ON THE SOLVABILITY OF GRADED NOVIKOV ALGEBRAS

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ABSTRACT. We show that the right ideal of a Novikov algebra generated by the square of a right nilpotent subalgebra is nilpotent. We also prove that a G-graded Novikov algebra N over a field K with solvable 0-component N_0 is solvable, where G is a finite additive abelean group and the characteristic of K does not divide the order of the group G. We also show that any Novikov algebra N with a finite solvable group of automorphisms Gis solvable if the algebra of invariants N^G is solvable.

Mathematics Subject Classification (2010): 17D25, 17B30, 17B70

Key words: Novikov algebra, graded algebra, solvability, nilpotency, automorphism, the ring of invariants

1. INTRODUCTION

A nonassociative algebra N over a field K is called a *Novikov* algebra [17] if it satisfies the following identities:

(1)
$$(x, y, z) = (y, x, z)$$
 (left symmetry),

(2)
$$(xy)z = (xz)y$$
 (right commutativity),

where (x, y, z) = (xy)z - x(yz) is the associator of elements x, y, z.

The defining identities of a Novikov algebra first appeared in the study of Hamiltonian operators in the formal calculus of variations by I.M. Gelfand and I.Ya. Dorfman [8]. These identities played a crucial role in the classification of linear Poisson brackets of hydrodynamical type by A.A. Balinskii and S.P. Novikov [1].

In 1987 E.I. Zelmanov [25] proved that all finite-dimensional simple Novikov algebras over a field K of characteristic 0 are one-dimensional. V.T. Filippov [6] constructed a wide class of simple Novikov algebras of characteristic $p \ge 0$. J.M. Osborn [17, 18, 19] and X. Xu [23, 24] continued the study of simple finite dimensional algebras over fields of positive characteristic and simple infinite dimensional algebras over fields of characteristic zero. A complete classification of finite dimensional simple Novikov algebras over algebraically closed fields of characteristic p > 2 is given in [23].

E.I. Zelmanov also proved that if N is a finite dimensional right nilpotent Novikov algebra then N^2 is nilpotent [25]. In 2001 V.T. Filippov [7] proved that any left-nil Novikov algebra of bounded index over a field of characteristic zero is nilpotent. A.S. Dzhumadildaev and K.M. Tulenbaev [5] proved that any right-nil Novikov algebra of

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bounded index n is right nilpotent if the characteristic p of the field K is 0 or p > n. In 2020 I. Shestakov and Z. Zhang proved [21] that for any Novikov algebra N over a field the following conditions are equivalent:

(i) N is solvable;

(*ii*) N^2 is nilpotent;

(iii) N is right nilpotent.

The Freiheitssatz for Novikov algebras over fields of characteristic 0 was proven by L. Makar-Limanov and U. Umirbaev [15]. L.A. Bokut, Y. Chen, and Z. Zhang [3] proved that every Novikov algebra is a subalgebra of a Novikov algebra obtained from some differential algebra by Gelfand-Dorfman construction [8].

This paper is devoted to the study of solvable, nilpotent, and right nilpotent Novikov algebras and graded Novikov algebras. Notice that an algebra A over a field containing all nth roots of unity admits an automorphism of order n if and only if A admits a \mathbb{Z}_n -grading. For this reason the study of graded algebras is related to the study of actions of finite groups. First we recall some definitions and classical results.

Let R be an algebra over a field K. For any automorphism ϕ of R the set of fixed elements

$$R^{\phi} = \{x \in R | \phi(x) = x\}$$

is a subalgebra of R and is called the subalgebra of *invariants* of ϕ . An automorphism ϕ is called *regular* if $R^{\phi} = 0$. For any group G of automorphisms of R the subalgebra of invariants

$$R^G = \{ x \in R | \phi(x) = x \text{ for all } \phi \in G \}$$

is defined similarly.

In 1957 G. Higman [10] published a classical result on Lie algebras which says that if a Lie algebra L has a regular automorphism ϕ of prime order p, then L is nilpotent. It was also shown that the index of nilpotency h(p) of L depends only on p. An explicit estimation of the function h(p) was found by A.I. Kostrikin and V.A. Kreknin [12] in 1963. A little later, V.A. Kreknin proved [13] that a finite dimensional Lie algebra with a regular automorphism of an arbitrary finite order is solvable. In 2005 N. Yu. Makarenko [14] proved that if a Lie algebra L admits an automorphism of prime order p with a finite-dimensional fixed subalgebra of dimension t, then L has a nilpotent ideal of finite codimension with the index of nilpotency bounded in terms of p and the codimension bounded in terms of t and p.

In 1973 G. Bergman and I. Isaacs [2] published a classical result on the actions of finite groups on associative algebras. Let G be a finite group of automorphisms of an associative algebra R and suppose that R has no |G|-torsion. If the subalgebra of invariants R^G is nilpotent then the Bergman-Isaacs Theorem [2] states that R is also nilpotent. Since then a very large number of papers have been devoted to the study of automorphisms of associative rings. The central problem of these studies was to identify the properties of rings that can be transformed from the ring of invariants to the whole ring. In 1974 V. K. Kharchenko [11] proved if R^G is a PI-ring then R is a PI-ring under the conditions of the Bergman-Isaacs Theorem. The Bergman-Isaacs Theorem was partially generalized by W.S. Martindale and S. Montgomery [16] in 1977 to the case of a finite group of *Jordan* automorphisms, that is a finite group of automorphisms of the adjoint Jordan algebra $R^{(+)}$.

An analogue of Kharchenko's result for Jordan algebras was proven by A. P. Semenov [20] in 1991. In particular, A. P. Semenov proved that if J^G is a solvable algebra over a field of characteristic zero, then so is the Jordan algebra J. His proof uses a deep result by E.I. Zel'manov [26] which says that every Jordan nil-algebra of bounded index over a field of characteristic zero is solvable. If a Jordan algebra J over a field of characteristic not equal to 2, 3 admits an automorphism ϕ of order 2 with solvable J^{ϕ} , then J is solvable [27].

In the case of alternative algebras one cannot expect that nilpotency of the invariant subalgebra implies the nilpotency of the whole algebra. There is an example (see [4, 30]) of a solvable non-nilpotent alternative algebra with an automorphism of order two such that its subalgebra of invariants is nilpotent. A combination of Semenov's result [20] and Zhevlakov's theorem [29] gives that, for an alternative algebra A over a field of characteristic zero, the solvability of the algebra of invariants A^G for a finite group Gimplies the solvability of A. It is also known [22] that if A is an alternative algebra over a field of characteristic not equal to 2 with an automorphism ϕ of order two, then the solvability of the algebra of invariants A^{ϕ} implies the solvability of A. In [9] M. Goncharov proved that an alternative \mathbb{Z}_3 - graded algebra $A = A_0 \oplus A_1 \oplus A_2$ over a field of characteristic not equal to 2, 3, 5 is solvable if A_0 is solvable.

It was shown in [28] for every n of the form $n = 2^k 3^l$ that a \mathbb{Z}_n -graded Novikov

$$N = N_0 \oplus \ldots \oplus N_{n-1}$$

over a field of characteristic not equal to 2, 3 is solvable if N_0 is solvable.

In this paper we first prove that if L is a right nilpotent subalgebra of a Novikov algebra N then the right ideal of N generated by L^2 is right nilpotent (Theorem 1). This result gives a deeper explanation of the results on the nilpotency of N^2 mentioned above. The main result of the paper (Theorem 2) says that if N is a G-graded Novikov algebra with solvable 0-component N_0 , where G is a finite additive abelian group, then N is solvable. This result allows us to prove (Theorem 3) that if N is a Novikov algebra with solvable algebra of invariants N^G , where G is a finite solvable group of automorphisms of N, then N is solvable. Theorems 2 and 3 are formulated for fields of characteristic 0 or positive characteristic p that does not divide |G|. Notice that the solvability and the right nilpotency of Novikov algebras are equivalent by the result of I. Shestakov and Z. Zhang mentioned above.

The paper is organized as follows. In Section 2 we prove some identities and Theorem 1. Sections 3–5 are devoted to the study of the structure of \mathbb{Z}_n -graded Novikov algebras. Theorems 2 and 3 are formulated and proven in Section 6.

2. RIGHT NILPOTENT SUBALGEBRAS

The identities (1) and (2) easily imply the identities

$$(3) \qquad (xy,z,t) = (x,z,t)y$$

and (4)

(x, yz, t) = (x, y, t)z.

Let A be an arbitrary algebra. The powers of A are defined inductively by $A^1 = A$ and

$$A^{m} = \sum_{i=1}^{m-1} A^{i} A^{m-i}$$

for all positive integers $m \ge 2$. The algebra A is called *nilpotent* if $A^m = 0$ for some positive integer m.

The right powers of A are defined inductively by $A^{[1]} = A$ and $A^{[m+1]} = A^{[m]}A$ for all integers $m \ge 1$. The algebra A is called *right nilpotent* if there exists a positive integer m such that $N^{[m]} = 0$. In general, the right nilpotency of an algebra does not imply its nilpotency. This is also true in the case of Novikov algebras.

Example 1. [25] Let N = Fa + Fb be a vector space of dimension 2. The product on N is defined as

$$ab = b, a^2 = b^2 = ba = 0.$$

It is easy to check that N is a right nilpotent Novikov algebra, but not nilpotent.

The derived powers of A are defined by $A^{(0)} = A$, $A^{(1)} = A^2$, and $A^{(m)} = A^{(m-1)}A^{(m-1)}$ for all positive integers $m \ge 2$. The algebra A is called *solvable* if $A^{(m)} = 0$ for some positive integer m. Every right nilpotent algebra is solvable, and, in general, the converse is not true. But every solvable Novikov algebra is right nilpotent [21].

It is well known that if I and J are ideals of a Novikov algebra N, then IJ is also an ideal of N. Consequently, if N is a Novikov algebra then N^m , $N^{[m]}$, and $N^{(m)}$ are ideals of N. If S is a subset of a Novikov algebra N, then denote by $\langle S \rangle$ the right ideal of N generated by S. Notice that if I is a right ideal of N, then IS is a right ideal of N for any subset $S \subseteq N$ by (2).

In any algebra we denote by $x_1x_2...x_k$ the right normed product $(...(x_1x_2)...)x_k$ of elements $x_1, x_2, ..., x_k$. For any x, y denote by $x \circ y = xy + yx$ the Jordan product.

Lemma 1. Any Novikov algebra satisfies the following identities:

(5)
$$a(bx_1 \dots x_t) = abx_1 x_2 \dots x_t - \sum_{i=1}^t (a, b, x_i) x_1 \dots x_{i-1} x_{i+1} \dots x_t$$

for each positive integer $t \geq 1$,

(6)
$$(ax_1 \dots x_s) \circ (bx_{s+1} \dots x_t) = (a \circ b)x_1 x_2 \dots x_t - \sum_{i=1}^k (a, b, x_i) x_1 \dots x_{i-1} x_{i+1} \dots x_t$$

for all nonnegative integers $0 \leq s < t$, and

(7)
$$(ax_1 \dots x_s) \circ (bx_{s+1} \dots x_t) = a \circ (bx_1 \dots x_t).$$

Proof. We prove (5) by induction on t. If t = 1, then (5) is true by the definition of the associator. By (4), we have

$$a(bx_1...x_t) = a(bx_1...x_{t-1})x_t - (a, bx_1...x_{t-1}, x_t)$$

= $a(bx_1...x_{t-1})x_t - (a, b, x_t)x_1...x_{t-1}$.

Using this and the induction proposition, we get

$$a(bx_1 \dots x_t) = (abx_1 x_2 \dots x_{t-1} - \sum_{i=1}^{t-1} (a, b, x_i) x_1 \dots x_{i-1} x_{i+1} \dots x_{t-1}) x_t$$
$$-(a, b, x_t) x_1 \dots x_{t-1} = abx_1 x_2 \dots x_t - \sum_{i=1}^{t} (a, b, x_i) x_1 \dots x_{i-1} x_{i+1} \dots x_t.$$

By (2), (3), and (5), we get

$$(ax_1 \dots x_s)(bx_{s+1} \dots x_t) = (ab)x_1 x_2 \dots x_t - \sum_{i=s+1}^t (a, b, x_i) x_1 \dots x_{i-1} x_{i+1} \dots x_t.$$

This implies (6). The identity (7) is a direct consequence of (2), (5), and (6). \Box

Let N be a Novikov algebra and let L be a subalgebra of N. Set $L_0 = N$ and $L_k = \langle L^{[k]} \rangle$ for each positive integer k.

Consider the descending sequence of right ideals

$$N = L_0 \supseteq L_1 \supseteq L_2 \supseteq \ldots \supseteq L_k \supseteq \ldots$$

of the algebra N.

Lemma 2. $L_sL_t \subseteq L_{s+t-1}$ for all positive integers s, t.

Proof. We prove the lemma by induction on t. It is true for t = 1 by the definition of L_s . Notice that

$$L_s = L_1 \underbrace{L \dots L}_{s-1}$$

for each $s \ge 1$ by (2).

Suppose that $t \ge 2$ and let $x \in L_s$ and $y = za_1 \dots a_{t-1} \in L_t$, where $z \in L_1$ and $a_1, \dots, a_{t-1} \in L$. By (5), we get

$$xy = xza_1 \dots a_{t-1} - \sum_{i=1}^{t-1} (x, z, a_i)a_1 \dots \widehat{a_i} \dots a_{t-1}$$

where $\widehat{a_i}$ means that a_i is absent. Notice that $xz \in L_s$ and

$$xza_1 \dots a_{t-1} \in L_s \underbrace{L \dots L}_{t-1} = L_{s+t-1}.$$

Moreover, (x, z, a_i) belongs to the right ideal generated by $(L^{[s]}, L, a_i)$ by (3) and (4). Consequently, $(x, z, a_i) \in L_{s+1}$ and $(x, z, a_i)a_1 \dots \widehat{a_i} \dots a_{t-1} \in L_{s+t-1}$. \Box

In general, L_1 is not an ideal of $L_0 = N$.

Example 2. Let K[x, y] be the polynomial algebra over K in the variables x, y. Define a new product \cdot on K[x, y] by

$$f \cdot g = f \frac{\partial g}{\partial x}, \ f, g \in K[x, y].$$

Then $N = (K[x, y], \cdot)$ is a Novikov algebra.

Let L = Kx. Then L is a subalgebra of N since $x \cdot x = x$. Let $L_1 = \langle L \rangle$. It is clear that $L_1 \subseteq xK[x, y]$. Hence,

$$y \cdot x = y \frac{\partial x}{\partial x} = y \notin L_1$$

Consequently, L_1 is not an ideal of $L_0 = N$.

But for each $r \ge 2$ the right ideal L_r is an ideal of L_1 by Lemma 2.

Corollary 1. $L_2^n \subseteq L_{n+1}$ for all $n \ge 1$.

Proof. It is trivial for n = 1 and true for n = 2 by Lemma 2. If $L_2^i \subseteq L_{2+i-1}$ and $L_2^j \subseteq L_{2+j-1}$, then $L_2^i L_2^j \subseteq L_{i+1} L_{j+1} \subseteq L_{i+j+1}$. Leading an induction on n we get

$$L_2^n = \sum_{i+j=n, i,j \ge 1} L_2^i L_2^j \subseteq L_{n+1}. \quad \Box$$

Theorem 1. Let L be a right nilpotent subalgebra of a Novikov algebra N over a field K. Then the right ideal $L_2 = \langle L^2 \rangle$ of N generated by L^2 is nilpotent.

Proof. Suppose that $L^{[n]} = 0$ for some $n \ge 2$. Then $L_n = 0$. By Corollary 1, we have $L_2^{n-1} \subseteq L_n = 0$. This means that L_2 is nilpotent. \Box

3. \mathbb{Z}_n -graded Novikov Algebras

Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the additive cyclic group of order n. Let

(8)
$$N = N_0 \oplus N_1 \oplus N_2 \oplus \ldots \oplus N_{n-1}, \quad N_i N_j \subseteq N_{i+j}, \ i, j \in \mathbb{Z}_n,$$

be a \mathbb{Z}_n -graded Novikov algebra over K.

If $f \in N_i$ then we say that f is a homogeneous element of degree i. Notice that i is an element of \mathbb{Z}_n . Sometimes we consider the subscripts i of N_i as integers satisfying the condition $0 \leq i \leq n-1$.

Obviously, $A = N_0$ is a subalgebra of N. Recall that $A^{[r]}$ is the right rth power of A.

Lemma 3. Let $i_1, i_2, ..., i_k \in \mathbb{Z}_n$ and $i_1 + i_2 + ... + i_k = 0$. Then

$$A^{[r]}N_{i_1}N_{i_2}\ldots N_{i_k} \subseteq A^{[r]}.$$

Proof. By the definition of a \mathbb{Z}_n -graded algebra, we have

$$AN_{i_1}N_{i_2}\ldots N_{i_k}\subseteq A$$

Using this and (2), we get

$$A^{[r]}N_{i_1}N_{i_2}\dots N_{i_k} = AN_{i_1}N_{i_2}\dots N_{i_k}\underbrace{A\dots A}_{r-1} \subseteq A\underbrace{A\dots A}_{r-1} = A^{[r]}. \quad \Box$$

Set $A^{\{0\}} = N$ and for any integer $r \ge 1$ denote by $A^{\{r\}} = \langle A^{[r]} \rangle$ the right ideal of N generated by $A^{[r]}$. Obviously, $A^{\{r\}}$ is a \mathbb{Z}_n -graded algebra, i.e.,

$$A^{\{r\}} = A_0^{\{r\}} \oplus A_1^{\{r\}} \oplus A_2^{\{r\}} \oplus \ldots \oplus A_{n-1}^{\{r\}}.$$

Corollary 2. If $r \ge 1$ and $0 \le i \le n-1$, then

$$A_i^{\{r\}} = \sum_{i_1, i_2, \dots, i_k} A^{[r]} N_{i_1} N_{i_2} \dots N_{i_k},$$

where $0 \le i_1, i_2, \dots, i_k \le n - 1$, $i_1 + i_2 + \dots + i_k \equiv i \pmod{n}$ and $i_1 + i_2 + \dots + i_k < n$. In particular, $A_0^{\{r\}} = A^{[r]}$.

Consider the descending sequence of right ideals

(9)
$$N = A^{\{0\}} \supseteq A^{\{1\}} \supseteq \ldots \supseteq A^{\{r\}} \supseteq \ldots$$

of the algebra N and the quotient algebra

(10)
$$B = A^{\{1\}}/A^{\{2\}} = B_0 \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_{n-1}, \quad B_i B_j \subseteq B_{i+j}, \ i, j \in \mathbb{Z}_n$$

Notice that B is a right N-module. We establish some properties of the algebra B.

Lemma 4. Let B be the Novikov algebra defined by (10). Then

(i) $B_0 = A/A^2$; (ii) $BB_0 = 0$; (iii) $x \circ y = xy + yx = 0$ for any $x \in B_i$ and $y \in B_{n-i}$.

Proof. The statement (i) is true since $A_0^{\{r\}} = A^{[r]}$ by Corollary 2. The statement (ii) is a direct corollary of the inclusion $A^{\{r\}}A \subseteq A^{\{r+1\}}$.

Let $x = ax_1x_2...x_s \in A_i^{\{1\}}$ and $y = bx_{s+1}x_{s+2}...x_t \in A_{n-i}^{\{1\}}$, where $a, b \in A$ and $x_r \in N_{k_r}$ for all $1 \leq r \leq t$. If i = 0, then $A_i = A_{n-i} = A$ and $xy, yx \in A^2$. Suppose that $i, n-i \neq 0$. Then $\sum_{r=1}^{s} k_r = i \neq 0$, $\sum_{r=s+1}^{t} l_r = n - i \neq 0$, and $\sum_{r=1}^{t} k_r = 0$. In particular, $t > s \geq 1$. By (7), we have

$$x \circ y = (ax_1 \dots x_s) \circ (bx_{s+1} \dots x_t) = a \circ (bx_1 \dots x_t).$$

The condition $\Sigma_{r=1}^t k_r = 0$ implies that $bx_1 \dots x_t \in A$. Consequently, $x \circ y \in A^2$. This proves *(iii)*. \Box

4. Right nilpotency modulo $A^{\{1\}}$

In this section we show that if the 0-component $A = N_0$ of a \mathbb{Z}_n -graded Novikov algebra N of the form (8) is right nlpotent, then $N^{[m]} \subseteq A^{\{1\}}$ for some positive integer m.

Lemma 5. Let N be an arbitrary Novikov algebra and let V be a subspace of N. Then for any $r \ge 1$ we have

$$NV^{[r]}V \subseteq \langle V^{[r]} \rangle + NV^{[r+1]}.$$

Proof. By (1), we get

$$(NV^{[r]})V \subseteq (N, V^{[r]}, V) + NV^{[r+1]} \subseteq (V^{[r]}, N, V) + NV^{[r+1]} \subseteq \langle V^{[r]} \rangle + NV^{[r+1]}. \quad \Box$$

Corollary 3. If $r \ge 1$, then

$$N\underbrace{V\ldots V}_{r+1} \subseteq \langle V^{[r]} \rangle + NV^{[r+1]}.$$

Proof. It is true for r = 1 by Lemma 5. If it is true for some $r \ge 1$, then we get

$$N\underbrace{V\dots V}_{r+2} \subseteq \langle V^{[r]} \rangle V + NV^{[r+1]}V \subseteq \langle V^{[r+1]} \rangle + NV^{[r+2]}$$

by (2) and Lemma 5. \Box

Lemma 6. Let N be an arbitrary \mathbb{Z}_n -graded Novikov algebra N from (8) and suppose that the 0-component $A = N_0$ of N is right nilpotent. Then there exists a positive integer m such that $N^{[m]} \subseteq A^{\{1\}}$.

Proof. Suppose that $A^{[r]} = 0$ for some positive integer r. By Corollary 3,

$$N\underbrace{A\ldots A}_{r+1} \subseteq \langle A^{[r]} \rangle + NA^{[r+1]} = 0$$

Again, by Corollary 3, we get

$$N\underbrace{N_i\dots N_i}_n \subseteq \langle N_i^{[n-1]} \rangle + NN_i^{[n]}$$

Notice that $N_i^{[n]} \subseteq A$. Consequently,

$$N\underbrace{N_i\dots N_i}_{n+1} \subseteq (\langle N_i^{[n-1]}\rangle + NA)N_i \subseteq \langle N_i^{[n]}\rangle + NN_iA \subseteq A^{\{1\}} + NN_iA.$$

Using this, we can easily show by induction on s that

$$N\underbrace{N_i\dots N_i}_{sn+1} \subseteq A^{\{1\}} + NN_i\underbrace{A\dots A}_s.$$

Cosequently,

$$N\underbrace{N_i\dots N_i}_{(r+1)n+1} \subseteq A^{\{1\}} + NN_i\underbrace{A\dots A}_{r+1} \subseteq A^{\{1\}}$$

since $N \underbrace{A \dots A}_{r+1} = 0.$

Thus, every N_i acts on N nilpotently modulo $A^{\{1\}}$ from the right hand side. Moreover, by (2), this action is commutative. This easily implies the existence of an integer m such that $N^{[m]} \subseteq A^{\{1\}}$. \Box

5. Right nilpotency of B

In this section we prove that any \mathbb{Z}_n -graded Novikov algebra B defined by (10) is right nilpotent if the characteristic of K does not divide n. Suppose that N is a \mathbb{Z}_n -graded Novikov algebra of the form (8) satisfying the conditions

(a) NA = 0 and

(b)
$$x \circ y = xy + yx = 0$$
 for any $x \in N_i$ and $y \in N_{n-i}$ and for any $i \in \mathbb{Z}_n$.

All statements in this section are formulated for the algebra N.

First we prove the following lemma.

Lemma 7. Let $x \in N_{n-i}$, $u \in N_i^{[k]}$, $i \in \mathbb{Z}_n$, and $k \ge 1$. Then xu = -kux.

Proof. We prove the statement of the lemma by induction on k. If k = 1, then it is true by (b). Suppose that k > 1 and u = vy, where $v \in N_i^{[k-1]}$ and $y \in N_i$. Using (1), (2), and the induction proposition, we get

$$xu = x(vy) = -(x, v, y) + (xv)y = -(v, x, y) - (k - 1)(vx)y$$
$$= -(vx)y + v(xy) - (k - 1)(vx)y = -k(vy)x + v(xy) = -kux + v(xy).$$

Notice that $xy \in N_{n-i}N_i \subseteq A$ and v(xy) = 0 by the condition (a). Consequently, xu = -kux. \Box

Corollary 4. If the characteristic of the field K does not divide n, then $N_i^{[n]}N_{n-i} = 0$ for any $i \in \mathbb{Z}_n$.

Proof. Notice that $N_i^{[n]} \subseteq A$ and $N_{n-i}N_i^{[n]} = 0$ by the condition (a). Then Lemma 7 gives that $nN_i^{[n]}N_{n-i} = 0$. If the characteristic of K does not divide n, then this gives $N_i^{[n]}N_{n-i} = 0$. \Box

Lemma 8. If the characteristic of the field K does not divide n, then

$$N\underbrace{N_i\dots N_i}_{2n} = 0$$

for any $i \in \mathbb{Z}_n$.

Proof. Corollary 3 and the condition (a) give that

$$N\underbrace{N_i\dots N_i}_n \subseteq \langle N_i^{[n-1]} \rangle + NN_i^{[n]} \subseteq \langle N_i^{[n-1]} \rangle$$

since $N_i^{[n]} \subseteq A$. Notice that i(n-1) = -i = n - i in \mathbb{Z}_n . This means $N_i^{[n-1]} \subseteq N_{n-i}$. Cosequently,

$$N\underbrace{N_i\ldots N_i}_n \subseteq \langle N_{n-i} \rangle$$

Using (2) and (a), we get

$$N\underbrace{N_{i}\ldots N_{i}}_{n+1} \subseteq \langle N_{n-i}\rangle N_{i} = \langle N_{n-i}N_{i}\rangle = \langle N_{i}N_{n-i}\rangle.$$

Then

$$N\underbrace{N_i\dots N_i}_{2n} \subseteq = \langle N_i N_{n-i} \rangle \underbrace{N_i\dots N_i}_{n-1} = \langle N_i^{[n]} N_{n-i} \rangle.$$

Corollary 4 implies the statement of the lemma. \Box

Proposition 1. Let N be a \mathbb{Z}_n -graded Novikov algebra of the form (8) satisfying the conditions

(a) NA = 0 and

(b) $x \circ y = xy + yx = 0$ for any $x \in N_i$ and $y \in N_{n-i}$ and for any $i \in \mathbb{Z}_n$.

If the characteristic of the field K does not divide n, then N is right nilpotent.

Proof. By Lemma 8, every N_i acts nilpotently on the right N-module N. Moreover, this action is commutative by (2). Consequently, N acts nilpotently on N. \Box

6. Solvability and right nilpotency

The solvability and the right nilpotency of Novikov algebras are equivalent [21]. In this section we use these notions as synonyms.

Proposition 2. Let N be a \mathbb{Z}_n -graded Novikov algebra of the form (8) such that $A = N_0$ is solvable. If the characteristic of the field K does not divide n, then N is solvable.

Proof. Consider the descending sequence of right ideals (9). By Lemma 6 there exists a positive integer m such that $N^{[m]} \subseteq A^{\{1\}}$. The algebra B from (10) satisfies all conditions of Proposition 1 by Lemma 4. By Proposition 1 there exists a positive integer t such that $B^{[t]} = 0$. This means that $(A^{\{1\}})^{[t]} \subseteq A^{\{2\}}$. By Theorem 1, the algebra $A^{\{2\}}$ is nilpotent. Consequently, $A^{\{1\}}$ and N are both solvable. \Box

Let G be an additive abelian group. We say that

$$N = \bigoplus_{g \in G} N_g$$

is a G-graded algebra if $N_g N_h \subseteq N_{g+h}$ for all $g, h \in G$.

Theorem 2. Let G be a finite additive abelian group and let N be a G-graded Novikov algebra with solvable 0-component N_0 . If the characteristic of the field K does not divide the order of the group G, then N is solvable.

Proof. We prove the statement of the theorem by induction on the order |G| of G. If $G = \mathbb{Z}_n$, then N is solvable by Proposition 2.

Every finite abelian group is a direct sum of cyclic subgroups. Suppose that $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}$, where $n_i > 1$ for all i and $k \ge 2$. Then $G = \mathbb{Z}_{n_1} \oplus G_1$, where $G_1 = \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}$. Denote by pr the projection of G onto the group \mathbb{Z}_{n_1} . Set

$$N_i' = \sum_{g \in G, pr(g)=i} N_g$$

where $i = 0, 1, \ldots, n_1 - 1$. It is easy to show that

$$N = N'_0 \oplus \dots N'_{n_1 - 1}$$

and N is a \mathbb{Z}_{n_1} -graded algebra.

It is also clear that

$$N_0' = \sum_{g \in G, pr(g)=0} N_g$$

is a G_1 -graded algebra and the 0-component of N'_0 is N_0 . Since $|G_1| < |G|$ it follows that N'_0 is solvable by the induction proposition. Now we can apply Proposition 2 to the \mathbb{Z}_{n_1} -graded algebra N. Hence N is solvable. \Box

The statement of the next lemma is well known.

Lemma 9. Let G be a group of automorphisms of an arbitrary algebra A and let H be a normal subgroup of G. Then A^H is G-invariant, the quotient group G/H acts on A^H by automorphisms, and $(A^H)^{G/H} = A^G$.

Proof. Let $a \in A^H$ and let $g \in G$. Then $ghg^{-1} \in H$ for any $h \in H$. Therefore, $a^{ghg^{-1}} = a$ and

$$(a^g)^h = a^{gh} = a^{ghg^{-1}g} = (a^{ghg^{-1}})^g = a^g.$$

Consequently, the algebra A^H is *G*-invariant. Let $g \in G$ and let \overline{g} be the image of g in G/H. Then \overline{g} defines an automorphism of the algebra A by the rule $a^{\overline{g}} = a^g$. This action is well defined. Hence the quotient group G/H acts on A^H . It is easy to check that $(A^H)^{G/H} = A^G$. \Box

Corollary 5. Let N be a Novikov algebra and let G be a finite abelian group of automorphisms of N. If the algebra N^G is solvable and the characteristic of the field K does not divide the order of the group G, then N is solvable.

Proof. We may assume that K is algebraically closed. We prove the statement of the corollary by induction on the order |G| of G. If G is a simple group, then $G \cong \mathbb{Z}_p$, where p is a prime number. Let ϕ be a generating element of the group G. Then $\phi^p = e$, where e is the identity element of G. Let ϵ be a primitive pth root of unity and let $N_i = \ker(\phi - \epsilon^i)$ for all $0 \leq i \leq p - 1$. The indexes i may be considered as elements of \mathbb{Z}_p since $\epsilon^p = 1$. Obviously,

$$N = N_0 \oplus \ldots \oplus N_{p-1}$$

and it is easy to check that $N_i N_j \subseteq N_{i+j}$ for all $i, j \in \mathbb{Z}_p$, i.e., N is a \mathbb{Z}_p -graded algebra. Moreover, $N_0 = N^G$. By Proposition 2, N is solvable.

Let H be a proper subgroup of G. Then, by Lemma 9, the quotient group G/H acts on N^H by automorphisms and $(N^H)^{G/H} = N^G$. We get that N^H is solvable by the induction proposition since |G/H| < |G|. Now we can apply the induction proposition to the group H and get that N is solvable. \Box

Theorem 3. Let N be a Novikov algebra and let G be a finite solvable group of automorphisms of N. If the algebra N^G is solvable and the characteristic of the field K does not divide the order of the group G, then N is solvable.

Proof. We prove the statement of the theorem by induction on |G|. The case of abelian groups is considered in Corollary 5. Suppose that G is not abelian. Then the commutator subgroup G' of the solvable finite group G is a proper normal subgroup.

By Lemma 9, $(N^{G'})^{G/G'} = N^{G}$. Then the algebra $N^{G'}$ is solvable by the induction proposition since |G/G'| < |G|. Applying the induction proposition to G', we get that N is solvable. \Box

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