

Algebras of slowly growing length *

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Abstract

We investigate the class of finite dimensional not necessary associative algebras that have slowly growing length, that is, for any algebra in this class its length is less than or equal to its dimension. We show that this class is considerably big, in particular, finite dimensional Lie algebras as well as many other important classical finite dimensional algebras belong to this class, for example, Leibniz algebras, Novikov algebras, and Zinbiel algebras. An exact upper bounds for the length of these algebras is proved. To do this we transfer the method of characteristic sequences to non-unital algebras and find certain polynomial conditions on the algebra elements that guarantee the slow growth of the length function.

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1 Introduction

Let \mathbb{F} be an arbitrary field. In this paper \mathcal{A} always denotes a finite dimensional not necessarily unital not necessarily associative \mathbb{F} -algebra with the operation (\cdot) usually denoted by the concatenation. Let $SS = \{a_1, \dots, a_k\}$ be a finite generating set of \mathcal{A} . Any product of a finite number of elements from SS is a *word* in SS . The *length* of the word w , denoted $l(w)$, equals to the number of letters in the corresponding product. If \mathcal{A} is unital, we consider 1 as a word in SS with the *length* 0. It is worth noting that different choices of brackets provide different words of the same length due to the non-associativity of \mathcal{A} .

The set of all words in SS with the lengths less than or equal to i is denoted by SS^i , here $i \geq 0$.

Note that similarly to the associative case, $m < n$ implies that $SS^m \subseteq SS^n$.

The set $L_i(SS) = \langle SS^i \rangle$ is the linear span of the set SS^i (the set of all finite linear combinations with coefficients belonging to \mathbb{F}). We write L_i instead of $L_i(SS)$ if SS is clear

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from the context. It should be noted that for unital algebras $L_0(SS) = \langle 1 \rangle = \mathbb{F}$ for any SS , and for non-unital algebras $L_0 = \emptyset$. We denote $L(SS) = \bigcup_{i=0}^{\infty} L_i(SS)$.

Since the set SS is generating for \mathcal{A} , we have $\mathcal{A} = L(SS)$.

Definition 1.1. The *length of a generating set* SS of a finite-dimensional algebra \mathcal{A} is defined as follows: $l(SS) = \min\{k \in \mathbb{Z}_+ : L_k(SS) = \mathcal{A}\}$.

Definition 1.2. The *length of an algebra* \mathcal{A} is $l(\mathcal{A}) = \max\{l(SS) : L(SS) = \mathcal{A}\}$.

The problem of the associative algebra length computation was first discussed in [19, 20] for the algebra of 3×3 matrices in the context of the mechanics of isotropic continua.

It is straightforward to see that the length of an associative algebra is strictly less than its dimension, and this bound is sharp. Namely, one-generated associative algebra of the dimension d has the length $d - 1$. The first non-trivial result in this direction is going back to Paz [16]. More results on abstract associative algebras can be found, for example, in [11, 14, 15]. However, in general most of the known results on the length function are just bounds that are not sharp. Even the sharp upper bound for the length of the matrix algebra is not known, see [14]. However, a great deal of work has been done investigating the related notion of length for given generating sets of matrices, see [10, 12, 13] and references therein.

Recent results on the lengths of non-associative algebras were obtained in the works [8, 9]. In particular a strict upper bound on the length of a general non-associative unital algebra is provided.

Theorem 1.3 ([8, Theorem 2.7]). *Let \mathcal{A} be a unital \mathbb{F} -algebra, $\dim \mathcal{A} = n \geq 2$. Then $l(\mathcal{A}) \leq 2^{n-2}$.*

To prove this and several other results the method of characteristic sequences was introduced, see [8, 9].

Definition 1.4. [8, Definition 3.1] Consider a unital \mathbb{F} -algebra \mathcal{A} of the dimension $\dim \mathcal{A} = n$, and its generating set SS . By the *characteristic sequence* of SS in \mathcal{A} we understand a monotonically non-decreasing sequence of non-negative integers (f_1, f_2, \dots, f_N) , constructed by the following rules:

1. $f_1 = 0$.
2. Denoting $s_1 = \dim L_1(SS)$, we define $f_2 = \dots = f_{s_1} = 1$.
3. Let for some $r > 0$, $k > 1$ the elements f_1, \dots, f_r be already defined and the sets $L_1(SS), \dots, L_{k-1}(SS)$ are considered. Then we inductively continue the process in the following way. Denote $s_k = \dim L_k(SS) - \dim L_{k-1}(SS)$. We define $f_{r+1} = \dots = f_{r+s_k} = k$.

It is proved in [8, Lemma 3.5] that $N = \dim \mathcal{A}$ and $f_N = l(SS)$.

The main focus of this paper is the algebras with slowly growing length.

Definition 1.5. We say that a class of algebras has *slowly growing length*, if for any representative \mathcal{A} of this class it holds that $l(\mathcal{A}) \leq \dim(\mathcal{A})$.

For example, associative algebras are of this type since the sequence $L_k(SS)$ grows strictly monotone with k , and hence can not have more than $\dim(\mathcal{A}) - 1$ elements.

The main purpose of our paper is to show that the class of algebras with slowly growing length is rather big, namely, a number of important classes of non-associative finite dimensional algebras have slowly growing length, in particular, Lie algebras and more general classes such as Leibniz algebras, Novikov algebras, and Zinbiel algebras are of this type. To proceed we generalized the notion of characteristic sequences introduced in [8] to non-unital algebras. It is straightforward to see that if the characteristic sequence (m_1, \dots, m_d) of an algebra satisfies the condition $m_{j+1} - m_j \leq 1$ for all $j = 1, \dots, d - 1$ then the length of this algebra is slowly growing. We find two combinatorial properties for algebras which ensure the aforesaid condition for the characteristic sequences to be fulfilled. We call the corresponding algebras

sliding and *mixing* due to the nature of these combinatorial properties, and investigate their interrelations. After that we examine these combinatorial properties for the major classes of algebras. We prove that associative and Lie algebras, and moreover, Leibniz algebras, are both sliding and mixing. Novikov algebras are mixing, but in general they are not sliding. Zinbiel algebras are sliding, but in general they are not mixing. However, there are algebras that are neither sliding nor mixing, but have slowly growing length. We provide an example of such algebras. Finally we discuss algebras that are neither sliding nor mixing and in general are not algebras with slowly growing length. In particular, Valya and Vinberg algebras are among them.

Our paper is organized as follows. In Section 2 we transfer the method of characteristic sequences to non-unital algebras. In Section 3 polynomial properties which guarantee a slow growth of length are introduced and length of corresponding algebras is estimated by means of the characteristic sequences. In Section 4 we examine which important classes of non-associative algebras have slowly growing length.

2 Characteristic sequences for non-unital algebras

We begin with several definitions and auxiliary results, inherited from the unital case. Let \mathcal{A} be an \mathbb{F} -algebra of the dimension $\dim \mathcal{A} = n$, $n > 2$, and SS be a generating set for \mathcal{A} . The algebra \mathcal{A} can be either unital or non-unital.

Definition 2.1. A word w from a generating set SS of an algebra \mathcal{A} is *irreducible*, if for each integer m , $0 \leq m < l(w)$, it holds that $w \notin L_m(SS)$.

Lemma 2.2. [8, Lemma 2.14] *Any irreducible word w , $l(w) > 1$, is a product of two irreducible words of non-zero lengths.*

To work with algebras which are not necessarily unital, we need to generalize Definition 1.4 for non-unital case.

Definition 2.3. By the *characteristic sequence* of SS in \mathcal{A} we understand a monotonically non-decreasing sequence of non-negative integers (m_1, \dots, m_N) , constructed by the following rules:

1. If $s_0 = \dim L_0(SS) = 1$, we set $m_1 = 0$. Otherwise $s_0 = 0$.
2. Denoting $s_1 = \dim L_1(SS) - \dim L_0(SS)$, we define $m_{s_0+1} = \dots = m_{s_0+s_1} = 1$.
3. Let for some $r > 0$, $k > 1$ the elements m_1, \dots, m_r be already defined and the sets $L_0(SS), \dots, L_{k-1}(SS)$ be considered. Then we inductively continue the process in the following way. Denote $s_k = \dim L_k(SS) - \dim L_{k-1}(SS)$. We define $m_{r+1} = \dots = m_{r+s_k} = k$.

For a unital algebra we have in respective notations $f_i = m_{i+1}$. The main difference between these sequences is that for non-unital algebra the characteristic sequence starts with 1, while in the unital case it starts with 0.

Lemma 2.4. *Consider a generating set SS of an algebra \mathcal{A} . There exists a finite series of sets $E_1, \dots, E_{l(SS)}$, satisfying the following properties:*

1. $E_h \subset E_{h+1}$, $h = 1, \dots, l(SS) - 1$
2. E_h is a basis of $L_h(SS)$.
3. E_h consists of irreducible words in SS of lengths $0, \dots, h$, with exactly $s_j = \dim L_j(SS) - \dim L_{j-1}(SS)$ words of length j for $j = 1, \dots, h$ and s_0 words of length 0, where s_0 is 1 for unital algebra and 0 otherwise.

Proof. We will construct E_h sequentially using induction on h .

The base: $h = 1$. Assume that \mathcal{A} is unital. Then we choose the basis E_1 as $\{1\} \cup SS_0$, where SS_0 is the maximal subset of SS , linearly independent modulo \mathbb{F} . If \mathcal{A} is non-unital,

then we choose the basis E_1 as the maximal linearly independent subset of SS . In both cases there are exactly s_0 irreducible words of length 0 and s_1 words of length 1 in E_1 .

The step. Assume we have constructed E_h for all $h \leq k-1$, $2 \leq k \leq l(SS)$.

By Definition 2.1 $L_k(SS)$ is the linear span of all irreducible words of length less than or equal to k . We will construct E_k expanding E_{k-1} by a set E using the following algorithm:

- Let $\{w_1, \dots, w_r\}$ be the set of all irreducible words of length k . It is finite since the number of words of length k is finite. Set $t = 1$, $E = \emptyset$.
- If $w_t \in \langle E \cup E_{k-1} \rangle$, increase t by 1. Otherwise, expand E with w_t , making it $E \cup \{w_t\}$, and increase t by 1.
- If $t < r$, return to step 2. If $t = r$, end the algorithm.

We will show that $E_k = E_{k-1} \cup E$ is the desired set. Note that $E \cap E_{k-1} = \emptyset$.

1. $E_k = E_{k-1} \cup E \supset E_{k-1}$.
2. E_k is a basis of $L_k(SS)$. Firstly, it is linearly independent by construction. Secondly, $\langle E_k \rangle = L_k(SS)$, since every irreducible word of length $l \leq k-1$ lies in $L_{k-1}(SS) = \langle E_{k-1} \rangle \subset \langle E \cup E_{k-1} \rangle$, and by construction every irreducible word of length k is in $\langle E \cup E_{k-1} \rangle$. Additionally, E_k being a basis of $L_k(SS)$ means that $|E| = s_k$ due to $\dim L_k(SS) = \dim \langle E_k \rangle = |E_k| = |E| + |E_{k-1}| = |E| + \dim \langle E_{k-1} \rangle = |E| + \dim L_{k-1}(SS)$.
3. Since E consists only of irreducible words of length k by construction, $|E| = s_k$ as noted above, and E_{k-1} consists of irreducible words of lengths $0, \dots, k-1$, with exactly s_j words of length j for $j = 0, \dots, k-1$, we have E_k being composed of irreducible words of lengths $0, \dots, k$, with exactly s_j words of length j for $j = 0, \dots, k$. □

The following statements provide the analogs of [8, Lemma 3.4] in non-unital case.

- Corollary 2.5.**
1. For any term m_h of the characteristic sequence of SS there is an irreducible word in $L(SS)$ of the length m_h .
 2. If there is an irreducible word in letters from SS of the length k , then k is included into the characteristic sequence of SS .

Proof. Consider set $\mathcal{E} = E_{l(SS)}$, constructed by Lemma 2.4.

Item 1. If m_h belongs to the characteristic sequence of SS , then $s_{m_h} \neq 0$ and there is at least one irreducible word of length m_h in the set \mathcal{E} .

Item 2. The existence of an irreducible word of length k guarantees that $s_k = \dim L_k(SS) - \dim L_{k-1}(SS) > 0$, which means that k is included in the characteristic sequence by definition. □

Lemma 2.6. *The characteristic sequence of SS contains exactly $\dim \mathcal{A}$ terms. Moreover, for the last term we have $m_N = l(SS)$.*

Proof. There are $s_0 + s_1 + \dots + s_{l(SS)}$ terms in the characteristic sequence since for $j > l(SS)$ it holds that $s_j = \dim L_j(SS) - \dim L_{j-1}(SS) = \dim \mathcal{A} - \dim \mathcal{A} = 0$. This sum can be rewritten as $\dim L_1(SS) + (\dim L_2(SS) - \dim L_1) + \dots + (\dim L_{l(SS)} - \dim L_{l(SS)-1}) = \dim L_{l(SS)} = \dim \mathcal{A}$.

By Corollary 2.5, Item 2, there is an element of characteristic sequence of SS which is equal to $l(SS)$. This implies $m_N \geq l(SS)$ as the sequence is non-decreasing. However by Corollary 2.5, Item 1 there exists an irreducible word of length m_N which means $m_N \leq l(SS)$. Thus, $m_N = l(SS)$. □

3 Mixing and sliding algebras

In this section we study two properties of multiplication which can guarantee a slow growth of length.

Let x, y, z be variables. To introduce the following definition we need the special sets Q_l and Q_r of monomials:

$$Q_l(x, y, z) = \{x(zy), x(yz), y(xz), y(zx), xy, yx, xz, zx, yz, zy, x, y, z\},$$

here we consider those monomials of degree three where z is an argument of the first multiplication and the multiplier with z is the second factor of the second multiplication,

$$Q_r(x, y, z) = \{(xz)y, (zx)y, (yz)x, (zy)x, xy, yx, xz, zx, yz, zy, x, y, z\},$$

here we consider those monomials of degree three where z is an argument of the first multiplication and the multiplier with z is the first factor of the second multiplication.

Definition 3.1. Let \mathcal{A} be an \mathbb{F} -algebra such that at least one of the following statements holds:

1. $z(xy) \in \langle Q_r(x, y, z) \rangle$ for all $x, y, z \in \mathcal{A}$, if \mathcal{A} is non-unital; $z(xy) \in \langle Q_r(x, y, z), 1 \rangle$ for all $x, y, z \in \mathcal{A}$, if \mathcal{A} is unital.
2. $(xy)z \in \langle Q_l(x, y, z) \rangle$ for all $x, y, z \in \mathcal{A}$, if \mathcal{A} is non-unital; $(xy)z \in \langle Q_l(x, y, z), 1 \rangle$ for all $x, y, z \in \mathcal{A}$, if \mathcal{A} is unital.

Then we call \mathcal{A} a *sliding* algebra.

To introduce the next class of algebras we need the monomial set: $P(x, y, z) =$

$$= Q_l(x, y, z) \cup Q_r(x, y, z) = \left\{ \begin{array}{c} (xz)y, (zx)y, (yz)x, (zy)x, x(zy), x(yz), y(xz), y(zx), \\ xy, yx, xz, zx, yz, zy, x, y, z \end{array} \right\},$$

i.e. we consider those monomials of degree 3 that have z inside the brackets.

Definition 3.2. Let \mathcal{A} be an \mathbb{F} -algebra such that for all $x, y, z \in \mathcal{A}$ it holds that $(xy)z, z(xy) \in \langle P(x, y, z), 1 \rangle$ if \mathcal{A} is unital, and $(xy)z, z(xy) \in \langle P(x, y, z) \rangle$ if \mathcal{A} is non-unital. Then we call \mathcal{A} a *mixing* algebra.

Remark 3.3. Associative algebras are both mixing and sliding.

We are going to prove the main properties of mixing and sliding algebras that guarantee that these algebras have slowly growing length.

Let $\hat{P}(x, y, z) \subset P(x, y, z)$ be the subset of degree 3 monomials. For a mixing algebra \mathcal{A} let $T_l(x, y, z) \subseteq \hat{P}(x, y, z)$, respectively $T_r(x, y, z) \subseteq \hat{P}(x, y, z)$, be the set of monomials that are included with non-zero coefficients in at least one of the representations of $(xy)z$, respectively $z(xy)$, as linear combinations of the elements of $P(x, y, z) \cup \{1\}$ or $P(x, y, z)$ in \mathcal{A} .

Lemma 3.4. Let \mathcal{A} be a mixing algebra, SS be a generating set of \mathcal{A} , and $M = (m_1, \dots, m_d)$ be a characteristic sequence of SS . Then $m_{j+1} - m_j \leq 1$ for all $j = 1, \dots, d - 1$.

Proof. Assume the contrary. Let \mathcal{A} be a mixing algebra, SS be its generating set, and assume that there exists j such that $1 \leq j \leq d - 1$ and the inequality $m_{j+1} - m_j \leq 1$ does not hold. Let k be the smallest index such that $m_{k+1} - m_k \geq 2$.

Consider a word w of length at least two. It can be uniquely represented as $w = w' \cdot w''$, where w' and w'' have non-zero lengths. We denote $s(w) = \min(l(w'), l(w''))$.

1. Consider an irreducible word w in SS of length m_{k+1} . Then $s(w) > 1$. Indeed, if $s(w) = 1$ then w is a product of irreducible words of length 1 and $m_{k+1} - 1$ by Lemma 2.2. Hence by Corollary 2.5, Item 2, there is an element equal to $m_{k+1} - 1$ in the characteristic sequence M . This is impossible, since M is non-decreasing and $m_k < m_{k+1} - 1$ by the assumption.

2. Let us choose an irreducible word w_0 of length m_{k+1} in SS such that $s(w_0)$ is the smallest. If there are several such words, we take any one of them. The chosen word can be represented as a product of two irreducible words, w'_0 and w''_0 , such that $l(w'_0) = s(w_0)$. Thus we have the following 2 cases:

Case 1: $w_0 = w'_0 \cdot w''_0$. By Item 1 of the proof $s(w_0) \geq 2$. Hence $w'_0 = w'_1 \cdot w'_2$, where w'_1, w'_2 are both irreducible and $l(w'_1) < l(w'_0)$, $l(w'_2) < l(w'_0)$. The algebra \mathcal{A} is mixing, which means that the irreducible word $w_0 = (w'_1 \cdot w'_2) \cdot w''_0 \in \langle P(w'_1, w'_2, w''_0) \rangle$, and from

this follows that at least one element of $T_l(w'_1, w'_2, w''_0)$ is an irreducible word, as elements of $P(w'_1, w'_2, w''_0) \setminus \widehat{P}(w'_1, w'_2, w''_0)$ have strictly lesser length than w_0 . Assume that $(w'_1 \cdot w''_0) \cdot w'_2$ is irreducible. Then we have $s((w'_1 \cdot w''_0) \cdot w'_2) = l(w'_2) < l(w'_0) = s(w_0)$, which contradicts our choice of w_0 . For other elements of T_l a similar reasoning holds. Thus the initial assumption is false, i.e. mixing algebra cannot have a generating set with such a characteristic sequence that the difference between neighboring element is greater than 1.

Case 2: $w_0 = w''_0 \cdot w'_0$. We obtain the same contradiction similarly, considering an irreducible element of $T_r(w'_1, w'_2, w''_0)$ instead of $T_l(w'_1, w'_2, w''_0)$. \square

In the next lemma we need the following sets of monomials. Let $\widehat{Q}_l(x, y, z) \subset Q_l(x, y, z)$ and $\widehat{Q}_r(x, y, z) \subset Q_r(x, y, z)$ be the subsets of degree 3 monomials. For a sliding algebra \mathcal{A} let $S_l(x, y, z) \subseteq \widehat{Q}_l(x, y, z)$, respectively $S_r(x, y, z) \subseteq \widehat{Q}_r(x, y, z)$, be the set of monomials that are included with non-zero coefficients in at least one of the representations of $(xy)z$, respectively $z(xy)$, as linear combinations of elements of $Q_l(x, y, z) \cup \{1\}$ or $Q_l(x, y, z)$, respectively $Q_r(x, y, z) \cup \{1\}$ or $Q_r(x, y, z)$, in the algebra \mathcal{A} .

Lemma 3.5. *Let \mathcal{A} be a sliding algebra, SS be a generating set of \mathcal{A} , and $M = (m_1, \dots, m_d)$ be a characteristic sequence of SS in \mathcal{A} . Then $m_{j+1} - m_j \leq 1$ for all $j = 1, \dots, d-1$.*

Proof. Assume the contrary: let \mathcal{A} be a sliding algebra satisfying Item 1 of Definition 3.1. The other cases can be considered similarly. Let SS be a generating set of \mathcal{A} such that for its characteristic sequence M the inequality $m_{j+1} - m_j \leq 1$ does not hold for all $j = 1, \dots, d-1$. Let k be the smallest index such that $m_{k+1} - m_k \geq 2$.

Consider a word w of length at least two. It can be uniquely represented as $w = w' \cdot w''$, where w' and w'' have non-zero lengths. We denote $l_r(w) = l(w'')$.

Let us choose such an irreducible word w_0 of length m_{k+1} in SS that $l_r(w)$ is minimal (if there are multiple possible candidates, we can choose one at random). By Lemma 2.2 w_0 is equal to $w'_0 \cdot w''_0$, where terms are irreducible and of lesser length.

1. $l_r(w_0) = l(w'_0) = 1$ cannot hold: this would mean that $l(w'_0) = m_{k+1} - 1$ which by Corollary 2.5, Item 2 would mean that there is an element of characteristic sequence equal to $m_{k+1} - 1$, and that is impossible: M is non-decreasing and $m_k < m_{k+1} - 1$ already.

2. If $l(w'_0) > 1$, we can represent w'_0 as $w''_1 \cdot w'_2$, where both w''_1, w'_2 are of positive length and irreducible. The algebra \mathcal{A} is sliding, which means that the irreducible word $w_0 = w'_0 \cdot (w''_1 \cdot w'_2) \in \langle Q_r(w''_1, w'_2, w'_0) \rangle$ (or $\langle Q_r(w''_1, w'_2, w'_0), 1 \rangle$), and from this follows that at least one element of $S_r(w''_1, w'_2, w'_0)$ is an irreducible word as elements of $Q_r(w'_1, w'_2, w''_0) \setminus \widehat{Q}_r(w'_1, w'_2, w''_0)$ have strictly lesser length than w_0 . Assume it is $(w'_1 \cdot w'_0) \cdot w'_2$. Then we have a contradiction as $l((w'_1 \cdot w'_0) \cdot w'_2) = l(w_0)$, but $l(w'_2) < l(w'_0)$. For other elements of T_r a similar observation holds. Thus the initial assumption is false, i.e. sliding algebra cannot have a generating set with such a characteristic sequence that the difference between neighboring element is greater than 1. \square

Theorem 3.6. *The length of a mixing or a sliding algebra \mathcal{A} of dimension $d \geq 2$ is less than or equal to d .*

Proof. Follows directly from Lemma 3.4 or Lemma 3.5: for a generating set SS of \mathcal{A} with $l(SS) = l(\mathcal{A})$ and characteristic sequence (m_1, \dots, m_d) we have $m_1 \leq 1$ and $l(SS) = m_d \leq m_{d-1} + 1 \leq \dots \leq m_1 + (d-1) \leq d$. \square

However, it is not necessary for an algebra \mathcal{A} to be mixing or sliding to satisfy $l(\mathcal{A}) \leq \dim(\mathcal{A})$ as the following example shows.

Example 3.7. Consider an algebra \mathcal{A} over field \mathbb{F} with basis $e_0 = 1_{\mathbb{F}}, e_1, \dots, e_4$ and the following multiplication law:

$$e_1 e_1 = e_2, \quad e_2 e_2 = e_3, \quad e_1 e_3 = e_4,$$

and other products equal to 0. This is a so-called bare algebra of the sequence $(0, 1, 2, 4, 5)$ and its length is equal to 5, as is its dimension, see [9, Definition 3.3, Theorem 3.15].

However, it is neither mixing nor sliding. To prove this, consider

$$P(e_1, e_1, e_2) = \left\{ \begin{array}{c} (e_1 e_2) e_1, (e_2 e_1) e_1, e_1 (e_2 e_1), e_1 (e_1 e_2), \\ e_1 e_1, e_1 e_2, e_2 e_1, e_1, e_2 \end{array} \right\} =$$

$= \{0, e_1, e_2\}$. Since $(e_1 e_1) e_2 = e_3 \notin \langle P(e_1, e_1, e_2) \cup \{1\} \rangle$, \mathcal{A} is not mixing. As $P(e_1, e_1, e_2) \supset Q_l(e_1, e_1, e_2)$, this also means that the first property of Definition 3.1 does not hold. To demonstrate that the second property does not hold, note that $(e_1 e_1) e_2 = e_2 (e_1 e_1)$ and $P(e_1, e_1, e_2) \supset Q_r(e_1, e_1, e_2)$, which means $e_2 (e_1 e_1) \notin \langle Q_r(e_1, e_1, e_2) \cup \{1\} \rangle$.

4 Important classes of non-associative algebras and slowly growing length

Non-associative algebras are very important in mathematics and its applications, see [6, 18, 22] and their bibliography. Now we examine standard classes of algebras of slowly growing length. Recall that in this paper all algebras are finite dimensional.

The following lemma is useful in establishing various examples for algebras with polynomial identities.

Lemma 4.1. *Consider a finite-dimensional algebra \mathcal{A} over field \mathbb{F} , its basis $\{e_1, \dots, e_d\}$ and multilinear function G of k arguments such that $G(e_{i_1}, \dots, e_{i_k}) = 0$ for all $i_t \in \{1, \dots, d\}$. For all $a_1, \dots, a_k \in \mathcal{A}$ holds $G(a_1, \dots, a_k) = 0$.*

Proof. As $\{e_1, \dots, e_d\}$ is a basis of \mathcal{A} , there exist such $r_{ij} \in \mathbb{F}$ that $a_i = r_{i1}e_1 + \dots + r_{id}e_d$ for all $i \in 1, \dots, k$. We have

$$\begin{aligned} G(a_1, \dots, a_k) &= G(r_{11}e_1 + \dots + r_{1d}e_d, \dots, r_{k1}e_1 + \dots + r_{kd}e_d) = \\ &= \sum_{j_1, \dots, j_k \in \{1, \dots, d\}} r_{1j_1} \dots r_{kj_k} G(e_{j_1}, \dots, e_{j_k}) = 0. \end{aligned}$$

□

4.1 Lie and Leibniz algebras

Definition 4.2. An algebra \mathcal{A} is called a *Lie algebra* if

1. $xy = -yx$ for all $x, y \in \mathcal{A}$,
2. $(xy)z + (yz)x + (zx)y = 0$ for all $x, y, z \in \mathcal{A}$.

Trivially, Lie algebras are non-unital.

One possible generalization of Lie algebras are Leibniz algebras.

Definition 4.3. An algebra \mathcal{A} is called a *Leibniz algebra* if $(xy)z = x(yz) + (xz)y$ for all $x, y, z \in \mathcal{A}$.

An overview of Leibniz algebras can be found in [7].

Proposition 4.4. *Leibniz algebras are both mixing and sliding.*

Proof. Required properties of Definitions 3.1 and 3.2 are evident from the definition. □

Corollary 4.5. *Leibniz algebras have slowly growing length.*

Below we provide an example that this bound is sharp for the class of Leibniz algebras.

Example 4.6. Consider the algebra \mathcal{B}_d with the basis x_1, \dots, x_d , $d \geq 3$ and the following multiplication law:

$$x_i x_1 = x_{i+1}, \quad i = 1, \dots, d-1,$$

with the other products being zero. Since $l(\mathcal{B}_d) \leq d$ and $l(\mathcal{B}_d) \geq l(\{x_1\}) = d$, we have $l(\mathcal{B}_d) = d$.

Let us show that \mathcal{B}_d is indeed a Leibniz algebra. We consider arbitrary elements $u, y, z \in \mathcal{B}_d$ and their representations via basis above. Let c_y and c_z be the coefficients at x_1 of y and z , correspondingly.

Due to the multiplication rules, the coefficient at x_1 of yz is equal to zero, which implies that $u(yz) = 0$. Meanwhile $(uy)z = (uy)(c_z x_1) = c_z(uy)x_1 = c_z(u(c_y x_1))x_1 = c_z c_y (u x_1)x_1 = c_y(u(c_z x_1))x_1 = c_y(uz)x_1 = (uz)(c_y x_1) = (uz)y$. Combining these two formulas we achieve $(uy)z = u(yz) + (uz)y$.

For Lie algebras the above bound can be slightly improved.

Proposition 4.7. *The length of a Lie algebra \mathcal{A} of dimension $d \geq 2$ is not greater than $d - 1$.*

Proof. Follows directly from Lemma 3.4 and the fact that \mathcal{A} cannot be 1-generated. Indeed, $a^2 = 0$ for any $a \in \mathcal{A}$. Thus 1-generated algebra can not be 2-dimensional. Now for a generating set SS of \mathcal{A} with $l(SS) = l(\mathcal{A})$ and characteristic sequence (m_1, \dots, m_d) we have $m_1 = m_2 = 1$ and $l(SS) = m_d \leq m_{d-1} + 1 \leq \dots \leq m_2 + (d-2) = d-1$. \square

This bound is sharp as well.

Example 4.8. Consider so-called filiform Lie algebra \mathcal{A}_d with basis x_1, \dots, x_d , $d \geq 3$ and the following multiplication law:

$$x_1 x_i = x_{i+1} = -x_i x_1, \quad i = 2, \dots, d-1,$$

with other products being zero. Since $l(\mathcal{A}_d) \leq d-1$ and $l(\mathcal{A}_d) \geq l(\{x_1, x_2\}) = d-1$, we have $l(\mathcal{A}_d) = d-1$.

Lie algebras arise from associative algebras by changing the product $x \cdot y$ into $[x, y] = x \cdot y - y \cdot x$, and the following statement provides the connections between these two related algebras.

Proposition 4.9. *Let \mathcal{A} be an associative algebra over a field \mathbb{F} with the multiplication $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, and $\mathcal{A}^{(-)}$ be its adjoint Lie algebra, i.e., $\mathcal{A}^{(-)} = (\mathcal{A}, [\cdot, \cdot])$, where $[x, y] = x \cdot y - y \cdot x$ for any $x, y \in \mathcal{A}$. Then any generating set SS of $\mathcal{A}^{(-)}$ is a generating set of \mathcal{A} and $l(SS) \leq l^{Lie}(SS)$, where $l^{Lie}(SS)$ is the length of SS in $\mathcal{A}^{(-)}$ and $l(SS)$ is its length in \mathcal{A} .*

Proof. Both statements follow from the fact that for set $SS \subset \mathcal{A}$ we have $L_n^{Lie}(SS) \subset L_n(SS)$. Here $L_n^{Lie}(SS)$ is a linear span of all words of length less than or equal to n in SS with respect to the product $[\cdot, \cdot]$. We prove this fact by induction.

The base. For $n = 1$ we have $L_1^{Lie}(SS) = L_1(SS)$ as the set of all linear combinations of elements from SS .

The step. Assume that the statement holds for $n = 1, \dots, N-1$. For $n = N$ we have $L_N^{Lie}(SS) = \bigcup_{i \in \{1, \dots, N-1\}} [L_i^{Lie}(SS), L_{N-i}^{Lie}(SS)]$. Applying induction hypothesis to each component we have $L_N^{Lie}(SS) \subseteq \bigcup_{i \in \{1, \dots, N-1\}} [L_i(SS), L_{N-i}(SS)]$. Then using that $[L_i(SS), L_{N-i}(SS)] \subseteq L_N(SS)$, we have the desired inclusion.

Since SS is a generating set of $\mathcal{A}^{(-)}$, there exists n_0 such that $\mathcal{A} = L_{n_0}^{Lie}(SS) \subset L_{n_0}(SS) \subset \mathcal{A}$. Thus, $L_{n_0}(SS) = \mathcal{A}$ and $l^{Lie}(SS) \geq l(SS)$. \square

Note that the above proposition does not mean that the length of $\mathcal{A}^{(-)}$ is greater than or equal to the length of \mathcal{A} . Actually, any mutual behavior of these numerical invariants is possible, as the following examples show.

Example 4.10. Consider the algebra $\mathcal{A}_1 = \mathbb{R}^2$ over \mathbb{R} with the addition and the multiplication defined coordinate-wise. Then \mathcal{A}_1 is a unital algebra of the dimension 2. Hence, $l(\mathcal{A}_1) = 1$. Also $l^{Lie}(\mathcal{A}_1^{(-)}) = 1$ since a product of any two elements in $\mathcal{A}_1^{(-)}$ equals 0.

Example 4.11. Consider $\mathcal{A}_2 = \mathbb{R}^3$ over \mathbb{R} with coordinate-wise addition and multiplication. Then \mathcal{A}_2 is a unital algebra of the dimension 3. Hence $l(\mathcal{A}_2) \leq 2$. Since $l(\{(0, 1, 2)\}) = 2$ in \mathcal{A}_2 it follows that $l(\mathcal{A}_2) = 2$. Meanwhile, $l(\mathcal{A}_2^{(-)}) = 1$ since any product in $\mathcal{A}_2^{(-)}$ is equal to 0.

Example 4.12. Consider $\mathcal{A}_3 = M_2(\mathbb{R})$. Then $l(\mathcal{A}_3) = 2$, see, for example, [16]. Let us prove that $l^{Lie}(\mathcal{A}_3^{(-)}) = 3$. To do this we consider the set $\{G_1, G_2\}$, where $G_1 = E_{11} - E_{12}$ and $G_2 = E_{21} + E_{22}$, here E_{ij} is the matrix with 1 in (i, j) -th position and 0 elsewhere. Then we have

$$\begin{aligned} [G_1, G_2] &= (E_{11} - E_{12})(E_{21} + E_{22}) - (E_{21} + E_{22})(E_{11} - E_{12}) = \\ &= -E_{11} - E_{12} - E_{21} + E_{22} =: G_3, \\ [G_3, G_1] &= (-E_{11} - E_{12} - E_{21} + E_{22})(E_{11} - E_{12}) - (E_{11} - E_{12})(-E_{11} - E_{12} - E_{21} + E_{22}) = \\ &= -E_{11} + 3E_{12} - E_{21} + E_{22} =: G_4. \end{aligned}$$

As $[G_1, G_1] = [G_2, G_2] = 0$, $[G_2, G_1] = -[G_1, G_2]$ and G_1, G_2, G_3 and G_4 are linearly independent, the set $\{G_1, G_2\}$ is a generating system of length 3. As $l(\mathcal{A}_3^{(-)}) \leq 3$ by Proposition 4.7, we have $l(\mathcal{A}_3^{(-)}) = 3$.

4.2 Novikov algebras

Another well-known class of non-associative algebras is the class of Novikov algebras. Their properties can be found, for example, in [3].

Definition 4.13. An algebra \mathcal{A} is called a *Novikov algebra* if

1. $x(yz) - (xy)z = y(xz) - (yx)z$ for all $x, y, z \in \mathcal{A}$,
2. $(xy)z = (xz)y$ for all $x, y, z \in \mathcal{A}$.

Proposition 4.14. *Novikov algebras are mixing and hence they have slowly growing length.*

Proof. Required properties of Definition 3.2 are evident from definition of the class. \square

The following example shows that this bound is sharp.

Example 4.15. Consider algebra C_d with basis x_1, \dots, x_d , $d \geq 3$ and the following multiplication law:

$$x_1 x_i = x_{i+1}, \quad i = 1, \dots, d-1,$$

with other products being zero. Since $l(C_d) \leq d$ and $l(C_d) \geq l(\{x_1\}) = d$, we have $l(C_d) = d$.

C_d is indeed a Novikov algebra. To prove this, consider elements $u, y, z \in C_d$ and their representations via basis above. Let c_u and c_y be coefficients of u and y at x_1 .

Coefficient at x_1 of uy and uz are zero, which means $(uy)z = (uz)y = (yu)z = 0$, and the second property of Definition 4.13 holds, while the first is reduced to $u(yz) = y(uz)$. For the latter we have $u(yz) = u((c_y x_1)z) = c_y u(x_1 z) = c_u c_y x_1(x_1 z) = c_u y(x_1 z) = y((c_u x_1)z) = y(uz)$.

Another example demonstrates that Novikov algebras are not necessarily sliding.

Example 4.16. Consider algebra C over field \mathbb{F} with basis x_1, x_2, x_3, x_4 and the following multiplication law:

$$x_1 x_1 = x_2, \quad x_1 x_2 = x_3, \quad x_2 x_1 = x_4,$$

with other products being zero.

C is a Novikov algebra. To prove this we will check the properties of Definition 4.13 on basis elements and infer it for other elements by Lemma 4.1.

For a triple $u, y, z \in \{x_1, x_2, x_3, x_4\}$ any product of the elements u, y, z in any order is zero if at least one of them is not equal to x_1 , which means that both properties hold in this case. If $u = y = z = x_1$, they also trivially hold.

However, C is not sliding as $(x_1x_1)x_1 = x_4$ cannot be represented as a linear combination of elements of $Q_l(x_1, x_1, x_1) = \{x_1(x_1x_1), x_1x_1, x_1\} = \{x_3, x_2, x_1\}$ and vice versa $x_1(x_1x_1) = x_3$ cannot be represented as a linear combination of elements of $Q_r(x_1, x_1, x_1) = \{(x_1x_1)x_1, x_1x_1, x_1\} = \{x_4, x_2, x_1\}$.

4.3 Zinbiel algebras

We also consider Zinbiel algebras, for further information on which we direct the reader to [1].

Definition 4.17. An algebra \mathcal{A} is called a (right)-Zinbiel algebra if $x(yz) = (xy + yx)z$ for all $x, y, z \in \mathcal{A}$.

Proposition 4.18. Zinbiel algebras are sliding and hence they have slowly growing length.

Proof. Required properties of Definition 3.1 are evident from definition of the class. \square

The following example shows that the above bound is sharp.

Example 4.19. Consider the algebra \mathcal{Z}_d over the field \mathbb{R} with the basis x_1, \dots, x_d , $d \geq 3$ and the following multiplication law:

$$x_i x_j = \frac{j}{i+j} x_{i+j}, \quad i, j = 1, \dots, d, \quad i+j \leq d,$$

with other products being zero. Since $l(\mathcal{Z}_d) \leq d$ and $l(\mathcal{Z}_d) \geq l(\{x_1\}) = d$, we have $l(\mathcal{Z}_d) = d$.

\mathcal{Z}_d is indeed a Zinbiel algebra. To prove this we will demonstrate its defining property on basis elements and infer it for other elements by Lemma 4.1.

We have for i, j, k such that $i+j+k \leq d$

$$x_i(x_j x_k) = \frac{k}{j+k} x_i x_{j+k} = \frac{k}{j+k+i} x_{i+j+k} = (x_i x_j + x_j x_i) x_k,$$

and for i, j, k such that $i+j+k > d$

$$x_i(x_j x_k) = 0 = (x_i x_j + x_j x_i) x_k.$$

Now we demonstrate that Zinbiel algebras are not necessarily mixing.

Example 4.20. Consider algebra \mathcal{Z} over field \mathbb{F} with basis x_1, x_2, x_3, x_4, x_5 and the following multiplication law:

$$x_1 x_2 = x_4 = -x_2 x_1, \quad x_4 x_3 = x_5,$$

with other products being zero.

\mathcal{Z}_d is a Zinbiel algebra. To prove this we will demonstrate its defining property on basis elements and infer it for other elements by Lemma 4.1.

For a triple $u, y, z \in \{x_1, x_2, x_3, x_4, x_5\}$ any product of u, y, z in any order is zero if at least one of them is equal to x_4, x_5 , which means that the property holds in this case. If $\{u, y, z\} \subseteq \{x_1, x_2, x_3\}$, the possible products are zero as well. For the remaining possibilities see the table below (the last column checking $u(yz) = (uy + yu)z$).

u	y	z	Result
x_1	x_2	x_3	$0 = (x_4 - x_4)x_3$
x_2	x_1	x_3	$0 = (-x_4 + x_4)x_3$
x_1	x_3	x_2	$0 = (0 + 0)x_2$
x_3	x_1	x_2	$0 = (0 + 0)x_2$
x_2	x_3	x_1	$0 = (0 + 0)x_1$
x_3	x_2	x_1	$0 = (0 + 0)x_1$

However, the algebra is not mixing as $(x_1 x_2) x_3 = x_5$ cannot be represented as a linear combination of elements of $P(x_1, x_2, x_3) =$

$$\begin{aligned} &= \left\{ \begin{array}{l} (x_1 x_3) x_2, (x_3 x_1) x_2, (x_2 x_3) x_1, (x_3 x_2) x_1, \\ x_1 (x_3 x_2), x_1 (x_2 x_3), x_2 (x_1 x_3), x_2 (x_3 x_1), \\ x_1 x_2, x_2 x_1, x_1 x_3, x_3 x_1, x_2 x_3, x_3 x_2, x_1, x_2, x_3 \end{array} \right\} = \\ &= \{0, x_1, x_2, x_3, x_4\}. \end{aligned}$$

4.4 Some classes of algebras that do not have slowly growing length

A class of algebras closely connected with Novikov algebras are Vinberg algebras, also known as right-symmetric algebras (RSA), which are the algebras satisfying just the first one of the two conditions determining Novikov algebras, i.e.

Definition 4.21. An algebra \mathcal{A} is called a *Vinberg algebra* if $(xy)z - x(yz) = (xz)y - x(zy)$ for all $x, y, z \in \mathcal{A}$.

An overview of such algebras can be found in [2, 4, 5].

It can be shown that Vinberg algebras, which are in general neither mixing nor sliding, do not have slowly growing length universally. Below we present an example of such algebra.

Example 4.22. Consider a non-unitary algebra \mathcal{R} with basis e_1, e_2, e_3, e_4 and the following multiplication table (the operation being concatenation):

$$e_1e_1 = e_2, \quad e_1e_2 = e_3, \quad e_3e_2 = e_4.$$

with the other products being zero. The characteristic sequence of the set $\{e_1\}$ is 1, 2, 3, 5, while \mathcal{R} belongs to the class of Vinberg algebras. To prove the latter, by Lemma 4.1 it is enough to check that for $x, y, z \in \{e_1, e_2, e_3, e_4\}$ it holds that

$$(xy)z - x(yz) = (xz)y - x(zy).$$

If either of x, y, z is e_4 , then every term is obviously zero. After substitution of e_i every term has the same length as words in $\{e_1\}$. This length is greater or equal to 3 (as there are three sub-terms of positive length).

For words of lengths 3 and 5 consider the table below.

x	y	z	Result
e_1	e_1	e_1	$0 - e_3 = 0 - e_3$
e_1	e_1	e_3	$0 - 0 = 0 - 0$
e_1	e_3	e_1	$0 - 0 = 0 - 0$
e_3	e_1	e_1	$0 - e_4 = 0 - e_4$
e_2	e_2	e_1	$0 - 0 = 0 - 0$
e_2	e_1	e_2	$0 - 0 = 0 - 0$
e_1	e_2	e_2	$e_4 - 0 = e_4 - 0$

Words of length 4 or 6 and higher are equal to zero, which means that the desired property holds trivially, and the algebra under consideration is a Vinberg algebra.

Definition 4.23. An algebra \mathcal{A} is called a *Valya algebra* if

1. $xy = -yx$ for all $x, y \in \mathcal{A}$,
2. $J(x_1x_2, x_3x_4, x_5x_6) = 0$, where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ for all $x_1, x_2, \dots, x_6 \in \mathcal{A}$

An overview of Valya algebras can be found in [21].

Universally Valya algebras are neither mixing nor sliding, and they do not necessarily have slow growing length.

Example 4.24. Consider algebra \mathbb{V} over a field \mathbb{F} with basis $e_1, e_2, e_3, e_4, e_5, e_6$ and the following multiplication laws:

$$\begin{aligned} e_1e_2 &= e_3 = -e_2e_1, \quad e_2e_3 = e_4 = -e_3e_2, \\ e_3e_4 &= e_5 = -e_4e_3, \quad e_4e_5 = e_6 = -e_5e_4, \end{aligned}$$

with other products being zero.

It is a Valya algebra: for the first property multiplication is clearly anti-commutative and for the second it is enough to check it on any six basis elements $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}, e_{i_5}, e_{i_6}$ by Lemma 4.1.

$J(e_{i_1}e_{i_2}, e_{i_3}e_{i_4}, e_{i_5}e_{i_6})$ is a sum of three words of similar length in letters $\{e_1, e_2\}$, and this length is at least 6 as each e_{i_j} is a word of positive length.

If this length is other than 8, then every summand is zero as there are no non-zero words in this alphabet of such length.

Otherwise consider the summands in $J(e_{i_1}, \dots, e_{i_6})$. They are represented as products of three words of length at least 2 in $\{e_1, e_2\}$. A non-zero word of length 8, equal to $\pm e_6$ can be represented this way only as a product of $\pm e_3, \pm e_4, \pm e_4$ in correct order (one of $\pm e_4$ being in the outer product).

However, $J(e_3, e_4, e_4) = 0$. Since J is linear and symmetric by its arguments, all other combinations of $\pm e_3, \pm e_4, \pm e_4$ as arguments of J will result in 0 as well.

The generating set $\{e_1, e_2\}$ has characteristic sequence $(1, 1, 2, 3, 5, 8)$. It follows that $l(\mathcal{A}) = 8 > 6 = \dim \mathbb{V}$.

In the previous sections we discussed classes of algebras with slowly growing lengths. We remark that there are many algebras such that their length is not bounded by the dimension but is bounded by a certain linear function of the dimension. Below we present a certain family of such algebras.

Proposition 4.25. *Let $r \geq 2$ be an integer and \mathcal{A}_r be an algebra satisfying the property: for all $x, y_1, \dots, y_r \in \mathcal{A}_r$ and any product $v = y_1 \cdots y_r$ (with any placement of parentheses) the equality $xv = 0$ holds. Then $l(\mathcal{A}_r) \leq (r-1)\dim \mathcal{A}_r$.*

Proof. Consider a generating set SS of \mathcal{A}_r such that $l(SS) = l(\mathcal{A}_r)$ and its characteristic sequence $M = (m_1, \dots, m_d)$, where $d = \dim \mathcal{A}_r$. We are going to prove that $m_{j+1} - m_j \leq r-1$ for all $1 \leq j \leq d-1$.

Assume the contrary. Let k be the smallest index such that $m_{k+1} - m_k \geq r$.

Consider a word w of length at least two. It can be uniquely represented as $w = w' \cdot w''$, where w' and w'' have non-zero lengths. We denote $s(w) = \min(l(w'), l(w''))$.

Consider an irreducible word w in SS of length m_{k+1} . There are two possibilities.

Case 1: $s(w) \leq r-1$. Then w is a product of irreducible words of length $s(w)$ and $m_{k+1} - s(w)$ by Lemma 2.2. Hence by Corollary 2.5, Item 2, there is an element equal to $m_{k+1} - s(w)$ in the characteristic sequence M . This is impossible, since M is non-decreasing and $m_k < m_{k+1} - s(w)$ by the assumption.

Case 2: $s(w) \geq r$. If $s(w) = l(w'')$ then $l(w'') \geq r$. Hence $w = 0$ by the condition on the products of $(r+1)$ factors in \mathcal{A}_r . Otherwise $s(w) = l(w')$. Then $l(w'') \geq l(w') \geq r$ and again $w = 0$. Both of these possibilities contradict the fact that w is irreducible.

Thus, the initial assumption is incorrect and $m_{j+1} - m_j \leq r-1$ for all $1 \leq j \leq d-1$. This allows us to conclude that $l(\mathcal{A}_r) = l(SS) = m_d \leq m_{d-1} + (r-1) \leq \dots \leq m_1 + (r-1)(d-1) < (r-1)d$. \square

Let us note that if $r = 2$ then the algebras \mathcal{A}_r are sliding, and therefore, the bound is sharp. If $r > 2$ then the resulting bound is not sharp. However, for any r there exist algebras which provide growth of length which is linear in dimension with coefficient $r-1$.

Example 4.26. Consider an algebra \mathcal{E}_d with the basis x_1, \dots, x_d , $d \geq r \geq 2$ and the following multiplication law:

$$\begin{aligned} x_j x_1 &= x_{j+1}, \quad j = 1, \dots, r-2, \\ x_i x_{r-1} &= x_{i+1}, \quad i = r-1, \dots, d-1, \end{aligned}$$

with other products being zero. We have

$$l(\mathcal{E}_d) \geq l(\{x_1\}) = (r-1)d - (r-2)(r-1).$$

Let us prove that \mathcal{E}_d satisfies the conditions of Proposition 4.25.

At first, we consider $x, y_1, \dots, y_r \in \{x_1, \dots, x_d\}$. A word xv , where v is a product of y_1, \dots, y_r , is indeed zero as v cannot be neither x_1 nor x_{r-1} . So, the required condition holds for the basis of \mathcal{E}_d . Then by Lemma 4.1 it is satisfied for other elements as well.

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