On the density of Cayley graphs of R. Thompson's group F in symmetric generators

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Abstract

By the density of a finite graph we mean its average vertex degree. For an m-generated group, the density of its Cayley graph in a given set of generators, is the supremum of densities taken over all its finite subgraphs. It is known that a group with m generators is amenable iff the density of the corresponding Cayley graph equals 2m.

A famous problem on the amenability of R. Thompson's group F is still open. What is known due to the result by Belk and Brown, is that the density of its Cayley graph in the standard set of group generators $\{x_0, x_1\}$, is at least 3.5. This estimate has not been exceeded so far.

For the set of symmetric generators $S = \{x_1, \bar{x}_1\}$, where $\bar{x}_1 = x_1 x_0^{-1}$, the same example gave the estimate only 3. There was a conjecture that for this generating set the equality holds. If so, F would be non-amenable, and the symmetric generating set had doubling property. This means that for any finite set $X \subset F$, the inequality $|S^{\pm 1}X| \geq 2|X|$ holds.

In this paper we disprove this conjecture showing that the density of the Cayley graph of F in symmetric generators S strictly exceeds 3. Moreover, we show that even larger generating set $S_0 = \{x_0, x_1, \bar{x}_1\}$ does not have doubling property.

Introduction

Some introductory information here repeats the one of [20].

 $^{^*}$ This work is partially supported by the Russian Foundation for Basic Research, project no. 19-01-00591 A.

The Richard Thompson group F can be defined by the following infinite group presentation

$$\langle x_0, x_1, x_2, \dots \mid x_j x_i = x_i x_{j+1} \ (i < j) \ \rangle. \tag{1}$$

This group was found by Richard J. Thompson in the 60s. We refer to the survey [8] for details. (See also [5, 6, 7].) It is easy to see that for any $n \ge 2$, one has $x_n = x_0^{-(n-1)} x_1 x_0^{n-1}$ so the group is generated by x_0 , x_1 . It can be given by the following presentation with two defining relations

$$\langle x_0, x_1 \mid x_1^{x_0^2} = x_1^{x_0 x_1}, x_1^{x_0^3} = x_1^{x_0^2 x_1} \rangle,$$
 (2)

where $a^b = b^{-1}ab$ by definition. Also we define a commutator $[a, b] = a^{-1}a^b = a^{-1}b^{-1}ab$ and notation $a \leftrightarrow b$ whenever a commutes with b, that is, ab = ba.

Each element of F can be uniquely represented by a *normal form*, that is, an expression of the form

$$x_{i_1}x_{i_2}\cdots x_{i_s}x_{i_t}^{-1}\cdots x_{i_2}^{-1}x_{i_1}^{-1},$$
 (3)

where $s, t \geq 0$, $0 \leq i_1 \leq i_2 \leq \cdots \leq i_s$, $0 \leq j_1 \leq j_2 \leq \cdots \leq j_t$ and the following is true: if (3) contains both x_i and x_i^{-1} for some $i \geq 0$, then it also contains x_{i+1} or x_{i+1}^{-1} (in particular, $i_s \neq j_t$).

An equivalent definition of F can be given in the following way. Let us consider all strictly increasing continuous piecewise-linear functions from the closed unit interval onto itself. Take only those of them that are differentiable except at finitely many dyadic rational numbers and such that all slopes (derivatives) are integer powers of 2. These functions form a group under composition. This group is isomorphic to F. Another useful representation of F by piecewise-linear functions can be obtained if we replace [0,1] by $[0,\infty)$ in the previous definition and impose the restriction that near infinity all functions have the form $t\mapsto t+c$, where c is an integer.

The group F has no free subgroups of rank > 1. It is known that F is not elementary amenable (EA). However, the famous problem about amenability of F is still open. If F is amenable, then it is an example of a finitely presented amenable group, which is not EA. If it is not amenable, then this gives an example of a finitely presented group, which is not amenable and has no free subgroups of rank > 1. Note that the first example of a non-amenable group without free non-abelian subgroups has been constructed by Ol'shanskii [23]. (The question about such groups was formulated in [10], it is also often attributed to von Neumann [22].) Adian [1] proved that free Burnside groups with m > 1 generators of odd exponent $n \ge 665$ are not amenable. The first example of a finitely presented non-amenable group without free non-abelian subgroups has been recently constructed by Ol'shanskii and Sapir [24]. Grigorchuk [15] constructed the first example of a finitely presented amenable group not in EA.

It is not hard to see that F has an automorphism given by $x_0 \mapsto x_0^{-1}$, $x_1 \mapsto x_1 x_0^{-1}$. To check that, one needs to show that both defining relators of F in (2) map to the identity. This is an easy calculations using normal forms. After that, we have an endomorphism

of F. Aplying it once more, we have the identity map. So this is an automorphism of order 2.

Notice that F has no non-Abelian homomorphic images [8]. So in order to check that an endomorphism of F is a monomorphism, it suffices to show that the image of the commutator $[x_0, x_1] = x_0^{-1} x_1^{-1} x_0 x_1 = x_2^{-1} x_1 = x_1 x_3^{-1}$ is nontrivial.

Later we will add more arguments to the importance of the symmetric set $S = \{x_1, \bar{x}_1 = x_1x_0^{-1}\}$. Obviously, it also generates F. It is easy to apply Tietze transormation to get a presentation of F in the new generating set from (2). So we let $\alpha = x_1^{-1}$, $\beta = \bar{x}_1^{-1} = x_0x_1^{-1}$. It follows that $x_0 = \beta\alpha^{-1}$. The first defining relation of (2) says that $x_1^{x_0} \leftrightarrow x_1x_0^{-1}$ so $\alpha^{\beta^{\alpha^{-1}}} \leftrightarrow \beta$. Therefore, $\alpha^{\beta} \leftrightarrow \beta^{\alpha}$. From this relation we can derive $x_1^{x_0^2} = x_1^{x_0x_1}$ in the opposite direction.

Now the second defining relation of (2) means that $x_1^{x_0^2} \leftrightarrow x_1 x_0^{-1}$, that is. $\alpha^{\beta\alpha^{-1}\beta\alpha^{-1}} \leftrightarrow \beta$. Conjugating by α , we get $\alpha^{\beta\alpha^{-1}\beta} \leftrightarrow \beta^{\alpha}$. Conjugation by α once more implies that $\alpha^{\beta\alpha^{-1}\beta\alpha} \leftrightarrow \beta^{\alpha^2}$. Since α^{β} commutes with $\beta^{\alpha} = \alpha^{-1}\beta\alpha$, we conclude that the left-hand side is α^{β} so we get the relation $\alpha^{\beta} \leftrightarrow \beta^{\alpha^2}$. Clearly, from this relation we can derive $x_1^{x_0^3} = x_1^{x_0^2x_1}$. Therefore, by standard Tietze transformations we obtain the following presentation of F in terms of symmetric generating set:

$$\langle \alpha, \beta \mid \alpha^{\beta} \leftrightarrow \beta^{\alpha}, \alpha^{\beta} \leftrightarrow \beta^{\alpha^{2}} \rangle.$$
 (4)

Of course, from the symmetry reasons we know that $\beta^{\alpha} \leftrightarrow \alpha^{\beta^2}$ also holds in F. Therefore, it is a consequence of the two relations of (4). Moreover, one can check that for any positive integers m, n it holds $\alpha^{\beta^m} \leftrightarrow \beta^{\alpha^n}$ as a consequence of the defining relations.

1 Density

By the *density* of a finite graph Γ we mean the average value of the degree of a vertex in Γ . More precisely, let v_1, \ldots, v_k be all vertices of Γ . Let $\deg_{\Gamma}(v)$ denote the degree of a vertex v in the graph Γ , that is, the number of oriented edges of Γ that come out of v. Then

$$\delta(\Gamma) = \frac{\deg_{\Gamma}(v_1) + \dots + \deg_{\Gamma}(v_k)}{k} \tag{5}$$

is the density of Γ .

Let G be a group generated by a finite set A. Let C(G,A) be the corresponding (right) Cayley graph. Recall that the set of vertices of this graph is G and the set of edges is $G \times A^{\pm 1}$. For an edge e = (g, a), its initial vertex is g, its terminal vertex is ga, and the inverse edge is $e^{-1} = (ga, a^{-1})$. The *label* of e equals a by definition. For the Cayley graph C = C(G, A) we define the number

$$\bar{\delta}(C) = \sup_{\Gamma} \delta(\Gamma), \tag{6}$$

where Γ runs over all finite subgraphs of C = C(G, A). So this number is the least upper bound of densities of all finite subgraphs of C. If C is finite, then it is obvious that $\delta(C) = \bar{\delta}(C)$. So we may call $\bar{\delta}(C)$ the density of the Cayley graph C.

This concept was used in [2] to study densities of the Cayley graphs of F.

Recall that a group G is called *amenable* whenever there exists a finitely additive normalized invariant mean on G, that is, a mapping $\mu: \mathcal{P}(G) \to [0,1]$ such that $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subseteq G$, $\mu(G) = 1$, and $\mu(Ag) = \mu(gA) = \mu(A)$ for any $A \subseteq G$, $g \in G$. One gets an equivalent definition of amenability if only one-sided invariance of the mean is assumed, say, the condition $\mu(Ag) = \mu(A)$ ($A \subseteq G$, $g \in G$). The proof can be found in [13].

The class of amenable groups includes all finite groups and all abelian groups. It is invariant under taking subgroups, quotient groups, group extensions, and ascending unions of groups. The closure of the class of finite and abelian groups under these operations is the class EA of elementary amenable groups. A free group of rank > 1 is not amenable. There are many useful criteria for (non)amenability [12, 21, 14]. We need to mention the two properties of a finitely generated group G that are equivalent to non-amenability.

 $\mathbf{NA_1}$. If G is generated by m elements and C is the corresponding Cayley graph, then the density of C does not have the maximum value, that is, $\bar{\delta}(C) < 2m$.

Note that if NA_1 holds for at least one finite generating set, then the group is not amenable and so the same property holds for any finite generating set. For the proof of this property, we need to use the well-known $F \emptyset lner\ condition\ [12]$. For our reasons it is convenient to formulate this condition as follows.

Let C be the Cayley graph of a group. By dist(u, v) we denote the distance between two vertices in C, that is, the length of a shortest path in C that connects vertices u, v. For any vertex v and a number r let $B_r(v)$ denote the ball of radius r around v, that is, the set of all vertices in C at distance $\leq r$ from v. For any set Y of vertices, by $B_r(Y)$ we denote the r-neighbourhood of Y, that is, the union of all balls $B_r(v)$, where v runs over Y. By ∂Y we denote the (outer) boundary of Y, that is, the set $B_1(Y) \setminus Y$. The Følner condition (for the case of a finitely generated group) says that G is amenable whenever inf $\#\partial Y/\#Y=0$, where the infimum is taken over all non-empty finite subsets of G for a Cayley graph of G in finite number of generators (this property does not depend on the choice of a finite generating set). Any finite set Y of vertices in C defines a finite subgraph (also denoted by Y). The degree of any vertex v in C equals 2m, where m is the number of generators. We know that exactly $\deg_Y(v)$ of the 2m edges that come out of v, connect the vertex v to a vertex from Y. The other $2m - \deg_{V}(v)$ edges connect v to a vertex from ∂Y . Note that each vertex of ∂Y is connected by an edge to at least one vertex in Y. This implies that the cardinality of ∂Y does not exceed the sum $\sum (2m - \deg_Y(v))$ over all vertices of Y. Dividing by #Y (the number of vertices in Y) implies the inequality $\#\partial Y/\#Y \leq 2m - \delta(Y)$. If $\bar{\delta}(C) = 2m$, then Y can be chosen such that $\delta(Y)$ is arbitrarily close to 2m so $\#\partial Y/\#Y$ will be arbitrarily close to 0. On the other hand, for any vertex v in Y there are at most 2m edges that connect v to a vertex in Y. Therefore, the sum

 $\sum (2m - \deg_Y(v))$ does not exceed $2m \# \partial Y$. So $2m - \delta(Y) \leq 2m \# \partial Y / \# Y$. If the right hand side can be made arbitrarily close to 0, then $\delta(Y)$ approaches 2m so $\bar{\delta}(C) = 2m$.

NA₂. If C is the Cayley graph of G in a finite set of generators, then there exists a function $\phi: G \to G$ such that a) for all $g \in G$ the distance $\operatorname{dist}(g, \phi(g))$ is bounded from above by a constant K > 0, b) any element $g \in G$ has at least two preimages under ϕ .

An elegant proof of this criterion based on the Hall – Rado theorem can be found in [9], see also [11]. Note that this property also does not depend on the choice of a finite generating set. A function ϕ from NA₂ will be called a *doubling function* on the Cayley graph C.

We need a definition. Suppose that NA₂ holds for the Cayley graph of a group G for the case K=1. Then we say that the Cayley graph C is strongly non-amenable. The function $\phi\colon G\to G$ will be called a strong doubling function on the Cayley graph C. Note that each vertex is either invariant under ϕ or it maps into a neighbour vertex. We know that NA₂ holds if and only if the group is not amenable, that is, $\bar{\delta}(C)<2m$. Now we would like to find out what happens if the Cayley graph of a 2-generated group is strongly non-amenable.

The following fact was proved in [20].

Theorem. The Cayley graph of a group with two generators is strongly non-amenable if and only if the density of this graph does not exceed 3.

It is also convenient to use the concept of Cheeger boundary $\partial^* Y$ of a finite subgraph in the Cayley graph of a group regarded as a set of vertices, as above. It consists of all directed edges that start at a vertex in Y and end at a vertex outside Y. Clearly, the density of Y as a subgraph equals $2m\#Y - \#\partial^*Y$.

We have to mention that the density of a Cayley graph of a group is closely related to an *isoperimetric constant* ι_* of a graph defined as $\#\partial^*Y/\#Y$; see also [9]). Namely, one has the equality $\iota_*(C) + \bar{\delta}(C) = 2m$ for the Cayley graph C of an m-generated group.

The above Theorem applied to the Cayley graph \mathcal{C} of F in any two generators $(x_0, x_1, \text{ or } \alpha, \beta)$ means that if we cannot find a subgraph in with density greater than 3, then there exists a strong doubling function on \mathcal{C} . One can imagine this doubling function in the following way. Suppose that a bug lives in each vertex of \mathcal{C} . We allow these bugs to jump at the same time such that each bug either returns to its initial position or it jumps to a neighbour vertex. As a result, we must have at least two bugs in each vertex.

It is natural to ask how much the value of $\delta(Y)$ can be for the finite subgraphs we are able to construct. In [20] it was constructed a family of finite subgraphs with density approaching 3. In the Addendum yo the same paper, there was a modification of the above construction showing that there are subgraphs with density strictly greater than 3. A much stronger result was obtained in [4]. This was a family of finite subgraphs with density approaching 3.5. We will describe this example in the next Section. Before that, we present a technical lemma.

First of all, we regard finite subgraphs in Cayley graphs of groups as automata, that is, labelled oriented graphs. Let v be a vertex and let a be a group generator or its inverse.

We say that the automaton accepts a whenever it has an edge labelled by a starting at v. If the automaton does not accept a, then the edge labelled by a starting at v in the Cayley graph, belongs to the Cheeger boundary. We claim that the number of such edges labelled by a is the same that the number of edges labelled by a^{-1} .

Lemma 1 Let G be a finitely generated group and let C = C(G, A) be its Cayley graph. Let Y be a nonempty finite subgraph of C. Then for any $a \in A^{\pm 1}$ the number of edges in the Cheeger boundary $\partial^* Y$ labelled by a is the same as the number of edges in $\partial^* Y$ labelled by a^{-1} .

Proof. We establish a natural bijection between edges of both types. Let e be an edge labelled by a in ∂Y . Its starting vertex v belongs to Y. Let $v_0 = v$, and for any $n \geq 0$ let v_{n+1} be the starting point of an edge in \mathcal{C} labelled by a whose terminal point is v_n . If a has an infinite order in G, then all vertices of the form v_n ($n \geq 0$) differ from each other. In this case, since Y is finite, there is the smallest n > 0 such that v_n does not belong to Y. So v_{n-1} belongs to Y, and the automaton dots not accept the egde from v_{n-1} to v_n with label a^{-1} . This edge f will correspond to e.

Suppose that a has finite order in G. Then there is a loop in \mathcal{C} at v labelled by a power of a. This loop has vertices outside Y. So, as in the previous paragraph, we can choose the smallest n with the same property. In this case we also let $e \mapsto f$, as above.

It is clear that we have a bijection between edges in $\partial^* Y$ labelled by a and a^{-1} . The inverse mapping $f \mapsto e$ is the same as above if we replace a in the beginning by a^{-1} .

The proof is complete.

To find the density of a subgraph, we will need to know the number of edges in its Cheeger boundary. If we found this number for a generator a, then we automatically know the number of edges for a^{-1} due to the above Lemma.

2 The Brown – Belk Construction

Let us recall the concept of a rooted binary tree. Formally, the definition of a rooted binary tree can be done be induction.

- 1) A dot . is a rooted binary tree.
- 2) If T_1 , T_2 are rooted binary trees, then $(T_1 \hat{T}_2)$ is a rooted binary tree.
- 3) All rooted binary trees are constructed by the above rules.

Instead of formal expressions, we will use their formal realizations. A dot will be regarded as a point. It coincides with the root of that tree. If $T = (T_1 \hat{\ } T_2)$, then we draw a caret for $\hat{\ }$ as a union of two closed intervals AB (goes left down) and AC (goes right down). The point A is the roof of T. After that, we draw trees for T_1 , T_2 and attach their roots to B, C respectively in such a way that they have no intersections. It is standard that for any $n \geq 0$ the number of rooted binary trees with n carets is equal to the nth Catalan number $c_n = \frac{(2n)!}{n!(n+1)!}$.

Each rooted binary trees has *leaves*. Formally they are defined as follows: for the one-vertex tree (which is called *trivial*) the only leaf coincides with the root. In case

 $T = (T_1 \hat{T}_2)$, the set of leaves equals the union of the sets of leaves for T_1 and T_2 . In this case the leaves are exactly vertices of degree 1.

We also need the concept of a *height* of a rooted binary tree. For the trivial tree, its height equals 0. For $T = (T_1 \hat{T}_2)$, its height is ht $T = \max(\operatorname{ht} T_1, \operatorname{ht} T_2) + 1$.

Now we define a rooted binary forest as a finite sequence of rooted binary trees T_1 , ..., T_m , where $m \geq 1$. The leaves of it are the leaves of the trees. It is standard from combinatorics that the number of rooted binary forests with n leaves also equals c_n . The trees are enumerated from left to right and they are drawn in the same way.

A marked (rooted binary) forest if the the above forest where one of the trees is marked.

Let $n \geq 1$, $k \geq 0$ be integer parameters. By BB(n,k) we denote the set of marked forests that have n leaves, and each tree has height at most k. The group F has a *left* partial action on this set. Namely, x_0 acts by shifting the marker left if this is possible. The action of x_1 is as follows. If the marked tree is trivial, this is not applied. If the marked tree is $T = (T_1 \hat{\ } T_2)$, then we remove its caret and mark the tree T_1 . It is easy to see that applying $\bar{x}_1 = x_1 x_0^{-1}$ means the same replacing T_1 by T_2 for the marked tree. The action of x_1^{-1} and \bar{x}_1^{-1} are defined analogously. Namely, if the marked tree of a

The action of x_1^{-1} and \bar{x}_1^{-1} are defined analogously. Namely, if the marked tree of a forest is rightmost, then x_1^{-1} cannot be applied. Otherwise, if the marked tree T has a tree T'' to the right of it, then we add a caret to these trees and the tree $T \cap T''$ will be marked in the result. Notice that if we are inside B(n,k), then both trees T, T'' must have height < k: otherwise x_1^{-1} cannot be applied. For the action of \bar{x}_1^{-1} , it cannot be applied if T is leftmost. Otherwise the marked tree T has a tree T' to the left of it. Here we add a caret to these trees and the tree $T' \cap T$ will be marked in the result. As above, both trees T', T' must have height < k to be possible to stay inside B(n,k).

We have to emphasize that the definition of these actions is very important for Section 3. So the reader has to keep in mind these rules. We will use them without reference.

It can be checked directly that applying defining relations of F leads to the trivial action (in case when the action of each letter is possible. For details we refer to [4]. So one can regard BB(n,k) as a set of vertices of the Cayley graph of F. This can be done for each of the three generating sets $\{x_0, x_1\}$, $\{x_1, \bar{x}_1\}$, and $\{x_1, \bar{x}_1, x_0\}$.

For any fixed k, let $n \gg k$. Since any tree of height k has at most 2^k leaves, any forest in B(n,k) contains at least $\frac{n}{2^k}$ trees. Therefore, if we randomly take a marked forest, the probability for this vertex of an automaton to accept both x_0 , x_0^{-1} approaches 1. Now look at the probability to accept x_1^{-1} . The contrary holds if and only if the marked tree is trivial. We may assume this tree is not the rightmost one of the forest. Then we remove the trivial tree and move the marker to the right. As a result, we obtain an element of B(n-1,k). The inverse operation is always possible. So the probability we are interested in, equals #B(n-1,k)/#B(n,k). It approaches some number ξ_k as $n \to \infty$. If k is big enough, then ξ_k is close to $\frac{1}{4}$. Indeed, for large k the number of elements in the set B(n,k) grows almost like 4^n , as Catalan numbers do.

For the inverse letter x_1^{-1} , straightforward estimating the probability not to accept it is more complicated. However, it is the same as for x_1 due to Lemma 1. We see that the number of outer edges in the subgraph (that is, the edges in its Cheeger boundary of B(n, k) approaches one half of the cardinality of this set. This means that the density of

the set B(n,k) approaches 3.5.

To be more precise, let us add some calculations. First of all, let $\Phi_k(z)$ be the generating function of the set of rooted binary trees of height lek with n leaves. Clearly, $\Phi_0(z) = z$. For k > 0 we have either a trivial tree that correspond to the summand z, or it has an upper caret. Removing it, we have an ordered pair of trees of height $\leq k - 1$. Hence $\Phi_k(z) = z + \Phi_{k-1}(z)^2$.

So we have a sequence of polynomials with positive integer coefficients. All of them are increasing functions on $z \geq 0$ and approach infinity as $z \to \infty$. So there exists a unique solution of the equation $\Phi_k(z) = 1$. We denote it by ξ_k . This is a decreasing sequence. Let us show that $\xi_k \to \frac{1}{4}$ as $k \to \infty$.

First of all, by induction on k one can easily check that $\Phi_k(\frac{1}{4}) < \frac{1}{2}$ for all $k \geq 0$. Thus $\frac{1}{4} < \xi_k$. On the other hand, every tree with $n \leq k$ carets (so n+1 leaves) has height $\leq k$. Hence the first terms of $\Phi_k(z)$ coincide with Catalan numbers: the coefficient on x_{n+1} equals c_n for $n \leq k$. It is known that the series $\Phi(z) = c_0 z + c_1 z^2 + \cdots + c_n z^{n+1} + \cdots = \frac{1-\sqrt{1-4z}}{2}$ has radius of convergence $\frac{1}{4}$. So for any $z > \frac{1}{4}$, the partial sums of the series approach infinity. Thus $c_0 z + c_1 z^2 + \cdots + c_n z^{n+1} > 1$ if n is sufficiently large. In particular, $\Phi_k(z) > 1$ whenever k is large enough. So $\frac{1}{4} < \xi_k < z = \frac{1}{4} + \varepsilon$ for $k \gg 1$. This proves what we claim.

3 Main Results

Theorem 1 The density of the Cayley graph of Thompson's group F in symmetric generating set $S = \{x_1, \bar{x}_1 = x_1x_0^{-1}\}$ is strictly greater than 3.

Proof. First we consider the Brown – Belk set B(n, k). It gives a subgraph in the Cayley graph \mathcal{C} of the group F in generating set S. Let us find the generating function of this set for any k. The coefficient on z^n will show the number of marked forests with n leaves where all trees of this forest have height $\leq k$.

The marked tree of the forest has generating function $\Phi_k(z)$. To the left of it, we may have any number of trees including zero. Thus for this part we get generating function $1 + \Phi_k(z) + \Phi_k^2(z) + \cdots = \frac{1}{1 - \Phi_k(z)}$. The same for the trees to the left of the marker. Therefore, we get a function $\Psi_k(z) = \frac{\Phi_k(z)}{(1 - \Phi_k(z))^2}$. Its coefficient on z^n in the series expansion is exactly the cardinality of B(n, k). We shall denote it by β_{nk} .

The radius of convergence of the series for $\Psi_k(z)$ equals ξ_k^{-1} . On the other hand, the quotient $\frac{\beta_{n-1,k}}{\beta_{nk}}$ approaches the reciprocal of the radius, that is, for any k one has

$$\frac{\beta_{n-1,k}}{\beta_{nk}} \to \xi_k \tag{7}$$

as $n \to \infty$.

The automaton corresponding to B(n,k) does not accept x_1 whenever the marked tree is trivial or it is the rightmost one in the forest. The former case happens with probability $\leq \frac{2^k}{n}$ since each forest has at least $\frac{n}{2^k}$ trees. So for any k, the probability to be rightmost is o(1) as $n \to \infty$. If the tree is not rightmost, then we remove the trivial

tree and move the marker right. The number of these new marked trees we obtain is exactly $\beta_{n-1,k}$. Indeed, an inverse operation of inserting the trivial tree and moving the marker left is always possible. So the total probability for a marked forest in B(n,k) not to accept x_1 equals $\frac{\beta_{n-1,k}}{\beta_{nk}} + o(1) = \xi_k + o(1)$ according to (7).

As for symmetric generator \bar{x}_1 , the probability has exactly the same value (replace "rightmost" by "leftmost"). Lemma 1 allows us to conclude that the same happens for inverse letters. Therefore, the cardinality of the Cheeger boundary of B(n,k) divided by the cardinality of B(n,k) itself, is $4\xi_k + o(1)$. It approaches 1 as $k \to \infty$, so the density of B(n,k) approaches 3. At the present time, the sets B(n,k) give the best density estimate for the generating set $\{x_0, x_1\}$. So there was a conjecture that for S the density 3 could be an optimal value.

However, this conjecture is not true. There is an essential difference between the standard generating set and the symmetric one. If we take a random marked forest, it always accepts both x_0 and x_0^{-1} if the marked tree is neither leftmost nor rightmost. We already know that the probability to be leftmost (rightmost) does not exceed $\frac{2^k}{n}$ so it is almost zero for $n \gg 1$. Thus the degree of any vertex of a graph is at least 2 for almost all cases if we work with standard generating set.

Now look at the vertices of the Cayley graph C. It turns out that they can be isolating. Indeed, let we have a marked forest ..., T', T, T'', \ldots where T is marked. Suppose that T is trivial. Then the vertex does not accept x_1 as well as \bar{x}_1 (we cannot remove a caret). Additionally suppose that both trees T', T'' have height k. This means that we cannot apply neither \bar{x}_1^{-1} (adding a caret to T' and T), nor x_1^{-1} (adding a caret to T and T''). So in this case we get an isolated vertex.

What is the probability for a vertex (that is, a random marked tree from B(n,k)) to be isolated? If it is small, then we have no profit from that. But it turns out that the probability is uniformly positive. That is, there exists a global positive constant $p_0 > 0$ such that the probability of a vertex to be isolated will be at least p_0 for all our graphs.

The fact we claim is sufficient to prove the theorem. Indeed, if we remove the isolated vertices from B(n,k), then we get a subgraph, say, B'(n,k), where the number of its edges is the same and the number of vertices will be less than $(1-p_0)\beta_{nk}$. Since the density is an average degree of a vertex, then the density of the new subgraph will be at least $\frac{1}{1-p_0}$ multiplied by the density of B(n,k), which is $3-\varepsilon$ for arbitrarily small $\varepsilon > 0$. This means that we can approach density $\frac{3}{1-p_0} > 3$ of the Cayley graph \mathcal{C} .

So let us show that the value $p_0 = \frac{1}{260}$ can be established (so that the density of C will exceed 3.011). (Recall that strict inequality here makes useless the idea to find any kind of a "doubling structure" on C, in the sense we have mentioned in the Introduction.) Direct calculations with generating functions do not give us a clear way to prove the statement. So we will prefer a probabilistic approach.

Let ..., $T_{-1}, T_0, T_1, ...$ be a random marked forest. Assume that all the three trees T_{-1}, T_0, T_1 are trivial. What is the probability of that? If T_1 is rightmost, then we already know that the probability is o(1) so we can ignore this case. If we remove the three trivial trees and move the marker to the tree that goes after T_1 (let it be T_2 in the above notation), then we obtain a marked forest from B(n-3,k). The inverse operation

is always possible to do. So our probability is $\frac{\beta n-3,k}{\beta_{nk}}+o(1)=\xi_k^3+o(1)$ as $n\to\infty$. Now we start add carets. The first one is added to T_1 and T_2 . Then we add a caret

Now we start add carets. The first one is added to T_1 and T_2 . Then we add a caret to obtain $((T_1 \hat{\ } T_2) \hat{\ } T_3)$ and so on. At some step we will not be able to add a new caret. This happens if we reach the rightmost tree (for what case the probability is very small), or we cannot add a new caret to two trees because at least one of them has height k. To be more precise, let us assume that the trees T_2, \ldots, T_{k+1} do exist in our marked forest. If not, the probability for a marked tree T be close to the right border does not exceed $\frac{(k+1)2^k}{n} = o(1)$ as $n \to \infty$. So the process of adding carets to the right of T will get us $\ldots, T, T_1'', T_2'', \ldots$ where at least one of the trees T_1'', T_2'' has height k.

The same process can be done to the left of T. There we get ..., $T'_2, T'_1, T, ...$, where at least one of the trees T'_2, T'_1 has height k.

Suppose that both T'_1 , T''_1 have height k. Then the marked forest ..., T'_1 , T, T''_1 , ... gives an isolated vertex as we have seen before. If T'_1 does not have height k then T'_2 has height k so we can swap the trees T'_2 and T'_1 in the forest. Both of these forest will have the same probability. Also if T''_1 does not have height k then T''_2 has height k and we swap these trees. Then the probability of our event (when T'_1 and T''_1 have height k) is at least $\frac{1}{4}$ of the probability of the event: $(T'_2 \text{ OR } T'_1$ has height k) AND $(T''_1 \text{ OR } T''_2$ has height k). The former is $\xi_k^3 + o(1)$ since the process of adding carets is unique and the inverse operations are possible to do. This will lead back to the case of three trivial trees for which the probability is already known.

So we proved that the probability of a random vertex to be isolated is at least $\frac{1}{4}\xi_k^3 + o(1)$. It approaches $\frac{1}{4^4} = \frac{1}{256} > \frac{1}{260} = p_0$. This completes the proof.

At the end of this Section we will obtain one more result. Let us add x_0 to the generating set S. We will get three generators $\{x_1, \bar{x}_1, x_0\}$. What is the density of the Cayley graph here, is not known. We only know that the isoperimetric constant ι^* is at least 1 but we cannot prove or disprove the strict inequality. The idea to remove isolated vertex does not work here since in the new graph the former isolated vertex will have degree 2 because of edges labelled by $x_0^{\pm 1}$.

However, we can say something about the outer boundary ∂ instead of the Cheeger boundary ∂^* . The question from the previous paragraph is equivalent to the following: is there a finite set $Y \subset F$ such that $\#\partial^*Y < \#Y$? We do not know the answer but we are able to prove the following.

Theorem 2 For the symmetric generating set $S = \{x_0, x_1, \bar{x}_1 = x_1 x_0^{-1}\}$, there exists finite subsets $Y \subset F$ in the Cayley graph of Thompson's group F such that $\#\partial Y < \#Y$.

Equivalently, the generating set S does not have doubling property, that is, there are finite subsets Y in F such that the 1-neighbourhood $\mathcal{N}_1(Y) = Y(\{1\} \cup S)$ has cardinality strictly less than 2#Y.

Proof. The second statement follows to the first one (and in fact it is equivalent) since the 1-neighbourhood of Y is the disjoint union of Y itself and its outer boundary.

The proof of the first statement will be easier that the proof of Theorem 1 since in this case it suffices to take Y = B(n,k). For every vertex v in its outer boundary, we choose an edge e connecting it to a vertex u in Y. If there are several edges with this

property, we fix one of them. The aim is to estimate the number of fixed edges, which is equal to ∂Y .

If the label of a fixed edge is x_0^{-1} , we already know that the probability is o(1). Here we think in terms of probabilities dividing the number of edges by #Y.

Suppose that the edge e has label x_1^{-1} . Then u as a vertex of the automaton Y does not accept x_1 . This means that we cannot remove a caret of the marked tree corresponding to u. This means that the tree is empty. So the number of edges e with label x_1^{-1} does not exceed the number of marked forests with trivial marked tree. In terms of probabilities, this gives the estimate $\xi_k + o(1)$. Exactly the same holds for edges e labelled by \bar{x}_1^{-1} because of symmetry.

Now look at the number of vertices v in the outer boundary for which the label of e is x_1 or \bar{x}_1 . The vertex v can be represented as a marked forest. After we apply x_1^{-1} or \bar{x}_1^{-1} to it removing the upper caret, we get to u which is a forest with all trees of height $\leq k$. Therefore, the tree $T = T_1 \hat{\,\,\,} T_2$ which is marked for the vertex v, has height k+1. Applying x_1^{-1} to it means that the caret is removed and the marked tree becomes T_1 .

So the vertices v in the outer boundary for which the label of e is x_1 or \bar{x}_1 are connected to a vertex in Y by both of these edges. So in the process of choosing edges, we may assume that the label of e is x_1 . Hence the number of vertices with this property does not exceed the number of vertices in Y for which x_1^{-1} cannot be applied. By Lemma 1, this number equals the one for the generator x_1 . The probability for that is $\xi_k + o(1)$ already know.

Summing the numbers, we obtain that $\#\partial Y/\#Y = 3\xi_k + o(1) < 1$ for $k \gg 1$. In fact, the constant 2 in the statement of the Theorem can be replaced by $\frac{7}{4} + \varepsilon$ for any positive ε .

The proof is complete.

Notice that in order to prove non-amenability of Thompson's group F (if it is the fact), it suffices to find a kind of a doubling structure on the Cayley graph of the group. If the generating set is "small" then we have no chances to find this structure as the above results show. If it is very "large", then it is more difficult to work with the graph. So we would like to offer the generating set $\{x_0, x_1, x_2\}$ for which there are chances to find a doubling structure on the Cayley graph.

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