BINOMIAL EDGE IDEALS OF UNICYCLIC GRAPHS

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ABSTRACT. Let G be a connected graph on the vertex set $[n]$. Then $\operatorname{depth}(S/J_G) \leq n+1$. In this article, we prove that if G is a unicyclic graph, then the depth of S/J_G is bounded below by n. Also, we characterize G with $\operatorname{depth}(S/J_G) = n$ and $\operatorname{depth}(S/J_G) = n+1$. We then compute one of the distinguished extremal Betti numbers of S/J_G. If G is obtained by attaching whiskers at some vertices of the cycle of length k, then we show that $k-1 \leq \operatorname{reg}(S/J_G) \leq k+1$. Furthermore, we characterize G with $\operatorname{reg}(S/J_G) = k-1$, $\operatorname{reg}(S/J_G) = k$ and $\operatorname{reg}(S/J_G) = k+1$. In each of these cases, we classify the uniqueness of the extremal Betti number of these graphs.

1. Introduction

Let $R = \mathbb{K}[x_1, \ldots, x_m]$ be the standard graded polynomial ring over an arbitrary field $\mathbb{K}$ and M be a finitely generated graded R-module. Let

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-p-j)^{\beta_{p,p+j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0,$$

be the minimal graded free resolution of M. The number $\beta_{i,j}(M)$ is called the (i,j)-th *graded Betti number* of M. From the minimal free resolution of a graded module, one can obtain two important invariants, namely the projective dimension and the Castelnuovo-Mumford regularity. The *projective dimension* of M, denoted by $\operatorname{pd}(M)$, is defined as

$$\operatorname{pd}(M) := \max\{i : \beta_{i,i+j}(M) \neq 0 \text{ for some } j\}$$

and the *Castelnuovo-Mumford regularity* (or simply, *regularity*) of M, denoted by $\operatorname{reg}(M)$, is defined as

$$\operatorname{reg}(M) := \max\{j : \beta_{i,i+j}(M) \neq 0 \text{ for some } i\}.$$

If $\beta_{i,i+j}(M) \neq 0$ and for all pairs $(k,l) \neq (i,j)$ with $k \geq i$ and $l \geq j$, $\beta_{k,k+l}(M) = 0$, then $\beta_{i,i+j}(M)$ is called an *extremal Betti number* of M. If $p = \operatorname{pd}(M)$ and $r = \operatorname{reg}(M)$, then there exist unique numbers i and j such that $\beta_{p,p+i}(M)$ and $\beta_{j,j+r}(M)$ are extremal Betti numbers. These extremal Betti numbers are called the *distinguished extremal Betti numbers* of M, see [13]. Note that M admits a unique extremal Betti number if and only if $\beta_{p,p+r}(M) \neq 0$.

Let G be a simple graph on the vertex set $V(G) = [n] := \{1, \ldots, n\}$ and the edge set $E(G)$. Let $S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring on $2n$ variables over an arbitrary field $\mathbb{K}$. The *binomial edge ideal* of G, denoted by J_G, defined as $J_G = (x_i y_j - x_j y_i : i < j \text{ and } \{i,j\} \in E(G)) \subseteq S$ was introduced by Herzog et al. in [12] and independently by Ohtani in [26]. In the recent past, there has been considerable interest in computing algebraic invariants such as depth and regularity of J_G in terms of combinatorial invariants

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such as clique number, number of vertices, length of a longest induced path and number of internal vertices of G, see [1, 2, 3, 8, 18, 22, 24, 31, 32, 33] for a partial list.

It is well known (see, for example [4, Proposition 1.2.13]) that $\operatorname{depth}(S/J_G) \leq \dim(S/P)$ for all $P \in \operatorname{Ass}(J_G)$. It follows from [12, Theorem 3.2] that $P = P_\emptyset(G) \in \operatorname{Ass}(J_G)$ with $\dim(S/P) = n+1$. Therefore, $\operatorname{depth}(S/J_G) \leq n+1$ for every connected graph G on n vertices. In general, there is no lower bound for the depth of S/J_G. In [37, Theorem 4.5], Zafar proved that $\operatorname{depth}(S/J_G) = n$ where G is a cycle on n vertices. If $G = G_1 * G_2$, the join product of G_1 and G_2, then Kumar and the present author gave a formula for the depth of S/J_G in terms of the depths of S_{G_1}/J_{G_1} and S_{G_2}/J_{G_2}, see [21, Theorems 4.1, 4.3 and 4.4]. Recently, Rouzbahani Malayeri, Saeedi Madani and Kiani studied the depth of S/J_G and they characterized all graphs G such that $\operatorname{depth}(S/J_G) = 4$ in [30]. Let G be a connected unicyclic graph of girth k on n vertices with $n > k$ for $k \geq 3$. If $k = 3$, then G is a chordal graph with the property that any two maximal cliques intersect in at most one vertex. In [7], Ene, Herzog and Hibi proved that $\operatorname{depth}(S/J_G) = n+1$ for such graphs. For $k \geq 4$, we compute the depth of S/J_G in a slightly more general setting.

Theorem 3.6. *Let $k \geq 3$ and $m \geq 2$. Let G be the clique sum of $H = C_k \cup_e K_m$ and a forest along some vertices of H. Then* $\operatorname{depth}(S/J_G) \geq n$. *Let $A = \{u \in V(C_k) :$ there is a tree incident on $u\}$. If $e \cap A \neq \emptyset$ and $G[A]$ is connected with $k-2 \leq |A|$, then* $\operatorname{depth}(S/J_G) = n+1$.

Considering $m = 2$, we obtain our results for unicyclic graph. Moreover, we prove that if there are trees attached to $k-2$ consecutive vertices of the cycle in G, then $\operatorname{depth}(S/J_G) = n+1$ and otherwise, $\operatorname{depth}(S/J_G) = n$, see Corollary 3.12.

In [24], Matsuda and Murai proved that $\ell(G) \leq \operatorname{reg}(S/J_G) \leq n-1$, where $\ell(G)$ is the length of a longest induced path in G. It is evident that $\ell(G)$ is not a sharp lower bound. If G is assumed to be a tree, then Chaudhry et al. [5] proved that $\operatorname{reg}(S/J_G) = \ell(G)$ if and only if G is a caterpillar. An improved lower bound for trees was obtained by Jayanthan et al. in [16], where they proved that $\operatorname{iv}(G) + 1 \leq \operatorname{reg}(S/J_G)$. In the case of G being a block graph, Herzog and Rinaldo [13] generalized their result and proved that $\operatorname{iv}(G)+1 \leq \operatorname{reg}(S/J_G)$ and they also characterized G admitting a unique extremal Betti number. There have been some other works as well on the computation of extremal Betti numbers of binomial edge ideals. In [6], de Alba and Hoang studied extremal Betti numbers of binomial edge ideals of closed graphs, and Kumar studied extremal Betti numbers of binomial edge ideals of generalized block graphs, [19]. Recently, Mascia and Rinaldo [23] studied extremal Betti numbers of some Cohen-Macaulay bipartite graphs. In this article, we study extremal Betti numbers of J_G, and as a consequence, we obtain a lower bound for the regularity of J_G, where G is a unicyclic graph.

Corollary 3.13. *Let G be a unicyclic graph of girth $k \geq 4$ with* $\operatorname{pd}(S/J_G) = p$. *If trees are attached to every vertex of the cycle in G, then $\beta_{p,p+\operatorname{iv}(G)+1}(S/J_G)$ is an extremal Betti number of S/J_G, and hence* $\operatorname{iv}(G)+1 \leq \operatorname{reg}(S/J_G)$. *Otherwise, either $\beta_{p,p+\operatorname{iv}(G)-1}(S/J_G)$ or $\beta_{p,p+\operatorname{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G, and hence* $\operatorname{iv}(G)-1 \leq \operatorname{reg}(S/J_G)$.

There are only few classes of graphs for which the regularity of their binomial edge ideals are known, see [9, 14, 34, 35, 38]. In the last section, we study the regularity and behavior of extremal Betti number of graphs G, where G is obtained by attaching whiskers to some vertices of the cycle of length k. We first prove that the regularity of S/J_G is bounded below by $k-1$ and bounded above by $k+1$. We then characterize G such that $\operatorname{reg}(S/J_G) = k+1$, $\operatorname{reg}(S/J_G) = k-1$ and $\operatorname{reg}(S/J_G) = k$.

Corollary 4.10. *Let $G = C_k \cup (\cup_{i=1}^{k} W^{r_i}(v_i))$ for $r_i \geq 0$ and $k \geq 4$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$ and suppose that $A \neq \emptyset$. Then $k-1 \leq \operatorname{reg}(S/J_G) \leq k+1$. Moreover,*

(1) $\operatorname{reg}(S/J_G) = k+1$ *if and only if $A = V(C_k)$,*
(2) $\operatorname{reg}(S/J_G) = k-1$ *if and only if $|A| = 1$ or $|A| = 2$ and vertices of A are adjacent,*
(3) $\operatorname{reg}(S/J_G) = k$ *if and only if A contains at least two non-adjacent vertices and $A \subsetneq V(C_k)$.*

Furthermore, we show that if $\operatorname{reg}(S/J_G) = k+1$ and $\operatorname{reg}(S/J_G) = k-1$, then S/J_G admits a unique extremal Betti number. If $\operatorname{reg}(S/J_G) = k$, then S/J_G does not always admit a unique extremal Betti number. In this case, we classify G such that S/J_G admits a unique extremal Betti number.

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2. Preliminaries

Let G be a simple graph with the vertex set $[n]$ and edge set $E(G)$. A graph G on the vertex set $[n]$ is said to be a *complete graph*, if $\{i,j\} \in E(G)$ for all $1 \leq i < j \leq n$. We denote the complete graph on n vertices by K_n. For $A \subseteq V(G)$, the *induced subgraph* of G on the vertex set A, denoted by $G[A]$, is the graph such that for $i,j \in A$, $\{i,j\} \in E(G[A])$ if and only if $\{i,j\} \in E(G)$. For a vertex $v \in V(G)$, let $G \setminus v$ denote the induced subgraph of G on the vertex set $V(G) \setminus \{v\}$. For a subset $U \subseteq V(G)$, if the induced subgraph $G[U]$ is a complete graph then U is called a *clique*. A vertex v of G is said to be a *simplicial vertex* if v belongs to only one maximal clique. If v is not a simplicial vertex, then v is called an *internal vertex*. The number of internal vertices of G is denoted by $\operatorname{iv}(G)$. For a vertex v in G, $N_G(v) := \{u \in V(G) : \{u,v\} \in E(G)\}$ denotes the *neighborhood* of v in G and G_v is the graph with the vertex set $V(G)$ and edge set $E(G_v) = E(G) \cup \{\{u,w\} : u,w \in N_G(v)\}$ i.e., G_v is obtained from G by making a complete graph on $N_G(v) \cup \{v\}$ in G. Set $N_G[v] := N_G(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg_G(v)$, is $|N_G(v)|$. A *cycle* on the vertex set $[k]$, denoted by C_k, is a graph with the edge set $\{i, i+1 : 1 \leq i \leq k-1\} \cup \{1,k\}$ for $k \geq 3$. A graph is said to be a *unicyclic* graph if it contains exactly one cycle as a subgraph. The *girth* of a graph G is the length of a shortest cycle in G. A graph G is called *chordal* if every induced cycle of G has 3 vertices. A connected graph is a *tree* if it does not have a cycle. A *forest* is a disconnected graph whose components are trees. A vertex $v \in V(G)$ is said to be a *cut vertex* if $G \setminus v$ has more components than G. A *block* of a graph is a maximal nontrivial connected subgraph which has no cut vertex. A graph G is called a *block graph* if every block of G is a complete graph. It is easy to see that G is a block graph if and only if G is a chordal graph with the property that any two maximal cliques intersect in at most one vertex. A connected chordal graph G is said to be a *generalized block graph* if three maximal cliques of G intersect non-trivially, then the intersection of each pair of them is the same.

For $T \subseteq V(G)$, let $c(T)$ denote the number of components of $G[\bar{T}]$, where $\bar{T} = V(G) \setminus T$. Also, let $G_1, \cdots, G_{c(T)}$ be the components of $G[\bar{T}]$ and for every i, $\tilde{G}_i$ denotes the complete graph on the vertex set $V(G_i)$. Moreover, we set $P_T(G) := (\underset{i \in T}{\cup}\{x_i, y_i\}, J_{\tilde{G}_1}, \cdots, J_{\tilde{G}_{c(T)}})$. In [12], Herzog et al. proved that $J_G = \underset{T \subseteq [n]}{\cap} P_T(G)$, which in particular, implies that J_G is a radical ideal. A set $T \subseteq V(G)$ is said to have *cut point property* if for every $i \in T$, i is a cut vertex of the graph $G[\bar{T} \cup \{i\}]$ i.e., $c(T \setminus \{i\}) < c(T)$. They also showed that $P_T(G)$ is a minimal prime of J_G if and only if either $T = \emptyset$ or $T \subset V(G)$ has the cut point property, see [12, Corollary 3.9].

Let $R = \mathbb{K}[x_1, \ldots, x_m]$, $R' = \mathbb{K}[x_{m+1}, \ldots, x_n]$ and $R'' = \mathbb{K}[x_1, \ldots, x_n]$ be polynomial rings. Let $I \subseteq R$ and $J \subseteq R'$ be homogeneous ideals. Then it is well known that the minimal free resolution of $R''/(I+J)$ is the tensor product of the minimal free resolutions of R/I and R'/J. Therefore, we have for all i, j,

$$\beta_{i,i+j}\left(\frac{R''}{I+J}\right) = \sum_{\substack{i_1+i_2=i \\ j_1+j_2=j}} \beta_{i_1,i_1+j_1}\left(\frac{R}{I}\right)\beta_{i_2,i_2+j_2}\left(\frac{R'}{J}\right). \tag{1}$$

The following depth lemma and regularity lemma can be easily derived from the long exact sequence of Tor and Ext corresponding to given short exact sequence.

Lemma 2.1. *Let R be a standard graded ring and M, N, P be finitely generated graded R-modules. If $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is a short exact sequence with f, g graded homomorphisms of degree zero, then*
(1) $\operatorname{depth}(M) \geq \min\{\operatorname{depth}(N), \operatorname{depth}(P) + 1\}$,
(2) $\operatorname{depth}(N) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(P)\}$,
(3) $\operatorname{depth}(P) \geq \min\{\operatorname{depth}(M) - 1, \operatorname{depth}(N)\}$,
(4) $\operatorname{depth}(M) = \operatorname{depth}(P) + 1$, if $\operatorname{depth}(N) > \operatorname{depth}(P)$ and
(5) $\operatorname{depth}(M) = \operatorname{depth}(N)$, if $\operatorname{depth}(N) < \operatorname{depth}(P)$.

Lemma 2.2. *Let R be a standard graded ring and M, N, P be finitely generated graded R-modules. If $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is a short exact sequence with f, g graded homomorphisms of degree zero, then*
(1) $\operatorname{reg}(M) \leq \max\{\operatorname{reg}(N), \operatorname{reg}(P) + 1\}$,
(2) $\operatorname{reg}(N) \leq \max\{\operatorname{reg}(M), \operatorname{reg}(P)\}$,
(3) $\operatorname{reg}(P) \leq \max\{\operatorname{reg}(M) - 1, \operatorname{reg}(N)\}$,
(4) $\operatorname{reg}(M) = \operatorname{reg}(P) + 1$, if $\operatorname{reg}(N) < \operatorname{reg}(M)$ and
(5) $\operatorname{reg}(M) = \operatorname{reg}(N)$, if $\operatorname{reg}(N) > \operatorname{reg}(P)$.

The following is a crucial lemma due to Ohtani, which is used repeatedly throughout this article.

Lemma 2.3. ([26, Lemma 4.8]) *Let G be a graph on $V(G)$ and $v \in V(G)$ such that v is not a simplicial vertex. Then $J_G = (J_{G\setminus v} + (x_v, y_v)) \cap J_{G_v}$.*

Thus, we get the following short exact sequence:

$$0 \longrightarrow \frac{S}{J_G} \longrightarrow \frac{S}{(x_v, y_v) + J_{G\setminus v}} \oplus \frac{S}{J_{G_v}} \longrightarrow \frac{S}{(x_v, y_v) + J_{G_v \setminus v}} \longrightarrow 0, \tag{2}$$

and correspondingly the long exact sequence of Tor modules:

$$\cdots \longrightarrow \operatorname{Tor}_i^S\left(\frac{S}{J_G},\mathbb{K}\right)_{i+j} \longrightarrow \operatorname{Tor}_i^S\left(\frac{S}{(x_v,y_v)+J_{G\setminus v}},\mathbb{K}\right)_{i+j} \oplus \operatorname{Tor}_i^S\left(\frac{S}{J_{G_v}},\mathbb{K}\right)_{i+j}$$
$$\longrightarrow \operatorname{Tor}_i^S\left(\frac{S}{(x_v,y_v)+J_{G_v\setminus v}},\mathbb{K}\right)_{i+j} \longrightarrow \operatorname{Tor}_{i-1}^S\left(\frac{S}{J_G},\mathbb{K}\right)_{i+j} \longrightarrow \cdots \quad (3)$$

3. Unicyclic Graph

Let G be a connected unicyclic graph (which is not a cycle) of girth k on n vertices for $k \geq 4$. In this section, we prove that $n \leq \operatorname{depth}(S/J_G)$ and we characterize unicyclic graphs G such that $\operatorname{depth}(S/J_G) = n$. We also compute one distinguished extremal Betti number of S/J_G.

Notation 3.1. Let G be a graph on $V(G) = [n]$. We reserve the notation S for the polynomial ring $\mathbb{K}[x_i, y_i : i \in [n]]$ and for $v \in V(G)$, S' for the polynomial ring $\mathbb{K}[x_i, y_i : i \in V(G) \setminus \{v\}]$. If H is any other graph with the vertex set $V(H)$, then we set $S_H = \mathbb{K}[x_i, y_i : i \in V(H)]$ and for $v \in V(H)$, set $S'_H = \mathbb{K}[x_i, y_i : i \in V(H) \setminus \{v\}]$.

To re-emphasize: Unless stated otherwise, G always denotes a graph on n vertices.

Definition 3.2. *Let G_1 and G_2 be two subgraphs of a graph G. If $G_1 \cap G_2 = K_m$, $V(G_1) \cup V(G_2) = V(G)$ and $E(G_1) \cup E(G_2) = E(G)$ with $G_1 \neq K_m$ then G is called the clique sum of G_1 and G_2 along the complete graph K_m, denoted by $G_1 \cup_{K_m} G_2$. Sometimes we call this clique sum that G_1 is attached to G_2 along K_m. If $G_2 = K_m$, then $G_1 \cup_{K_m} G_2 = G_1$.*

Notation 3.3. *Let $k \geq 3$. For the rest of the article, we fix the following notation for the cycle graph on k-vertices. Let $V(C_k) = \{v = v_1, v_2, \ldots, v_k\}$ be such that $E(C_k) = \{\{v_i, v_{i+1}\}, \{v_1, v_k\} : 1 \leq i \leq k-1\}$.*

Let H be the clique sum of C_k and a complete graph K_m along an edge e for $m \geq 2$. If $m = 2$, then $H = C_k$ and in this case, Zafar and Zahid [38, Corollary 16] proved that $\operatorname{reg}(S_H/J_H) = k-2$ and $\beta_{k,2k-2}(S_H/J_H)$ is the unique extremal Betti number of S_H/J_H. For $m \geq 3$, Jayanthan et al. proved that $\operatorname{reg}(S_H/J_H) = k-1$, see [15, Proposition 3.11]. Here, we prove that S_H/J_H admits a unique extremal Betti number, namely $\beta_{n,n+k-1}(S_H/J_H)$, where $|V(H)| = n$.

Proposition 3.4. *Let $k, m \geq 3$ and $H = C_k \cup_e K_m$, with $|V(H)| = n$, be the clique sum of a cycle C_k and a complete graph K_m along an edge e. Then $\operatorname{depth}(S_H/J_H) = n$ and $\beta_{n,n+k-1}(S_H/J_H)$ is the unique extremal Betti number of S_H/J_H.*

Proof. Let $e = \{v, v_2\}$. We proceed by induction on k. Suppose first that $k = 3$. Then $H = C_3 \cup_e K_m$, which is a generalized block graph. Thus by [17, Theorem 3.2], $\operatorname{depth}(S_H/J_H) = n$, and hence it follows from [19, Theorem 3.7] that $\beta_{n,n+2}(S_H/J_H)$ is the unique extremal Betti number of S_H/J_H.

Now suppose that $k \geq 4$. By Lemma 2.3, $J_H = J_{H_v} \cap ((x_v, y_v) + J_{H\setminus v})$, where $H_v = C_{k-1} \cup_{e'} K_{m+1}$ and $H \setminus v = P_{k-1} \cup_{v_2} K_{m-1}$ with $e' = \{v_2, v_k\}$, and $V(P_{k-1}) = V(C_{k-1}) = \{v_2, \ldots, v_k\}$. Note that $H_v \setminus v = C_{k-1} \cup_{e'} K_m$. Therefore by induction, $\operatorname{depth}(S_H/J_{H_v}) = n$, $\operatorname{depth}(S'_H/J_{H_v\setminus v}) = n-1$ and $\beta_{n,n+k-2}(S_H/J_{H_v})$, $\beta_{n-1,n+k-3}(S'_H/J_{H_v\setminus v})$ are the unique extremal Betti numbers. Thus it follows from (1) that $\beta_{n+1,n+k-1}(S_H/((x_v, y_v)+J_{H_v\setminus v}))$ is the unique extremal Betti number. Since $\operatorname{iv}(H \setminus v) = k-2$, it follows from [7, Theorem 1.1] that

$\operatorname{depth}(S'_H/J_{H\setminus v}) = n$, and hence by [13, Theorem 8] and (1), $\beta_{n,n+k-1}(S_H/((x_v,y_v)+J_{H\setminus v}))$ is the unique extremal Betti number. As v is not a simplicial vertex, we apply Lemma 2.1 on the short exact sequence (2) for the pair (H,v) and get that $\operatorname{depth}(S_H/J_H) \geq n$. Considering the long exact sequence of Tor (3) for $i=n$ and in graded degree $j=k-1$, we get

$$0 \longrightarrow \operatorname{Tor}_{n+1}^{S_H}\left(\frac{S_H}{((x_v,y_v)+J_{H_v\setminus v})},\mathbb{K}\right)_{n+k-1} \longrightarrow \operatorname{Tor}_n^{S_H}\left(\frac{S_H}{J_H},\mathbb{K}\right)_{n+k-1} \longrightarrow \cdots$$

which implies that $\beta_{n,n+k-1}(S_H/J_H) \neq 0$. Therefore by Auslander-Buchsbaum formula, $\operatorname{depth}(S_H/J_H) \leq n$. Hence, $\operatorname{depth}(S_H/J_H) = n$ and $\beta_{n,n+k-1}(S_H/J_H)$ is the unique extremal Betti number of S_H/J_H as $\operatorname{reg}(S_H/J_H) = k-1$, by [15, Proposition 3.11]. $\square$

Let M be a graded S-module. Then the *Betti polynomial* of M is defined as $\sum_{i,j}\beta_{i,j}(M)s^it^j$ and denoted by $B_M(s,t)$.

A graph G is said to be a *decomposable graph* if G is the clique sum of subgraphs G_1 and G_2 along a simplicial vertex i.e., $G = G_1 \cup_v G_2$, where v is a simplicial vertex of G_1 and G_2. If G is not decomposable, then it is called an *indecomposable graph*. We now recall the following result due to Herzog and Rinaldo.

Proposition 3.5. [13, Proposition 3] *Let $G = G_1 \cup G_2$ be a decomposable graph. Then*

$$B_{S/J_G}(s,t) = B_{S_{G_1}/J_{G_1}}(s,t)B_{S_{G_2}/J_{G_2}}(s,t).$$

As a corollary of the above Proposition, we get that if $G = G_1 \cup \cdots \cup G_l$ is a decomposition into indecomposable graphs G_i, then $\operatorname{pd}(S/J_G) = \sum_{i=1}^{l}\operatorname{pd}(S_{G_i}/J_{G_i})$ and $\operatorname{reg}(S/J_G) = \sum_{i=1}^{l}\operatorname{reg}(S_{G_i}/J_{G_i})$. These two equalities follow from [27, Theorem 2.7] and [16, Theorem 3.1], respectively, as well. Also, if for each $i = 1,\ldots,l$, $\beta_{p_i,p_i+r_i}(S_{G_i}/J_{G_i})$ is an extremal Betti number of S_{G_i}/J_{G_i}, then $\beta_{p,p+r}(S/J_G) = \prod_{i=1}^{l}\beta_{p_i,p_i+r_i}(S_{G_i}/J_{G_i})$ is an extremal Betti number of S/J_G, where $p = \sum_{i=1}^{l}p_i$ and $r = \sum_{i=1}^{l}r_i$. Therefore to find the regularity, projective dimension, and extremal Betti number of G, it is enough to consider G to be an indecomposable graph. So for the rest of the section, we assume that G is an indecomposable graph.

For a connected graph G, $\kappa(G) \geq 1$, and so by [2, Theorems 3.19 and 3.20], $\operatorname{depth}(S/J_G) \leq n+1$. Let $H = C_k \cup_e K_m$ be the clique sum of C_k and K_m along an edge e for $k \geq 3$, $m \geq 2$. Let G be the clique sum of H and a forest along some vertices of H. We now study the depth of S/J_G and prove that $n \leq \operatorname{depth}(S/J_G)$. Therefore, $\operatorname{depth}(S/J_G) \in \{n, n+1\}$. We then characterize G with $\operatorname{depth}(S/J_G) = n$ and $\operatorname{depth}(S/J_G) = n+1$. Also, we obtain a lower bound for the regularity and show that if there are trees attached to each vertex of C_k, then $\operatorname{iv}(G)+1 \leq \operatorname{reg}(S/J_G)$, otherwise $\operatorname{iv}(G)-1 \leq \operatorname{reg}(S/J_G)$ by computing its one distinguished extremal Betti number.

Theorem 3.6. *Let $k \geq 3$ and $m \geq 2$. Let G be the clique sum of $H = C_k \cup_e K_m$ and a forest along some vertices of H. Then* $\operatorname{depth}(S/J_G) \geq n$. *Let $A = \{u \in V(C_k) :$ there is a tree incident on $u\}$. If $e \cap A \neq \emptyset$ and $G[A]$ is connected with $k-2 \leq |A|$, then* $\operatorname{depth}(S/J_G) = n+1$.

Proof. Let $e = \{v, v_2\}$. By Lemma 2.3, $J_G = J_{G_v} \cap ((x_v,y_v)+J_{G\setminus v})$, where $G \setminus v$ is a block graph on $n-1$ vertices. So by [7, Theorem 1.1], $\operatorname{depth}(S'/J_{G\setminus v}) \geq n$. We prove the theorem by induction on k. For the case $k=3$, it can be noted that G_v and $G_v \setminus v$ are both block graphs on n and $n-1$ vertices, respectively. Thus it follows from [7, Theorem 1.1] that

$\operatorname{depth}(S/J_{G_v}) = n+1$ and $\operatorname{depth}(S'/J_{G_v\setminus v}) = n$. Consider the short exact sequence (2) and apply Lemma 2.1 to get that $\operatorname{depth}(S/J_G) \geq n$. Now suppose that there is one tree attached to v. Then $G \setminus v$ is a disconnected block graph, and hence by [7, Theorem 1.1], $\operatorname{depth}(S'/J_{G\setminus v}) \geq n+1$. Therefore, by Lemma 2.1 and the short exact sequence (2), we have $\operatorname{depth}(S/J_G) = n+1$.

We now assume that $k \geq 4$. Let K' and K'' be complete graphs on vertex sets $N_G[v]$ and $N_G(v)$, respectively. Also, let $H' = C_{k-1} \cup_{e'} K'$ and $H'' = C_{k-1} \cup_{e'} K''$, where $e' = \{v_2, v_k\}$ and $V(C_{k-1}) = \{v_2, \ldots, v_k\}$. Then it can be observed that G_v is the clique sum of H' and a forest along some vertices of G. Also, $G_v \setminus v$ is the clique sum of H'' and a forest along some vertices of G. Therefore, by induction $\operatorname{depth}(S/J_{G_v}) \geq n$ and $\operatorname{depth}(S'/J_{G_v\setminus v}) \geq n-1$. Thus, by applying Lemma 2.1 on the short exact sequence (2), we get $\operatorname{depth}(S/J_G) \geq n$. Suppose now that $v \in A$ and $G[A]$ is connected with $k-2 \leq |A|$. Then $G_v[A \setminus \{v\}]$ and $G_v \setminus v[A \setminus \{v\}]$ are both connected with $k-3 \leq |A \setminus \{v\}|$. Hence, by induction $\operatorname{depth}(S/J_{G_v}) = n+1$ and $\operatorname{depth}(S'/J_{G_v\setminus v}) = n$. By [7, Theorem 1.1], we have $\operatorname{depth}(S'/J_{G\setminus v}) \geq n+1$. Therefore it follows from Lemma 2.1 and the short exact sequence (2) that $\operatorname{depth}(S/J_G) = n+1$. $\square$

Let $H = C_3 \cup_e K_m$ for $m \geq 2$. Let G be the clique sum of H and a forest along some vertices of H. If $m = 2$, then G is a block graph. Ene, Herzog and Hibi [7, Theorem 1.1] proved that in this case $\operatorname{depth}(S/J_G) = n+1$. Let $m \geq 3$. In Theorem 3.6 we proved that $\operatorname{depth}(S/J_G) \geq n$, and if there are trees attached to either one vertex of e or both the vertices of e, then $\operatorname{depth}(S/J_G) = n+1$. If there are no trees attached to any vertex of e, then G is a generalized block graph. Hence it follows from [17, Theorem 3.2] and [19, Theorem 3.4] that $\operatorname{depth}(S/J_G) = n$ and $\beta_{n,n+\operatorname{iv}(G)}(S/J_G)$ is an extremal Betti number. Now we consider the case when trees are attached to at least one of the vertices of e.

Theorem 3.7. *Let $H = C_3 \cup_e K_m$ for $m \geq 3$. Let G be the clique sum of H and a forest along some vertices of H. If there are trees attached to one vertex of e, then $\beta_{n-1,n-1+\operatorname{iv}(G)}(S/J_G)$ is an extremal Betti number and if there are trees attached to both the vertices of e, then $\beta_{n-1,n-1+\operatorname{iv}(G)+1}(S/J_G)$ is an extremal Betti number. In particular,* $\operatorname{iv}(G) \leq \operatorname{reg}(S/J_G)$.

Proof. Let $e = \{v, v_2\}$ and suppose that there are trees attached to v in G. Then $G \setminus v$ is a disconnected block graph on $n-1$ vertices. By virtue of [7, Theorem 1.1] and (1), we have $p = \operatorname{pd}(S/((x_v, y_v) + J_{G\setminus v})) \leq n-1$. By Lemma 2.3, $J_G = J_{G_v} \cap ((x_v, y_v) + J_{G\setminus v})$. Note that G_v and $G_v \setminus v$ are block graphs on n and $n-1$ vertices respectively. Therefore it follows from [13, Theorem 6] and (1) that $\beta_{p,p+\operatorname{iv}(G\setminus v)+1}(S/((x_v, y_v) + J_{G\setminus v}))$, $\beta_{n-1,n-1+\operatorname{iv}(G_v)+1}(S/J_{G_v})$ and $\beta_{n,n+\operatorname{iv}(G_v\setminus v)+1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are extremal Betti numbers. We consider the long exact sequence (3) for $i = n-1$

$$0 \longrightarrow \operatorname{Tor}_n^S\left(\frac{S}{(x_v, y_v) + J_{G_v\setminus v}}, \mathbb{K}\right)_{n-1+j} \longrightarrow \operatorname{Tor}_{n-1}^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n-1+j} \longrightarrow$$
$$\longrightarrow \operatorname{Tor}_{n-1}^S\left(\frac{S}{(x_v, y_v) + J_{G\setminus v}}, \mathbb{K}\right)_{n-1+j} \oplus \operatorname{Tor}_{n-1}^S\left(\frac{S}{J_{G_v}}, \mathbb{K}\right)_{n-1+j} \longrightarrow \cdots \quad (4)$$

It is known [20, Lemma 3.2] that $\operatorname{iv}(G) > \operatorname{iv}(G_v) = \operatorname{iv}(G_v \setminus v)$ and $\operatorname{iv}(G) > \operatorname{iv}(G \setminus v)$. Thus $\beta_{n-1,n-1+j}(S/J_{G_v}) = 0 = \beta_{n-1,n-1+j}(S/((x_v, y_v) + J_{G\setminus v}))$ for $j \geq \operatorname{iv}(G) + 1$. Therefore, we obtain

$$\operatorname{Tor}_n^S\left(\frac{S}{(x_v, y_v) + J_{G_v\setminus v}}, \mathbb{K}\right)_{n-1+j} \simeq \operatorname{Tor}_{n-1}^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n-1+j} \quad \text{for } j \geq \operatorname{iv}(G) + 1.$$

If there is no tree attached to v_2, then $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v) + 2$. Therefore it follows from the equation (4) that $\beta_{n-1,n-1+\mathrm{iv}(G)}(S/J_G) \neq 0$ and $\beta_{n-1,n-1+j}(S/J_G) = 0$ for $j \geq \mathrm{iv}(G) + 1$. If there is a tree attached to v_2, then $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v) + 1$, and similarly, we have $\beta_{n-1,n-1+\mathrm{iv}(G)+1}(S/J_G) \neq 0$ and $\beta_{n-1,n-1+j}(S/J_G) = 0$ for $j \geq \mathrm{iv}(G) + 2$. By Theorem 3.6, $\mathrm{pd}(S/J_G) = n - 1$, and hence, either $\beta_{n-1,n-1+\mathrm{iv}(G)}(S/J_G)$ or $\beta_{n-1,n-1+\mathrm{iv}(G)+1}(S/J_G)$ is an extremal Betti number of S/J_G, as desired. □

For $k = 3$, we proved that $\mathrm{depth}(S/J_G) = n + 1$ if and only if there are trees attached to at least one vertex of e and in this case either $\beta_{n-1,n-1+\mathrm{iv}(G)}(S/J_G)$ or $\beta_{n-1,n-1+\mathrm{iv}(G)+1}(S/J_G)$ is an extremal Betti number. From now on, we assume that $k \geq 4$. Let $H = C_k \cup_e K_m$ for $m \geq 2$. Let G be the clique sum of H and a forest along some vertices H. First, we compute one distinguished extremal Betti number for the class of graphs G with $\mathrm{depth}(S/J_G) = n+1$, considered in Theorem 3.6.

Theorem 3.8. *Let $H = C_k \cup_e K_m$ for $k \geq 4$ and $m \geq 2$. Also, let G be the clique sum of H and a forest along some vertices of H. Let $A = \{u \in V(C_k) : \text{there is a tree incident on } u\}$. If $e \cap A \neq \emptyset$ and $G[A]$ is connected with $k-2 \leq |A| \leq k-1$, then either $\beta_{n-1,n-1+\mathrm{iv}(G)-1}(S/J_G)$ or $\beta_{n-1,n-1+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number. If $A = V(C_k)$, then $\beta_{n-1,n-1+\mathrm{iv}(G)+1}(S/J_G)$ is an extremal Betti number. In particular, $\mathrm{iv}(G) - 1 \leq \mathrm{reg}(S/J_G)$.*

Proof. Let $e = \{v, v_2\}$. Suppose that $A \cap e \neq \emptyset$, $G[A]$ is connected, and $k - 2 \leq |A| \leq k - 1$. Since $A \cap e \neq \emptyset$, we may assume that $v \in A$. Then $G \setminus v$ is a disconnected block graph on $n - 1$ vertices, and hence it follows from [13, Theorem 6] and (1) that $\beta_{p,p+\mathrm{iv}(G\setminus v)+1}(S/((x_v, y_v)+J_{G\setminus v}))$ is an extremal Betti number of $S/((x_v, y_v)+J_{G\setminus v})$, where $p = \mathrm{pd}(S/((x_v, y_v) + J_{G\setminus v}))$. We prove the assertion by induction on k. First assume that $k = 4$. By Lemma 2.3, $J_G = J_{G_v} \cap ((x_v, y_v) + J_{G\setminus v})$ where G_v belong to the class of graphs considered in Theorem 3.7. Therefore, $G_v \setminus v$ also belong to the class of graphs considered in Theorem 3.7. Hence, either $\beta_{n-1,n-1+\mathrm{iv}(G_v)}(S/J_{G_v})$ or $\beta_{n-1,n-1+\mathrm{iv}(G_v)+1}(S/J_{G_v})$ is an extremal Betti number of S/J_{G_v} and either $\beta_{n,n+\mathrm{iv}(G_v\setminus v)}(S/((x_v, y_v) + J_{G_v\setminus v}))$ or $\beta_{n,n+\mathrm{iv}(G_v\setminus v)+1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ is an extremal Betti number of $S/((x_v, y_v) + J_{G_v\setminus v})$. It is known [20, Lemma 3.2] that $\mathrm{iv}(G) > \mathrm{iv}(G_v) = \mathrm{iv}(G_v \setminus v)$ and $\mathrm{iv}(G) > \mathrm{iv}(G \setminus v)$. By virtue of [7, Theorem 1.1], $p \leq n - 1$. Hence, we have $\beta_{n-1,n-1+j}(S/J_{G_v}) = 0 = \beta_{n-1,n-1+j}(S/((x_v, y_v) + J_{G\setminus v}))$ for $j \geq \mathrm{iv}(G) + 1$. Therefore, it follows from the equation (4) that for $j \geq \mathrm{iv}(G) + 1$,

$$\mathrm{Tor}_n^S\left(\frac{S}{(x_v, y_v) + J_{G_v\setminus v}}, \mathbb{K}\right)_{n-1+j} \simeq \mathrm{Tor}_{n-1}^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n-1+j}. \tag{5}$$

Case 1: Let $A = \{v, v_2\}$ or $A = \{v, v_4\}$. By Theorem 3.7 and the equation (1), we get that $\beta_{n,n+\mathrm{iv}(G_v\setminus v)}(S/((x_v, y_v) + J_{G_v\setminus v}))$ is an extremal Betti number. In this case, $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v) + 2$. Therefore, it follows from (4) that $\beta_{n-1,n-1+\mathrm{iv}(G)-1}(S/J_G) \neq 0$.

Case 2: If $A = \{v, v_2, v_3\}$ or $A = \{v, v_4, v_3\}$, then by Theorem 3.7 and (1), we get that $\beta_{n,n+\mathrm{iv}(G_v\setminus v)}(S/((x_v, y_v) + J_{G_v\setminus v}))$ is an extremal Betti number. In this case, $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v) + 1$. Therefore, by putting $j = \mathrm{iv}(G)$ in (4), we have $\beta_{n-1,n-1+\mathrm{iv}(G)}(S/J_G) \neq 0$.

Case 3: If $A = \{v, v_2, v_4\}$, then by Theorem 3.7 and (1), $\beta_{n,n+\mathrm{iv}(G_v\setminus v)+1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ is an extremal Betti number. In this case, $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v) + 2$. Thus it follows from the equation (4) that $\beta_{n-1,n-1+\mathrm{iv}(G)}(S/J_G) \neq 0$.

For all the above three cases, it follows from (5) that $\beta_{n-1,n-1+j}(S/J_G) = 0$ for $j \geq \mathrm{iv}(G) + 1$. Hence, either $\beta_{n-1,n-1+\mathrm{iv}(G)-1}(S/J_G)$ or $\beta_{n-1,n-1+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G.

Case 4: If $A = V(C_4)$, then by Theorem 3.7 and the fact $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v) + 1$, we have $\beta_{n,n+\mathrm{iv}(G)}(S/((x_v, y_v) + J_{G_v\setminus v}))$ is an extremal Betti number of $S/((x_v, y_v) + J_{G_v\setminus v})$. Hence, it follows from (5) that $\beta_{n-1,n+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G.

Now assume that $k \geq 5$. Let K' and K'' denote complete graphs on vertex sets $N_G[v]$ and $N_G(v)$ respectively. Also, let $H' = C_{k-1} \cup_{e'} K'$ and $H'' = C_{k-1} \cup_{e'} K''$, where $e' = \{v_2, v_k\}$ and $V(C_{k-1}) = \{v_2, \dots, v_k\}$. Then G_v (resp. $G_v \setminus v$) is the clique sum of H' (resp. H'') and a forest along some vertices of G. Clearly, $G_v[A \setminus \{v\}]$ and $G_v \setminus v[A \setminus \{v\}]$ are both connected with $k-3 \leq |A \setminus \{v\}| \leq k-2$, and so G_v and $G_v \setminus v$ satisfy induction hypotheses. Therefore by induction either $\beta_{n-1,n-1+\mathrm{iv}(G_v)-1}(S/J_{G_v})$ or $\beta_{n-1,n-1+\mathrm{iv}(G_v)}(S/J_{G_v})$ is an extremal Betti number. Also, by induction and (1), either $\beta_{n,n+\mathrm{iv}(G_v\setminus v)-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ or $\beta_{n,n+\mathrm{iv}(G_v\setminus v)}(S/((x_v, y_v) + J_{G_v\setminus v}))$ is an extremal Betti number. Note that $\mathrm{iv}(G) = \mathrm{iv}(G_v) + 1 = \mathrm{iv}(G_v \setminus v) + 1$. Therefore the assertion follows from the equations (4) and (5). Suppose now that $A = V(C_k)$. Then clearly trees are attached to all the vertices of C_{k-1} in both G_v and $G_v \setminus v$. Thus by induction and (1), $\beta_{n-1,n-1+\mathrm{iv}(G_v)+1}(S/J_{G_v})$ and $\beta_{n,n+\mathrm{iv}(G_v\setminus v)+1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are extremal Betti numbers of S/J_{G_v} and $S/((x_v, y_v) + J_{G_v\setminus v})$ respectively. Therefore it follows from (5) and the fact $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v) + 1$ that $\beta_{n-1,n+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G. $\square$

By Theorem 3.6, we have that $\mathrm{depth}(S/J_G) \geq n$. We now characterize graphs attaining the lower bound. Also, we give a lower bound for the regularity of S/J_G by computing one distinguished extremal Betti number of S/J_G. First, we consider the case $m \geq 3$.

Theorem 3.9. *Let $H = C_k \cup_e K_m$ for $k \geq 4$ and $m \geq 3$. Let G be the clique sum of H and a forest along some vertices of H. Let $A = \{u \in V(C_k) : \text{there is a tree incident on } u\}$. Suppose either $A \cap e = \emptyset$ or if $A \cap e \neq \emptyset$, then A does not contain any $k-2$ consecutive vertices. Then either $\beta_{n,n+\mathrm{iv}(G)-1}(S/J_G)$ or $\beta_{n,n+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G. In particular,* $\mathrm{depth}(S/J_G) = n$ *and* $\mathrm{iv}(G) - 1 \leq \mathrm{reg}(S/J_G)$.

Proof. Let $e = \{v, v_2\}$. Then $N_{C_k}(v) = \{v_2, v_k\}$. We proceed by induction on k. Let $k = 4$. First assume that $A \cap e \neq \emptyset$ and A does not contain any 2 consecutive vertices. If $v \in A$, then v is an internal vertex in G and by Lemma 2.3, we can write $J_G = J_{G_v} \cap ((x_v, y_v) + J_{G\setminus v})$. It can be noted that G_v and $G_v \setminus v$ are generalized block graphs on n and $n-1$ vertices respectively. If $v_2 \in A$, then v_2 is an internal vertex in G and it follows from Lemma 2.3 that $J_G = J_{G_{v_2}} \cap ((x_{v_2}, y_{v_2}) + J_{G\setminus v_2})$. We can make similar conclusion about G_{v_2} and $G_{v_2} \setminus v_2$. Now, suppose $A \cap e = \emptyset$. If $A = \{v_3\}$, then G_v and $G_v \setminus v$ are generalized block graphs and if $A = \{v_4\}$, then G_{v_2} and $G_{v_2} \setminus v_2$ are generalized block graphs. If $A \cap e \neq \emptyset$ and $v \in A$ or $A = \{v_3\}$, then set $w = v$. If $A \cap e \neq \emptyset$ and $v_2 \in A$ or $A = \{v_4\}$, then set $w = v_2$. Then by [17, Theorem 3.2] and (1), $\mathrm{pd}(S/J_{G_w}) = n$ and $\mathrm{pd}(S/((x_w, y_w) + J_{G_w\setminus w})) = n+1$. Hence it follows from [19, Theorem 3.4] that $\beta_{n,n+\mathrm{iv}(G_w)}(S/J_{G_w})$ and $\beta_{n+1,n+1+\mathrm{iv}(G_w\setminus w)}(S/((x_w, y_w) + J_{G_w\setminus w}))$ are extremal Betti numbers of S/J_{G_w} and $S/((x_w, y_w) + J_{G_w\setminus w})$ respectively. Since w is not a simplicial vertex, we consider the long exact sequence (3) for $i = n$:

$$0 \longrightarrow \mathrm{Tor}_{n+1}^S\left(\frac{S}{(x_w, y_w) + J_{G_w\setminus w}}, \mathbb{K}\right)_{n+j} \longrightarrow \mathrm{Tor}_n^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n+j} \longrightarrow$$
$$\longrightarrow \mathrm{Tor}_n^S\left(\frac{S}{(x_w, y_w) + J_{G\setminus w}}, \mathbb{K}\right)_{n+j} \oplus \mathrm{Tor}_n^S\left(\frac{S}{J_{G_w}}, \mathbb{K}\right)_{n+j} \longrightarrow \cdots \quad (6)$$

By virtue of [20, Lemma 3.2], we have $\operatorname{iv}(G) > \operatorname{iv}(G_w) = \operatorname{iv}(G_w \setminus w)$ and $\operatorname{iv}(G) > \operatorname{iv}(G \setminus w)$. Hence, $\beta_{n,n+j}(S/J_{G_w}) = 0$ for $j \geq \operatorname{iv}(G)$. Since $G \setminus w$ is a block graph on $n-1$, by [7, Theorem 1.1], $\operatorname{pd}(S/((x_w, y_w) + J_{G\setminus w})) \leq n$. Now it follows from [13, Theorem 6] and (1) that $\beta_{n,n+j}(S/((x_w, y_w) + J_{G\setminus w})) = 0$ for $j \geq \operatorname{iv}(G) + 1$. Therefore, we get the isomorphism:

$$\operatorname{Tor}_{n+1}^S\left(\frac{S}{(x_w, y_w) + J_{G_w\setminus w}}, \mathbb{K}\right)_{n+j} \simeq \operatorname{Tor}_n^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n+j} \quad \text{for} j \geq \operatorname{iv}(G) + 1. \tag{7}$$

If $v_3 \in A$ or $v_4 \in A$, then note that $\operatorname{iv}(G) = \operatorname{iv}(G_w \setminus w) + 1$, otherwise $\operatorname{iv}(G) = \operatorname{iv}(G_w \setminus w) + 2$. Therefore it follows from (6) and (7) that either $\beta_{n,n+\operatorname{iv}(G)}(S/J_G) \neq 0$ or $\beta_{n,n+\operatorname{iv}(G)-1}(S/J_G) \neq 0$ and $\beta_{n,n+j}(S/J_G) = 0$ for $j \geq \operatorname{iv}(G)+1$. Hence, either $\beta_{n,n+\operatorname{iv}(G)-1}(S/J_G)$ or $\beta_{n,n+\operatorname{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G.

Now the last case is $A = \{v_3, v_4\}$. Then G_v and $G_v \setminus v$ belong to class of graphs considered in Theorem 3.7. Hence, $\beta_{n-1,n-1+\operatorname{iv}(G_v)}(S/J_{G_v})$ and $\beta_{n,n+\operatorname{iv}(G_v\setminus v)}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are extremal Betti numbers. Note that $\operatorname{iv}(G) = \operatorname{iv}(G_v \setminus v) + 1$. Therefore, $\beta_{n,n+j}(S/((x_v, y_v) + J_{G_v\setminus v})) = 0$ for $j \geq \operatorname{iv}(G)$. Since $G \setminus v$ is a block graph on $n-1$, by [7, Theorem 1.1], $\operatorname{pd}(S/((x_v, y_v) + J_{G\setminus v})) = n$. Therefore, it follows from the long exact sequence (3) that

$$\operatorname{Tor}_n^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n+j} \simeq \operatorname{Tor}_n^S\left(\frac{S}{(x_v, y_v) + J_{G\setminus v}}, \mathbb{K}\right)_{n+j} \quad \text{for } j \geq \operatorname{iv}(G).$$

Since $\operatorname{iv}(G) = \operatorname{iv}(G \setminus v) + 1$, by the help of [13, Theorem 6] and the equation (1), we get that $\beta_{n,n+\operatorname{iv}(G)}(S/((x_v, y_v) + J_{G\setminus v}))$ is an extremal Betti number of $S/((x_v, y_v) + J_{G\setminus v})$. Therefore, $\beta_{n,n+\operatorname{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G.

Now we assume that $k \geq 5$. Suppose, either $A \cap e = \emptyset$ or if $A \cap e \neq \emptyset$, then A does not contain any $k-2$ consecutive vertices. Let K' and K'' be complete graphs on vertex sets $N_G[v]$ and $N_G(v)$ respectively. Also, let $H' = C_{k-1} \cup_{e'} K'$ and $H'' = C_{k-1} \cup_{e'} K''$, where $e' = \{v_2, v_k\}$ and $V(C_{k-1}) = \{v_2, \ldots, v_k\}$. Then G_v (resp. $G_v \setminus v$) is the clique sum of H' (resp. H'') and a forest along some vertices of G. Obviously, $A \setminus \{v\} \subseteq V(C_{k-1})$ is the set of vertices at which trees are attached in both G_v and $G_v \setminus v$.

Case 1: If $|A| \leq k-4$, then clearly G_v and $G_v \setminus v$ satisfy induction hypotheses.

Case 2: Let $|A| = k-3$. If $v \in A$, then also G_v and $G_v \setminus v$ satisfy induction hypotheses. If $v \notin A$ and $v_2 \in A$, then G_{v_2} and $G_{v_2} \setminus v_2$ satisfy induction hypotheses. Let $v, v_2 \notin A$. Then $v_3 \in A$ or $v_k \in A$. If $v_3 \in A$, then G_v, $G_v \setminus v$ and if $v_k \in A$, then G_{v_2}, $G_{v_2} \setminus v_2$ satisfy induction hypotheses.

Case 3: Let $|A| = k-2$ with $A \cap e \neq \emptyset$. If $v \in A$, then G_v and $G_v \setminus v$ satisfy induction hypotheses, and if $v \notin A$, then G_{v_2} and $G_{v_2} \setminus v_2$ satisfy induction hypotheses.

If G_v satisfies induction hypotheses, then set $w = v$, and if G_{v_2} satisfies induction hypotheses then set $w = v_2$. Now we apply induction on G_w and $G_w \setminus w$. Therefore, either $\beta_{n,n+\operatorname{iv}(G_w)-1}(S/J_{G_w})$ or $\beta_{n,n+\operatorname{iv}(G_w)}(S/J_{G_w})$ is an extremal Betti number of S/J_{G_w}. Also, by induction and the equation (1), either $\beta_{n+1,n+1+\operatorname{iv}(G_w\setminus w)-1}(S/((x_w, y_w) + J_{G_w\setminus w}))$ or $\beta_{n+1,n+1+\operatorname{iv}(G_w\setminus w)}(S/((x_w, y_w) + J_{G_w\setminus w}))$ is an extremal Betti number of $S/((x_w, y_w) + J_{G_w\setminus w})$. Here, it can be observed that $\operatorname{iv}(G) = \operatorname{iv}(G_w) + 1 = \operatorname{iv}(G_w \setminus w) + 1$. Hence, $\beta_{n,n+j}(S/J_{G_w}) = 0$ for $j \geq \operatorname{iv}(G)$. Since $G \setminus w$ is a block graph, it follows from [7, Theorem 1.1] and [13, Theorem 6] that $\beta_{n,n+j}(S/((x_w, y_w) + J_{G\setminus w})) = 0$ for $j \geq \operatorname{iv}(G) + 1$. Therefore, we get from (6) that:

$$\operatorname{Tor}_{n+1}^S\left(\frac{S}{(x_w, y_w) + J_{G_w\setminus w}}, \mathbb{K}\right)_{n+j} \simeq \operatorname{Tor}_n^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n+j} \quad \text{for } j \geq \operatorname{iv}(G) + 1. \tag{8}$$

Hence, it follows from (6) and (8) that either $\beta_{n,n+\mathrm{iv}(G)-1}(S/J_G)$ or $\beta_{n,n+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G.

Case 4: The last case is $A \cap e = \emptyset$ and $|A| = k-2$. Then G_v and $G_v \setminus v$ are graphs such that there are trees attached to $k-2$ consecutive vertices with $v_k \in e' \cap A \setminus \{v\}$. Then by Theorem 3.8, we have that either $\beta_{n-1,n-1+\mathrm{iv}(G_v)-1}(S/J_{G_v})$ or $\beta_{n-1,n-1+\mathrm{iv}(G_v)}(S/J_{G_v})$ is an extremal Betti number of S/J_{G_v} and either $\beta_{n,n+\mathrm{iv}(G_v\setminus v)-1}(S/((x_v,y_v)+J_{G_v\setminus v}))$ or $\beta_{n,n+\mathrm{iv}(G_v\setminus v)}(S/((x_v,y_v)+J_{G_v\setminus v}))$ is an extremal Betti number of $S/((x_v,y_v)+J_{G_v\setminus v})$. Since, $\mathrm{iv}(G) > \mathrm{iv}(G_v \setminus v)$, $\beta_{n,n+j}(S/((x_v,y_v)+J_{G_v\setminus v})) = 0$ for $j \geq \mathrm{iv}(G)$. By [13, Theorem 6] and (1), $\beta_{n,n+\mathrm{iv}(G\setminus v)+1}(S/((x_v,y_v)+J_{G\setminus v}))$ is an extremal Betti number as $G \setminus v$ is a block graph on $n-1$ vertices. Therefore, it follows from the long exact sequence (3) that:

$$\mathrm{Tor}_n^S\left(\frac{S}{J_G},\mathbb{K}\right)_{n+j} \simeq \mathrm{Tor}_n^S\left(\frac{S}{(x_v,y_v)+J_{G\setminus v}},\mathbb{K}\right)_{n+j} \quad \text{for } j \geq \mathrm{iv}(G).$$

Note that $\mathrm{iv}(G) = \mathrm{iv}(G\setminus v)+1$. Hence, $\beta_{n,n+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G. Therefore, $\mathrm{pd}(S/J_G) \geq n$, and so we have $\mathrm{depth}(S/J_G) \leq n$. Hence, by Theorem 3.6, $\mathrm{depth}(S/J_G) = n$, as desired. $\square$

Remark 3.10. Let $H = C_k \cup_e K_m$ for $k \geq 3$ and $m \geq 3$. Let G be the clique sum of H and a forest along some vertices of H. Let $A = \{u \in V(C_k) : \text{there is a tree incident on u}\}$. By [2, Theorems 3.19 and 3.20] and Theorem 3.6, $n \leq \mathrm{depth}(S/J_G) \leq n+1$. Moreover, it follows from Theorems 3.6, 3.7 and 3.9 that $A \cap e \neq \emptyset$ and $G[A]$ is connected with $k-2 \leq |A|$ if and only if $\mathrm{depth}(S/J_G) = n+1$.

Now we consider the case $m = 2$. Let G be a unicyclic graph of girth $k(\geq 4)$. If there are trees attached to $k-2$ consecutive vertices, then by Theorem 3.6, $\mathrm{depth}(S/J_G) = n+1$. We now characterize unicyclic graph G such that $\mathrm{depth}(S/J_G) = n$.

Theorem 3.11. *Let G be a unicyclic graph of girth k for $k \geq 4$. Let $A = \{u \in V(C_k) :$ there is a tree incident on $u\}$. If A does not contain any $k-2$ consecutive vertices, then either $\beta_{n,n+\mathrm{iv}(G)-1}(S/J_G)$ or $\beta_{n,n+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G. In particular,* $\mathrm{depth}(S/J_G) = n$ *and* $\mathrm{iv}(G) - 1 \leq \mathrm{reg}(S/J_G)$.

Proof. Suppose that A does not contain any $k-2$ consecutive vertices. Since $A \neq \emptyset$, we may assume that $v \in A$. Then $G \setminus v$ is a disconnected block graph, and hence by [7, Theorem 1.1] and (1), $p = \mathrm{pd}(S/((x_v,y_v)+J_{G\setminus v})) \leq n-1$. We prove the theorem by induction on k. First assume that $k = 4$. Then $v_2, v_k \notin A$, and hence G_v and $G_v \setminus v$ are generalized block graphs on n and $n-1$ vertices respectively. By virtue of [17, Theorem 3.2] and (1), we get that $\mathrm{pd}(S/J_{G_v}) = n$ and $\mathrm{pd}(S/((x_v,y_v)+J_{G_v\setminus v})) = n+1$. Hence, by [19, Theorem 3.4], $\beta_{n,n+\mathrm{iv}(G_v)}(S/J_{G_v})$ is an extremal Betti number of S/J_{G_v} and $\beta_{n+1,n+1+\mathrm{iv}(G_v\setminus v)}(S/((x_v,y_v)+J_{G_v\setminus v}))$ is an extremal Betti number of $S/((x_v,y_v)+J_{G_v\setminus v})$. Since $\mathrm{iv}(G) > \mathrm{iv}(G_v)$, $\beta_{n,n+j}(S/J_{G_v}) = 0$ for $j \geq \mathrm{iv}(G)$. Thus, we have the following isomorphism from the long exact sequence (3):

$$\mathrm{Tor}_{n+1}^S\left(\frac{S}{(x_v,y_v)+J_{G_v\setminus v}},\mathbb{K}\right)_{n+j} \simeq \mathrm{Tor}_n^S\left(\frac{S}{J_G},\mathbb{K}\right)_{n+j} \quad \text{for} j \geq \mathrm{iv}(G). \tag{9}$$

If $v_3 \in A$, then $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v)+1$, otherwise $\mathrm{iv}(G) = \mathrm{iv}(G_v\setminus v)+2$. Therefore, it follows from (3) and (9) that either $\beta_{n,n+\mathrm{iv}(G)}(S/J_G)$ or $\beta_{n,n+\mathrm{iv}(G)-1}(S/J_G)$ is an extremal Betti number of S/J_G.

Now we assume that $k \geq 5$. Let $n_v = |N_G(v)|$. Then $n_v \geq 3$. Let $H' = C_{k-1} \cup_{e'} K_{n_v+1}$ and $H'' = C_{k-1} \cup_{e'} K_{n_v}$ where $e' = \{v_2, v_k\}$ and C_{k-1} is a cycle on $\{v_2, \dots, v_k\}$. Therefore, G_v (resp. $G_v \setminus v$) is the clique sum of H' (resp. H'') and a forest along some vertices of G. It can be easily seen that for G_v (resp. $G_v \setminus v$), $A \setminus v \subset V(C_{k-1})$ is the set of vertices along which trees are attached to H' (resp. H''). Set $A' = A \setminus v$. If $v_2, v_k \notin A'$, then $A' \cap e' = \emptyset$. Otherwise, if $A' \cap e' \neq \emptyset$, then clearly A' does not contain any $k-3$ consecutive vertices. Therefore, by Theorem 3.9, either $\beta_{n,n+\mathrm{iv}(G_v)-1}(S/J_{G_v})$ or $\beta_{n,n+\mathrm{iv}(G_v)}(S/J_{G_v})$ is an extremal Betti number of S/J_{G_v} and either $\beta_{n+1,n+1+\mathrm{iv}(G_v\setminus v)-1}(S/((x_v, y_v) + J_{G_v \setminus v}))$ or $\beta_{n+1,n+1+\mathrm{iv}(G_v\setminus v)}(S/((x_v, y_v) + J_{G_v \setminus v}))$ is an extremal Betti number of $S/((x_v, y_v) + J_{G_v \setminus v})$. Note that $\mathrm{iv}(G) = \mathrm{iv}(G_v \setminus v) + 1$. Now the assertion follows from (3) and (9). □

Therefore, from Theorems 3.6 and 3.11, we can conclude the following result for unicyclic graphs.

Corollary 3.12. *Let G be a unicyclic graph on the vertex set $[n]$ of girth $k \geq 4$. Then $n \leq \mathrm{depth}(S/J_G) \leq n+1$. Moreover, if there are trees attached to $k-2$ consecutive vertices of the cycle in G, then $\mathrm{depth}(S/J_G) = n+1$, otherwise $\mathrm{depth}(S/J_G) = n$.*

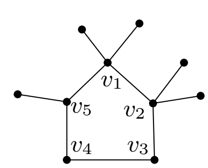
Also, we can conclude from Theorems 3.8 and 3.11 that.

Corollary 3.13. *Let G be a unicyclic graph of girth $k \geq 4$ with $\mathrm{pd}(S/J_G) = p$. If trees are attached to every vertex of the cycle in G, then $\beta_{p,p+\mathrm{iv}(G)+1}(S/J_G)$ is an extremal Betti number of S/J_G, and hence $\mathrm{iv}(G) + 1 \leq \mathrm{reg}(S/J_G)$. Otherwise, either $\beta_{p,p+\mathrm{iv}(G)-1}(S/J_G)$ or $\beta_{p,p+\mathrm{iv}(G)}(S/J_G)$ is an extremal Betti number of S/J_G, and hence $\mathrm{iv}(G) - 1 \leq \mathrm{reg}(S/J_G)$.*

4. Cycles with whiskers

Let G be a graph with the vertex set $V(G)$ and $v \in V(G)$. Let $u_1, \dots, u_r$ be new vertices and we define $G \cup W^r(v)$ to be the graph with vertex set $V(G \cup W^r(v)) = V(G) \cup \{u_1, \dots, u_r\}$ and edge set $E(G \cup W^r(v)) = E(G) \cup \{\{u_i, v\} : 1 \leq i \leq r\}$ i.e., $G \cup W^r(v)$ is the graph obtained from G by attaching r whiskers or pendant edges at the vertex v. If $r = 0$, then $G \cup W^0(v) = G$. Let $v_1, \dots, v_s \in V(G)$. Then in a similar way, we can attach r_i whiskers at v_i for $1 \leq i \leq s$. We denote this graph by $G \cup (\cup_{i=1}^s W^{r_i}(v_i))$. Algebraic effect of attaching whiskers to a graph has already been studied for the case of monomial edge ideals, see [10], [25] and [36]. The algebraic effect of attaching whiskers to a graph has been studied also for binomial edge ideals. You can see, for instance, [5, 28, 29, 38]. Here we study the binomial edge ideals of graphs with whiskers attached.

The graph G, given on the right, is $G = C_5 \cup W^2(v_1) \cup W^2(v_2) \cup W^1(v_5)$.



G

Let $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ for $r_i \geq 0$. Then $n = k + \sum_{i=1}^k r_i$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $A = \emptyset$, then $G = C_k$ and in this case, Zafar and Zahid [38, Corollary 16] proved that $\mathrm{reg}(S/J_G) = k-2$. So, in the rest of the section we assume that $A \neq \emptyset$. If $k = 3$, then $1 \leq \mathrm{iv}(G) \leq 3$, and hence by [13, Theorem 8], $2 \leq \mathrm{reg}(S/J_G) = 1 + \mathrm{iv}(G) \leq 4$. In this section, we generalize their result for $k \geq 4$ and prove that the regularity of S/J_G is bounded below by $k-1$ and bounded above by $k+1$. We then characterize graphs G with $\operatorname{reg}(S/J_G) = k-1$, $\operatorname{reg}(S/J_G) = k$ and $\operatorname{reg}(S/J_G) = k+1$. We also classify G which admits a unique extremal Betti number.

Theorem 4.1. *Let $H = K_m \cup_e C_k$ for $m \geq 2$, $k \geq 3$. Let $G = H \cup (\cup_{i=1}^{k} W^{r_i}(v_i))$ for $r_i \geq 0$, and suppose that $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $A \neq \emptyset$, then $k-1 \leq \operatorname{reg}(S/J_G) \leq k+1$.*

Proof. Note that G contains an induced path of length $k-1$. Then the lower bound follows from [24, Corollary 2.3]. Let $e = \{v, v_2\}$. Then $G \backslash v$ is the graph $K_{m-1} \cup_{v_2} P_{k-1} \cup (\cup_{i=2}^{k} W^{r_i}(v_i))$ with r_1 isolated vertices, where $V(P_{k-1}) = \{v_2, \ldots, v_k\}$. Since $\operatorname{iv}(G \setminus v) \leq k-1$, it follows from [13, Theorem 8] and (1) that $\operatorname{reg}(S/((x_v, y_v) + J_{G \backslash v})) \leq k$. We prove the upper bound by induction on k. Assume that $k = 3$. If $m = 2$, then the assertion follows from [13, Theorem 8]. Suppose now that $m \geq 3$. Since v is an internal vertex, by Lemma 2.3, $J_G = J_{G_v} \cap ((x_v, y_v) + J_{G \backslash v})$, where $G_v = K_{m+r_1+1} \cup W^{r_2}(v_2) \cup W^{r_3}(v_3)$. Therefore, $G_v \setminus v = K_{m+r_1} \cup W^{r_2}(v_2) \cup W^{r_3}(v_3)$. Hence, by [13, Theorem 8], $\operatorname{reg}(S/J_{G_v}) = \operatorname{iv}(G_v) + 1 = \operatorname{iv}(G_v \setminus v) + 1 = \operatorname{reg}(S/((x_v, y_v) + J_{G_v \backslash v})) \leq 3$. We apply Lemma 2.2 on the short exact sequence (2) to get that $\operatorname{reg}(S/J_G) \leq 4$. If $A = \{v\}$, then $G_v = K_{m+r_1+1}$, $G_v \setminus v = K_{m+r_1}$ and $G \setminus v = K_{m-1} \cup_{v_2} P_2$ with r_1 isolated vertices. Therefore, $\operatorname{reg}(S/J_{G_v}) = 1 = \operatorname{reg}(S/((x_v, y_v) + J_{G_v \backslash v}))$ and $\operatorname{reg}(S/((x_v, y_v) + J_{G \backslash v})) = 2$. Thus it follows from Lemma 2.2 and the short exact sequence (2) that $\operatorname{reg}(S/J_G) = 2$.

Now assume that $k \geq 4$. Note that $G_v = K_{m+r_1+1} \cup_{\{v_2, v_k\}} C_{k-1} \cup (\cup_{i=2}^{k} W^{r_i}(v_i))$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_2, v_k\}} C_{k-1} \cup (\cup_{i=2}^{k} W^{r_i}(v_i))$, where $V(C_{k-1}) = \{v_2, \ldots, v_k\}$. If $A = \{v\}$, then $G_v = K_{m+r_1+1} \cup_{\{v_2, v_k\}} C_{k-1}$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_2, v_k\}} C_{k-1}$. Thus, by [15, Theorem 3.12], $\operatorname{reg}(S/J_{G_v}) = \operatorname{reg}(S/((x_v, y_v) + J_{G_v \backslash v})) = k-2$. Also, in this case $G \setminus v = K_{m-1} \cup_{v_2} P_{k-1}$ with r_1 isolated vertices, and hence by Proposition 3.5 and (1), $\operatorname{reg}(S/((x_v, y_v) + J_{G \backslash v})) \leq k-1$. Therefore, by Lemma 2.2 and the short exact sequence (2), we have that $\operatorname{reg}(S/J_G) \leq k-1$. Hence, if $A = \{v\}$, then $\operatorname{reg}(S/J_G) = k-1$. Let $v_i \in A$ for some $2 \leq i \leq k$. Then by induction, $\operatorname{reg}(S/J_{G_v}) \leq k$ and $\operatorname{reg}(S/((x_v, y_v) + J_{G_v \backslash v})) \leq k$. Thus, Lemma 2.2 and the short exact sequence (2) together imply that $\operatorname{reg}(S/J_G) \leq k+1$. $\square$

Considering $m = 2$ in Theorem 4.1, we obtain bounds for cycles with whiskers.

Corollary 4.2. *Let $k \geq 3$ and $G = C_k \cup (\cup_{i=1}^{k} W^{r_i}(v_i))$, $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $A \neq \emptyset$, then $k-1 \leq \operatorname{reg}(S/J_G) \leq k+1$. Moreover, if $|A| = 1$, then $\operatorname{reg}(S/J_G) = k-1$.*

We now characterize G with $\operatorname{reg}(S/J_G) = k+1$. First, we prove the following Proposition.

Proposition 4.3. *Let $H = K_m \cup_e C_k$ for $m \geq 2$, $k \geq 3$. Let $G = H \cup (\cup_{i=1}^{k} W^{r_i}(v_i))$ for $r_i \geq 0$, and suppose that $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $e \nsubseteq A$, then $\operatorname{reg}(S/J_G) \leq k$.*

Proof. Let $e = \{v, v_2\}$. We assume that $v_2 \notin A$, i.e., $r_2 = 0$. We proceed by induction on k. Note that $G \setminus v = K_{m-1} \cup_{v_2} P_{k-1} \cup (\cup_{i=3}^{k} W^{r_i}(v_i))$ with r_1 isolated vertices. Hence by [13, Theorem 8] and (1), $\operatorname{reg}(S/((x_v, y_v) + J_{G \backslash v})) \leq k$. For $k = 3$, $G_v = K_{m+r_1+1} \cup W^{r_3}(v_3)$ and $G_v \setminus v = K_{m+r_1} \cup W^{r_3}(v_3)$. Then by virtue of [13, Theorem 8], $\operatorname{reg}(S/J_{G_v}) = \operatorname{reg}(S/((x_v, y_v) + J_{G_v \backslash v})) \leq 2$. Therefore, by applying Lemma 2.2 on the short exact sequence (2), we get $\operatorname{reg}(S/J_G) \leq 3$. Now suppose that $k \geq 4$. Then $G_v = K_{m+r_1+1} \cup_{\{v_2, v_k\}} C_{k-1} \cup (\cup_{i=3}^{k} W^{r_i}(v_i))$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_2, v_k\}} C_{k-1} \cup (\cup_{i=3}^{k} W^{r_i}(v_i))$. Hence by induction, $\operatorname{reg}(S/J_{G_v}) \leq k-1$ and $\operatorname{reg}(S/((x_v, y_v) + J_{G_v \backslash v})) \leq k-1$. Therefore, it follows from Lemma 2.2 and the short exact sequence (2) that $\operatorname{reg}(S/J_G) \leq k$. $\square$

Corollary 4.4. *Let $k \geq 3$ and $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$, $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. Then $A = V(C_k)$ if and only if $\operatorname{reg}(S/J_G) = k+1$. Moreover, in this case, S/J_G admits a unique extremal Betti number.*

Proof. First, we assume that whiskers are attached at every vertex of C_k i.e., $r_i \geq 1$ for all $1 \leq i \leq k$. For $k = 3$, by [13, Theorem 8], we have that $\beta_{n-1,n-1+4}(S/J_G)$ is the unique extremal Betti number. For $k \geq 4$, by Theorem 3.8, $\beta_{n-1,n-1+k+1}(S/J_G)$ is an extremal Betti number, which further implies that $k+1 \leq \operatorname{reg}(S/J_G)$. By Corollary 4.2, $\operatorname{reg}(S/J_G) \leq k+1$. Hence, $\operatorname{reg}(S/J_G) = k+1$ and S/J_G admits a unique extremal Betti number. For the converse part, suppose there exists $i \in [k]$ such that $r_i = 0$. Then by Proposition 4.3, $\operatorname{reg}(S/J_G) \leq k$, which is a contradiction. Hence, the assertion follows. $\square$

If $\emptyset \neq A \subsetneq V(C_k)$, then by Corollary 4.2 and Proposition 4.3, $k-1 \leq \operatorname{reg}(S/J_G) \leq k$. We now characterize G with $\operatorname{reg}(S/J_G) = k-1$ and $\operatorname{reg}(S/J_G) = k$.

Theorem 4.5. *Let $k \geq 4$ and $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ for $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If either $|A| = 1$ or $|A| = 2$ and vertices of A are adjacent, then $\operatorname{reg}(S/J_G) = k-1$. Moreover, in this case, S/J_G admits a unique extremal Betti number.*

Proof. Suppose $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ and whiskers are attached only at one vertex. Then by Corollary 4.2, $\operatorname{reg}(S/J_G) = k-1$. Now assume that whiskers are attached only at two adjacent vertices of C_k, say $G = C_k \cup W^{r_1}(v) \cup W^{r_2}(v_2)$ for $r_1, r_2 \geq 1$. Note that $G \setminus v$ is the graph $P_{k-1} \cup W^{r_2}(v_2)$ with r_1 isolated vertices, where $V(P_{k-1}) = \{v_2, \ldots, v_k\}$. Thus by [5, Theorem 4.1] and (1), $\operatorname{reg}(S/((x_v, y_v) + J_{G\setminus v})) = k-1$. Here, $G_v = K_{r_1+3} \cup_{\{v_2,v_k\}} (C_{k-1} \cup W^{r_2}(v_2))$ and $G_v \setminus v = K_{r_1+2} \cup_{\{v_2,v_k\}} (C_{k-1} \cup W^{r_2}(v_2))$. Then it follows from the proof of Theorem 4.1 that $\operatorname{reg}(S/J_{G_v}) = k-2$ and $\operatorname{reg}(S/((x_v, y_v) + J_{G_v\setminus v})) = k-2$. Therefore by Lemma 2.2 and the short exact sequence (2), $\operatorname{reg}(S/J_G) \leq k-1$. Hence, $\operatorname{reg}(S/J_G) = k-1$. Now it follows from Corollary 3.13 that $\beta_{p,p+k-1}(S/J_G)$ is the unique extremal Betti number of S/J_G, where $p = \operatorname{pd}(S/J_G)$. $\square$

Now we prove that if G does not belong to the class of graphs considered in Theorem 4.5, then $\operatorname{reg}(S/J_G) = k$. To prove this, we first need to compute extremal Betti number of some intermediate graphs.

Proposition 4.6. *Let $G = C_k \cup_e K_m \cup_{e'} K_{m'}$ for $k, m, m' \geq 3$. Then $\beta_{n,n+k}(S/J_G)$ is the unique extremal Betti number of S/J_G.*

Proof. Let $e = \{v, v_2\}$. It is enough to prove that $n \leq \operatorname{depth}(S/J_G)$, $\operatorname{reg}(S/J_G) \leq k$ and $\beta_{n,n+k}(S/J_G) \neq 0$. We prove this by induction on k. Assume that $k = 3$. Since $e \cap e' \neq \emptyset$, we assume that $e' = \{v, v_3\}$. Then, it can be seen that $G_v = K_n$, $G_v \setminus v = K_{n-1}$ and $G \setminus v = K_{m-1} \cup_{v_2} P_2 \cup_{v_3} K_{m'-1}$. Thus, we have that $\operatorname{depth}((S/J_{G_v})) = n+1$, $\operatorname{depth}(S/((x_v, y_v) + J_{G_v\setminus v})) = n$ and $\operatorname{reg}((S/J_{G_v})) = \operatorname{reg}(S/((x_v, y_v) + J_{G_v\setminus v})) = 1$. Moreover, $\beta_{n-1,n}(S/J_{G_v})$ and $\beta_{n,n+1}(S/((x_v, y_v)+J_{G_v\setminus v}))$ are the unique extremal Betti numbers of S/J_{G_v} and $S/((x_v, y_v)+J_{G_v\setminus v})$, respectively. By Proposition 3.5, $\operatorname{reg}(S/((x_v, y_v) + J_{G\setminus v})) = 3$. Also, it follows from [7, Theorem 3.1] that $J_{G\setminus v}$ is Cohen-Macaulay. Hence, $\operatorname{depth}(S/((x_v, y_v) + J_{G\setminus v})) = n$ and $\beta_{n,n+3}(S/((x_v, y_v) + J_{G\setminus v}))$ is the unique extremal Betti number. Therefore, by applying Lemmas 2.1 and 2.2 on the short exact sequence (2), we get that $\operatorname{depth}(S/J_G) \geq n$ and $\operatorname{reg}(S/J_G) \leq 3$. Now it follows from the long exact sequence (3) for $i = n$ and $j = 3$ that

$$\operatorname{Tor}_n^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n+3} \simeq \operatorname{Tor}_n^S\left(\frac{S}{(x_v, y_v) + J_{G\setminus v}}, \mathbb{K}\right)_{n+3} \neq 0.$$

Therefore, $\beta_{n,n+3}(S/J_G) \neq 0$. Now assume that $k \geq 4$.

Case 1: Let $e \cap e' \neq \emptyset$. Suppose $v \in e \cap e'$. Then $e' = \{v, v_k\}$. Note that $G_v = K_{m+m'-1} \cup_{\{v_2,v_k\}} C_{k-1}$, $G_v \setminus v = K_{m+m'-2} \cup_{\{v_2,v_k\}} C_{k-1}$ and $G \setminus v = K_{m-1} \cup_{v_2} P_{k-1} \cup_{v_k} K_{m'-1}$. Thus, by virtue of [15, Proposition 3.11], we have $\operatorname{reg}(S/J_{G_v}) = k-2 = \operatorname{reg}(S/((x_v, y_v) + J_{G_v\setminus v}))$. By Proposition 3.4, $\operatorname{depth}(S/J_{G_v}) = n$, $\operatorname{depth}(S/((x_v, y_v) + J_{G_v\setminus v})) = n-1$ and $\beta_{n,n+k-2}(S/J_{G_v})$, $\beta_{n+1,n+1+k-2}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are the unique extremal Betti numbers. It follows from Proposition 3.5 that $\operatorname{reg}(S/((x_v, y_v) + J_{G\setminus v})) = k$. By virtue of [7, Theorem 3.1], $J_{G\setminus v}$ is Cohen-Macaulay. Therefore, $\operatorname{depth}(S/((x_v, y_v) + J_{G\setminus v})) = n$ and $\beta_{n,n+k}(S/((x_v, y_v) + J_{G\setminus v}))$ is the unique extremal Betti number. Hence, by using Lemmas 2.1 and 2.2 on the short exact sequence (2), we have $n \leq \operatorname{depth}(S/J_G)$ and $\operatorname{reg}(S/J_G) \leq k$. Consider the long exact sequence (3) for $i = n$, $j = k$ and we get that

$$\operatorname{Tor}_n^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n+k} \simeq \operatorname{Tor}_n^S\left(\frac{S}{(x_v, y_v) + J_{G\setminus v}}, \mathbb{K}\right)_{n+k} \neq 0.$$

Case 2: Let $e \cap e' = \emptyset$. Let $e' = \{v_i, v_{i+1}\}$ for $i \geq 3$. It can be noted that $G_v = K_{m+1} \cup_{\{v_2,v_k\}} C_{k-1} \cup_{e'} K_{m'}$, $G_v \setminus v = K_m \cup_{\{v_2,v_k\}} C_{k-1} \cup_{e'} K_{m'}$ and $G \setminus v = K_{m-1} \cup_{v_2} P_{k-1} \cup_{e'} K_{m'}$. Thus, by induction and (1), $\beta_{n,n+k-1}(S/J_{G_v})$ and $\beta_{n+1,n+1+k-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are the unique extremal Betti numbers of S/J_{G_v} and $S/((x_v, y_v) + J_{G_v\setminus v})$ respectively. By virtue of Proposition 3.5, $\operatorname{reg}(S/((x_v, y_v) + J_{G\setminus v})) = k-1$. It is known [7, Theorem 3.1] that $J_{G\setminus v}$ is Cohen-Macaulay, and hence $\operatorname{depth}(S/((x_v, y_v) + J_{G\setminus v})) = n$ and $\beta_{n,n+k-1}(S/((x_v, y_v) + J_{G\setminus v}))$ is the unique extremal Betti number. Therefore, by applying Lemmas 2.1, 2.2 on the short exact sequence (2), we have $\operatorname{depth}(S/J_G) \geq n$ and $\operatorname{reg}(S/J_G) \leq k$. Also, it follows from the long exact sequence (3) for $i = n+1$ and in graded degree $j = k-1$ that

$$\operatorname{Tor}_{n+1}^S\left(\frac{S}{(x_v, y_v) + J_{G_v\setminus v}}, \mathbb{K}\right)_{n+1+k-1} \simeq \operatorname{Tor}_n^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n+1+k-1} \neq 0.$$

Therefore, $\beta_{n,n+k}(S/J_G) \neq 0$, as required. □

Proposition 4.7. *Let $m \geq 3$, $k \geq 4$ and $G = K_m \cup_e C_k \cup W^{r_1}(v)$ for $r_1 \geq 1$ with $v \notin e$. Then $\beta_{n,n+k}(S/J_G)$ is the unique extremal Betti number of S/J_G.*

Proof. As in the previous result, we prove that $n \leq \operatorname{depth}(S/J_G)$, $\operatorname{reg}(S/J_G) \leq k$ and $\beta_{n,n+k}(S/J_G) \neq 0$. Note that $G \setminus v = K_m \cup_e P_{k-1}$ with r_1 isolated vertices. By Proposition 3.5, $\operatorname{reg}(S/((x_v, y_v) + J_{G\setminus v})) = k-2$, and by [7, Theorem 1.1], $\operatorname{pd}(S/((x_v, y_v) + J_{G\setminus v})) = n - r_1 \leq n-1$. Here, $G_v = K_m \cup_e C_{k-1} \cup_{\{v_2,v_k\}} K_{r_1+3}$ and $G_v \setminus v = K_m \cup_e C_{k-1} \cup_{\{v_2,v_k\}} K_{r_1+2}$. By virtue of Proposition 4.6, we have that $\beta_{n,n+k-1}(S/J_{G_v})$ and $\beta_{n+1,n+1+k-1}(S/((x_v, y_v)+J_{G_v\setminus v}))$ are the unique extremal Betti numbers of S/J_{G_v} and $S/((x_v, y_v) + J_{G_v\setminus v})$ respectively. Then it follows from Lemmas 2.1, 2.2 and the short exact sequence (2) that $\operatorname{depth}(S/J_G) \geq n$ and $\operatorname{reg}(S/J_G) \leq k$. Now consider the long exact sequence (3) for $i = n+1$ and $j = k-1$, we get the isomorphism:

$$\operatorname{Tor}_{n+1}^S\left(\frac{S}{(x_v, y_v) + J_{G_v\setminus v}}, \mathbb{K}\right)_{n+1+k-1} \simeq \operatorname{Tor}_n^S\left(\frac{S}{J_G}, \mathbb{K}\right)_{n+1+k-1}.$$

This implies that $\beta_{n,n+k}(S/J_G) \neq 0$, as desired. □

Proposition 4.8. *Let $k \geq 4$, $m \geq 2$ and $G = K_m \cup_e C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ for $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $|A \cap e| = 1$, $|A| = 2$ and vertices of A are not adjacent, then $\operatorname{reg}(S/J_G) = k$. Moreover, S/J_G admits a unique extremal Betti number.*

Proof. Let $e = \{v, v_k\}$, $A = \{v, v_i\}$ such that v and v_i are non-adjacent. Then $3 \leq i \leq k-1$ and $G = K_m \cup_e C_k \cup W^{r_1}(v) \cup W^{r_i}(v_i)$ for $r_1, r_i \geq 1$. It follows from Theorem 3.9 and Proposition 4.3 that $\operatorname{depth}(S/J_G) = n$ and $\operatorname{reg}(S/J_G) \leq k$ respectively. So we only need to show that $\beta_{n,n+k}(S/J_G) \neq 0$. Note that $G \setminus v$ is a block graph with r_1 isolated vertices. Thus, by [7, Theorem 1.1] and (1), $\operatorname{pd}(S/((x_v, y_v) + J_{G \setminus v})) \leq n-1$. Also, it can be observed that $G_v = K_{m+r_1+1} \cup_{\{v_2,v_k\}} C_{k-1} \cup W^{r_i}(v_i)$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_2,v_k\}} C_{k-1} \cup W^{r_i}(v_i)$ with $v_i \notin \{v_2, v_k\}$. Then G_v and $G_v \setminus v$ belong to the class of graphs considered in Proposition 4.7. Hence, $\beta_{n,n+k-1}(S/J_{G_v})$ and $\beta_{n+1,n+1+k-1}(S/((x_v, y_v) + J_{G_v \setminus v}))$ are the unique extremal Betti numbers. Now it follows from the long exact sequence (3) for $i = n+1$ and $j = k-1$ that $\beta_{n,n+k}(S/J_G) \neq 0$. Hence, the assertion follows. □

Corollary 4.9. *Let $k \geq 4$ and $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ for $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $|A| = 2$ and vertices of A are non-adjacent or $3 \leq |A| \leq k-1$, then $\operatorname{reg}(S/J_G) = k$.*

Proof. Let $v_j, v_l \in A$ such that v_j and v_l are non-adjacent. Set $G' = C_k \cup W^{r_j}(v_j) \cup W^{r_l}(v_l)$. Then, clearly G' is an induced subgraph of G. By considering $m = 2$ in Proposition 4.8, we have $\operatorname{reg}(S_{G'}/J_{G'}) = k$. Hence it follows from [24, Corollary 2.2] and Proposition 4.3 that $\operatorname{reg}(S/J_G) = k$. □

We now combine the results from Theorem 4.5 and Corollaries 4.2, 4.4, 4.9 to get the following conclusion.

Corollary 4.10. *Let $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$, $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $A \neq \emptyset$, then $k-1 \leq \operatorname{reg}(S/J_G) \leq k+1$. Moreover,*

(1) $\operatorname{reg}(S/J_G) = k+1$ *if and only if* $A = V(C_k)$*,*
(2) $\operatorname{reg}(S/J_G) = k-1$ *if and only if* $|A| = 1$ *or* $|A| = 2$ *and vertices of A are adjacent,*
(3) $\operatorname{reg}(S/J_G) = k$ *if and only if A contains at least two non-adjacent vertices and* $A \subsetneq V(C_k)$*.*

Let $k \geq 4$ and $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$, $r_i \geq 0$. If $\operatorname{reg}(S/J_G) = k+1$ or $\operatorname{reg}(S/J_G) = k-1$, then we proved that S/J_G admits a unique extremal Betti number, see Corollary 4.4 and Theorem 4.5. From now, we suppose $\operatorname{reg}(S/J_G) = k$. We show that S/J_G does not always admit a unique extremal Betti number. In rest of the section, we study behavior of uniqueness of extremal Betti number for them. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. First, we consider the case when $G[A]$ is disconnected.

Proposition 4.11. *Let $k \geq 4$ and $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ for $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $2 \leq |A| \leq k-2$ and $G[A]$ is disconnected, then $\beta_{n,n+k}(S/J_G)$ is the unique extremal Betti number of S/J_G.*

Proof. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. Then $G = C_k \cup (\cup_{v_i \in A} W^{r_i}(v_i))$ for $r_i \geq 1$. If $|A| = 2$, then we choose e such that $A \cap e \neq \emptyset$. If $|A| \geq 3$, then choose $v_j \in A$ such that $G[A \setminus v_j]$ is disconnected and $e \cap A = \{v_j\}$. Let $H = K_m \cup_e G$ for $m \geq 2$. Set $n' = |V(H)|$. Then $n' = n + m - 2$. We claim that $\beta_{n',n'+k}(S_H/J_H) \neq 0$. We prove this by induction on $|A|$. If $|A| = 2$, then the assertion follows from Proposition 4.8. Suppose $|A| \geq 3$. Note that $H_{v_j} = K_{m+r_j+1} \cup_{\{v_{j-1},v_{j+1}\}} C_{k-1} \cup (\cup_{v_i \in A \setminus \{v_j\}} W^{r_i}(v_i))$ and $H_{v_j} \setminus v_j = K_{m+r_j} \cup_{\{v_{j-1},v_{j+1}\}} C_{k-1} \cup (\cup_{v_i \in A \setminus \{v_j\}} W^{r_i}(v_i))$. Since $G[A \setminus \{v_j\}]$ is disconnected with $|A \setminus \{v_j\}| \geq 2$, H_{v_j} and $H_{v_j} \setminus v_j$ satisfy induction hypotheses. Therefore, $\beta_{n',n'+k-1}(S_H/J_{H_{v_j}})$ and $\beta_{n'+1,n'+1+k-1}(S_H/((x_{v_j}, y_{v_j}) + J_{H_{v_j} \setminus v_j}))$ are the unique extremal Betti numbers of $S_H/J_{H_{v_j}}$ and $S_H/((x_{v_j}, y_{v_j}) + J_{H_{v_j} \setminus v_j})$ respectively. By [7, Theorem 1.1],

$\mathrm{pd}(S_H/((x_{v_j}, y_{v_j}) + J_{H\setminus v_j})) \leq n' - 1$. Now consider the long exact sequence (3) for the pair (H, v_j) to get that $\beta_{n',n'+k}(S_H/J_H) \neq 0$. Taking $m = 2$, we get $\beta_{n,n+k}(S/J_G) \neq 0$. By Corollaries 3.12 and 4.9, we have $\mathrm{depth}(S/J_G) = n$ and $\mathrm{reg}(S/J_G) = k$. Hence, $\beta_{n,n+k}(S/J_G)$ is the unique extremal Betti number of S/J_G. □

From now, we suppose $G[A]$ is connected.

Proposition 4.12. *Let $k \geq 5$, $m \geq 3$ and $G = K_m \cup_e C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ for $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $|A \cap e| = 1$, $2 \leq |A| \leq k-3$ and $G[A]$ is connected, then $\beta_{n,n+k-1}(S/J_G)$ is an extremal Betti number. In particular, if $k \geq 6$, $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ and $G[A]$ is connected with $3 \leq |A| \leq k-3$, then $\beta_{n,n+k-1}(S/J_G)$ is an extremal Betti number, i.e., S/J_G does not admit a unique extremal Betti number.*

Proof. Let $G = K_m \cup_e C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ for $r_i \geq 0$. By Theorem 3.9, we have either $\beta_{n,n+k-1}(S/J_G)$ or $\beta_{n,n+k}(S/J_G)$ is an extremal Betti number. So, it is enough to show that $\beta_{n,n+k}(S/J_G) = 0$. We prove this by induction on $|A|$. Let $e = \{v, v_k\}$. Since $|A \cap e| = 1$, assume that $v \in A$. Set $A = \{v, v_2, \ldots, v_t\}$ for some $2 \leq t \leq k-3$. Since, $G \setminus v$ is a disconnected block graph with $r_1 + 1$ components, by [7, Theorem 1.1] and (1), $\mathrm{pd}(S/((x_v, y_v) + J_{G\setminus v})) = n - r_1 \leq n-1$. Suppose $|A| = 2$. Then it can be noted that $G_v = K_{m+r_1+1} \cup_{\{v_k,v_2\}} C_{k-1} \cup W^{r_2}(v_2)$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_k,v_2\}} C_{k-1} \cup W^{r_2}(v_2)$. Now it follows from the proof of Theorem 4.1 that $\mathrm{reg}(S/J_{G_v}) = k-2 = \mathrm{reg}(S/((x_v, y_v) + J_{G_v\setminus v}))$. Therefore, by Theorem 3.9, $\beta_{n,n+k-2}(S/J_{G_v})$ and $\beta_{n+1,n+1+k-2}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are extremal Betti numbers, and hence $\beta_{n,n+k}(S/J_{G_v}) = 0 = \beta_{n+1,n+k}(S/((x_v, y_v) + J_{G_v\setminus v}))$. Thus, it follows from the long exact sequence (3) that $\beta_{n,n+k}(S/J_G) = 0$. Now assume that $|A| \geq 3$. Then $G_v = K_{m+r_1+1} \cup_{\{v_k,v_2\}} C_{k-1} \cup (\cup_{i=2}^t W^{r_i}(v_i))$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_k,v_2\}} C_{k-1} \cup (\cup_{i=2}^t W^{r_i}(v_i))$. Clearly, G_v and $G_v \setminus v$ satisfy induction hypotheses. Therefore, $\beta_{n,n+k-1}(S/J_{G_v}) = 0 = \beta_{n+1,n+1+k-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$. Hence, from the long exact sequence (3), we get $\beta_{n,n+k}(S/J_G) = 0$.

Let $G = C_k \cup (\cup_{i=1}^k W^{r_i}(v_i))$ for $r_i \geq 0$. Set $A = \{v, v_2, \ldots, v_t\}$ for some $3 \leq t \leq k-3$. Then $G = C_k \cup (\cup_{i=1}^t W^{r_i}(v_i))$ for $r_i \geq 1$. Observe that $G_v = K_{r_1+3} \cup_{\{v_k,v_2\}} C_{k-1} \cup (\cup_{i=2}^t W^{r_i}(v_i))$ and $G_v \setminus v = K_{r_1+2} \cup_{\{v_k,v_2\}} C_{k-1} \cup (\cup_{i=2}^t W^{r_i}(v_i))$. Thus, by the above part and (1), $\beta_{n,n+k-1}(S/J_{G_v}) = 0 = \beta_{n+1,n+1+k-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$. Since, $G \setminus v$ is a forest with $r_1 + 1$ trees, by [7, Theorem 1.1] and (1), $\mathrm{pd}(S/((x_v, y_v) + J_{G\setminus v})) = n - r_1 \leq n-1$. Hence, it follows from the long exact sequence (3) that $\beta_{n,n+k}(S/J_G) = 0$. By Corollary 4.9, we have $\mathrm{reg}(S/J_G) = k$. Therefore, $\beta_{n,n+k-1}(S/J_G)$ is not the unique extremal Betti number of S/J_G. □

Now we consider the case when $|A| = k-2$ and $G[A]$ is connected. We assume that $A = \{v, v_2, \ldots, v_{k-2}\}$. Then $G = C_k \cup (\cup_{i=1}^{k-2} W^{r_i}(v_i))$ for $r_i \geq 1$. Now, we investigate the uniqueness of extremal Betti number of S/J_G.

Proposition 4.13. *Let $k \geq 4$, $m \geq 3$ and $G = K_m \cup_{\{v,v_k\}} C_k \cup (\cup_{i=1}^{k-2} W^{r_i}(v_i))$ for $r_i \geq 1$. If $r_i \geq 2$ for all $1 \leq i \leq k-3$, then $\beta_{n-1,n-1+k-1}(S/J_G)$ is an extremal Betti number. In particular, if $k \geq 5$, $G = C_k \cup (\cup_{i=1}^{k-2} W^{r_i}(v_i))$ with $r_i \geq 2$ for all $2 \leq i \leq k-3$, then $\beta_{n-1,n-1+k-1}(S/J_G)$ is an extremal Betti number, i.e., S/J_G does not admit a unique extremal Betti number.*

Proof. Let $G = K_m \cup_{\{v,v_k\}} C_k \cup (\cup_{i=1}^{k-2} W^{r_i}(v_i))$, $r_i \geq 1$ and suppose that $r_i \geq 2$ for all $1 \leq i \leq k-3$. Then it follows from [7, Theorem 1.1] and (1) that $\mathrm{pd}(S/((x_v, y_v) + J_{G\setminus v})) = n - r_1 \leq n-2$. Due to Theorem 3.8, it is enough to show that $\beta_{n-1,n-1+k}(S/J_G) = 0$. We proceed it by induction on k. Assume that $k = 4$. Then $G_v = K_{m+r_1+1} \cup_{\{v_2,v_k\}} C_3 \cup W^{r_2}(v_2)$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_2,v_k\}} C_3 \cup W^{r_2}(v_2)$. Thus, G_v and $G_v \setminus v$ belong to the class of graphs considered in Theorem 3.7. Hence, $\beta_{n-1,n-1+2}(S/J_{G_v})$ and $\beta_{n,n+2}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are extremal Betti numbers. Therefore, $\beta_{n-1,n-1+4}(S/J_{G_v}) = 0 = \beta_{n,n-1+4}(S/((x_v, y_v) + J_{G_v\setminus v}))$. Now it follows from the long exact sequence (3) for $i = n-1$ that $\beta_{n-1,n-1+4}(S/J_G) = 0$. We assume that $k \geq 5$. Then $G_v = K_{m+r_1+1} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$. Thus, by induction and (1), $\beta_{n-1,n-1+k-1}(S/J_{G_v}) = 0 = \beta_{n,n+k-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$. Hence, from the long exact sequence (3) for $i = n-1$, we get $\beta_{n-1,n-1+k}(S/J_G) = 0$.

Let $G = C_k \cup (\cup_{i=1}^{k-2} W^{r_i}(v_i))$, where $r_1, r_{k-2} \geq 1$ and $r_i \geq 2$ for all $2 \leq i \leq k-3$. As in the above part, it is enough to show that $\beta_{n-1,n-1+k}(S/J_G) = 0$. Note that $G \setminus v$ is the graph $P_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$ with r_1 isolated vertices. So, $\mathrm{iv}(G \setminus v) = k-2$. Then by [13, Theorem 8] and (1) $\mathrm{reg}(S/((x_v, y_v)+J_{G\setminus v})) = k-1$, and hence $\beta_{n-1,n-1+k}(S/((x_v, y_v)+J_{G\setminus v})) = 0$. Here, $G_v = K_{r_1+3} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$ and $G_v \setminus v = K_{r_1+2} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$. Therefore, by the above part and (1), $\beta_{n-1,n-1+k-1}(S/J_{G_v}) = 0 = \beta_{n,n+k-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$. Now it follows from the long exact sequence (3) that $\beta_{n-1,n-1+k}(S/J_G) = 0$, as required. $\square$

Proposition 4.14. *Let $k \geq 4$ and $m \geq 3$. Let $G = K_m \cup_{\{v,v_k\}} C_k \cup (\cup_{i=1}^{k-2} W^{r_i}(v_i))$ for $r_i \geq 1$. If $r_i = 1$ for some $1 \leq i \leq k-3$, then $\beta_{n-1,n-1+k}(S/J_G)$ is an extremal Betti number of S/J_G. In particular, if $k \geq 5$, $G = C_k \cup (\cup_{i=1}^{k-2} W^{r_i}(v_i))$ and $r_i = 1$ for some $2 \leq i \leq k-3$, then $\beta_{n-1,n-1+k}(S/J_G)$ is the unique extremal Betti number of S/J_G.*

Proof. Due to Theorem 3.8, it is enough to show that $\beta_{n-1,n-1+k}(S/J_G) \neq 0$. To prove this we proceed by induction on k. Assume that $k = 4$. Then $G = K_m \cup_{\{v,v_k\}} C_4 \cup W^{r_1}(v) \cup W^{r_2}(v_2)$ for $r_1 = 1$ and $r_2 \geq 1$. In this case, $G \setminus v$ is the graph $K_{m-1} \cup_{v_k} P_3 \cup W^{r_2}(v_2)$ with one isolated vertex. Therefore, by [7, Theorem 1.1] and (1), $\mathrm{pd}(S/((x_v, y_v)+J_{G\setminus v})) = n-1$, and hence by [13, Theorem 8] and (1), $\beta_{n-1,n-1+4}(S/((x_v, y_v) + J_{G\setminus v}))$ is an extremal Betti number. Note that $G_v = K_{m+2} \cup_{\{v_2,v_k\}} C_3 \cup W^{r_2}(v_2)$ and $G_v \setminus v = K_{m+1} \cup_{\{v_2,v_k\}} C_3 \cup W^{r_2}(v_2)$. By Proposition 4.3, $\mathrm{reg}(S/J_{G_v}) \leq 3$ and $\mathrm{reg}(S/((x_v, y_v) + J_{G_v\setminus v})) \leq 3$. Therefore, $\beta_{n-1,n-1+j}(S/J_{G_v}) = 0 = \beta_{n-1,n-1+j}(S/((x_v, y_v) + J_{G_v\setminus v}))$ for $j \geq 4$. Hence it follows from the long exact sequence (3) that $\beta_{n-1,n-1+4}(S/J_G) = \beta_{n-1,n-1+4}(S/((x_v, y_v) + J_{G\setminus v})) \neq 0$.

Now, we assume that $k \geq 5$. Let $r_i = 1$ for some $1 \leq i \leq k-3$. Then $G_v = K_{m+r_1+1} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$.

Case 1: Let $r_1 = 1$ and $r_i \geq 2$ for all $2 \leq i \leq k-3$. Then $G_v = K_{m+2} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$ and $G_v \setminus v = K_{m+1} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$. By virtue of Proposition 4.3, we have $\mathrm{reg}(S/J_{G_v}) \leq k-1$ and $\mathrm{reg}(S/((x_v, y_v)+J_{G_v\setminus v})) \leq k-1$. Hence, $\beta_{n-1,n-1+j}(S/J_{G_v}) = 0 = \beta_{n-1,n-1+j}(S/((x_v, y_v) + J_{G_v\setminus v}))$ for $j \geq k$. In this case, $G \setminus v$ is the graph $K_{m-1} \cup_{v_k} P_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$ with one isolated vertex. Therefore, by [7, Theorem 1.1] and (1), $\mathrm{pd}(S/((x_v, y_v)+J_{G\setminus v})) = n-1$, and hence by [13, Theorem 8] and (1), $\beta_{n-1,n-1+k}(S/((x_v, y_v)+J_{G\setminus v}))$ is an extremal Betti number. Therefore, from the long exact sequence (3), we get that $\beta_{n-1,n-1+k}(S/J_G) = \beta_{n-1,n-1+k}(S/((x_v, y_v) + J_{G\setminus v})) \neq 0$.

Case 2: Let $r_i = 1$ for some $2 \leq i \leq k-3$. By [7, Theorem 1.1] and (1), $\mathrm{pd}(S/((x_v, y_v) + J_{G\setminus v})) = n - r_1 \leq n-1$. In this case, notice that G_v and $G_v \setminus v$ satisfy induction hypotheses. Therefore, $\beta_{n-1,n-1+k-1}(S/J_{G_v})$ and $\beta_{n,n+k-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are extremal Betti numbers. Then it follows from the long exact sequence (3) for $i = n-1$ and $j = k$ that $\beta_{n-1,n-1+k}(S/J_G) \neq 0$.

As in the above part, it is enough to show that $\beta_{n-1,n-1+k}(S/J_G) \neq 0$. Note that $G \setminus v$ is the graph $P_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$ with r_1 isolated vertices. Therefore, by [7, Theorem 1.1], $p = \operatorname{pd}(S/((x_v, y_v) + J_{G\setminus v})) = n - r_2 \leq n-1$. Observe that $G_v = K_{r_1+3} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$ and $G_v \setminus v = K_{r_1+2} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=2}^{k-2} W^{r_i}(v_i))$. Therefore, by the above part and (1), $\beta_{n-1,n-1+k-1}(S/J_{G_v})$ and $\beta_{n,n+k-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are extremal Betti numbers. Hence the assertion follows from the long exact sequence (3) for $i = n-1$ and $j = k$. ☐

Now we are left with the case that $|A| = k-1$. In this case, we prove that S/J_G admits a unique extremal Betti number.

Proposition 4.15. *Let $H = K_m \cup_e C_k$ for $k \geq 3$, $m \geq 2$. Let $G = H \cup (\cup_{i=1}^{k} W^{r_i}(v_i))$ for $r_i \geq 0$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$. If $|A| = k-1$ and $e \nsubseteq A$, then $\beta_{n-1,n-1+k}(S/J_G)$ is an extremal Betti number. In particular, if $k \geq 3$, $G = C_k \cup (\cup_{i=1}^{k} W^{r_i}(v_i))$ with $|A| = k-1$, then S/J_G admits a unique extremal Betti number.*

Proof. As in the previous result, it is enough to show that $\beta_{n-1,n-1+k}(S/J_G) \neq 0$. Let $e = \{v, v_2\}$. We may assume that $r_2 = 0$. Then $G = K_m \cup_e C_k \cup (\cup_{i=1, i\neq 2}^{k} W^{r_i}(v_i))$ for $r_i \geq 1$. We prove the first part by induction on k. If $k = 3$ and $m = 2$, then the result follows from [13, Theorem 8]. If $k = 3$ and $m \geq 3$, then the result follows from Theorem 3.7. Now assume that $k \geq 4$. Note that $G_v = K_{m+r_1+1} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=3}^{k} W^{r_i}(v_i))$, $r_i \geq 1$ and $G_v \setminus v = K_{m+r_1} \cup_{\{v_2,v_k\}} C_{k-1} \cup (\cup_{i=3}^{k} W^{r_i}(v_i))$, $r_i \geq 1$. Thus, by induction and (1), $\beta_{n-1,n-1+k-1}(S/J_{G_v})$ and $\beta_{n,n+k-1}(S/((x_v, y_v) + J_{G_v\setminus v}))$ are extremal Betti numbers. Also, by [7, Theorem 1.1], $\operatorname{pd}(S/((x_v, y_v) + J_{G\setminus v})) = n - r_1 \leq n-1$. Hence it follows from the long exact sequence (3) for $i = n-1$ and $j = k$ that $\beta_{n-1,n-1+k}(S/J_G) \neq 0$. Taking $m = 2$, we get the second assertion. ☐

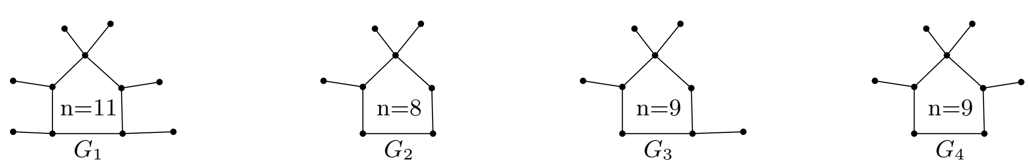
We now conclude the following result for the behavior of uniqueness of extremal Betti number for cycles with whiskers graphs.

Corollary 4.16. *Let $G = C_k \cup (\cup_{i=1}^{k} W^{r_i}(v_i))$, $r_i \geq 0$ with $\operatorname{reg}(S/J_G) = k$. Let $A = \{v_i \in V(C_k) : r_i \geq 1\}$.*

(1) If $2 \leq |A| \leq k-2$ and $G[A]$ is disconnected, then S/J_G admits a unique extremal Betti number.

(2) Suppose $G[A]$ is connected:
- *(a) If $3 \leq |A| \leq k-3$, then S/J_G does not admit a unique extremal Betti number.*
- *(b) Suppose $|A| = k-2$, $A = \{v_1, \ldots, v_{k-2}\}$. If $r_i \geq 2$ for all $2 \leq i \leq k-3$, then S/J_G does not admit a unique extremal Betti number.*
- *(c) Suppose $|A| = k-2$, $A = \{v_1, \ldots, v_{k-2}\}$. If $r_i = 1$ for some $2 \leq i \leq k-3$, then S/J_G admits a unique extremal Betti number*
- *(d) If $|A| = k-1$, then S/J_G admits a unique extremal Betti number.*

To get a better insight into our results, let us look at some of the following examples:



By Corollary 4.4, Theorem 4.5 and Corollary 4.9, $\operatorname{reg}(S_{G_1}/J_{G_1}) = 6$, $\operatorname{reg}(S_{G_2}/J_{G_2}) = 4$, $\operatorname{reg}(S_{G_3}/J_{G_3}) = 5$ and $\operatorname{reg}(S_{G_4}/J_{G_4}) = 5$. Also, we get that $\beta_{10,16}(S_{G_1}/J_{G_1})$ and $\beta_{8,12}(S_{G_2}/J_{G_2})$ are the unique extremal Betti numbers of S_{G_1}/J_{G_1} and S_{G_2}/J_{G_2} . By Proposition 4.11, $\beta_{9,14}(S_{G_3}/J_{G_3})$ is the unique extremal Betti number of S_{G_3}/J_{G_3}. By Proposition 4.13, $\beta_{8,12}(S_{G_4}/J_{G_4})$ is an extremal Betti number of S_{G_4}/J_{G_4}, i.e., S_{G_4}/J_{G_4} does not admit a unique extremal Betti number.

It will be interesting to obtain an answer to:

Question 4.17. *Characterize unicyclic graphs G such that S/J_G admits a unique extremal Betti number.*

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