symbolic powers, *J. Algebra* **390** (2013), 221–230.
- [10] E. Guardo, B. Harbourne and A. Van Tuyl, Asymptotic resurgences for ideals of positive dimensional subschemes of projective space, *Adv. Math.* **246** (2013), 114–127.
- [11] Y. Gu, H. T. Hà, J. L. O’Rourke and J. W. Skelton, Symbolic powers of edge ideals of graphs, *Comm. Algebra* **48** (2020), no. 9, 3743–3760.
- [12] H. T. Hà, N. V. Trung and T. N. Trung, Depth and regularity of powers of sums of ideals, *Math. Z.* **282** (2016), no. 3-4, 819-838.
- [13] H. T. Hà, K.N. Lin, S. Morey, E. Reyes and R. H. Villarreal, Edge ideals of oriented graphs, *Internat. J. Algebra Comput.* **29** (2019), no. 3, 535-559.
- [14] M. Janssen, T. Kamp and J. Vander Woude, Comparing Powers of Edge Ideals, *J. Alg. Appl.* **18** (2019), no. 10, 1950184.
- [15] J. Martínez-Bernal, Y. Pitones and R. H. Villarreal, Minimum distance functions of graded ideals and Reed-Muller-type codes, *J. Pure Appl. Algebra* **221** (2017), no. 2, 251-275.
- [16] Y. Pitones, E. Reyes and J. Toledo, Monomial ideals of weighted oriented graphs, *Electron. J. Combin.* **26** (2019), no. 3.
- [17] A. Simis, W. Vasconcelos and R. H. Villarreal, On the ideal theory of graphs, *J. Algebra* **167** (1994), 389–416.
- [18] A. Van Tuyl, A beginner’s guide to edge and cover ideals, *Monomial ideals, computations and applications* 63-94, Lecture Notes in Math., 2083, Springer, Heidelberg, 2013.
- [19] O. Zariski and P. Samuel, *Commutative Algebra*, vol. II, Springer, 1960.
- [20] G. Zhu, L. Xu, H. Wang and Z. Tang, Projective dimension and regularity of edge ideal of some weighted oriented graphs, *Rocky MT J. Math.* **49** (2019), no. 4, 1391-1406.

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