Maltsev bases for partially commutative nilpotent groups

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Abstract

We construct an ordered set of commutators in a partially commutative nilpotent group $F(X; \Gamma; \mathfrak{N}_m)$. This set allows us to define a canonical form for each element of $F(X; \Gamma; \mathfrak{N}_m)$. Namely, we construct a Maltsev basis for the group $F(X; \Gamma; \mathfrak{N}_m)$.

Keywords: partially commutative group, basis, partially commutative algebra Lie.

1. Introduction

Many classes of algebraic structures are defined through the category of simple graphs. One of these is the class of so-called partially commutative algebraic structures. These structures have well known applications both in mathematics and in computer sciences as well as in robotics.

In the paper, let $\Gamma = \langle X; E \rangle$ be a simple graph with the set of vertices X and the set of edges E.

___First partially commutative structures being studied were monoids. A free partially commutative monoid on X associated with Γ is the monoid denoted by $M(X;\Gamma)$ which is defined by the monoid presentation

$$M(X;\Gamma) = \langle X; xy = yx, \{x, y\} \in E \rangle.$$

The notion of partially commutative monoid was introduced by P. Cartier and D. Foata in 1969 (see [1]) to study combinatorial problems in connection with word rearrangements.

A free partially commutative group $F(X;\Gamma)$ is closely related to $M(X;\Gamma)$. It is defined by the group presentation

$$F(X;\Gamma) = \langle X; xy = yx, \{x,y\} \in E \rangle.$$

 $F(X;\Gamma) = \langle X; xy = yx, \{x, y\} \in E \rangle.$ The groups $F(X;\Gamma)$ were first introduced in the 1970's by A. Baudisch (see [2]) as "semifree groups" and then were studied in the 1980's by C. Droms (see [3, 4, 5]) calling these groups by "graph groups".

The class of free partially commutative groups contains free and free abelian groups. Free partially commutative groups possess a number of remarkable properties. For example, a group $F(X;\Gamma)$ is a residually torsion-free nilpotent group (see [6]). Therefore free partially commutative groups are torsion-free. These groups are linear (see [7]). Fundamental groups of almost all surfaces are subgroups of free partially commutative groups (see [8]). In [9], it is observed that two free partially commutative groups $F(X;\Gamma)$ and $F(Y;\Delta)$ are isomorphic iff their defining graphs Γ and Δ are isomorphic.

Free partially commutative groups have provided several crucial examples having shaped the theory of finitely presented groups; notably M. Bestvina and N. Brady's example of a homologically finite (of type FP) but not geometrically finite (in fact not of type F2) group; and Mikhailova's example of a group with unsolvable subgroup membership problem. Recently it was shown by work of M. Sageev, F. Haglund, D. Wise, I. Agol and others that many wellknown families of groups virtually embed into free partially commutative groups: among these are Coxeter groups, limit groups, and fundamental groups of closed 3-manifold groups (see for example [10]).

Consider a variety \mathfrak{M} of groups. A partially commutative group in \mathfrak{M} with a defining graph Γ is a group $F(X;\Gamma;\mathfrak{M})$ defined as

$$F(X;\Gamma;\mathfrak{M}) = \langle X; xy = yx, \{x,y\} \in E \rangle$$

in \mathfrak{M} .

Consequently a free partially commutative group $F(X;\Gamma)$ is a partially commutative group in the variety \mathfrak{G} of all groups.

Recall that the commutator (g_1, g_2) of two elements g_1, g_2 of a group G is defined by $(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$. By \mathfrak{A}^2 denote a variety of all metabelian groups, i.e., all groups satisfying an identity ((x, y), (z, v)) = 1.

Among all partially commutative groups $F(X; \Gamma; \mathfrak{M}), \mathfrak{M} \neq \mathfrak{G}$, the most studied case is the case of partially commutative metabelian groups, i.e. the groups $F(X; \Gamma; \mathfrak{A}^2)$.

Let $v(x_{i_1}, \ldots, x_{i_m})$ be a representation of an element $v \in F(X; \Gamma; \mathfrak{M})$ as a product of generators in X, where the vertices x_{i_1}, \ldots, x_{i_m} occur in this representation. Then set $\sigma(v) = \{x_{i_1}, \ldots, x_{i_m}\}$. Denote by Γ_v the subgraph of Γ generated by the set $\sigma(v)$ and by $\Gamma_{v,x}$ the connected component of the graph Γ_v such that this component contains a vertex $x \in \sigma(v)$. Let us order the set X as $x_1 < x_2 < \ldots < x_n$. By $\max(\Gamma_{v,x})$ denote the greatest vertex in the connected component $\Gamma_{v,x}$.

The following theorem describes a basis of the commutant G' of a partially commutative metabelian group $G = F(X; \Gamma; \mathfrak{A}^2)$.

Theorem 1. [11] Let the set $X = \{x_1, \ldots, x_n\}$ of vertices of a graph Γ be ordered as follows $x_1 < x_2 < \ldots < x_n$ and let $G = F(X; \Gamma; \mathfrak{A}^2)$. Then a basis of the commutant G' is the set consisting of all elements v of the form

$$v = u^{-1}(x_i, x_j)u, \text{ where } u = x_{j_1}^{t_1} \dots x_{j_m}^{t_m}, \{t_1, \dots, t_m\} \subset \mathbb{Z} \setminus \{0\},$$

such that the following conditions hold:

(a) $j \leq j_1 < j_2 \ldots < j_m \leq n, 1 \leq j < i \leq n;$ (b) the vertices x_i and x_j are in different connected components of the graph Γ_v ; (c) $x_i = \max(\Gamma_{v,x_i}).$

There are results obtained for centralizers and annihilators of groups $F(X; \Gamma; \mathfrak{A}^2)$ ([12]), embeddings these groups into matrix groups (see [13]), and their groups of automorphisms ([14]) and values of centralizer dimensions ([15, 16]). The universal and elementary theories of these groups are investigated in [12, 17, 18].

The lower central series of a group G is the sequence of subgroups $G_{(n)}$, $n \ge 1$, defined inductively as follows

$$G_{(1)} = G, \ G_{(i+1)} = (G_{(i)}, G),$$

where $(G_{(i)}, G)$ denotes the subgroup of G generated by the commutators (x, y) with $x \in G_{(i)}, y \in G$.

A variety \mathfrak{N}_c consists of all groups G such that $G_{(c+1)} = 1$.

The properties of partially commutative nilpotent groups $F(X; \Gamma; \mathfrak{N}_c)$ are much less studied. Even a canonical form for elements of groups $F(X; \Gamma; \mathfrak{N}_c)$ for $c \ge 4$ is not known yet (the cases c = 2, 3 are considered in [19]).

In this paper, we study partially commutative groups in \mathfrak{N}_c . All groups considered below are finitely generated. So, the set $X = \{x_1, \ldots, x_n\}$ is finite.

For a subset H of a group G denote by $gp\langle H \rangle$ the subgroup generated by H.

If G is a torsion-free finitely generated nilpotent group then G has a central series

$$G = G_1 > G_2 > \ldots > G_{s+1} = 1 \tag{1}$$

with infinite cyclic factors. Take elements a_1, \ldots, a_s such that $G_i = gp\langle a_i, G_{i+1} \rangle$.

Definition 1. (see [20]) An ordered system $\{a_1, \ldots, a_s\}$ of elements is called a Maltsev basis for G obtained by the central series (1).

The construction of a Maltsev basis of a group makes it possible to indicate a canonical form of its elements. Every element $g \in G$ can be uniquely represented in the form

$$g = a_1^{t_1} \dots a_s^{t_s}, \ t_i \in \mathbb{Z}.$$

A Maltsev basis for a group $F(X; \Gamma; \mathfrak{A}^2 \wedge \mathfrak{N}_c)$ was found in [19]. Let us recall its description. Define a commutator $c_m = (y_1, y_2, \dots, y_m)$, where $y_i \in X$, by induction: $c_2 = (y_1, y_2), c_m = (c_{m-1}, y_m)$.

Let B be the set of commutators of the form

$$v = (x_{j_1}, x_{j_2}, \dots, x_{j_m}), \ 2 \le m \le c,$$

in a group $F(X; \Gamma; \mathfrak{A}^2 \wedge \mathfrak{N}_c)$ such that the following conditions hold:

- (a) $1 \le j_2 \le j_3 \le j_m \le n, \ j_2 < j_1 \le n;$
- (b) the vertices x_{j_1} and x_{j_2} are in different connected components of the graph Γ_v ; (c) $x_{j_1} = \max(\Gamma_{v,x_{j_1}})$.

Theorem 2. [19] The set of elements $X \sqcup B$ is a Maltsev basis of $F(X; \Gamma, \mathfrak{A}^2 \land \mathfrak{N}_c)$ obtained by refining the lower central series of this group.

The group $F(X;\Gamma;\mathfrak{N}_c) \cong F(X;\Gamma)/F_{(c+1)}(X;\Gamma)$ is torsion-free (see [6], Theorem 2.1). This means that there exists a Maltsev basis for $F(X;\Gamma;\mathfrak{N}_c)$.

The aim of this paper is to find a Maltsev basis for the group $F(X; \Gamma; \mathfrak{N}_c)$.

The study of the free partially commutative Lie algebra was started by G. Duchamp in 1987 (see [21]). Let R be a domain. A free partially commutative Lie R-algebra $\mathcal{L}_R(X;\Gamma)$ is the Lie algebra defined by the Lie algebra presentation

$$\mathcal{L}_R(X;\Gamma) = \langle X; [x_i, x_j] = 0, \{x_i, x_j\} \in E \rangle.$$

Put $\mathcal{L}(X;\Gamma) = \mathcal{L}_{\mathbb{Z}}(X;\Gamma)$. In [6], the relation between the graded Lie Z-algebra associated with the quotients of the lower central series of $F(X;\Gamma)$ and the Lie algebra $\mathcal{L}(X;\Gamma)$ was established. We are going to use this relation.

The concept of basic commutators was introduced by Ph. Hall in [22]. Hall's commutators are usually used in group theory.

For convenience, we will use so called standard commutators (see [23]) for the description of a Maltsev basis.

Denote by X^* the set of all words in $X = \{x_1, \ldots, x_n\}$ including the empty word denoted by 1. We also denote by |u| the length of any $u \in X^*$. Let us extend an arbitrary linear order on X to a lexicographic order "<" on X^* as follows. Put u < 1 for each $1 \neq u \in X^*$ and by induction put $x_i u' < x_j v'$ if $x_i < x_j$ or $x_i = x_j, u' < v'$.

Definition 2. Let

$$ALS(X) = \{ u \in X^* \mid \forall u_1, u_2 \in X^* (u = u_1 u_2 \longrightarrow u_2 u_1 < u_1 u_2) \}.$$

A word $u \in ALS(X)$ is called an associative Lyndon-Shirshov word.

Let us define a set $\mathcal{G}(X)$ and a bar map $\mathcal{G}(X) \longrightarrow X^*$ as follows.

Definition 3. (a) $x_i \in \mathcal{G}(X)$ for all $x_i \in X$, $\overline{x_i} = x_i$. (b) If $u, v \in \mathcal{G}(X)$, then $(u, v) \in \mathcal{G}(X)$ and $\overline{(u, v)} = \overline{u} \overline{v}$.

The bar map erases all parentheses and commas. We put

$$\mathcal{G}_m(X) = \{ u \mid u \in \mathcal{G}(X), \ |\overline{u}| = m \}, \text{ then } \mathcal{G}(X) = \bigcup_{m \ge 1} \mathcal{G}_m(X).$$

Now we give a definition of the set (X^*) of standard commutators.

Definition 4. (a) $x_i \in (X^*)$ for i = 1, ..., n. (b) Let w = (u, v). Then $w \in (X^*)$ if and only if the following conditions are true: (b1) $\overline{w} \in ALS(X)$; (b2) $u, v \in (X^*), \overline{u} > \overline{v}$; (b2) if $u = (u_1, u_2)$ then $\overline{v} \ge \overline{u_2}$.

Let

$$(X^*)_m = \{ u \mid u \in (X^*), |\overline{u}| = m \}.$$

If F is the free group with the basis $X = \{x_1, \ldots, x_n\}$, and $(x, y) = x^{-1}y^{-1}xy$ for $x, y \in F$, then the set of commutators $(X^*)_m$ forms a basis of the free abelian group $F_{(m)}/F_{(m+1)}$ for $m = 1, 2 \ldots$ (see [23], Theorem 3.5).

Definition 5. Let $u \in X^*$. By $\delta_i(u)$ denote the number of occurrences of x_i in u. For $u \in X^*$, put

$$supp(u) = \{x_i \mid \delta_i(u) \neq 0\}$$

Finally, let us define by induction a subset $\mathcal{C}(X;\Gamma)$ of (X^*) .

Definition 6. (a) All elements of X belong to $C(X; \Gamma)$.

(b) An element $u \in (X^*)_m, m \ge 2$, belongs to $\mathcal{C}(X, \Gamma)$ if u = (v, w), where v and w are elements of $\mathcal{C}(X; \Gamma)$ and there is an element in supp(v) such that this element is not connected in Γ with the first letter of w.

(c) There are no other elements in $\mathcal{C}(X;\Gamma)$.

Let

$$\mathcal{C}_i(X;\Gamma) = \{ u \in \mathcal{C}(X;\Gamma) \mid |\overline{u}| = i, \ i = 0, 1, \ldots \}.$$

Let " \prec " on $\mathcal{C}(X; \Gamma)$ such that $u \prec v$ if $u \in \mathcal{C}_p(X; \Gamma)$, $v \in \mathcal{C}_q(X; \Gamma)$, $1 \leq p < q$. Let

$$\mathcal{C}^{(m)}(X;\Gamma) = \bigcup_{1 \le i \le m} \mathcal{C}_i(X;\Gamma).$$

Theorem 3. The set $\mathcal{C}^{(m)}(X;\Gamma)$ with respect to the order " \prec " is a Maltsev basis for the group $F(X;\Gamma;\mathfrak{N}_m)$ obtained by refining the lower central series.

2. Bases for partially commutative nilpotent Lie algebras

An explicit construction for bases of free partially commutative Lie algebras was obtained in [24]. To give this description let us first recall a definition of Lyndon—Shirshov words.

The lexicographic order "<" has been defined above (in the last paragraph before Definition 2as well as the set ALS(X) of associative Lyndon-Shirshov words on X (see Definition 2).

Let $\mathcal{L}(X)$ be the free Lie algebra on the set $X = \{x_1, \ldots, x_n\}$.

Let us give a definition of a set $[X^*]$, $[X^*] \subseteq \mathcal{L}(X)$, of non-associative Lyndon–Shirshov words.

Definition 7. (a) $x_i \in [X^*]$ for i = 1, ..., n; (b) Let [w] = [[u], [v]]. Then $[w] \in [X^*]$ if and only if the following conditions hold: (b1) $w \in ALS(X)$; (b2) $[u], [v] \in [X^*], u > v$, where u, v denote the words in X^* obtained from [u], [v] by omitting the Lie brackets [,]; (b3) if $[u] = [[u_1], [u_2]]$ then $v \ge u_2$.

It was shown in [25] that the set $[X^*]$ of all non-associative Lyndon–Shirshov words is a linear basis of the free Lie *R*-algebra $\mathcal{L}_R(X)$ over a domain *R*.

For a free partially commutative Lie algebra $\mathcal{L}_R(X; \Gamma)$ over a domain R define inductively the set of partially commutative Lyndon–Shirshow words (PCLS-words for short) by induction.

Definition 8. (a) All elements of X are PCLS-words.

(b) A Lyndon-Shirshov word [u] such that |u| > 1 is a PCLS-word if [u] = [[v], [w]], where [v] and [w] are PCLS-words and there is an element in supp(v) such that it is not connected in Γ with the first letter of w.

(c) There are no other PCLS-words.

Denote the set of all PCLS-words of a free partially commutative Lie *R*-algebra $\mathcal{L}_R(X;\Gamma)$ by $PCLS(X;\Gamma)$.

The first result on bases of free partially commutative Lie algebras $\mathcal{L}_R(X;\Gamma)$ was obtained by D. Duchamp and D. Krob in [26], but they did not give an explicit description of a basis.

Using the method of Gröbner—Shirshov bases E. Poroshenko in [24] obtained an explicit description of bases for free partially commutative Lie algebras.

Theorem 4. [24] Let R be a unital commutative ring and Γ be a graph. Then the set $PCLS(X;\Gamma)$ is a linear basis of the free partially commutative Lie R-algebra $\mathcal{L}_R(X;\Gamma)$.

Let \mathfrak{L}_m be a variety of all nilpotent Lie algebras of class at most m. Denote by $\mathcal{L}_R(X; \Gamma; \mathfrak{L}_m)$ the partially commutative m-nilpotent R-algebra Lie.

A linear basis for a partially commutative nilpotent Lie algebra can be easily obtained from a linear basis for the corresponding free partially commutative Lie algebra.

Theorem 5. [24] Let R be a unital commutative ring and Γ be a graph. Then a linear basis of the partially commutative nilpotent R-algebra $\mathcal{L}_R(X;\Gamma;\mathfrak{L}_m)$ consists of all elements of $PCLS(X;\Gamma)$ whose lengthes are not greater than m.

3. Proof of Theorem 3

Let G be a group. Define the associated graded abelian group gr(G) as follows

$$gr(G) = \bigoplus_{m \ge 1} gr_m(G),$$

where $gr_m(G) = G_{(m)}/G_{(m+1)}$. The group gr(G) has a structure of a graded Lie algebra over the ring \mathbb{Z} of integers with the bracket operation in gr(G) induced by the commutator operation in G.

By \mathcal{F} denote a graded \mathbb{Z} -module

$$\mathcal{F} = \bigoplus_{m \ge 1} F_{(m)}(X; \Gamma) / F_{(m+1)}(X; \Gamma).$$

The element $g^* \in \mathcal{F}$ is called a homogeneous element of degree m if this element is in $F_{(m)}(X;\Gamma)/F_{(m+1)}(X;\Gamma)$.

 ${\mathcal F}$ can be equipped with a Lie ${\mathbb Z}\text{-algebra structure as follows.}$

Let g^* and h^* be homogeneous elements of degrees m and n respectively. Denote by g a preimage of g^* in $F_{(m)}(X; \Gamma)$ and by h a preimage of h^* in $F_{(n)}(X; \Gamma)$. Then $(g, h) \in F_{(m+n)}(X; \Gamma)$ according to a property of the lower central series. Thus we can equip \mathcal{F} with the Lie bracket defined by the relation

$$[g^*, h^*] = (g, h)F_{(m+n+1)}(X; \Gamma).$$

This Lie bracket is well-defined. It does not depend on the choice of preimages g and h for elements g^* and h^* . We can extend the bracket operation to \mathcal{F} by distributivity.

Let vertices x_i and x_j be adjacent in Γ . Then

$$[x_i F_{(2)}(X; \Gamma), x_j F_{(2)}(X; \Gamma)] = (x_i, x_j) F_{(3)}(X; \Gamma) = F_{(3)}(X; \Gamma) = 0$$

in \mathcal{F} . Therefore, we can extend mapping

$$\alpha(x) = xF_{(2)}(X;\Gamma), \ x \in X,$$

to a homomorphism of the Lie \mathbb{Z} -algebras $\mathcal{L}(X; \Gamma)$ and \mathcal{F} :

$$\alpha: \mathcal{L}(X; \Gamma) \longrightarrow \mathcal{F}.$$

Let us now define a family $\mathcal{A}(X)$ of $\mathcal{L}(X, \Gamma)$ by induction. We set $\mathcal{A}_1(X) = X$. For $m \ge 2$, put

$$\mathcal{A}_m(X) = \{ [u, v] \mid u \in \mathcal{A}_p(X), v \in \mathcal{A}_q(X), p + q = m \},$$
$$\mathcal{A}(X) = \bigcup_{m \ge 1} \mathcal{A}_m(X).$$

Let $\mathcal{L}_m(X;\Gamma)$ be a submodule of $\mathcal{L}(X;\Gamma)$ generated by $\mathcal{A}_m(X)$.

In [6], Theorem 2.1, it was proved that α is an isomorphism of graded Lie algebras from $\mathcal{L}(X;\Gamma)$ graded by $(\mathcal{L}_m(X;\Gamma)_{m\geq 1})$ into \mathcal{F} .

Consequently

$$F_{(m)}(X;\Gamma)/F_{(m+1)}(X;\Gamma) \stackrel{\alpha}{\simeq} \mathcal{L}_m(X;\Gamma),$$

for $m \geq 1$.

By $PCLS_m(X;\Gamma)$ denote the set of all $PCLS(X;\Gamma)$ words of length m. As follows from Theorem 5, the set $PCLS_m(X;\Gamma)$ forms a basis for the additive abelian group $\mathcal{L}_m(X;\Gamma)$.

Comparing Definitions 4 and 7, and then Definition 6 and 8, we see that the isomorphism α maps the set $PCLS_m(X;\Gamma)$ onto the set $\mathcal{C}_m(X;\Gamma)$. Therefore, $\mathcal{C}_m(X;\Gamma)$ forms a basis for the abelian group $F_{(m)}(X;\Gamma)/F_{(m+1)}(X;\Gamma)$. This completes the proof.

Example. Let $\Gamma = \langle x_1, x_2, x_3; \{x_1, x_2\} \rangle$, $x_1 > x_2 > x_3$. By construction,

$$\mathcal{C}^{(3)}(X;\Gamma) = \{x_1, x_2, x_3; (x_1, x_3), (x_2, x_3); (x_1, (x_1, x_3)), (x_2, (x_2, x_3)), ((x_1, x_3), x_2), ((x_1, x_3), x_3), ((x_2, x_3), x_3)\}$$

is a Maltsev basis of the group $F(X; \Gamma; \mathfrak{N}_3)$.

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