NUMERICAL SEMIGROUPS OF SMALL AND LARGE TYPE

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ABSTRACT. A numerical semigroup is a sub-semigroup of the natural numbers that has a finite complement. Some of the key properties of a numerical semigroup are its Frobenius number F, genus g and type t. It is known that for any numerical semigroup $\frac{g}{F+1-g} \leq t \leq 2g - F$. Numerical semigroups with t = 2g - F are called almost symmetric, we introduce a new property that characterises them. We give an explicit characterisation of numerical semigroups with $t = \frac{g}{F+1-g}$. We show that for a fixed α the number of numerical semigroups with Frobenius number F and type $F - \alpha$ is eventually constant for large F. Also the number of numerical semigroups with genus g and type $g - \alpha$ is also eventually constant for large g.

1. INTRODUCTION

A numerical semigroup S is a subset of natural numbers that contains 0, is closed under addition and has a finite complement. The numbers in the complement are called gaps, and the number of gaps is called the genus of S, is denoted by g(S). The largest gap is called the Frobenius number, is denoted by F(S). The smallest nonzero element of S is called its multiplicity and is denoted by m(S).

A pseudo-Frobenius number of S is a gap x such that for any nonzero s in S, x + s is still in S. The collection of the pseudo-Frobenius numbers of S is denoted by PF(S), the number of pseudo-Frobenius numbers of S is called its type and is denoted by t(S).

For numerical semigroups in general it is known that

$$\frac{g(S)}{F(S) + 1 - g(S)} \le t(S) \le 2g(S) - F(S).$$

The first inequality was proved in [2], and the second in [5]. Numerical semigroups that satisfy t(S) = 2g(S) - F(S) are called almost symmetric, this notion was introduced in [6] and has been studied in several papers [5, 7, 8, 9, 10, 11, 12]. It is known that all irreducible numerical semigroups are almost symmetric.

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For a numerical semigroup S we define it's T-Set to be

$$T(S) = \{ x \in \mathbb{N} | F(S) - x \in ((\mathbb{Z} \setminus S) \cup \{0\}) \}.$$

Note that T(S) is a subset of natural numbers, it contains 0 and has a finite complement with respect to \mathbb{N} . However it is not necessarily closed under addition, such sets are called numerical sets. In fact it is closed under addition pricisely when S is almost symmetric, in Section 2 we prove that

Theorem 1.1. A numerical semigroup is almost symmetric if and only if its T-Set is itself a numerical semigroup.

If S is a numerical semigroup then $A^*(S) = S \cup PF(S)$ is another numerical semigroup. In [5] it is proven that

Theorem 1.2. [5, Theorem 3.7] Given an almost symmetric numerical semigroup S, $A^{\star}(S)$ is almost symmetric if and only if $t(A^{\star}(S)) =$ m(S) - t(S).

By using T-Sets we find another equivalent condition for this to happen, we prove in Corollary 4.8.3 that

Theorem 1.3. If S is an almost symmetric numerical semigroup then $A^{\star}(S)$ is almost symmetric if and only if $S \cup \{F(S)\}$ has max embedding dimension.

We also extend this result by applying the operation A^* multiple times. Note that $q(A^{\star}(S)) < q(S)$ so this operation can only be applied finitely many times. We define L(S) in Section 4, it measures how many times it can be applied. Let $m_i(S)$ be the i^{th} positive element and $F_i(S)$ be the i^{th} largest gap of S. We prove the following in Section 4.

Theorem 1.4. Given a numerical semigroup S, and k < L(S). $S, A^{\star}(S), \ldots, A^{\star k}(S)$ are all almost symmetric numerical semigroups if and only if

- $2i \in [0,k]$ implies $t(A^{\star 2i}(S)) = 2g(S) m_i(S) F_{i+1}(S)$ $2i + 1 \in [0,k]$ implies $t(A^{\star 2i+1}(S)) = F_{i+1}(S) 2g(S) + m_{i+1}(S)$

The set of all numerical semigroups can be arranged in a directed tree in which S' is the parent of S if $S' = S \cup \{F(S)\}$. This tree has been studied in many papers [16, 17, 18]. [16] studies the infinite chains of this tree and characterises them. The only numerical semigroups with F(S) < m(S) are those of the form $\{0, f+1 \rightarrow\}$, they form a chain and within this chain the type keeps increasing. Note that in a chain of the tree if we have a numerical semigroup with m(S) < F(S)

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then every numerical semigroup after that in the chain will also satisfy m(S) < F(S). We prove that if m(S) < F(S) then

$$t(S \cup \{F(S)\}) \ge t(S).$$

This implies that in any other chain the type of the numerical semigroup must eventually be constant. This is done in Section 4.

Theorem 1.5. Given a sequence S_i of numerical semigroups such that for each $i S_i = S_{i+1} \cup \{F(S_{i+1})\}$ and for at least one i we have $m(S_i) < F(S_i)$. Then their type $t(S_i)$ will eventually be constant.

We also consider the numerical semigroups with the smallest type given F, g, in Section 5 we classify them.

Theorem 1.6. A numerical semigroup S satisfies

$$t(S) = \frac{g(S)}{F(S) + 1 - g(S)}$$

if and only if either S is symmetric or it is of the following form for some m, n

$$S = \{0, m, 2m \dots nm \to \}.$$

Denote the number of numerical semigroups with Frobenius number F and type t by T(F,t). By $T_1(F,t)$ denote the number of almost symmetric numerical semigroups with Frobenius number F and type t. By L(g,t) the number of numerical semigroups with genus g and type t.

In [1] it was proved that for a fixed α , the function $T_1(F, F - 2\alpha)$ remains constant once $F > 4\alpha + 1$. This was done by finding a bijection between the set of almost symmetric numerical semigroups S with F(S) = F, $t(S) = F - 2\alpha$ and the set of numerical semigroups with genus g. Following a similar method we show in Theorem 3.2 and Theorem 3.4 that

Theorem 1.7. For a fixed α , the functions $T(F, F - \alpha)$ and $L(g, g - \alpha)$ are both eventually constant.

On the other hand for a fixed α the number of numerical semigroups of type α grow exponentially. By constructing explicit families of numerical semigroups we show in Theorems 6.2 and 6.3 that

Theorem 1.8. For a fixed positive integer α and $F > 3\alpha + 3$ such that $F \equiv \alpha \pmod{2}$

$$T_1(F,\alpha) \ge 2^{\frac{F}{6} - \frac{\alpha}{2}}$$

Also for a fixed positive integer α and $\epsilon > 0$, for sufficiently large F

$$T(F,\alpha) > 2^{(0.1-\epsilon)F}$$

2. The T-Set of a numerical semigroup

A numerical set is a subset of the natural numbers that contains 0 and has a finite complement. The genus (respectively Frobenius number) of a numerical set is defined to be the number of (respectively the largest) gaps. The multiplicity of a numerical set is it's smallest non-zero element. Every numerical set T has an associated numerical semigroup given by $A(T) = \{x \mid \forall y \in T, x + y \in T\}$. It is easily seen that $A(T) \subseteq T$ and they are equal if and only if T is a numerical semigroup.

Lemma 2.1. For any numerical semigroup S, F(T(S)) = F(S) - m(S)and g(T(S)) = F(S) - g(S).

Proof. Notice that the gaps of T(S) are F(S) - s for s in $S \cap [1, F(S)]$.

Theorem 2.2. Let S be a numerical semigroup, then

$$A(T(S)) = S \cup PF(S).$$

Proof. If x is not in $S \cup PF(S)$, then there is a $y \in S$, $y \neq 0$ such that $x + y \notin S$. Then $F(S) - x - y \in T(S)$, $F(S) - y \notin T(S)$ and hence $x \notin A(T(S))$.

Now for the other direction, if x is not in A(T(S)) then there is a $y \in T(S)$ such that $x + y \notin T(S)$. Then $F(S) - y \in ((\mathbb{Z} \setminus S) \cup \{0\})$, $F(S) - x - y \in (S \setminus \{0\})$. It follows that $x \notin S \cup PF(S)$. \Box

Corollary 2.2.1. t(S) = g(S) - g(A(T(S))).

Proof of Theorem 1.1. By Corollary 2.2.1 and Lemma 2.1 2g(S)-F(S)-t(S) = g(S)-F(S)+g(A(T(S))) = g(A(T(S)))-g(T(S)).Therefore, t(S) = 2g(S) - F(S) if and only if g(A(T(S))) = g(T(S))which happens if and only if T(S) is a numerical semigroup. \Box

For a numerical semigroup S, the quantity 2g(S) - F(S) - t(S)measures how far S is from being almost symmetric, this is equal to g(A(T(S))) - g(T(S)) which measures how far the *T*-set of S is from being a numerical semigroup.

Corollary 2.2.2. A numerical semigroup S is almost symmetric if and only if it satisfies the property that whenever $x, y \in Gap(S)$ and x + y > F(S) we have $x + y - F(S) \in Gap(S)$.

Proof. Consider $a, b \in T(S)$, such that a, b < F(S). Say a = F(S) - x, b = F(S) - y, so $x, y \in Gap(S)$. Now if $x + y \leq F(S)$ then $a + b = 2F(S) - x - y \geq F(S) > F(T(S))$ and hence $a + b \in T(S)$. On

the other hand if x + y > F(S) then $a + b \in T(S)$ if and only if $x + y - F(S) \in Gap(S)$.

3. Numerical Semigroups of type $F - \alpha$, $g - \alpha$

In this section we will show that for a fixed α the functions $T(F, F - \alpha)$ and $L(g, g - \alpha)$ are both eventually constant. We will do this by finding a bijection between numerical semigroups with Frobenius number F, type $F - \alpha$ and numerical sets T with $g(T) + g(A(T)) = \alpha$. And for $L(g, g - \alpha)$ we find a bijection between numerical semigroups with genus g, type $g - \alpha$ and numerical sets T with $g(A(T)) = \alpha$.

We use the notation that for a set X, $a - X = \{a - x \mid x \in X\}$.

Lemma 3.1. Given a numerical set T with F(T) = f and given an integer F larger than 2f. Consider

$$S = (\mathbb{N} \setminus (F - T)) \cup \{0\}$$

Then S is a numerical semigroup with Frobenius number F, T(S) = Tand type F - g(T) - g(A(T)).

Proof. It is clear that S has Frobenius number F. Next, we want to show that S is closed under addition. Consider x, y in S, assume that they are non-zero and x, y < F. Then F - x, F - y are gaps of T which implies $F - x, F - y \leq f$ and hence $x, y \geq F - f > \frac{F}{2}$. Therefore $x + y > F, x + y \in S$. Therefore S is a numerical semigroup.

It follows from the definition of T-Set that T(S) = T. Corollary 2.2.1 and Lemma 2.1 imply that

$$t(S) = g(S) - g(A(T)) = F - g(T) - g(A(T)).$$

Theorem 3.2. Given positive integers α , F such that $F > 4\alpha - 6$. There is a bijection between the set of numerical semigroups with Frobenius number F, type $F - \alpha$ and the set of numerical sets T for which $g(T) + g(A(T)) = \alpha$.

Proof. Given a numerical semigroup of Frobenius number F and type $F - \alpha$. We know by Corollary 2.2.1 and Lemma 2.1 that,

$$\alpha = F - t(S) = F - (g(S) - g(A(T(S)))) = g(T(S)) + g(A(T(S))).$$

Since the Frobenius number F is fixed, the map T(S) is injective.

Next, we need to show that it is surjective. Consider a numerical set T with $g(T) + g(A(T)) = \alpha$. Then $g(A(T)) \le \alpha - 1$ and hence

$$F(T) = F(A(T)) \le 2g(A(T)) - 1 \le 2\alpha - 3.$$

Therefore $2F(T) \leq 4\alpha - 6 < F$, and hence Lemma 3.1 gives a numerical semigroup S with Frobenius number F, type $F - \alpha$ and T(S) = T. It follows that the map is surjective and hence bijective.

Corollary 3.2.1. (Proved in [1]) Given positive integers F, β such that $F > 8\beta - 6$. There is a bijection between the set of almost symmetric numerical semigroups with Frobenius number F and type $F - 2\beta$ and the number of numerical semigroups with genus β .

Proof. Let $\alpha = 2\beta$, restrict the map to almost symmetric numerical semigroups and use the fact that S is almost symmetric if and only if T(S) is a numerical semigroup (Theorem 1.1).

Next we will obtain a similar bijection for the collection of numerical semigroups with genus g and type $g - \alpha$. But first we need a result from [13]. Given a numerical semigroup S they define

$$B(S) = \{ x \mid x \notin S, F(S) - x \notin S \}.$$

Notice that |B(S)| = 2g(S) - F(S) - 1.

Theorem 3.3. Given a numerical semigroup S, numerical set T such that A(T) = S, T must satisfy

$$S \subseteq T \subseteq S \cup B(S).$$

Proof. See [13].

Corollary 3.3.1. For any numerical set T

$$g(T) \ge F(T) - g(A(T)) + 1$$

Proof. Theorem 3.3 implies that

 $g(T) \ge g(A(T)) - |B(A(T))| = g(A(T)) - (2g(A(T)) - F(A(T)) - 1).$ Also F(T) = F(A(T)), so we are done.

Theorem 3.4. Given positive integers α , g such that $g \ge 3\alpha - 1$. There is a bijection between the set of numerical semigroups with genus g, type $g - \alpha$ and the set of numerical sets T for which $g(A(T)) = \alpha$.

Proof. Given a numerical semigroup S with genus g and type $g - \alpha$, we know by Corollary 2.2.1 that $g(A(T(S))) = \alpha$. We first show that this map is injective. Given two such S_1, S_2 , if $T(S_1) = T(S_2)$ by Lemma 2.1 we have

$$F(S_1) = g(S_1) + g(T(S_1)) = g(S_2) + g(T(S_2)) = F(S_2).$$

Since S_1, S_2 have the same Frobenius number and T-Sets it follows that $S_1 = S_2$.

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Next we show that the map is surjective. Consider a numerical set T such that $g(A(T)) = \alpha$. Let F = g + g(T). We know that

$$F(T) = F(A(T)) \le 2g(A(T)) - 1 = 2\alpha - 1.$$

Therefore by Corollary 3.3.1

$$F \ge 3\alpha - 1 + g(T) \ge 3\alpha - 1 + F(T) - \alpha + 1 = F(T) + 2\alpha > 2F(T).$$

Now Lemma 3.1 gives us a numerical semigroup S with Frobenius number F and T(S) = T. Finally lemma 2.1 implies that g(S) = F - g(T) = g and Corollary 2.2.1 implies that $t(S) = g - g(A(T)) = g - \alpha$. Therefore, the map is surjective and hence bijective.

4. The type and T-Set of a numerical set

In this section we will define the *T*-Set for a numerical set and study it's properties proving several interesting results like Theorem 1.3, Theorem 1.4 and Theorem 1.5 on the way. One of the ways of obtaining a numerical semigroup from a numerical set is the associated semigroup A(T). There is another useful way of obtaining a numerical semigroup from a numerical set. Given a numerical set $H \neq \mathbb{N}$ we define

$$A^{\star}(H) = \{a \mid a + (H \setminus \{0\}) \subseteq H\}$$

We make a note of some observations, given a numerical set $H \neq \mathbb{N}$

- $A^{\star}(H)$ is a numerical semigroup
- $A(H) = A^{\star}(H) \cap H$
- If H is itself a numerical semigroup then $A^*(H) = H \cup PF(H)$

We now define the pseudo-Frobenius numbers and type for numerical sets analogously to numerical semigroups. Our definition of type of a numerical set is different from the one given in [14]. For a numerical set H we define

$$PF(H) = \{a \in \mathbb{N} \mid a \notin H, a + (H \setminus \{0\}) \subseteq H\}$$

and the type of H is t(H) = |PF(H)|. It is clear that $F(H) \in PF(H)$, so $t(H) \ge 1$. Also

$$A^{\star}(H) = PF(H) \dot{\cup} A(H).$$

Lemma 4.1. For any numerical set H, $PF(H) \subseteq PF(A(H))$ and therefore $t(H) \leq t(A(H))$.

Proof. Given $P \in PF(H)$, we know that $P \notin H$ therefore $P \notin A(H)$. If $x \in A(H)$, $x \neq 0$ then we want to show that $P + x \in A(H)$. Pick $h \in H$, if $h \neq 0$ then $P + h \in H$ and hence $P + h + x \in H$. On the other hand if h = 0, then $P + x \in H$ as $x \in H \setminus \{0\}$. Therefore $P + x \in A(H)$, which in turn implies $P \in PF(A(H))$.

Definition 4.2. Let $H \neq \mathbb{N}$ be a numerical set with Frobenius number F, then it's T-Set is defined to be the following numerical set

 $T(H) = \{ x \in \mathbb{N} | F - x \in ((\mathbb{Z} \setminus H) \cup \{0\}) \}.$

We now find the analogues of the results from Section 2.

Lemma 4.3. If $H \neq \mathbb{N}$ is a numerical set then F(T(H)) = F(H) - m(H), g(T(H)) = F(H) - g(H).

Proof. Note that the gaps of T(H) are precisely F(H) - x for x in $H \cap [1, F(H)]$.

Proposition 4.4. For a numerical set $H \neq \mathbb{N}$, $A(T(H)) = A^{\star}(H)$.

Proof. Pick a natural number a. We know that a is not in A(T(H)) if and only if there is a $x \in T(H)$ such that $x + a \notin T(H)$. This means there is an x such that $F(H) - x \in (\mathbb{Z} \setminus H) \cup \{0\}$ and $F(H) - x - a \in$ $H \setminus \{0\}$. This is equivalent to saying that there is a $y \in H \setminus \{0\}$ such that $y + a \notin H$ i.e. $a \notin A^*(H)$. It follows that $A(T(H)) = A^*(H)$. \Box

Corollary 4.4.1. t(H) = g(A(H))) - g(A(T(H))).

Theorem 4.5. Given a numerical set $H \neq \mathbb{N}$, we have

 $t(H) \le g(H) + g(A(H)) - F(H).$

Moreover equality holds if and only if T(H) is a numerical semigroup.

Proof. We know by Lemma 4.3 and Corollary 4.4.1 that

$$g(A(T(H))) - g(T(H)) = g(A(H)) - t(H) - (F(H) - g(H))$$
$$= g(A(H)) + g(H) - F(H) - t(H).$$

This quantity must be non-negative and it is zero if and only if T(H) is a numerical semigroup.

A numerical set H is called ordinary if m(H) = F(H) + 1 i.e. $H = \{0, F(H) + 1 \rightarrow\}$. Note that $T(H) = \mathbb{N}$ if and only if H is ordinary. Since we have defined the T-Set for an arbitrary numerical set, we can apply in multiple times.

Theorem 4.6. For a non-ordinary numerical set H,

$$t(T(H)) + t(H) \le g(A(H)) - g(H) + m(H).$$

Moreover, equality holds if and only if T(T(S)) is a numerical semigroup.

Proof. We apply Theorem 4.5,

$$g(T(H)) + g(A(T(H))) - F(T(H))$$

= F(H) - g(H) + g(A(H)) - t(H) - (F(H) - m(H))
= g(A(H)) - g(H) + m(H) - t(H).

Corollary 4.6.1. If S is a non-ordinary numerical semigroup then T(T(S)) is a numerical semigroup if and only if t(S)+t(T(S)) = m(H).

Corollary 4.6.2 (From [5]). If S is an almost symmetric numerical semigroup then $A^*(S)$ is almost symmetric if and only if $t(S) + t(A^*(S)) = m(S)$.

The transformation obtained by applying the T-Set twice has a simpler description as follows.

Lemma 4.7. For a non-ordinary numerical set H,

 $T(T(H)) = \{x - m(H) \mid x \in H \cup \{F(H)\}, x \neq 0\}$

Proof. We know that T(T(H)) consists of elements of the form F(T(H)) - x for $x \in Gap(T(H))$ along with all natural numbers $\geq F(T(H))$. Also we know that F(T(H)) = F(H) - m(H) and the gaps of T(H) are F(H) - y for y in $T \cap [1, F(H)]$. The result follows. \Box

Lemma 4.7 implies that the gaps of T(T(H)) are x - m(H) for x in $(Gap(H) \setminus \{F(H)\}) \cap [m(H) + 1, F(H)]$. And therefore

$$g(T(T(H))) = g(H) - m(H).$$

Proposition 4.8. $A(T(T(H))) = A^{*}(H \cup \{F(H)\}).$

Proof. Pick a natural number a. By Lemma 4.7 $a \in A(T(T(H)))$ if and only if $x \in H \cup \{F(H)\} \setminus \{0\}$ implies $a + x \in H \cup \{F(H)\} \setminus \{0\}$. Which happens if and only if $a \in A^*(H \cup \{F(H)\})$. \Box

Corollary 4.8.1. If S is a non-ordinary numerical semigroup then

$$t(S \cup \{F(S)\}) = t(S) + t(T(S)) - 1 \ge t(S).$$

Proof. By applying Corollary 4.4.1 twice we see that a(A(T(T(H)))) = a(A(T(H))) = t(T(H)) = a(A(H)) = t(T(H))

$$g(A(I(I(H)))) = g(A(I(H))) - t(I(H)) = g(A(H)) - t(H) - t(I(H))$$

We now use the above proposition.

$$t(S \cup \{F(S)\}) = g(S \cup \{F(S)\}) - g(A^{\star}(S \cup \{F(S)\}))$$

= $g(S) - 1 - g(A(T(T(S)))) = g(S) - 1 - (g(S) - t(S) - t(T(S))).$

Corollary 4.8.2. If S is a non-ordinary numerical semigroup then T(T(S)) is a numerical semigroup if and only if $S \cup \{F(S)\}$ has max embedding dimension.

Proof. It is a known fact that a numerical semigroup has max embedding dimension if and only if its type is one less than the multiplicity. Sis not ordinary so $m(S \cup \{F(S)\}) = m(S)$. It follows that $S \cup \{F(S)\}$ is max embedding dimension if and only if m(S) = t(S) + t(T(S)). By Corollary 4.6.1 this happens if and only if T(T(S)) is a numerical semigroup.

Corollary 4.8.3. If S is an almost symmetric numerical semigroup then $A^*(S)$ is almost symmetric if and only if $S \cup \{F(S)\}$ has max embedding dimension.

Proof of Theorem 1.5. If S_n is not ordinary it follows that none of the S_i are ordinary for $i \ge n$. Therefore by Corollary 4.8.1 the sequence $t(S_{n+i})$ is non increasing and positive which implies that it must be eventually constant.

So far we have applied the *T*-Set operation twice, we now look at how many times it can be applied to a given numerical set. Given a numerical set *H*, define $l(H) = |H \cap [1, g(H)]|$. Moreover, if $g(H) \in H$ then define L(H) = 2l(H). And if $g(H) \notin H$ then define L(H) =2l(H) + 1. Note that L(H) = 0 if and only if $H = \mathbb{N}$, so T(H) is defined if and only if $L(H) \geq 1$.

Lemma 4.9. Given a numerical set $H \neq \mathbb{N}$, L(T(H)) = L(H) - 1.

Proof. Firstly notice that

 $|Gap(H) \cap [g(H) + 1, F(H)]| = g(H) - |Gap(H) \cap [1, g(H)]| = l(H).$

Therefore

$$l(T(H)) = |T(H) \cap [1, F(H) - g(H)]| = |Gap(H) \cap [g(H), F(H) - 1)|$$

= $l(H) - 1 + |Gap(H) \cap \{g(H)\}|$

Now if $g(H) \in H$ then $g(T(H)) = F(H) - g(H) \notin T(H)$. In this case l(T(H)) = l(H) - 1 and hence L(T(H)) = L(H) - 1.

On the other hand if $g(H) \notin H$ then $g(T(H)) = F(H) - g(H) \in T(H)$. In this case l(T(H)) = l(H) and L(T(H)) = L(H) - 1. \Box

It follows that given a numerical set H, we can construct

$$H, T(H), T^2(H), \dots, T^{L(H)}(H) = \mathbb{N}.$$

Now define $m_i(H)$ to be the i^{th} positive element of H. Let $m_0(H) = 0$. For $i \leq g(H)$ let $F_i(H)$ be the i^{th} largest gap of H.

Proposition 4.10. Given a numerical set H and positive integer i, such that $2i \leq L(H)$. We have

$$T^{2i}(H) = \{x - m_i(H) \mid x \ge m_i(H), x \in H \cup \{F_1(H), F_2(H), \dots, F_i(H)\}\}.$$

Proof. We induct on i, the base case i = 1 is already known from Lemma 4.7. Now assume this holds for some i, and $2i + 2 \leq L(H)$. We write m_j for $m_j(H)$ and F_j for $F_j(H)$.

$$T^{2(i+1)}(H) = T^{2i}(T^{2}(H))$$

= {x - m_i(T²(H)) | x ≥ m_i(T²(H)), x ∈ T²(H) ∪ {F_j(T²(H)) | 1 ≤ j ≤ i}}
= {x - (m_{i+1} - m) | x ≥ m_{i+1} - m, x + m ∈ H ∪ {F(H)} ∪ {F_j | 2 ≤ j ≤ i + 1}}
= {y - m_{i+1} | y ≥ m_{i+1}, y ∈ H ∪ {F₁, F₂, ..., F_{i+1}}}.

We therefore see that if $2i \leq L(H)$ then $F(T^{2i}(H)) = F_{i+1} - m_i$, $g(T^{2i}(H)) = g(H) - m_i$ and $m(T^{2i}(H)) = m_{i+1} - m_i$. And hence if $2i + 1 \leq L(H)$ then $F(T^{2i+1}(H)) = F_{i+1} - m_{i+1}$, $g(T^{2i+1}(H)) = F_{i+1} - g(H)$ and $m(T^{2i+1}(H)) = F_{i+1} - F_{i+2}$.

Proposition 4.11. Given k < L(H)

$$\sum_{i=0}^{k} t(T^{i}(H)) = g(A(H)) - g(A(T^{k+1}(H))).$$

In particular

$$g(A(H)) = \sum_{i=0}^{L(H)-1} t(T^{i}(H)).$$

Proof. This follows from the fact that $t(T^i(H)) = g(A(T^i(H))) - g(A(T^{i+1}(H)))$.

Corollary 4.11.1. If $2i \leq L(H)$ then

$$\sum_{j=0}^{2i-1} t(T^{j}(H)) \le g(A(H)) - g(H) + m_{i}(H).$$

Moreover equality holds if and only if $T^{2i}(H)$ is a numerical semigroup.

Proof. We use the fact that $g(A(T^{2i}(H))) \ge g(T^{2i}(H)) = g(H) - m_i(H)$. And equality holds if and only if $T^{2i}(H)$ is a numerical semigroup.

Corollary 4.11.2. *If* $2i + 1 \le L(H)$ *then*

$$\sum_{j=0}^{2i} t(T^{j}(H)) \le g(A(H)) + g(H) - F_{i+1}(H).$$

Moreover equality holds if and only if $T^{2i+1}(H)$ is a numerical semigroup.

Proof. This is similar,
$$g(T^{2i+1}(H)) = F_{i+1}(H) - g(H)$$
.

Proof of Theorem 1.4. If they are all almost symmetric then for each $j \in [0, k] A^{\star j}(S) = T^j(S)$, and hence $T(A^{\star j}(S)) = T^{j+1}(S)$ is a numerical semigroup. The result follows from Corollary 4.11.1 and Corollary 4.11.2.

Conversely if the $t(A^{\star j}(S))$ are given by these expressions then $t(S) = 2g(S) - m_0(S) - F_1(S) = 2g(S) - F(S)$. Therefore S is almost symmetric which implies that $T(S) = A^{\star}(S)$. Continuing by induction we see that for each $j \in [0, k]$ $T^j(S) = A^{\star j}(S)$ and it is almost symmetric. \Box

For a non-ordinary numerical set H, it's ordinarisation transform is $O(H) = H \cup \{F(H)\} \setminus \{m(H)\}$. This was defined in [15]. They also defined l(S) for a numerical semigroup S and called it the ordinarization number of S.

Proposition 4.12. For $2 \le 2i \le L(H)$, $A(T^{2i}(H)) = A^*(O^{i-1}(H \cup \{F(H)\})).$

Moreover for $0 \leq 2i < L(H)$,

$$A(T^{2i+1}(H)) = A^*(O^i(H)).$$

Proof. Proposition 4.10 implies that $T(T^{2i}(H)) = T(O^i(H))$. Therefore we have $A(T^{2i+1}(H)) = A^*(O^i(H))$. $A(T^{2i}(H))$ is computed directly using Proposition 4.10 analogously to Proposition 4.8.

Corollary 4.12.1. If S is a numerical semigroup then

$$t(O^{i}(S)) = \sum_{j=0}^{2i} t(T^{j}(S)) \le 2g(S) - F_{i+1}(S).$$

Equality holds if and only if $T^{2i+1}(S)$ is a numerical semigroup which happens if and only if $O^i(S)$ is almost symmetric.

Proof. We know that

$$t(O^{i}(S)) = g(O^{i}(S)) - g(A^{\star}(O^{i}(S))) = g(S) - g(A(T^{2i+1}(S))) = \sum_{j=0}^{2i} t(T^{j}(S))$$

The result follows from Corollary 4.11.2 and the fact that $F(O^i(S)) = F_{i+1}(S)$.

5. Numerical Semigroups of small type

We define n(S) to be the size of $S \cap [0, F(S)]$, i.e. n(S) = F(S) + 1 - g(S). In their Theorem 20 the authors of [2] prove that

$$\frac{g(S)}{n(S)} \le t(S)$$

In this section we state a slight variation of their proof and then classify the numerical semigroups for which equality holds. Given a numerical semigroup S define a function ϕ_S from Gap(S) to S,

$$\phi_S(x) = Max\{s \in S | x + s \in Gap(S)\}.$$

Lemma 5.1. For any numerical semigroup S and $x \in Gap(S)$

- (1) $0 \le \phi_S(x) < F(S)$
- (2) $\phi_S(x) = 0$ if and only if $x \in PF(S)$
- (3) $x + \phi_S(x) \in PF(S)$

Proof. The first part is clear, the second is essentially a restatement of the definition of a pseudo-Frobenius number. For the third part, given $s \in S$ with $s \neq 0$, we know that $\phi_S(x) + s$ is in S and it is bigger than $\phi_S(x)$. Therefore, the maximality of $\phi_S(x)$ implies that $x + \phi_S(x) + s$ is in S. Hence, $x + \phi_S(x)$ is a pseudo-Frobenius number of S. \Box

Theorem 5.2 (From [2]). Given a numerical semigroup S,

 $g(S) \le n(S)t(S).$

Proof. Consider the map ψ_S from Gap(S) to $(S \cap [0, F]) \times PF(S)$, $\psi_S(x) = (\phi(x), x + \phi(x))$. It is clear that ψ_S is injective and hence $g(S) \leq n(S)t(S)$.

Lemma 5.3. Given a numerical semigroup S with Frobenius number F and multiplicity m, [F - m + 1, F] is a subset of $S \cup PF(S)$.

Proof. By Theorem 2.2 we know that $S \cup PF(S) = A(T(S))$. Moreover F(A(T(S))) = F(T(S)) = F - m by Lemma 2.1. The result follows.

Proof of Theorem 1.6. First, if S is symmetric then t(S) = 1, g(S) = n(S) so the equation is satisfied. Next, for $S = \{0, m, 2m \dots nm \rightarrow\}$, F(S) = nm-1, n(S) = n and g(S) = F(S)+1-n(S) = n(m-1). Also, PF(S) = [(n-1)m+1, nm-1] so t(S) = m-1 and g(S) = n(S)t(S).

Now, assume that S is a numerical semigroup that satisfies g(S) = n(S)t(S), is not symmetric. Therefore, $t(S) \ge 2$ and $m(S) \ge t(S) + 1$.

Now, g(S) = n(S)t(S) implies that the map ψ_S from proof of Theorem 5.2 must be surjective. Therefore, for any $s \in S \cap [1, F(S)]$ and $P \in PF(S)$, s < P. Moreover, the map λ_S from $(S \cap [0, F]) \times PF(S)$ to Gap(S), $\lambda_S(s, P) = P - s$ is injective as it is the inverse of ψ_S .

By Lemma 5.3 the interval [F - (m - 1), F] consists entirely of elements of S and its pseudo-Frobenius numbers. It has m consecutive numbers and hence contains a multiple of m which will be in S. This tells us that PF(S) = [F - (t - 1), F] and $[F - (m - 1), F - t] \subseteq S$. If $m \ge t + 2$ then F - (t + 1) will be in S. But then

$$\lambda_S(F - (t+1), F - (t-1)) = 2 = \lambda_S(F - t, F - (t-2))$$

which contradicts the fact that λ_S is injective. Therefore, m = t + 1.

Now since every number in the interval [F - (m - 2), F] is a gap (in fact a pseudo-Frobenius number) none of them can be a multiple of m. Therefore, $F \equiv -1 \pmod{m}$ say F = nm - 1.

Finally if any $s \in S \cap [0, F]$ is not a multiple of m, then say $s + r \equiv -1 \pmod{m}$ with $0 \leq r \leq m-2$. Then $F - r \equiv s \pmod{m}$ and $F - r \geq F - (m-2) > s$ i.e. F - r = s + km for some $k \geq 1$, but this implies that $F - r \in S$ which is a contradiction. Therefore, $S \cap [0, F]$ consists entirely of multiples of m and hence $S = \{0, m, 2m \dots nm \rightarrow\}$. \Box

6. Numerical semigroups of type α

In this section we will give lower bounds for number of numerical semigroups of given type and Frobenius number. We will do so by constructing explicit families of numerical semigroups. Numerical semigroups of type 1 always have an odd Frobenius number. The authors of [2] showed that for odd F the number of type 1 numerical semigroups grows exponentially and is at least $2^{\lfloor \frac{F}{8} \rfloor}$. In [3] the following theorem is proved.

Theorem 6.1. For $i \in \{1, 3, 5\}$ the following limit exist and are positive

$$\lim_{F \equiv i \pmod{6}, F \to \infty} \frac{T(F, 1)}{2^{\frac{F}{6}}}.$$

Proof. See [3].

We will now look at type at least 2 and start with constructing a family of almost symmetric numerical semigroups of a given type.

Theorem 6.2. Assume F > 6k + 6, let A be subset of $\left(\frac{F}{3}, \lfloor \frac{F-1}{2} \rfloor - k\right)$

$$B = \left\{ x \mid \left\lceil \frac{F+1}{2} \right\rceil + k < x < F, F - x \notin A \right\}$$

$$S = \{0\} \cup A \cup B \cup \{F+1 \rightarrow\}$$

Then S is an almost symmetric numerical semigroup with Frobenius number F. Moreover if F is odd then t(S) = 2k + 1 and if F is even then t(S) = 2k + 2.

Proof. We first show that S is a numerical semigroup for which we need to check that S is closed under addition. Consider $x, y \in S$, assume that $x, y \neq 0$ and $x + y \leq F$ as otherwise we have nothing to prove. Also without loss of generality assume $x \leq y$, which implies that $x < \frac{F}{2}$ and hence $x \in A$. We know that $x + y > \frac{2F}{3} > \left\lceil \frac{F+1}{2} \right\rceil + k$, which implies that $F - (x + y) < \frac{F}{3}$ and hence $F - (x + y) \notin A$. We know x + y cannot equal F, so x + y < F. It follows that $x + y \in B$. Now, given a number x such that $\lfloor \frac{F-1}{2} \rfloor - k \leq x \leq \lfloor \frac{F+1}{2} \rceil + k$,

Now, given a number x such that $\lfloor \frac{F-1}{2} \rfloor - k \leq x \leq \lfloor \frac{F+1}{2} \rfloor + k$, we firstly know that x is not in S. We will show that x is a pseudo-Frobenius number of S. Consider for a non-zero s in S, assume that $x + s \leq F$ as otherwise we have nothing to prove. Note that x + s cannot be F, so x + s < F. Moreover,

$$x + s > \frac{F}{3} + \left\lfloor \frac{F - 1}{2} \right\rfloor - k \ge \frac{F}{3} + \frac{F}{2} - 1 - k > \frac{5F}{6} - \frac{F}{6} = \frac{2F}{3}$$

So, $F - (x + s) < \frac{F}{3}$ and $F - (x + s) \notin A$. Also it can be checked that F > 6k + 6 implies that $\left\lceil \frac{F+1}{2} \right\rceil + k < \frac{2F}{3}$. Therefore, $\left\lceil \frac{F+1}{2} \right\rceil + k < x+s < F$ and hence $x+s \in B$. This shows that x is a pseudo-Frobenius number of S.

Next, for any other x in [1, F - 1], either $x \in S$ or $F - x \in S$, either way x cannot be a pseudo-Frobenius number. Therefore

$$PF(S) = \left[\left\lfloor \frac{F-1}{2} \right\rfloor - k, \left\lceil \frac{F+1}{2} \right\rceil + k \right] \cup \{0\}$$

It is now clear that if F is odd then t(S) = 2k + 1 and if F is even then t(S) = 2k + 2.

Corollary 6.2.1. For a fixed positive integer α and $F > 3\alpha + 3$ such that $F \equiv \alpha \pmod{2}$

$$T_1(F,\alpha) \ge 2^{\frac{F}{6} - \frac{\alpha}{2}}$$

Theorem 6.3. Pick an irrational β such that $\frac{2}{5} < \beta < \frac{1}{2}$. And for $F > Max\left(\frac{k+1}{5\beta-2}, \frac{k}{1-2\beta}\right)$, pick a subset A of $(\beta F, \frac{F}{2})$. Let

$$B = \left\{ x \mid \frac{F}{2} < x < (1 - \beta)F, F - x \notin A \right\},\$$

 $S = \{0\} \cup A \cup B \cup [\lceil (1-\beta)F \rceil, \lfloor 2\beta F \rfloor - k] \cup [\lceil 2\beta F \rceil, F-1] \cup \{F+1 \rightarrow\}.$ Then S is a numerical semigroup of Frobenius number F. If F is odd then t(S) = k + 1 and if F is even then t(S) = k + 2.

Proof. We show that S is closed under addition, consider $x, y \in S$, assume that $x, y \neq 0$ and $x + y \leq F$ as otherwise we have nothing to prove. Clearly $x + y \neq F$, so x + y < F. We know that $x, y > \beta F$ and therefore $x + y > 2\beta F$ which means that $x + y \in \lceil 2\beta F \rceil, F - 1 \rceil$.

Now for $0 \leq i \leq k-1$, $\lfloor 2\beta F \rfloor - i \notin S$. Moreover for any non-zero $s \in S$, $s > \beta F$ and hence $\lfloor 2\beta F \rfloor - i + s > 2\beta F - k + \beta F > F$. This means that $\lfloor 2\beta F \rfloor - i$ is a pseudo-Frobenius number of S.

Let $n = F - (\lfloor 2\beta F \rfloor - i)$ and let $s = \lfloor 2\beta F \rfloor - n = 2\lfloor 2\beta F \rfloor - F - i$. This means that $s > (4\beta - 1)F - (k - 1) - 2 > (1 - \beta)F$. Also $s < (4\beta - 1)F < 2\beta - k$, which means $s \in [\lceil (1 - \beta)F \rceil, \lfloor 2\beta F \rfloor - k]$. So we have $s \in S, s \neq 0$ and $n + s = \lfloor 2\beta F \rfloor \notin S$. Therefore, n is not a pseudo-Frobenius number of S.

If F is even, then for any non-zero $s \in S$, $s > \beta F$ so $\frac{F}{2} + s > (\frac{1}{2} + \beta)F > 2\beta F$. Clearly $\frac{F}{2} + s \neq F$, so $\frac{F}{2} + s \in S$. Therefore $\frac{F}{2}$ is a pseudo-Frobenius number of S.

Finally for any other x in [1, F-1] either $x \in S$ or $F-x \in S$, which means that x cannot be a pseudo-Frobenius number of S. It follows that if F is odd then type of S is k+1 and if F is even then type of Sis k+2.

Corollary 6.3.1. For a fixed positive integer α and $\epsilon > 0$, for sufficiently large F

$$T(F,\alpha) > 2^{(0.1-\epsilon)F}.$$

7. Some further observations and conjectures

In this section we describe several observations about counting numerical semigroups of of a given type which are based on numerical evidence. We start by looking at numerical semigroups of a given Frobenius number and looking at how many of them have odd type, how many have even type. Table 1 has these values for $F \leq 30$, based on this we make the following conjectures.

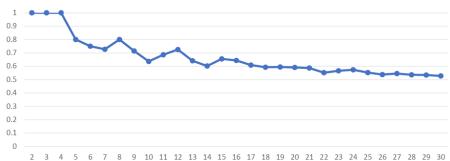
Conjecture 7.1. For every F

$$\#\{S \mid F(S) = F, t(S) \equiv F(mod \ 2)\} > \#\{S \mid F(S) = F, t(S) \not\equiv F(mod \ 2)\}\$$

Conjecture 7.2.

$$\lim_{F \to \infty} \frac{\#\{S \mid F(S) = F, t(S) \equiv F(mod \ 2)\}}{\#\{S \mid F(S) = F\}} = \frac{1}{2}.$$

The data in Table 1 has been used to plot the ratio of numerical semigroups in which $t(S) \equiv F(mod \ 2)$ among all numerical semigroups with Frobenius number F.



Frobenius number

F	Odd t	Even t	F	Odd t	Even t	F	Odd t	Even t
2	0	1	12	11	29	22	857	1056
3	2	0	13	68	38	23	2320	1776
4	0	2	14	41	62	24	1524	2054
5	4	1	15	131	69	25	4573	3700
6	1	3	16	73	132	26	3779	4396
7	8	3	17	283	182	27	8803	7329
8	2	8	18	165	240	28	7547	8720
9	15	6	19	571	390	29	18656	16247
10	8	14	20	368	532	30	15023	16799
11	35	16	21	1073	755			

Ratio of NS with t,F of same parity

TABLE 1. Number of numerical semigroups with odd, even type. Counted by Frobenius number.

Further we conjecture the following about the growth of $T(F, \alpha)$.

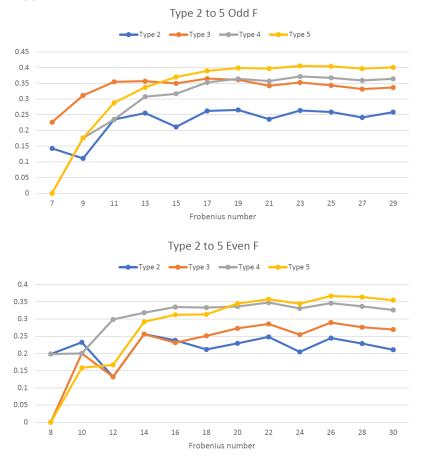
Conjecture 7.3. For any $\alpha \geq 2$ and $r \in \{0,1\}$ the following limits exist

$$\lim_{F \equiv r \pmod{2}, F \to \infty} \frac{\log_2(T(F, \alpha))}{F}$$

When r = 0 we denote the conjectured limit by a_{α} and when r = 1we denote it by b_{α} . Corollary 6.3.1 implies that a_{α}, b_{α} are at least 0.1. Moreover Corollary 6.2.1 implies that a_{α} for even α and b_{α} for odd α are at least $\frac{1}{6}$.

Conjecture 7.4. The two sequences a_{α} and b_{α} are both strictly increasing sequences.

We plot $\frac{\log_2(T(F,\alpha))}{F}$ for even and odd F and $\alpha = 2, 3, 4, 5$. The limits do appear to exist.



Similar patterns are observed when we count by genus. However this time numerical semigroups with odd type seem to outnumber the numerical semigroups with even type. Table 2 verifies Conjecture 7.5 for $g \leq 20$.

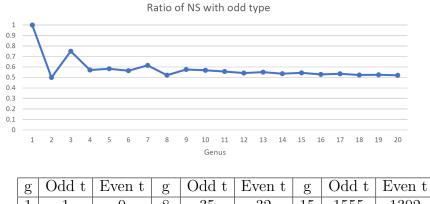
Conjecture 7.5. For every g we have

 $\#\{S \mid g(S) = g, t(S) \equiv 1 \pmod{2}\} \ge \#\{S \mid g(S) = g, t(S) \equiv 0 \pmod{2}\}.$

Conjecture 7.6.

$$\lim_{g \to \infty} \frac{\#\{S \mid g(S) = g, t(S) \equiv 1 \pmod{2}\}}{\#\{S \mid g(S) = g\}} = \frac{1}{2}$$

We plot the ratio of numerical semigroups with odd type among numerical semigroups with genus g based on Table 2.



18	Ouu v	LIVOII U	8	Ouu i	L'UUII U	6	Ouu i	L'VOII 0
1	1	0	8	35	32	15	1555	1302
2	1	1	9	68	50	16	2548	2258
3	3	1	10	116	88	17	4307	3738
4	4	3	11	191	152	18	7060	6407
5	7	5	12	321	271	19	11804	10660
6	13	10	13	552	449	20	19464	17932
7	24	15	14	908	785			

TABLE 2. Number of numerical semigroups with odd, even type. Counted by genus.

Conjecture 7.7. For any α the following limits exist

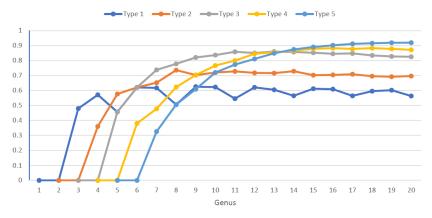
$$\lim_{g \to \infty} \frac{\log_{\phi}(L(g, \alpha))}{g}$$

Here ϕ is the golden ratio, we denote the limit by c_{α} .

Conjecture 7.8. The sequence c_{α} is monotonically increasing.

We plot $\frac{\log_{\phi}(L(g,\alpha))}{g}$ for $\alpha = 1, 2, 3, 4, 5$. The limits do appear to exist.

Type 1 to 5 by Genus



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