

HOLONOMY THEOREM FOR FINITE SEMIGROUPS

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ABSTRACT. We provide a simple proof of the Holonomy Theorem using a new Lyndon–Chiswell length function on the Karnofsky–Rhodes expansion of a semigroup. Unexpectedly, we have both a left and a right action on the Chiswell tree by elliptic maps.

1. INTRODUCTION

The following problem is at the heart of the global theory of finite semigroups. Let S, T, U be finite semigroups. Consider a product of the form $T \star U$ (direct, semidirect, wreath, block, triple – see [RS09]) admitting a projection morphism on U . When does S divide $T \star U$ (i.e. S is a homomorphic image of some subsemigroup of $T \star U$)? The standard technique consists of proving that, for a given generating set A of S , each $a \in A$ can be lifted to some $\bar{a} \in T \star U$. Examples include the $V \cup T$ proof of the Krohn–Rhodes Theorem (see [RS09]) or the alternative proof by Zeiger [Zei68, Til76a, Til76b], called the Holonomy Theorem proof.

Denoting by \bar{S} the subsemigroup of $T \star U$ generated by $\{\bar{a} \mid a \in A\}$, these theorems prove that S is a homomorphic image of \bar{S} . But sometimes it takes decades to discover the exact nature of \bar{S} . Rhodes’ first attempt for the case of the Holonomy theorem was the Rhodes expansion $(S, A)^R$ (see [Til76a, Til76b]) on the way to proving the Fundamental Lemma of Complexity of finite semigroups (surmorphisms which are one-to-one on subgroups preserve complexity, see [RS09]).

A better version is the right Karnofsky–Rhodes expansion $\text{KR}_{\text{right}}(S, A)$ used in this paper. If we denote by T_n the semigroup of transformations of an n -set, then, for $n \geq 3$, T_n is not a subsemigroup of any nontrivial product. So T_n must be expanded by taking preimages to become a subsemigroup of a wreath product and is intuitively the “smallest” one. We note that $\text{KR}_{\text{left}}(S, A)$ (for left acting semigroups, note the right/left reversal) relates to *coupling from the past* in Markov chains [PW96, RS19].

In this paper, the Holonomy Theorem is proved via Lyndon–Chiswell theory of semigroups acting on trees (created by Rhodes in [Rho91] and developed by Rhodes and Silva in [RS12]). We use that $\text{KR}_{\text{right}}(S, A)$ acts faithfully as elliptic maps on a tree, elliptic maps being an abstraction of wreath products. A key idea is that $\text{KR}_{\text{right}}(S, A)$ has many more useful surmorphisms, namely when $\text{KR}_{\text{right}}(S, A)$ is proved to consist of elliptic maps: restricting the elliptic maps to all vertices of bounded depth gives a surmorphism.

Another key idea is that the \mathcal{J} -order comes explicitly into the definition of the Lyndon–Chiswell function, and $\text{KR}_{\text{right}}(S, A)$ carries more information than the Rhodes expansion $(S, A)^R$. This extra information makes the proof in this paper much easier in comparison with [Rho91, RS12]. Also we obtain an extra action we need to understand more fully.

Passing from elliptic maps to wreath products is not difficult: one just needs to number the edges leaving level j with some index set X_j . The Lyndon–Chiswell construction does not yield uniform branching, but simply adds “fake” edges and leaves the function undefined on those. This leads to a wreath product of partial transformation semigroups.

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Zeiger coding (see [Zei68, Til76a, Til76b]) becomes the following. Suppose that v_1, v_2 are vertices of the tree at the same level such that $v_1 r_{12} = v_2$ and $v_2 r_{21} = v_1$, where r_{12} and r_{21} are elliptic maps representing elements of $\text{KR}_{\text{right}}(S, A)$. Then if real edges E_1, \dots, E_d descending from v_1 are labeled $1, \dots, d$ by the index set, then the distinct $E_1 r_{12}, \dots, E_d r_{12}$ edges get labeled $1, \dots, d$ also by the index function. As mentioned in the first two paragraphs, elliptic maps arise in studying the Zeiger proof of the Krohn–Rhodes Theorem and determining \bar{S} .

We plan to apply the results of this paper to Markov chains [RS19, ASST15b, ASST15a], where the tree is associated to a statistic and strings running through the tree determine the stationary distribution and mixing time.

At the moment, it is very mysterious how the \mathcal{J} -order plays such a key role in both the semigroup theory and the Markov chain theory. In a future paper [RSS] we will compare the Chiswell–Lyndon trees of the Karnofsky–Rhodes and Rhodes expansions, leading to various new statistics for the finite Markov chain.

This paper is organized as follows. In Section 2, we review the Karnofsky–Rhodes expansion of the Cayley graph of a semigroup with a finite set of generators. In Section 3, we introduce the Dedekind height function and our new Lyndon–Chiswell length function. The Lyndon–Chiswell length function is used in the Chiswell construction, which provides a rooted tree associated to the Karnofsky–Rhodes expansion of the semigroup. The Chiswell construction in turn establishes the Holonomy Theorem (see Theorem 3.11). We conclude in Section 4 with several examples.

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2. THE KARNOFSKY–RHODES EXPANSION

Let A be a finite alphabet and let A^+ (respectively A^*) denote the free semigroup (respectively the free monoid) on A . If $\theta: A^+ \rightarrow S$ is a semigroup morphism onto a semigroup S , we say that S is *generated by* A . We usually view A as a subset of S . The reference to the morphism is omitted whenever possible and we use the notation (S, A) to describe this situation.

We denote by $S^{\mathbb{1}}$ the monoid obtained by adjoining to S a (new) identity $\mathbb{1}$ (even if S is already a monoid). *Green’s quasi-orders* on S are defined by

- $a \leq_{\mathcal{R}} b$ if $a \in bS^{\mathbb{1}}$,
- $a \leq_{\mathcal{L}} b$ if $a \in S^{\mathbb{1}}b$,
- $a \leq_{\mathcal{J}} b$ if $a \in S^{\mathbb{1}}bS^{\mathbb{1}}$.

Then $\mathcal{X} = \leq_{\mathcal{X}} \cap \geq_{\mathcal{X}}$ for $\mathcal{X} = \mathcal{R}, \mathcal{L}, \mathcal{J}$.

We denote by S^{op} the *opposite* semigroup of S , where the binary operation \cdot on S is replaced by the binary operation $x \circ y = y \cdot x$. Note that the \mathcal{L} relation of S^{op} is the \mathcal{R} relation of S , and the \mathcal{R} relation of S^{op} is the \mathcal{L} relation of S .

We define the *left and right Cayley graphs* of (S, A) , denoted respectively by $\text{LCay}(S, A)$ and $\text{RCay}(S, A)$, as follows:

- $S^{\mathbb{1}}$ is the vertex set in both graphs,
- the edge set of $\text{LCay}(S, A)$ is $\{(s, a, as) \mid s \in S^{\mathbb{1}}, a \in A\}$,
- the edge set of $\text{RCay}(S, A)$ is $\{(s, a, sa) \mid s \in S^{\mathbb{1}}, a \in A\}$.

Note that these graphs are *complete* and *deterministic*: given a vertex s and $u \in A^+$, there exists a unique path with label u starting at s . The following remark, which follows from the definitions, will allow us to use left-right symmetries:

Remark 2.1. $\text{LCay}(S, A) = \text{RCay}(S^{\text{op}}, A)$.

An edge (p, q) of a directed graph is called a *transition edge* if there exists no path from q to p . This applies also to A -labeled graphs (in particular to left and right Cayley graphs), where (s, a, s') is a transition edge if there is no path from s' to s . Note that in a Cayley graph, edges of the form $(\mathbb{1}, a, a)$ are always transition edges.

If $s \xrightarrow{a} s'$ is an edge of $\text{RCay}(S, A)$, then $s' = sa$ and so $s' \leq_{\mathcal{R}} s$. Hence this edge is a transition edge if and only if $s' <_{\mathcal{R}} s$. Note also that if two transition edges occur in two different paths, they must occur *in the same order*.

The *right Karnofsky–Rhodes expansion* $\text{KR}_{\text{right}}(S, A)$ of (S, A) is defined as the quotient A^+/τ_r , where τ_r is the congruence on A^+ defined as follows: $u \tau_r v$ if $u = v$ holds in S and the paths $\mathbb{1} \xrightarrow{u} u$ and $\mathbb{1} \xrightarrow{v} v$ in $\text{RCay}(S, A)$ have the same transition edges. Then S is a homomorphic image of $\text{KR}_{\text{right}}(S, A)$ in the obvious way.

The *left Karnofsky–Rhodes expansion* of (S, A) can be defined by

$$\text{KR}_{\text{left}}(S, A) = \text{KR}_{\text{right}}(S^{\text{op}}, A).$$

We will be paying particular attention to $\text{KR}_{\text{right}}^{\mathbb{1}}(S, A)$, which is obtained by adjoining the (new) identity $\mathbb{1}$ to $\text{KR}_{\text{right}}(S, A)$. We can view $\text{KR}_{\text{right}}^{\mathbb{1}}(S, A)$ as the quotient $A^*/(\tau_r \cup \{(1, 1)\})$. Similarly, we define $\text{KR}_{\text{left}}^{\mathbb{1}}(S, A)$.

3. THE CHISWELL CONSTRUCTION

From now on, S is a fixed finite semigroup and A is a generating set of S .

3.1. The Dedekind height function. We shall write $T = \text{KR}_{\text{right}}(S, A)$ throughout this section, and let $\varphi: T^{\mathbb{1}} \rightarrow S^{\mathbb{1}}$ denote the canonical surmorphism.

The *Dedekind height function* $h: S^{\mathbb{1}} \rightarrow \mathbb{N}$ is defined as

$$h(s) = \max\{k \in \mathbb{N} \mid \text{there exists a chain } s_0 >_{\mathcal{J}} \cdots >_{\mathcal{J}} s_k = s \text{ in } S^{\mathbb{1}}\}.$$

This should be denoted h_S , but the semigroup S is usually understood, as in Proposition 3.4 below.

Finite semigroups are known to be *stable*: they satisfy the equalities

$$\leq_{\mathcal{R}} \cap \mathcal{J} = \mathcal{R}, \quad \leq_{\mathcal{L}} \cap \mathcal{J} = \mathcal{L}.$$

The following result will prove useful later.

Lemma 3.1. *If $s <_{\mathcal{K}} s'$ holds in S for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}\}$, then $h(s) > h(s')$.*

Proof. The result is immediate for \mathcal{J} . By symmetry, we may assume that $s <_{\mathcal{R}} s'$. It follows that $s \leq_{\mathcal{J}} s'$. Now since S is stable we cannot have $s\mathcal{J}s'$, thus $s <_{\mathcal{J}} s'$ and so $h(s) > h(s')$. \square

A semigroup S is *regular* if every $s \in S$ is regular. That is, for each $s \in S$ there exists an element $s' \in S$ such that $ss's = s$.

Lemma 3.2. *If $t, t' \in T^{\mathbb{1}}$ satisfy $\varphi(tt't) = \varphi(t)$, then $tt't = t$.*

Proof. Let $u, v \in A^*$ represent t and t' , respectively. We have paths $\mathbb{1} \xrightarrow{u} \varphi(t)$ and $\mathbb{1} \xrightarrow{uvu} \varphi(tt't)$ in $\text{RCay}(S, A)$. Since $\varphi(tt't) = \varphi(t)$ and $\text{RCay}(S, A)$ is deterministic, we actually have a loop labeled by vu at $\varphi(t)$. Since a loop cannot contain transition edges, it follows that $uvu \tau_r u$ and so $tt't = t$. \square

It follows that if S is regular, then $T^{\mathbb{1}}$ is also regular.

Lemma 3.3. *Assume that S is a regular semigroup and let $t, t' \in T^{\mathbb{1}}$.*

- (i) *If $\varphi(t) \leq_{\mathcal{J}} \varphi(t')$ if and only if $t \leq_{\mathcal{J}} t'$.*
- (ii) *$\varphi(t) <_{\mathcal{J}} \varphi(t')$ if and only if $t <_{\mathcal{J}} t'$.*

Proof. (i) If $\varphi(t) \leq_{\mathcal{J}} \varphi(t')$, there exist $p, q \in T^{\mathbb{1}}$ such that $\varphi(t) = \varphi(pt'q)$. On the other hand, since S is regular, we have $\varphi(t) = \varphi(tzt)$ for some $z \in T$. Hence

$$\varphi(t) = \varphi(tzt) = \varphi(tztzt) = \varphi(tzpt'qzt)$$

and it follows from Lemma 3.2 that $t = tzpt'qzt$. Therefore $t \leq_{\mathcal{J}} t'$.

The converse implication follows from φ being a homomorphism.

(ii) Assume that $\varphi(t) <_{\mathcal{J}} \varphi(t')$. By (i), we obtain $t \leq_{\mathcal{J}} t'$. Since $\leq_{\mathcal{J}}$ is preserved by homomorphisms, $t \mathcal{J} t'$ implies $\varphi(t) \mathcal{J} \varphi(t')$, a contradiction. Thus $t <_{\mathcal{J}} t'$.

Conversely, assume that $t <_{\mathcal{J}} t'$. Hence $\varphi(t) \leq_{\mathcal{J}} \varphi(t')$. Since $\varphi(t) \mathcal{J} \varphi(t')$ implies $t \mathcal{J} t'$ by (i), we get $\varphi(t) <_{\mathcal{J}} \varphi(t')$. \square

Proposition 3.4. *Assume that S is a regular semigroup and let $t \in T^{\mathbb{1}}$. Then $h(t) = h(\varphi(t))$.*

Proof. By Lemma 3.3(ii), we have a chain

$$t_1 >_{\mathcal{J}} \cdots >_{\mathcal{J}} t_k = t$$

in T if and only if we have a chain

$$\varphi(t_1) >_{\mathcal{J}} \cdots >_{\mathcal{J}} \varphi(t_k) = \varphi(t)$$

in S . Thus $h(t) = h(\varphi(t))$. \square

Remark 3.5. If (S, A) is not regular, computing $h(t)$ for $t \in T$ can be more challenging sometimes.

3.2. The Lyndon–Chiswell length function. Write

$$\ell = 2 \max\{h(s) \mid s \in S^{\mathbb{1}}\}.$$

Denote by $t(E)$ the endpoint of an edge E of a directed graph.

Let $\alpha, \beta \in T^{\mathbb{1}}$. Let (E_1, \dots, E_m) and (E'_1, \dots, E'_n) be the corresponding sequences of transition edges. Since any edge starting at $\mathbb{1}$ is a transition edge, we have $m = 0$ if and only if $\alpha = \mathbb{1}$. Let

$$\xi(\alpha, \beta) = \max\{i \in \{0, \dots, m\} \mid E_1 = E'_1, \dots, E_i = E'_i\}.$$

Hence $\xi(\alpha, \beta)$ counts the maximum number of transition edges consecutively shared by α and β , when we start with the first and proceed in order. If $\xi(\alpha, \beta) = k > 0$, we write also $\eta(\alpha, \beta) = E_k = E'_k$.

Lemma 3.6. *For all $\alpha, \beta, \gamma \in T^{\mathbb{1}}$, we have:*

- (i) $\xi(\alpha, \beta) = \xi(\beta, \alpha)$;
- (ii) $\xi(\alpha\gamma, \beta\gamma) \geq \xi(\alpha, \beta)$;
- (iii) $\xi(\alpha, \gamma) \geq \min(\xi(\alpha, \beta), \xi(\beta, \gamma))$.

Proof. (i) follows from the symmetry of equality.

(ii) follows from the following fact: the sequence of transition edges of $\alpha\gamma$ starts with the sequence of transition edges of α .

For (iii), we may assume that $\min(\xi(\alpha, \beta), \xi(\beta, \gamma)) = k > 0$. Let (E_1, \dots, E_m) , (E'_1, \dots, E'_n) and (E''_1, \dots, E''_p) be the sequences of transition edges corresponding to α , β and γ . Then $E_1 = E'_1, \dots, E_k = E'_k$ and also $E'_1 = E''_1, \dots, E'_k = E''_k$. Hence $E_1 = E''_1, \dots, E_k = E''_k$ and so $\xi(\alpha, \gamma) \geq k$ as required. \square

We prove also the following result:

Lemma 3.7. *Let $\alpha, \beta, \gamma \in T^{\mathbb{1}}$ be such that $\xi(\alpha, \beta) > 0$. Then:*

- (i) $\xi(\gamma\alpha, \gamma\beta) > 0$;
- (ii) $t(\eta(\gamma\alpha, \gamma\beta)) \leq_{\mathcal{R}} \varphi(\gamma)t(\eta(\alpha, \beta)) \leq_{\mathcal{L}} t(\eta(\alpha, \beta))$.

Proof. We may assume that $\gamma \neq \mathbb{1}$.

(i) Since the first letter of a word representing γ must necessarily label a transition edge of γ (or $\gamma\alpha$, or $\gamma\beta$), it follows that $\xi(\gamma\alpha, \gamma\beta) > 0$.

(ii) Let (E_1, \dots, E_m) , (E'_1, \dots, E'_n) and (E''_1, \dots, E''_p) be the sequences of transition edges corresponding to α , β and γ , respectively. Let $\xi(\alpha, \beta) = k$, so that $\eta(\alpha, \beta) = E_k = E'_k$. What are the possible transition edges of $\gamma\alpha$? Clearly, E''_1, \dots, E''_p are all transition edges of $\gamma\alpha$.

Write $S^{\mathbb{1}} = A^*/\sigma$ and $T^{\mathbb{1}} = A^*/\tau$. Let $u = e_1u_1 \dots e_mu_m$ and $u' = e'_1u'_1 \dots e'_nu'_n$ be words representing α and β respectively, where e_i and e'_j denote the labels of E_i and E'_j . Write $v_i = e_1u_1 \dots e_iu_i$ and $v'_j = e'_1u'_1 \dots e'_ju'_j$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Let $\gamma = w\tau$.

Since the letters occurring in the u_i label no transition edges in the path $\mathbb{1} \xrightarrow{u} u\sigma$ in $\text{RCay}(S, A)$, there exists a path $v_i\sigma \xrightarrow{x_i} (v_{i-1}e_i)\sigma$ in $\text{RCay}(S, A)$ for $i = 1, \dots, m$. Hence $(v_ix_i)\tau = (v_{i-1}e_i)\tau$ and so $(wv_ix_i)\tau = (wv_{i-1}e_i)\tau$ for $i = 1, \dots, m$. Thus the only possible transition edges of $\gamma\alpha$ beyond E''_1, \dots, E''_p are of the form $(wv_{i-1})\sigma \xrightarrow{e_i} (wv_{i-1}e_i)\sigma$ for some $i \in \{1, \dots, m\}$. Similarly, the only possible transition edges of $\gamma\beta$ beyond E''_1, \dots, E''_p are of the form $(wv'_{i-1})\sigma \xrightarrow{e'_i} (wv'_{i-1}e'_i)\sigma$ for some $i \in \{1, \dots, n\}$.

Let

$$I = \{i \in \{1, \dots, k\} \mid (wv_{i-1})\sigma \xrightarrow{e_i} (wv_{i-1}e_i)\sigma \text{ is a transition edge of } \text{RCay}(S, A)\}.$$

Note that I needs not to contain all the integers between 1 and r . Suppose first that $I \neq \emptyset$ and let $r = \max I$. Since

$$(v_{i-1}\sigma, e_i, (v_{i-1}e_i)\sigma) = E_i = E'_i = (v'_{i-1}\sigma, e'_i, (v'_{i-1}e'_i)\sigma)$$

for $i = 1, \dots, k$, we get $((wv_{i-1})\sigma, e_i, (wv_{i-1}e_i)\sigma) = ((wv'_{i-1})\sigma, e'_i, (wv'_{i-1}e'_i)\sigma)$ as well. Hence $\gamma\alpha$ and $\gamma\beta$ share the same transition edges up to $(wv_{r-1}\sigma, e_r, (wv_{r-1}e_r)\sigma) = (wv'_{r-1}\sigma, e'_r, (wv'_{r-1}e'_r)\sigma)$ at least. Now since $r = \max I$ there are no more transition edges between $(wv_{r-1}e_r)\sigma$ and $(wv_{k-1}e_k)\sigma$ in $\text{RCay}(S, A)$. Hence

$$t(\eta(\gamma\alpha, \gamma\beta)) \leq_{\mathcal{R}} (wv_{r-1}e_r)\sigma \mathcal{R} (wv_{k-1}e_k)\sigma = \varphi(\gamma)t(\eta(\alpha, \beta)).$$

We reach the same conclusion in the case $I = \emptyset$, replacing the edge E_r in the above argument by E''_p (note that $p \geq 1$ since we are assuming $\gamma \neq \mathbb{1}$).

Finally, $\gamma t(\eta(\alpha, \beta)) \leq_{\mathcal{L}} t(\eta(\alpha, \beta))$ holds trivially. \square

Let (E_1, \dots, E_m) and (E'_1, \dots, E'_n) be the sequences of transition edges corresponding to α and β , respectively. Let $k = \xi(\alpha, \beta)$. If $k > 0$, we have $\eta(\alpha, \beta) = E_k = E'_k$. We define the *Lyndon-Chiswell length function* $D: T^{\mathbb{1}} \times T^{\mathbb{1}} \rightarrow \mathbb{N}$ by

$$D(\alpha, \beta) = \begin{cases} \ell & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta \text{ and } k = 0, \\ 2h(t(E_k)) & \text{if } 0 < k < m, n \text{ and } t(E_{k+1}) = t(E'_{k+1}), \\ 2h(t(E_k)) - 1 & \text{in all remaining cases.} \end{cases}$$

Note that $\text{im}(h) = \{0, 1, \dots, \frac{\ell}{2}\}$ implies $\text{im}(D) \subseteq \{0, 1, \dots, \ell\}$. We show now that

$$(3.1) \quad (\alpha \neq \beta \wedge \xi(\alpha, \beta) < \xi(\alpha, \gamma)) \Rightarrow D(\alpha, \beta) < D(\alpha, \gamma)$$

holds for all $\alpha, \beta, \gamma \in T^{\mathbb{1}}$.

Assume that $\xi(\alpha, \beta) < \xi(\alpha, \gamma)$. We may assume that $\xi(\alpha, \beta) > 0$, otherwise $D(\alpha, \beta) = 0$. Then $t(\eta(\alpha, \beta)) >_{\mathcal{R}} t(\eta(\alpha, \gamma))$ because there exists in $\text{RCay}(S, A)$ a path from $t(\eta(\alpha, \beta))$ to $t(\eta(\alpha, \gamma))$ containing transition edges. By Lemma 3.1, we get $h(t(\eta(\alpha, \beta))) < h(t(\eta(\alpha, \gamma)))$, yielding $D(\alpha, \beta) < D(\alpha, \gamma)$. Thus (3.1) holds.

The following properties go a little beyond those of [Rho91, Fact 1.9]. We provide a full proof.

Lemma 3.8. *The Lyndon-Chiswell length function satisfies the following properties for all $\alpha, \beta, \gamma \in T^{\mathbb{1}}$:*

- (i) $D(\alpha, \beta) = D(\beta, \alpha)$;
- (ii) $D(\alpha\gamma, \beta\gamma) \geq D(\alpha, \beta)$;
- (iii) $D(\gamma\alpha, \gamma\beta) \geq D(\alpha, \beta)$;
- (iv) (**isoperimetric inequality**) $D(\alpha, \gamma) \geq \min(D(\alpha, \beta), D(\beta, \gamma))$.

Proof. (i) It follows easily from Lemma 3.6(i).

(ii) We may assume that $\alpha\gamma \neq \beta\gamma$, otherwise $D(\alpha\gamma, \beta\gamma) = \ell$ is maximum. Hence $\alpha \neq \beta$ as well. We may also assume that $\xi(\alpha, \beta) > 0$, otherwise $D(\alpha, \beta) = 0$.

Let (E_1, \dots, E_m) and (E'_1, \dots, E'_n) be the sequences of transition edges corresponding to α and β . Let $k = \xi(\alpha, \beta) > 0$, so that $E_1 = E'_1, \dots, E_k = E'_k$. By the proof of Lemma 3.6(ii), the sequences of transition edges corresponding to $\alpha\gamma$ and $\beta\gamma$ are of the form $(E_1, \dots, E_m, F_1, \dots, F_r)$ and $(E'_1, \dots, E'_n, F'_1, \dots, F'_s)$. Suppose that $\xi(\alpha\gamma, \beta\gamma) > \xi(\alpha, \beta) = k$. Then either $m = k < n$ and $F_1 = E'_{k+1}$, or $n = k < m$ and $E_{k+1} = F'_1$, or $m = n = k$ and $F_1 = F'_1$. In any case, we have $t(\eta(\alpha\gamma, \beta\gamma)) <_{\mathcal{R}} t(\eta(\alpha, \beta))$. Now it follows from Lemma 3.1 that $h(t(\eta(\alpha\gamma, \beta\gamma))) > h(t(\eta(\alpha, \beta)))$. Thus $D(\alpha\gamma, \beta\gamma) > D(\alpha, \beta)$.

Therefore we may assume by Lemma 3.6(ii) that $\xi(\alpha\gamma, \beta\gamma) = \xi(\alpha, \beta) = k$, which means that $D(\alpha\gamma, \beta\gamma)$ and $D(\alpha, \beta)$ differ by at most 1. Hence we may also assume that $t(E_{k+1}) = t(E'_{k+1})$ (the case where we do not subtract 1). Since E_1, \dots, E_{k+1} and E'_1, \dots, E'_{k+1} are the first $k+1$ transition edges corresponding to $\alpha\gamma$ and $\beta\gamma$, respectively, we immediately get $D(\alpha\gamma, \beta\gamma) = D(\alpha, \beta)$.

(iii) Similarly to the proof of (ii), we may assume that $\gamma\alpha \neq \gamma\beta$ and $\xi(\alpha, \beta) > 0$. By Lemma 3.7(ii), we have $t(\eta(\gamma\alpha, \gamma\beta)) \leq_{\mathcal{J}} t(\eta(\alpha, \beta))$.

Suppose first that $t(\eta(\gamma\alpha, \gamma\beta)) <_{\mathcal{J}} t(\eta(\alpha, \beta))$. Then Lemma 3.1 yields $h(t(\eta(\gamma\alpha, \gamma\beta))) > h(t(\eta(\alpha, \beta)))$ and so

$$D(\gamma\alpha, \gamma\beta) \geq 2h(t(\eta(\gamma\alpha, \gamma\beta))) - 1 > 2h(t(\eta(\alpha, \beta))) \geq D(\alpha, \beta).$$

Thus we may assume that $t(\eta(\gamma\alpha, \gamma\beta)) \mathcal{J} t(\eta(\alpha, \beta))$, so that $h(t(\eta(\gamma\alpha, \gamma\beta))) = h(t(\eta(\alpha, \beta)))$. We may also assume that $D(\alpha, \beta)$ is even, so that $k < m, n$ and $t(E_{k+1}) = t(E'_{k+1})$.

Recall from the proof of Lemma 3.7 the words $u = e_1 u_1 \dots e_m u_m$ and $u' = e'_1 u'_1 \dots e'_n u'_n$ representing α and β , and all the associated notation. By Lemma 3.7(ii), we get

$$t(\eta(\gamma\alpha, \gamma\beta)) \leq_{\mathcal{R}} \varphi(\gamma)t(\eta(\alpha, \beta)) = (wv_{k-1}e_k)\sigma.$$

Thus

$$(wv_k e_{k+1})\sigma \leq_{\mathcal{L}} (v_k e_{k+1})\sigma <_{\mathcal{R}} (v_{k-1} e_k)\sigma = t(\eta(\alpha, \beta)) \mathcal{J} t(\eta(\gamma\alpha, \gamma\beta)) \leq_{\mathcal{R}} (wv_{k-1} e_k)\sigma \mathcal{R} (wv_k)\sigma.$$

Since finite semigroups are stable, the relation $<_{\mathcal{R}}$ is contained in $<_{\mathcal{J}}$, hence $(wv_k e_{k+1})\sigma <_{\mathcal{J}} (wv_k)\sigma$. Since $(wv_k e_{k+1})\sigma \leq_{\mathcal{R}} (wv_k)\sigma$, it follows that $(wv_k e_{k+1})\sigma <_{\mathcal{R}} (wv_k)\sigma$. Hence

$$(wv_k)\sigma \xrightarrow{e_{k+1}} (wv_k e_{k+1})\sigma$$

is a transition edge, in fact the first transition edge of $\gamma\alpha$ which is not shared with $\gamma\beta$. Similarly,

$$(wv'_k)\sigma \xrightarrow{e'_{k+1}} (wv'_k e'_{k+1})\sigma$$

is the first transition edge of $\gamma\beta$ which is not shared with $\gamma\alpha$. Now

$$(v_k e_{k+1})\sigma = t(E_{k+1}) = t(E'_{k+1}) = (v'_k e'_{k+1})\sigma$$

yields $(wv_k e_{k+1})\sigma = (wv'_k e'_{k+1})\sigma$ and so

$$D(\gamma\alpha, \gamma\beta) = 2h(t(\eta(\gamma\alpha, \gamma\beta))) = 2h(t(\eta(\alpha, \beta))) = D(\alpha, \beta).$$

Therefore (iii) holds.

(iv) We may assume that α, β, γ are all different. Let (E''_1, \dots, E''_p) be the sequence of transition edges corresponding to γ .

If $\xi(\alpha, \beta) > \xi(\beta, \gamma)$, we can exchange α and γ in view of (i), and so Lemma 3.6(i) allows us to assume that $\xi(\alpha, \beta) \leq \xi(\beta, \gamma)$. Now Lemma 3.6(iii) yields $\xi(\alpha, \gamma) \geq \xi(\alpha, \beta)$. Since $\xi(\alpha, \gamma) > \xi(\alpha, \beta)$ immediately implies our objective in view of (3.1), we are now restricted to the case

$$0 < k = \xi(\alpha, \gamma) = \xi(\alpha, \beta) \leq \xi(\beta, \gamma).$$

It follows that $E_i = E'_i = E''_i$ for $i = 1, \dots, k$.

Unless $t(E_{k+1}) = t(E'_{k+1})$, we get $D(\alpha, \beta) = 2h(t(E_k)) - 1$ and we are done. Hence we may assume that $t(E_{k+1}) = t(E'_{k+1})$. Now, unless $t(E'_{k+1}) = t(E''_{k+1})$, we get $D(\beta, \gamma) = 2h(t(E_k)) - 1$ and we are done as well. But then $t(E_{k+1}) = t(E''_{k+1})$ and so $D(\alpha, \gamma) = 2h(t(E_k)) = D(\alpha, \beta)$. Therefore (iv) holds. \square

Remark 3.9. In view of these properties, D can indeed be called a length function for (unexpectedly) both a left and right action because of Lemma 3.8 (ii) and (iii).

3.3. Representations as elliptic maps on a rooted tree. Let $\Gamma = (V, E)$ be a simple undirected graph. Then Γ is a tree if it is connected and admits no cycles (i.e. no closed paths of the form $v_1 - \dots - v_n - v_1$ with $n \geq 3$ different vertices). If we distinguish a vertex $v_0 \in V$, we get the *rooted tree* (Γ, v_0) .

Given a rooted tree (Γ, v_0) , we get a *depth function* $\delta : V \rightarrow \mathbb{N}$ as follows: $\delta(v)$ is the edge length of the shortest path connecting v to v_0 . An *endomorphism* of the rooted tree (Γ, v_0) is a function $\varphi : V \rightarrow V$ such that:

- $\delta(\varphi(v)) = \delta(v)$ for every $v \in V$;
- if $v - w$ is an edge of Γ , so is $\varphi(v) - \varphi(w)$.

Endomorphisms of rooted trees are also known as *elliptic maps*. We denote by $\text{EM}(\Gamma, v_0)$ the monoid of all elliptic maps of (Γ, v_0) .

A *representation* of a monoid M as elliptic maps on a rooted tree (Γ, v_0) is a monoid homomorphism $\theta : M \rightarrow \text{EM}(\Gamma, v_0)$. The representation is *faithful* if φ is one-to-one.

3.4. The Chiswell construction and the holonomy theorem. We adapt next the Chiswell construction described in [Rho91, Proof of Theorem 1.12] and [RS12, Proof of Theorem 4.7] (see also [Chi76]).

Let $T = \text{KR}_{\text{right}}(S, A)$ and let $D : T^{\mathbb{1}} \times T^{\mathbb{1}} \rightarrow \mathbb{N}$ be the Lyndon–Chiswell length function defined before (with maximum value ℓ). Write

$$C = \{(k, \alpha) \mid 0 \leq k \leq \ell, \alpha \in T^{\mathbb{1}}\}.$$

We define a relation \sim on C by $(k, \alpha) \sim (k', \beta)$ if:

- $k = k'$;
- $D(\alpha, \beta) \geq k$.

It follows from Lemma 3.8(i) and (iv) that \sim is indeed an equivalence relation on C . Note that $(0, \alpha) \sim (0, \beta)$ for all $\alpha, \beta \in T^{\mathbb{1}}$.

Denote by $[k, \alpha]$ the equivalence class of $(k, \alpha) \in C$. Define an undirected graph \mathcal{C} with vertices $[k, \alpha]$ and edges $[k, \alpha] - [k+1, \alpha]$ when $0 \leq k < \ell$ and $\alpha \in T^{\mathbb{1}}$. Note that

$$(3.2) \quad \text{if } [k, \beta] - [k+1, \alpha] \text{ is an edge of } \mathcal{C} \text{ then } [k, \beta] = [k, \alpha].$$

Indeed, if there exists such an edge then there exists some $\gamma \in T^{\mathbb{1}}$ such that $[k, \beta] = [k, \gamma]$ and $[k+1, \alpha] = [k+1, \gamma]$. It follows that $D(\alpha, \gamma) \geq k+1 > k$. Hence $[k, \alpha] = [k, \gamma]$ and (3.2) holds.

With minimal adaptations from [Rho91] and [RS12], we prove the following lemma for the sake of completeness.

Lemma 3.10. $(\mathcal{C}, [0, \mathbb{1}])$ is a rooted tree.

Proof. We have a path

$$[0, \mathbb{1}] = [0, \alpha] \text{ --- } [1, \alpha] \text{ --- } \cdots \text{ --- } [k, \alpha]$$

for every vertex $[k, \alpha]$, hence \mathcal{C} is connected.

Suppose that

$$[k_0, \alpha_0] \text{ --- } [k_1, \alpha_1] \text{ --- } \cdots \text{ --- } [k_n, \alpha_n] = [k_0, \alpha_0]$$

is a cycle in \mathcal{C} . We may assume that $k_0 \geq k_i$ for every $0 \leq i \leq n$. Then $k_1 = k_{n-1} = k_0 - 1$ and it follows from (3.2) that $[k_1, \alpha_1] = [k_0 - 1, \alpha_0] = [k_{n-1}, \alpha_{n-1}]$, a contradiction. Therefore \mathcal{C} is a tree as required. \square

This rooted tree is the *Chiswell tree* induced by the Lyndon–Chiswell length function $D: T^{\mathbb{1}} \times T^{\mathbb{1}} \rightarrow \mathbb{N}$. Note that the depth function is given by $\delta([k, \alpha]) = k$.

Theorem 3.11 (*Holonomy Theorem*). *Let (S, A) be a finite semigroup S with generators A . Then $\text{KR}_{\text{right}}^{\mathbb{1}}(S, A)$ and $\text{KR}_{\text{left}}^{\mathbb{1}}(S, A)$ are faithfully represented as elliptic maps on a finite rooted tree.*

Proof. Once again, we adapt the proof from [Rho91, RS12].

Let

$$\begin{aligned} \epsilon : \text{KR}_{\text{right}}(S, A) &\rightarrow \text{EM}(\mathcal{C}, [0, \mathbb{1}]) \\ \alpha &\mapsto \epsilon_{\alpha} \end{aligned}$$

be defined by

$$\epsilon_{\alpha}([k, \beta]) = [k, \alpha\beta].$$

First, we show that ϵ_{α} is well defined. Suppose that $[k, \beta] = [k', \beta']$. Then $k = k'$ and $D(\beta, \beta') \geq k$. Then $D(\alpha\beta, \alpha\beta') \geq k$ by Lemma 3.8(iii) and so $[k, \alpha\beta] = [k', \alpha\beta']$. Thus ϵ_{α} is well defined.

It is obvious that $\delta(\epsilon_{\alpha}([k, \beta])) = k = \delta([k, \beta])$. On the other hand, if $[k, \beta] \text{ --- } [k+1, \beta]$ is an edge of \mathcal{C} , so is $\epsilon_{\alpha}([k, \beta]) \text{ --- } [k+1, \alpha\beta] = \epsilon_{\alpha}([k+1, \beta])$. Therefore ϵ_{α} is an elliptic map on the finite rooted tree $(\mathcal{C}, [0, \mathbb{1}])$ and so ϵ is well defined.

Given $\alpha, \alpha' \in \text{KR}_{\text{right}}^{\mathbb{1}}(S, A)$, we have

$$\epsilon_{\alpha\alpha'}([k, \beta]) = [k, \alpha\alpha'\beta] = \epsilon_{\alpha}(\epsilon_{\alpha'}([k, \beta])),$$

hence $\epsilon_{\alpha\alpha'} = \epsilon_{\alpha}\epsilon_{\alpha'}$. On the other hand, $\epsilon_{\mathbb{1}}([k, \beta]) = [k, \beta]$ and so $\epsilon_{\mathbb{1}}$ is the identity map. Thus ϵ is a monoid homomorphism.

Finally, assume that $\epsilon_{\alpha} = \epsilon_{\alpha'}$. Then in particular

$$[\ell, \alpha] = \epsilon_{\alpha}([\ell, \mathbb{1}]) = \epsilon_{\alpha'}([\ell, \mathbb{1}]) = [\ell, \alpha'],$$

hence $D(\alpha, \alpha') \geq \ell = \max(\text{im}(D))$.

Suppose that $\alpha \neq \alpha'$. Let (E_1, \dots, E_m) be the sequence of transition edges corresponding to α . Since

$$D(\alpha, \alpha') = \ell = 2 \max\{h(s) \mid s \in S^{\mathbb{1}}\},$$

we are not subtracting 1, which implies that α possesses transition edges beyond $E_{\xi(\alpha, \alpha')}$, i.e. $\xi(\alpha, \alpha') < m$. In view of Lemma 3.1, this contradicts the fact that h should reach its maximum value at $t(E_{\xi(\alpha, \alpha')})$. Thus $\alpha = \alpha'$ and so ϵ is one-to-one. Therefore the representation is faithful.

Recall now that $\text{KR}_{\text{left}}(S, A) = \text{KR}_{\text{right}}(S^{\text{op}}, A)$. It follows from the first part that $\text{KR}_{\text{left}}^{\mathbb{1}}(S, A)$ is also faithfully represented as elliptic maps on a finite rooted tree. \square

Remark 3.12. It follows easily from Lemma 3.8(ii) that we can consider a right action of $\text{KR}_{\text{right}}(S, A)$ on the Chiswell tree $(\mathcal{C}, [0, \mathbb{1}])$ given by

$$[k, \beta]\alpha = [k, \beta\alpha].$$

A straightforward adaptation of the proof of Theorem 3.11 shows that we obtain an injective monoid homomorphism

$$\epsilon' : \text{KR}_{\text{right}}(S, A) \rightarrow (\text{EM}(\mathcal{C}, [0, \mathbb{1}]))^{\text{op}},$$

since here the elliptic mappings must compose from left to right.

	$a \cdot$	$b \cdot$
aaa	aaa	baa
aab	aaa	baa
aba	aab	bab
baa	aba	$bbaa$
bab	aba	$bbab$
$bbaa$	$abba$	$bbba$
$abba$	aab	bab
$bbba$	$abbb$	$bbbb$
$bbab$	$abba$	$bbba$
$abbb$	aab	bab
$bbbb$	$abbb$	$bbbb$

TABLE 1. The left action semaphore code in $\{a, b\}^4$ associated to the ideal generated by aaa, aab, aba, baa, bab .

Remark 3.13. Note that the Lyndon-Chiswell length function on $\text{KR}_{\text{right}}^1(S^{\text{op}}, A) \times \text{KR}_{\text{right}}^1(S^{\text{op}}, A)$ would be the version of the Lyndon-Chiswell length function built from S when we replace its right Cayley graph by its left Cayley graph. And Lemma 3.8(ii) ensures that left-right symmetry is preserved at all levels of the proofs, so we could replicate all the preceding proofs using $\text{LCay}(S, A)$ and $\text{KR}_{\text{left}}^1(S, A)$.

In the next paper we will expand this theory and apply it to mixing times.

4. EXAMPLES

4.1. Left action on semaphore codes. Let A be a finite alphabet and let $k \geq 1$. Consider the left action $A^+ \times A^k \rightarrow A^k$ defined as follows: given $v \in A^+$ and $u \in A^k$, let $v \cdot u$ denote the prefix of length k of vu . An equivalence relation ρ on A^k is a *left congruence* if

$$u\rho v \Rightarrow (w \cdot u)\rho(w \cdot v)$$

holds for all $u, v \in A^k$ and $w \in A^+$. Then the set $\rho \backslash A^k$ of all ρ -classes becomes a left zero semigroup under the operation $(\rho u)(\rho v) = \rho u$. We denote by $\text{LC}(A^k)$ the set of left congruences on A^k . As shown in [RSS16a, RSS16b], every left congruence can be approximated by a special left congruence and special left congruences are in bijection with semaphore codes. A *semaphore code* [BPR10] is a prefix code \mathcal{S} over A (i.e., all elements in the code are incomparable in prefix order) for which there is a left action in the following sense: If $a \in A$ and $u \in \mathcal{S}$, then au has a prefix in \mathcal{S} . The left action $a \cdot u$ is the prefix of au that is in \mathcal{S} .

Remark 4.1. In fact, [RSS16a, RSS16b] use right congruences and semaphore codes are suffix codes. But as outlined in the introduction, in applications the left action is usually used. See also [RS19].

Given a semaphore code \mathcal{S} and the left action by A^+ on \mathcal{S} , consider the functions $\mathcal{S} \rightarrow \mathcal{S}$ induced by this action. This yields a semigroup (\mathcal{S}, A) . The right Cayley graph of (\mathcal{S}, A) is equal to its Karnofsky–Rhodes expansion.

Example 4.2. Let $A = \{a, b\}$ be a two letter alphabet and I the ideal in A^* generated by aaa, aab, aba, baa, bab . Then the left action prefix semaphore code in A^4 is given in Table 1. Hence S has 11 elements. To compute $\text{KR}_{\text{right}}(S, A)$, we compute the action of the various subwords of the elements in S on S and record the images, see Figure 1. We have $\ell = 8$, so the Chiswell tree has 9 levels. However, the Lyndon–Chiswell length function D cannot take on the values 2, 4, 6, 8. It follows that these levels are equal to their predecessors and can be omitted. The elliptic left action of a on the Chiswell tree is given in Figure 2, whereas the elliptic left action of b is given in Figure 3.

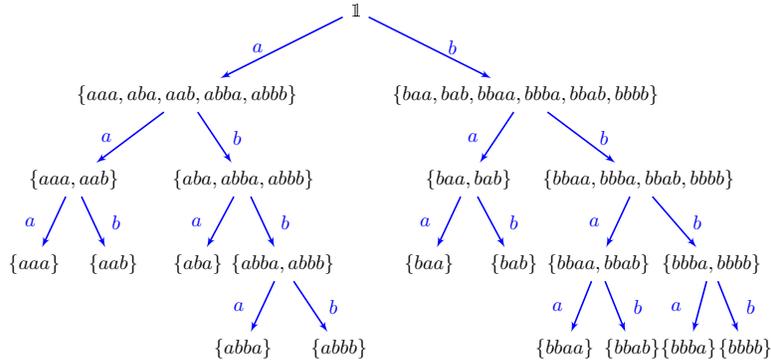


FIGURE 1. The right Karnofsky–Rhodes expansion of $\text{RCay}(S, A)$ of Example 4.2, where the action on the leaves is omitted.

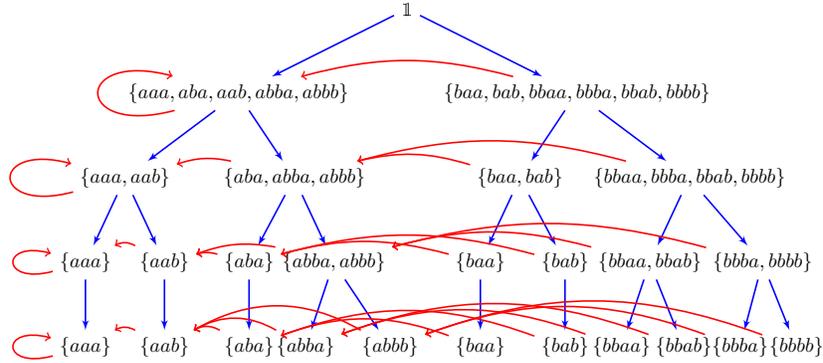


FIGURE 2. The action of $a \cdot$ on the semaphore code induces the action level-by-level on the the Chiswell tree of Example 4.2 (in red).

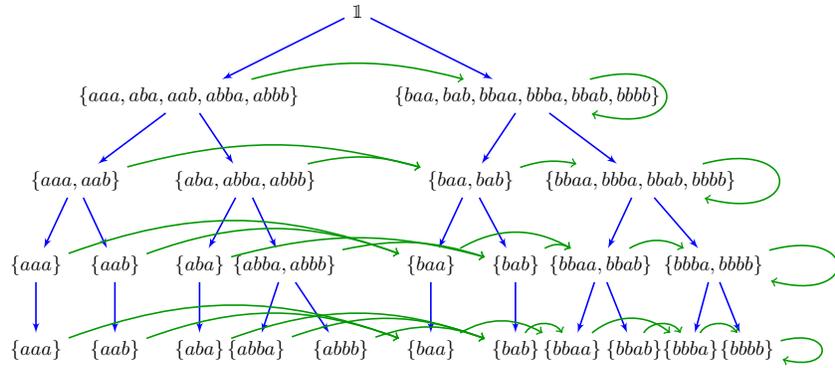


FIGURE 3. The action of $b \cdot$ on the semaphore code induces the action level-by-level on the the Chiswell tree of Example 4.2 (in green).

4.2. **Right zero semigroup with two generators.** Let (S, A) be the right zero semigroup $\text{RZ}(2)$ (that is $xy = y$ for all $x, y \in \text{RZ}(2)$) with two generators $A = \{a, b\}$. The Karnofsky–Rhodes expansion of the right Cayley graph of (S, A) is depicted in Figure 4. Then the Chiswell construction

is given in Figure 5. The right and left actions of a and b on the Chiswell construction are given in Figures 6, 7, 8 and 9, respectively.

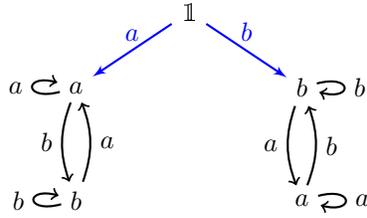


FIGURE 4. $KR_{\text{right}}(\text{RZ}(2), \{a, b\})$.

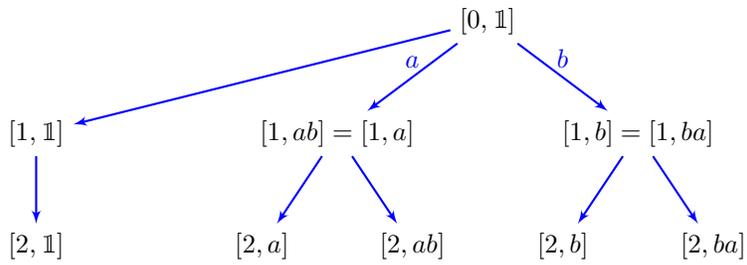


FIGURE 5. The Chiswell construction for $KR_{\text{right}}(\text{RZ}(2), \{a, b\})$.

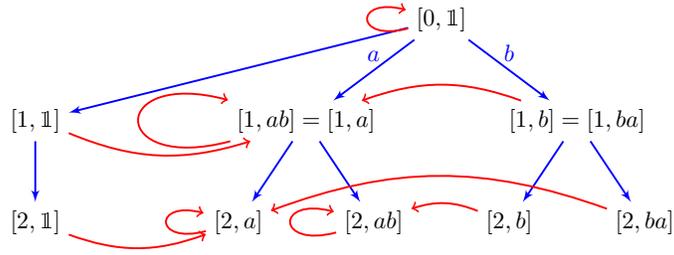


FIGURE 6. The left action of a on the Chiswell tree for $\text{KR}_{\text{right}}(\text{RZ}(2), \{a, b\})$.

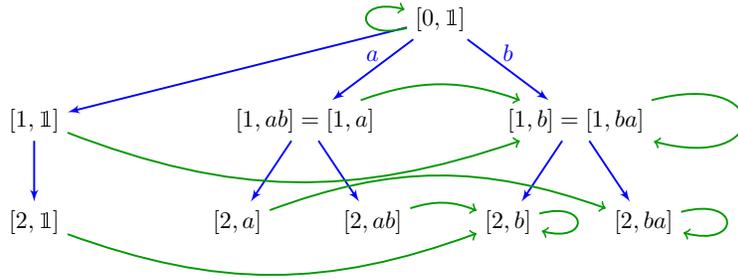


FIGURE 7. The left action of b on the Chiswell tree for $\text{KR}_{\text{right}}(\text{RZ}(2), \{a, b\})$.

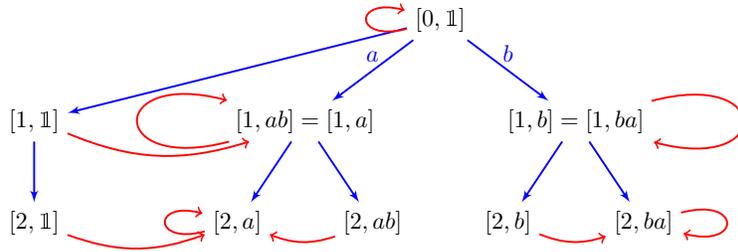


FIGURE 8. The right action of a on the Chiswell tree for $\text{KR}_{\text{right}}(\text{RZ}(2), \{a, b\})$.

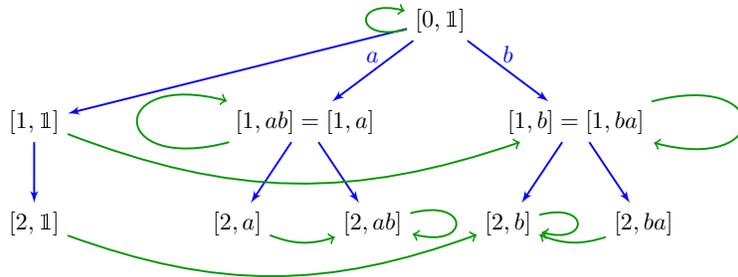


FIGURE 9. The right action of b on the Chiswell tree for $\text{KR}_{\text{right}}(\text{RZ}(2), \{a, b\})$.

4.3. The monoid T_2 of total transformations. Let T_2 denote the monoid of total transformations on the set $\{1, 2\}$. We denote $\varphi \in T_2$ by $(\varphi_1 \ \varphi_2)$ (so in $\psi\varphi$ the map φ acts first). Let $A = \{a, b\}$ and let $\varphi: A^* \rightarrow S^1$ be the monoid homomorphism defined by $\varphi(a) = (2 \ 1)$ and $\varphi(b) = (1 \ 1)$. It is routine to check that φ is onto and $\text{RCay}(T_2, A) = \text{LCay}(T_2^{\text{op}}, A)$ is depicted in Figure 10. The Karnofsky–Rhodes expansion is given in Figure 11, the Chiswell construction is drawn in Figure 12, and the left action of a on the Chiswell tree is depicted in Figure 13.

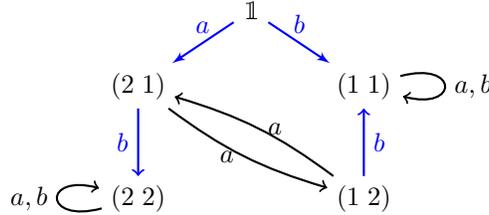


FIGURE 10. $\text{RCay}(T_2, A)$ with the transition edges in blue.

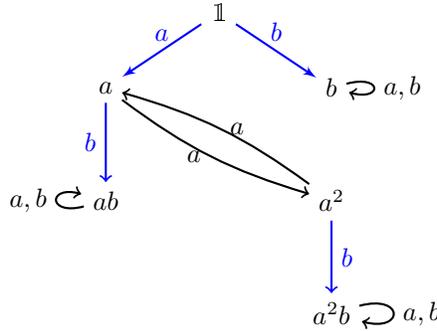


FIGURE 11. $\text{KR}_{\text{right}}(T_2, A)$.

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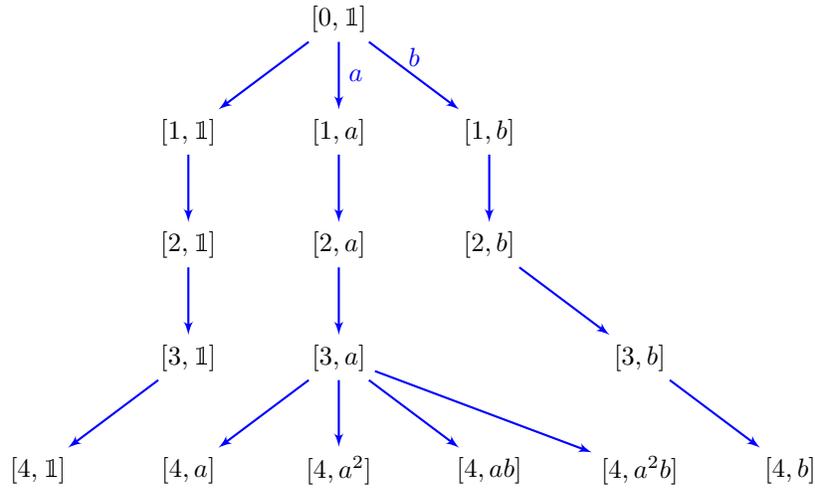
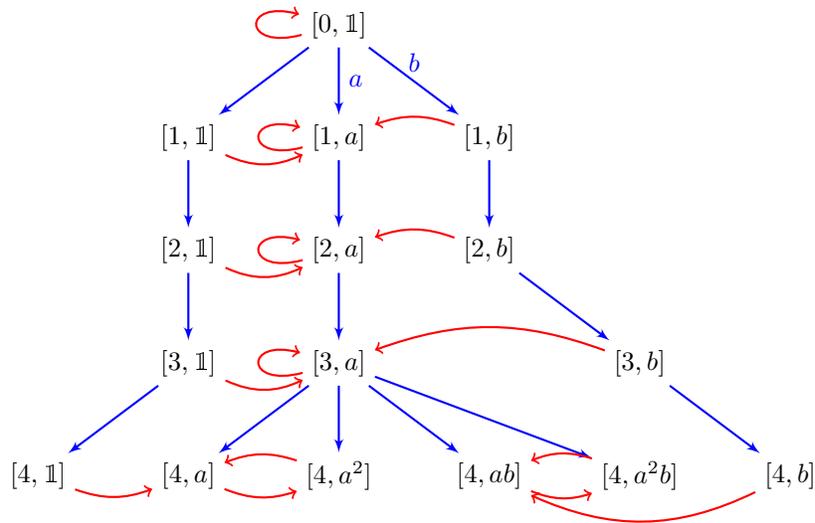
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FIGURE 12. The Chiswell construction for $\text{KR}_{\text{right}}(T_2, A)$.FIGURE 13. The (left) action of $a \cdot$ on the Chiswell tree for $\text{KR}_{\text{right}}(T_2, A)$.

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