RATIONAL FUNCTION SEMIFIELDS OF TROPICAL CURVES ARE FINITELY GENERATED OVER THE TROPICAL SEMIFIELD

SONG JUAE

Abstract. We prove that the rational function semifield of a tropical curve is finitely generated as a semifield over the tropical semifield $\boldsymbol{T} := (\boldsymbol{R} \cup \{-\infty\}, \max, +)$ by giving a specific finite generating set. Also, we show that for a finite harmonic morphism between tropical curves $\varphi : \Gamma \to \Gamma'$, the rational function semifield of Γ is finitely generated as a $\varphi^*(\mathrm{Rat}(\Gamma'))$-algebra, where $\varphi^*(\mathrm{Rat}(\Gamma'))$ stands for the pull-back of the rational function semifield of Γ' by φ.

1. Introduction

This paper gives a tropical analogue of the fact that the function field of an algebraic curve over $\boldsymbol{C}$ is generated by two elements over $\boldsymbol{C}$:

Theorem 1.1. *Let Γ be a tropical curve. Then, the rational function semifield $\mathrm{Rat}(\Gamma)$ of Γ is finitely generated as a semifield over the tropical semifield $\boldsymbol{T} := (\boldsymbol{R} \cup \{-\infty\}, \max, +)$.*

Here, a tropical curve is a metric graph that may have edges of length ∞, and a rational function on a tropical curve is a piecewise affine continuous function with integer slopes and with a finite number of pieces or a constant $-\infty$ function. The set $\mathrm{Rat}(\Gamma)$ of all rational functions on a tropical curve Γ has a natural structure of a semifield over $\boldsymbol{T}$, where the addition $\oplus$ is defined as the pointwise maximum operation and the multiplication $\odot$ as the pointwise usual addition.

The following lemma is our key to prove Theorem 1.1:

Lemma 1.2 ([2, Lemma 2.4.2])**.** *Let Γ be a tropical curve. Then, $\mathrm{Rat}(\Gamma)$ is generated by all chip firing moves and all constant functions as a group with tropical multiplication $\odot$ as its binary operation.*

Here, a chip firing move $\mathrm{CF}(\Gamma_1, l)$ is the rational function defined by the pair of a subgraph Γ_1 and a number $l \in \boldsymbol{R}_{>0} \cup \{\infty\}$ as follows: $\mathrm{CF}(\Gamma_1, l)(x) := -\min\{\mathrm{dist}(\Gamma_1, x), l\}$, where Γ_1 has no connected components consisting only of a point at infinity and $\mathrm{dist}(\Gamma_1, x)$ denotes the distance between Γ_1 and x. By this lemma, it is enough to find a

2020 *Mathematics Subject Classification.* Primary 14T10; Secondary 14T20.

Key words and phrases. rational function semifields of tropical curves, chip firing moves on tropical curves.

finite set of rational functions which generates all chip firing moves as a semifield over $\boldsymbol{T}$.

The following example suggests that all chip firing moves defined by one point generate all chip firing moves. Here, the valence $\mathrm{val}(x)$ of a point x of a tropical curve is the minimum number of the connceted components $U \setminus \{x\}$ with all neighborhoods U of x.

Example 1.3. Let Γ be a tropical curve. Let Γ_1 be a proper subgraph of Γ that has no connected components consisting only of a point at infinity. For any $l \in \boldsymbol{R} \cup \{\infty\}$ and any l' such that $0 < l' \leq l$, we can *cut* the bottom side of the chip firing move $\mathrm{CF}(\Gamma_1, l)$:

$$\mathrm{CF}(\Gamma_1, l') = \mathrm{CF}(\Gamma_1, l) \oplus (-l').$$

Let $\Gamma_2 := \{x \in \Gamma \mid \mathrm{dist}(\Gamma_1, x) \leq l'\}$. We can *cut* the top side of the chip firing move $\mathrm{CF}(\Gamma_1, l)$:

$$\mathrm{CF}(\Gamma_2, l - l') = \left\{\mathrm{CF}(\Gamma_1, l)^{\odot(-1)} \oplus l'\right\}^{\odot(-1)} \odot l'.$$

We can *extend* the chip firing move $\mathrm{CF}(\Gamma_1, l')$:

$$\mathrm{CF}(\Gamma_1, l) = \mathrm{CF}(\Gamma_1, l') \odot \mathrm{CF}(\Gamma_2, l - l').$$

Let x be a boundary point of Γ_1 in Γ. Let ε be a sufficiently small positive real number and $y \in \Gamma \setminus \Gamma_1$ such that $\mathrm{dist}(x, y) = \varepsilon$. Then, we can *connect* two chip firing moves $\mathrm{CF}(\Gamma_1, \varepsilon)$ and $\mathrm{CF}(\{y\}, \varepsilon)$:

$$\mathrm{CF}(\Gamma_1 \cup [x, y], \varepsilon) = \mathrm{CF}(\Gamma_1, \varepsilon) \odot \mathrm{CF}(\{y\}, \varepsilon) \odot \varepsilon.$$

Let Γ_3, Γ_4 be any two proper subgraphs of Γ whose intersection is empty and both that have no connected components consisting only of a point at infinity. Let l be a positive real number such that the intersection of $\{x \in \Gamma \mid \mathrm{dist}(\Gamma_3, x) \leq l\}$ and $\{x \in \Gamma \mid \mathrm{dist}(\Gamma_4, x) \leq l\}$ is finite. Then, we have

$$\mathrm{CF}(\Gamma_3 \sqcup \Gamma_4, l) = \mathrm{CF}(\Gamma_3, l) \oplus \mathrm{CF}(\Gamma_4, l).$$

Note that Algorithm 1 in Section 3 gives a range of "ε is sufficiently small". If Γ has no edges of length ∞, then in fact all chip firing moves defined by one point generate all chip firing moves. By the following lemma, we may assume that Γ has no edges of length ∞.

Lemma 1.4. *Let Γ be a tropical curve. Let Γ' be a tropical curve which is obtained from Γ by contracting edges of length ∞. If $\mathrm{Rat}(\Gamma')$ is finitely generated as a semifield over $\boldsymbol{T}$, then so is $\mathrm{Rat}(\Gamma)$.*

Hence our next target is to find a finite set of rational functions which generates all chip firing moves defined by one point as a semifield over $\boldsymbol{T}$. Let $(G_\circ, l_\circ)$ be the canonical model for Γ, i.e., the pair of the underlying graph $G_\circ$ of Γ whose set $V(G_\circ)$ of vertices is $\{x \in \Gamma \mid \mathrm{val}(x) \neq 2\}$ and the length function $l_\circ$ defined by Γ and $G_\circ$ (for more precisely, see Subsection 2.2). Fix a direction on edges of $G_\circ$. Let each $e \in E(G_\circ)$ be identified with the interval $[0, l_\circ(e)]$ with this direction, where $E(G_\circ)$ denotes the set of edges of $G_\circ$. For each edge $e \in E(G_\circ)$, let $x_e = \frac{l_\circ(e)}{4}$, $y_e = \frac{l_\circ(e)}{2}$, and $z_e = \frac{3l_\circ(e)}{4}$. We define rational functions

$$f_e := \mathrm{CF}\left(\{y_e\}, \frac{l_\circ(e)}{2}\right), g_e := \mathrm{CF}\left(\{x_e\}, \frac{l_\circ(e)}{4}\right), h_e := \mathrm{CF}\left(\{z_e\}, \frac{l_\circ(e)}{4}\right).$$

Note that the semifield generated by g_e, h_e over $\boldsymbol{T}$ coincides with the semifield generated by $g_e \odot h_e^{\odot(-1)}$ over $\boldsymbol{T}$ since

$$g_e = \left(-\frac{l_\circ(e)}{4}\right) \odot \left(g_e \odot h_e^{\odot(-1)} \oplus 0\right)$$

and

$$h_e = \left(-\frac{l_\circ(e)}{4}\right) \odot \left\{\left(g_e \odot h_e^{\odot(-1)}\right)^{\odot(-1)} \oplus 0\right\}.$$

Let R be the semifield generated by f_e, g_e, h_e for any $e \in E(G_\circ)$ and $\mathrm{CF}(\{v\}, \infty)(= -\mathrm{dist}(v, \cdot))$ for any $v \in V(G_\circ)$ over $\boldsymbol{T}$. This semifield R is finitely generated, and in fact, coincides with $\mathrm{Rat}(\varGamma)$. Hence, we have Theorem 1.1.

In the setting that a finite harmonic morphism between tropical curves is given, we have the following proposition:

Proposition 1.5. *Let $\varphi : \varGamma \to \varGamma'$ be a finite harmonic morphism between tropical curves. Then,* $\mathrm{Rat}(\varGamma)$ *is finitely generated as a* $\varphi^*(\mathrm{Rat}(\varGamma'))$*-algebra, where* $\varphi^*(\mathrm{Rat}(\varGamma'))$ *stands for the pull-back of* $\mathrm{Rat}(\varGamma')$ *by* φ*.*

Note that $\mathrm{Rat}(\varGamma)$ may not be finitely generated as a $\varphi^*(\mathrm{Rat}(\varGamma'))$-module. See Example 3.15.

This paper is organized as follows. In Section 2, we give basic definitions related to semirings and tropical curves which we need later. Section 3 gives proofs of Theorem 1.1, Lemma 1.4, and Proposition 1.5. In that section, we also show that there exists a generating set of rational function semifield of any tree whose elements are fewer than that of the above generating set and that rational function semifields of tropical curves other than a singleton are not finitely generated as a $\boldsymbol{T}$-algebra.

Acknowledgements

The author thanks my supervisor Masanori Kobayashi, Yuki Kageyama, Yasuhito Nakajima, Ken Sumi, and Daichi Miura for helpful comments. This work was supported by JSPS KAKENHI Grant Number 20J11910.

2. Preliminaries

In this section, we prepare basic definitions related to semirings and tropical curves which we need later. For an introduction to the theory of tropical geometry, for example, see [4]. We employ definitions in [2] for tropical curves.

2.1. **Semirings.** In this paper, a *semiring* is a commutative semiring with the absorbing neutral element 0 for addition and the identity 1 for multiplication such that $0 \neq 1$. If every nonzero element of a semiring S is multiplicatively invertible, then S is called a *semifield*.

A map $\varphi : S_1 \to S_2$ between semirings is a *semiring homomorphism* if for any $x, y \in S_1$,

$$\varphi(x+y) = \varphi(x)+\varphi(y),\ \varphi(x\cdot y) = \varphi(x)\cdot\varphi(y),\ \varphi(0) = 0,\ \text{and}\ \varphi(1) = 1.$$

Given a semiring homomorphism $\varphi : S_1 \to S_2$, we call the pair (S_2, φ) (for short, S_2) a S_1-*algebra*.

The set $\boldsymbol{T} := \boldsymbol{R} \cup \{-\infty\}$ with two tropical operations:

$$a \oplus b := \max\{a, b\} \text{ and } a \odot b := a + b,$$

where both a and b are in $\boldsymbol{T}$, becomes a semifield. Here, for any $a \in \boldsymbol{T}$, we handle $-\infty$ as follows:

$$a \oplus (-\infty) = (-\infty) \oplus a = a \text{ and } a \odot (-\infty) = (-\infty) \odot a = -\infty.$$

$\boldsymbol{T}$ is called the *tropical semifield*.

2.2. **Tropical curves.** In this paper, a *graph* is an unweighted, undirected, finite, connected nonempty multigraph that may have loops. For a graph G, the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$. The *valence* of a vertex v of G is the number of edges incident to v, where each loop is counted twice. A vertex v of G is a *leaf end* if v has valence one. A *leaf edge* is an edge of G incident to a leaf end.

An *edge-weighted graph* (G, l) is the pair of a graph G and a function $l : E(G) \to \boldsymbol{R}_{>0} \cup \{\infty\}$, where l can take the value ∞ on only leaf edges. A *tropical curve* is the underlying topological space of an edge-weighted graph (G, l) together with an identification of each edge e of G with the closed interval $[0, l(e)]$. The interval $[0, \infty]$ is the one point compactification of the interval $[0, \infty)$. We regard $[0, \infty]$ not just as a topological space but as almost a metric space. The distance between ∞ and any other point is infinite. When $l(e) = \infty$, the leaf end of e must be identified with ∞. If $E(G) = \{e\}$ and $l(e) = \infty$, then we can identify either leaf ends of e with ∞. When a tropical curve Γ is obtained from (G, l), the edge-weighted graph (G, l) is called a *model* for Γ. There are many possible models for Γ. A model (G, l) is *loopless* if G is loopless. We frequently identify a vertex (resp. an edge) of G with the corresponding point (resp. the corresponding closed subset) of Γ. For a point x on a tropical curve Γ obtained from (G, l), if x is identified with ∞, then x is called a *point at infinity*, else, x is called a *finite point*. Γ_∞ denotes the set of all points at infinity of Γ. If Γ_∞ is empty, i.e. $l : E(G) \to \boldsymbol{R}_{>0}$, then Γ is called a *metric graph*. If x is a finite point, then the *valence* $\mathrm{val}(x)$ is the number of connected components of $U \setminus \{x\}$ with any sufficiently small connected neighborhood U of x, if x is a point at infinity, then $\mathrm{val}(x) := 1$. Remark that this "valence" is defined for a point of a tropical curve and the "valence" in the first paragraph of this subsection is defined for a vertex of a graph, and these are compatible with each other. We construct a model $(G_\circ, l_\circ)$ called the *canonical model* for Γ as follows. We determine $V(G_\circ) := \{x \in \Gamma \mid \mathrm{val}(x) \neq 2\}$ except following two cases. When Γ is homeomorphic to a circle S^1, we determine $V(G_\circ)$ as the set consisting of one arbitrary point of Γ. When Γ has the edge-weighted graph (T, l) as its model, where T is the tree consisting of three vertices and two edges and $l(E(T)) = \{\infty\}$, we determine $V(G_\circ)$ as the set of two points at infinity and any finite point of Γ. The *relative interior* e° of an edge e is $e \setminus \{v, w\}$ with the endpoint(s) v, w of e. The *genus* $g(\Gamma)$ of Γ is the first Betti number of Γ, which coincides with $\#E(G) - \#V(G) + 1$ for any model (G, l) for Γ. A *tree* is a tropical curve of genus zero. The word "an edge of Γ" means an edge of G with some model (G, l) for Γ.

2.3. **Rational functions and chip firing moves.** Let Γ be a tropical curve. A continuous map $f : \Gamma \to \boldsymbol{R} \cup \{\pm\infty\}$ is a *rational function* on Γ if f is a piecewise affine function with integer slopes, with a finite number of pieces and that can take the value $\pm\infty$ only at points at infinity, or a constant function of $-\infty$. For a point x of Γ and a rational function $f \in \mathrm{Rat}(\Gamma) \setminus \{-\infty\}$, x is a *pole* of f if the sign of the sum of outgoing slopes of f at x is minus. The absolute value of the sum is its degree. Let $\mathrm{Rat}(\Gamma)$ denote the set of all rational functions on Γ. For rational functions $f, g \in \mathrm{Rat}(\Gamma)$ and a point $x \in \Gamma \setminus \Gamma_\infty$, we define

$$(f \oplus g)(x) := \max\{f(x), g(x)\} \text{ and } (f \odot g)(x) := f(x) + g(x).$$

We extend $f \oplus g$ and $f \odot g$ to points at infinity to be continuous on whole Γ. Then both are rational functions on Γ. Note that for any $f \in \mathrm{Rat}(\Gamma)$, $f \odot (-\infty) = (-\infty) \odot f = -\infty$. Then $\mathrm{Rat}(\Gamma)$ becomes a semifield with these two operations. Also, $\mathrm{Rat}(\Gamma)$ becomes a $\boldsymbol{T}$-algebra with the natural inclusion $\boldsymbol{T} \hookrightarrow \mathrm{Rat}(\Gamma)$. Note that for $f, g \in \mathrm{Rat}(\Gamma)$, $f = g$ means that $f(x) = g(x)$ for any $x \in \Gamma$.

A *subgraph* of a tropical curve is a compact nonempty subset of the tropical curve with a finite number of connected components. Let Γ_1 be a subgraph of a tropical curve Γ which does not have any connected components consisting of only points at infinity and l a positive real number or infinity. The *chip firing move* by Γ_1 and l is defined as the rational function $\mathrm{CF}(\Gamma_1, l)(x) := -\min(\mathrm{dist}(x, \Gamma_1), l)$.

2.4. **Finite harmonic morphisms.** Let $\varphi : \Gamma \to \Gamma'$ be a continuous map between tropical curves. φ is a *finite harmonic morphism* if there exist loopless models (G, l) and (G', l') for Γ and Γ', respectively, such that (1) $\varphi(V(G)) \subset V(G')$ holds, (2) $\varphi(E(G)) \subset E(G')$ holds, (3) for any edge e of G, there exists a positive integer $\deg_e(\varphi)$ such that for any points x, y of e, $\mathrm{dist}(\varphi(x), \varphi(y)) = \deg_e(\varphi) \cdot \mathrm{dist}(x, y)$ holds, and (4) for every vertex v of G, the sum $\sum_{e \in E(G):\, e \mapsto e',\, v \in e} \deg_e(\varphi)$ is independent of the choice of $e' \in E(G')$ incident to $\varphi(v)$. This sum is denoted by $\deg_v(\varphi)$. Then, the sum $\sum_{v \in V(G):\, v \mapsto v'} \deg_v(\varphi)$ is independent of the choice of a vertex v' of G', and is called the *degree* of φ. If both Γ and Γ' are singletons, we regard φ as a finite harmonic morphism that can have any number as its degree.

Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves. The *pull-back* $\varphi^* : \mathrm{Rat}(\Gamma') \to \mathrm{Rat}(\Gamma)$ is defined by $f' \mapsto f' \circ \varphi$. Note that on each $e \in E(G)$ with the model (G, l) above, $\varphi^*(f')$ has only multiples of $\deg_e(\varphi)$ as its slopes for any $f' \in \mathrm{Rat}(\Gamma')$.

3. Main results

In this section, we give proofs of Theorem 1.1 and Proposition 1.5.

First, we prove Theorem 1.1. To do it, we will prepare multiple lemmas and an algorithm. Algorithm 1 gives a range of values for a proper connected subgraph of a metric graph to connect the chip firing move defined by it and another chip firing move (see Example 1.3).

Algorithm 1

Input: Γ : a metric graph
  $E(G_\circ) = \{e_1, \dots, e_n\}$: a labeling of edges of the canonical model for Γ
  S : a proper connected subgraph of Γ
Output: l_S

$i \leftarrow 1$
while $i \leq n$ **do**
  if $e_i \cap S = \varnothing$ **then**
    $l_i \leftarrow (\text{the diameter of } \Gamma), i \leftarrow i + 1$
  else $\{S \supset e_i\}$
    $l_i \leftarrow (\text{the diameter of } \Gamma), i \leftarrow i + 1$
  else $\{S \supset \partial e_i\}$
    $l_i \leftarrow (\text{the length of } \overline{e_i \setminus S})/2, i \leftarrow i + 1$
  else $\{S \subset e_i^\circ\}$
    $l_i \leftarrow \min\{\mathrm{dist}(S, x) \,|\, x \text{ is one of the endpoints of } e_i\}, i \leftarrow i + 1$
  else
    $l_i \leftarrow (\text{the length of } \overline{e_i \setminus S}), i \leftarrow i + 1$
  end if
end while
$l_S \leftarrow \min\{l_1, \dots, l_n\}$
return l_S

In Algorithm 1, $\overline{e_i \setminus S}$ denotes the closure of $e_i \setminus S$, and if S consists of only one point x, then we write l_x instead of $l_{\{x\}}$.

Remark 3.1. Let Γ be a metric graph and S_1 a proper connceted subgraph of Γ. Let $l \leq l_{S_1}$ and $S_2 := \{x \in \Gamma \,|\, \mathrm{dist}(S_1, x) \leq l\}$. With $a := \min\{k \in \boldsymbol{Z}_{>0} \,|\, l/k \leq l_{S_2}\}$, $m := \min\{k \in \boldsymbol{Z}_{>0} \,|\, l_{S_2}/k \leq l/a\}$ and any $l' > 0$, by the definition of chip firing moves, we have

$$\begin{aligned}\mathrm{CF}\left(S_2, \frac{l}{a}\right) &= \mathrm{CF}\left(S_1, \frac{l}{a}\right)\\ &\odot \bigodot_{k=1}^{a} \bigodot_{\substack{x' \in \Gamma: \\ \mathrm{dist}(S_1, x') = \frac{kl}{a}}} \left\{\mathrm{CF}\left(\{x'\}, \frac{l}{a}\right) \odot \frac{l}{a}\right\},\end{aligned}$$

$$\begin{aligned}&\mathrm{CF}\left(\left\{x \in \Gamma \,|\, \mathrm{dist}(S_2, x) \leq \frac{kl_{S_2}}{m}\right\}, \frac{l_{S_2}}{m}\right)\\ = \;&\mathrm{CF}\left(\left\{x \in \Gamma \,|\, \mathrm{dist}(S_2, x) \leq \frac{(k-1)l_{S_2}}{m}\right\}, \frac{l_{S_2}}{m}\right)\\ \odot \;&\bigodot_{\substack{x' \in \Gamma: \\ \mathrm{dist}(S_2, x') = \frac{kl_{S_2}}{m}}} \left\{\mathrm{CF}\left(\{x'\}, \frac{l_{S_2}}{m}\right) \odot \frac{l_{S_2}}{m}\right\},\end{aligned}$$

$$\begin{aligned}\mathrm{CF}(S_2, l_{S_2}) &= \left\{\mathrm{CF}\left(S_2, \frac{l}{a}\right) \oplus \left(-\frac{l_{S_2}}{m}\right)\right\}\\ &\odot \bigodot_{k=1}^{m-1} \mathrm{CF}\left(\left\{x \in \Gamma \,|\, \mathrm{dist}(S_2, x) \leq \frac{kl_{S_2}}{m}\right\}, \frac{l_{S_2}}{m}\right),\end{aligned}$$

and

$$\mathrm{CF}(S_1, l + l') = \mathrm{CF}(S_1, l) \odot \mathrm{CF}(S_2, l').$$

Let Γ be a metric graph. Let R be as in Section 1. Let $(G_\circ, l_\circ)$ be the canonical model for Γ.

Lemma 3.2. *Let e be an edge of $G_\circ$. Let x be in e°. Then,* $\mathrm{CF}(\{x\}, l_x) \in R$.

Proof. If x is the midpoint of e, then $\mathrm{CF}(\{x\}, l_x) = f_e \in R$. Suppose that x is not the midpoint of e. Assume that $0 < l_x \leq \frac{l_\circ(e)}{4}$ and $g_e(x) = -\frac{l_\circ(x)}{4}$. Then

$$\begin{aligned}\mathrm{CF}(\{x\}, l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)}\\ &\odot \left(-\frac{l_\circ(e)}{4}\right) \odot g_e^{\odot(-1)} \oplus (-l_x) \in R.\end{aligned}$$

Similarly, if $0 < l_x \le \frac{l_\circ(e)}{4}$ and $h_e(x) = -\frac{l_\circ(e)}{4}$, then

$$\begin{aligned}\mathrm{CF}(\{x\}, l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)} \\ &\quad \odot \left(-\frac{l_\circ(e)}{4}\right) \odot h_e^{\odot(-1)} \oplus (-l_x) \in R.\end{aligned}$$

When $\frac{l_\circ(e)}{4} < l_x \le \frac{l_\circ(e)}{3}$ and $g_e(x) = -\frac{l_\circ(e)}{4}$, we have

$$\begin{aligned}\mathrm{CF}(\{x\}, l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)} \\ &\quad \odot \left\{\left(-\frac{l_\circ(e)}{4}\right) \odot g_e^{\odot(-1)}\right\}^{\odot 2} \oplus (-l_x) \in R.\end{aligned}$$

Similarly, if $\frac{l_\circ(e)}{4} < l_x \le \frac{l_\circ(e)}{3}$ and $h_e(x) = -\frac{l_\circ(e)}{4}$, then

$$\begin{aligned}\mathrm{CF}(\{x\}, l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)} \\ &\quad \odot \left\{\left(-\frac{l_\circ(e)}{4}\right) \odot h_e^{\odot(-1)}\right\}^{\odot 2} \oplus (-l_x) \in R.\end{aligned}$$

When $\frac{l_\circ(e)}{3} < l_x < \frac{l_\circ(e)}{2}$ and $g_e(x) = -\frac{l_\circ(e)}{4}$, we have

$$\begin{aligned}\mathrm{CF}(\{x\}, l_\circ(e) - 2l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)} \\ &\quad \odot \left\{\left(-\frac{l_\circ(e)}{4}\right) \odot g_e^{\odot(-1)}\right\}^{\odot 2} \oplus (2l_x - l_\circ(e)) \in R.\end{aligned}$$

Similarly, if $\frac{l_\circ(e)}{3} < l_x < \frac{l_\circ(e)}{2}$ and $h_e(x) = -\frac{l_\circ(e)}{4}$, then

$$\begin{aligned}\mathrm{CF}(\{x\}, l_\circ(e) - 2l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)} \\ &\quad \odot \left\{\left(-\frac{l_\circ(e)}{4}\right) \odot h_e^{\odot(-1)}\right\}^{\odot 2} \oplus (2l_x - l_\circ(e)) \in R.\end{aligned}$$

Let x be in the fifth case. Since

$$\begin{aligned}&\mathrm{CF}\left(\left\{x_1 \in \varGamma \mid \mathrm{dist}(x, x_1) \le \frac{l_\circ(e)}{2} - l_x\right\}, \frac{l_\circ(e)}{2} - l_x\right) \\ = \ &\mathrm{CF}\left(\{x\}, \frac{l_\circ(e)}{2} - l_x\right) \odot \bigodot_{\substack{x_1 \in e: \\ \mathrm{dist}(x,x_1) = \frac{l_\circ(e)}{2} - l_x}} \left\{\mathrm{CF}\left(\{x_1\}, \frac{l_\circ(e)}{2} - l_x\right) \odot \left(\frac{l_\circ(e)}{2} - l_x\right)\right\} \in R,\end{aligned}$$

with inputs $l = \frac{l_\circ(e)}{2} - l_x$, $S_1 = \{x_1 \in \varGamma \mid \mathrm{dist}(x, x_1) \le l_\circ(e)/2 - l_x\}$ in Remark 3.1, we have $\mathrm{CF}(\{x\}, l_x) \in R$.

When x is in the sixth case, by the same argument, we have $\mathrm{CF}(\{x\}, l_x) \in R$. □

Note that l_x coincides with $\min(\mathrm{dist}(x, v), \mathrm{dist}(x, w))$ in the setting of Lemma 3.2.

By Remark 3.1 and Lemma 3.2, we prove the following three lemmas. Let d be the diameter of Γ, i.e., $d = \sup\{\mathrm{dist}(x, y) \mid x, y \in \Gamma\} = \max\{\mathrm{dist}(x, y) \mid x, y \in \Gamma\}$.

Lemma 3.3. *For any $x \in \Gamma$ and any positive real number l, the chip firing move* $\mathrm{CF}(\{x\}, l)$ *is in R.*

Proof. For any $x \in \Gamma$ and $l > 0$, by the definition of chip firing moves, we have $\mathrm{CF}(\{x\}, l) = \mathrm{CF}(\{x\}, d) \oplus (-l)$. Hence it is sufficient to check that $\mathrm{CF}(\{x\}, d) \in R$. If $x \in V(G_\circ)$, then $\mathrm{CF}(\{x\}, d) \in R$.

Suppose that there exists an edge $e \in E(G_\circ)$ such that $x \in e^\circ$. Considering Remark 3.1 with $l = l_x, S_1 = \{x\}$, by Lemma 3.2, we have

$$\mathrm{CF}(S_2, l_{S_2}) \in R$$

and

$$\mathrm{CF}(S_1, l + l_{S_2}) = \mathrm{CF}(S_1, l) \odot \mathrm{CF}(S_2, l_{S_2}) \in R.$$

Since S_2 contains a lot of whole edges of $G_\circ$ more than S_1 and the set of edges of $G_\circ$ is finite, by repeating inputs of $l = l_{S_2}$, $S_1 = S_2$ in Remark 3.1, we have $\mathrm{CF}(\{x\}, d) \in R$. $\square$

Lemma 3.4. *For any proper connected subgraph Γ_1 and any positive real number l, the chip firing move* $\mathrm{CF}(\Gamma_1, l)$ *is in R.*

Proof. By Lemma 3.3, if Γ_1 consists of only one point, then we have the conclusion. Assume that Γ_1 does not consist of only one point.

Suppose that Γ_1 contains no whole edges of $G_\circ$ and that there exists an edge $e \in E(G_\circ)$ containing Γ_1. Let x_1 and x_2 be the endpoints of Γ_1. Let x be the midpoint of Γ_1. By Lemma 3.3, for any positive real number l, we have

$$\mathrm{CF}(\Gamma_1, l) = \left[\{\mathrm{CF}(\{x\}, l + \mathrm{dist}(x_1, x)) \odot \mathrm{dist}(x_1, x)\}^{\odot(-1)} \oplus 0\right]^{\odot(-1)} \in R.$$

Note that $\mathrm{CF}(\Gamma_1, l)$ is also obtained as follows with a sufficiently large $b \in \boldsymbol{Z}_{>0}$:

$$\mathrm{CF}\left(\Gamma_1, \frac{l_x}{b}\right) = \mathrm{CF}\left(\{x\}, \frac{l_x}{b}\right) \odot \bigodot_{k=1}^{b} \bigodot_{\substack{x' \in \Gamma: \\ \mathrm{dist}(x, x') = \frac{k l_x}{b}}} \left\{\mathrm{CF}\left(\{x'\}, \frac{l_x}{b} \odot \frac{l_x}{b}\right)\right\},$$

and inputs $l = l_x/b$ and $S_1 = \Gamma_1$, and repeating this process with inputs $l = l_{S_2}$, $S_1 = S_2$ in Remark 3.1.

Suppose Γ_1 contains p edges. Let $\partial\Gamma_1 \cup (V(G_\circ) \cap \Gamma_1) = \{x_1, \ldots, x_q\}$. We may assume that $x_1, \ldots, x_q$ are distinct. Let $\Gamma_{11}, \ldots, \Gamma_{1s}$ be connected components of $\Gamma_1 \setminus \{x_1, \ldots, x_q\}$. For a sufficiently small positive real number ε, let Γ'_{1i} be the connected subgraph $\{x \in \Gamma_{1i} \,|\, \text{for any } j, \mathrm{dist}(x, x_j) \geq \varepsilon\}$ of Γ. Then, we have

$$\mathrm{CF}(\Gamma_1, \varepsilon) = \left\{ \bigoplus_{k=1}^{q} \mathrm{CF}(\{x_k\}, \varepsilon) \right\} \odot \bigodot_{k=1}^{s} (\varepsilon \odot \mathrm{CF}(\Gamma'_{1k}, \varepsilon)) .$$

The last divisor is in the first case, and thus it is in R. By inputting $l = \varepsilon$, $S_1 = \Gamma_1$ and by repeating inputs $l = l_{S_2}$, $S_1 = S_2$ in Remark 3.1, we have $\mathrm{CF}(\Gamma_1, d) \in R$. From this, for any $l > 0$, we have $\mathrm{CF}(\Gamma_1, l) = \mathrm{CF}(\Gamma_1, d) \oplus (-l) \in R$. □

Lemma 3.5. *For any proper subgraph Γ_1 and any positive real number l, the chip firing move $\mathrm{CF}(\Gamma_1, l)$ is in R.*

Proof. Let Γ_1 be a proper subgraph of Γ. Let s be the number of connected components of Γ_1. If $s = 1$, then the conclusion follows Lemma 3.4. Assume $s \geq 2$. Let $\Gamma'_1, \ldots, \Gamma'_s$ be all the distinct connected components of Γ_1. For $l' > 0$, let $\Gamma'_k(l') := \{x \in \Gamma \mid \mathrm{dist}(\Gamma'_k, x) \leq l'\}$. If l' is sufficiently small, then the intersection of $\Gamma'_1(l'), \ldots, \Gamma'_s(l')$ is empty. Let l'_1 be the minimum value of l' such that this intersection is nonempty. By induction on s, $\mathrm{CF}\left(\bigcup_{k=1}^{s} \Gamma'_k(l'_1), d\right) \in R$. On the other hand,

$$\mathrm{CF}(\Gamma_1, l'_1) = \bigoplus_{k=1}^{s} \mathrm{CF}(\Gamma'_k, l'_1) \in R.$$

Hence

$$\mathrm{CF}(\Gamma_1, d) = \mathrm{CF}(\Gamma_1, l'_1) \odot \mathrm{CF}\left(\bigcup_{k=1}^{s} \Gamma'_k(l'_1), d\right) \in R.$$

In conclusion, for any $l > 0$, we have

$$\mathrm{CF}(\Gamma_1, l) = \mathrm{CF}(\Gamma_1, d) \oplus (-l) \in R.$$ □

From Lemmas 1.2, 3.3, 3.4, 3.5, we have the following proposition.

Proposition 3.6. *Let Γ be a metric graph. Then, $\mathrm{Rat}(\Gamma)$ coincides with R. In particular, it is finitely generated as a semifield ovar $\boldsymbol{T}$.*

Let us show Lemma 1.4:

Proof of Lemma 1.4. There exists a natural inclusion $\iota : \Gamma' \hookrightarrow \Gamma$ (cf. [1]). With this inclusion ι, we have a natural inclusion $\kappa : \mathrm{Rat}(\Gamma') \hookrightarrow \mathrm{Rat}(\Gamma)$, i.e., for any $f' \in \mathrm{Rat}(\Gamma')$ and $x' \in \Gamma'$, $\kappa(f')(\iota(x')) = f'(x')$ and $\kappa(f')$ is extended to be constant on each connected component of $\Gamma \setminus \iota(\Gamma')$. Let $\{f'_1, \ldots, f'_n\}$ be a finite generating set of $\mathrm{Rat}(\Gamma')$. Let $L_1, \ldots, L_m$ be all the connected components of $\Gamma \setminus \iota(\Gamma)$. Then $\{\kappa(f'_1), \ldots, \kappa(f'_n), \mathrm{CF}(\overline{\Gamma \setminus L_1}, \infty), \ldots, \mathrm{CF}(\overline{\Gamma \setminus L_m}, \infty)\}$ is a finite generating set of $\mathrm{Rat}(\Gamma)$. In fact, for any $f \in \mathrm{Rat}(\Gamma) \setminus \{-\infty\}$, since f is a piecewice affine function with a finite number of pieces, it breaks each L_i into a finite number of pieces $L_{i1}, \ldots, L_{is_i}$ on each which it has a constant slope. We may assume that $\overline{L_{ij}} \cap \{\iota(\Gamma') \cup \bigcup_{k=1}^{j-1} \overline{L_{ik}}\} \neq \varnothing$. Let x_{i1} be the unique point of $\overline{L_{i1}} \cap \iota(\Gamma')$. For any $j = 2, \ldots, s_i$, let x_{ij} be the unique point of $\overline{L_{ij}} \cap \overline{L_{i,j-1}}$. Let x_{i,s_i+1} be the point at infinity of $\overline{L_{is_i}}$. Let a_{ij} be the slope of f on L_{ij} in the direction from x_{ij} to $x_{i,j+1}$. Since the restriction $f|_{\iota(\Gamma')}$ can be regarded as a rational function on Γ', it is written as $g(\kappa(f_1'), \ldots, \kappa(f_n')) \odot h(\kappa(f_1'), \ldots, \kappa(f_n'))^{\odot(-1)}$ with polynomials $g, h \in \boldsymbol{T}[X_1, \ldots, X_n]$. Let b_{ij} be the value $\mathrm{CF}(\overline{\Gamma \setminus L_i}, \infty)(x_{ij})$. Then, we have

$$\begin{aligned} f \quad = \quad & g(\kappa(f_1'), \ldots, \kappa(f_n')) \odot h(\kappa(f_1'), \ldots, \kappa(f_n'))^{\odot(-1)} \\ & \odot \bigodot_{i=1}^{m} \bigodot_{j=1}^{s_i} \left[(-b_{ij}) \odot \left\{ \left(\mathrm{CF}(\overline{\Gamma \setminus L_i}, \infty) \oplus b_{i,j+1} \right)^{\odot(-1)} \oplus (-b_{ij}) \right\}^{\odot(-1)} \right]^{\odot(-a_{ij})}, \end{aligned}$$

which completes the proof. Here, when $a_{ij} = 0$, then the last divisor means the zero function $0 \in \mathrm{Rat}(\Gamma)$. □

In conclusion, we have Theorem 1.1.

Remark 3.7. By the proof of Theorem 1.1, we have the following: for a tropical curve Γ, all chip firing moves defined by one finite point, and of the form $\mathrm{CF}(\Gamma \setminus (y, x], \infty)$ with $x \in \Gamma_\infty$ and a finite point y on the unique edge incident to x generate $\mathrm{Rat}(\Gamma)$ as a semifield over $\boldsymbol{T}$. This assertion is used in the proof of [3, Corollary 3.9].

Since the pull-back of the rational function semifield of a tropical curve by a finite harmonic morphism contains $\boldsymbol{T}$, the following corollary follows from Theorem 1.1:

Corollary 3.8. *Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves. Then,* $\mathrm{Rat}(\Gamma)$ *is finitely generated as a semifield over* $\varphi^*(\mathrm{Rat}(\Gamma'))$.

By the proof of Lemma 1.4, we have the following corollary:

Corollary 3.9. *Let Γ be a tropical curve. Let $(G_\circ, l_\circ)$ be the canonical model for Γ and E_∞ the subset of $E(G_\circ)$ cosisting of all edges of length ∞. Then, there exists a generating set of* $\mathrm{Rat}(\Gamma)$ *consisting of at most $\#V(G_\circ) + 2(\#E(G_\circ) - \#E_\infty)$ elements.*

Proof. $V(G_\circ)$ contains $\#V(G_\circ) - \#E_\infty$ vertices which are finite points. Thus R for the metric graph obtained from Γ by contracting all edges in E_∞ is generated by $\#V(G_\circ) - \#E_\infty + 2(\#E(G_\circ) - \#E_\infty)$ elements. From the proof of Lemma 1.4, $\mathrm{Rat}(\Gamma)$ is generated by $\#V(G_\circ) - \#E_\infty + 2(\#E(G_\circ) - \#E_\infty) + \#E_\infty = \#V(G_\circ) + 2(\#E(G_\circ) - \#E_\infty)$ elements. □

Second, we consider rational function semifields of trees.

Lemma 3.10. *Let T be a tree. Let $(G_\circ, l_\circ)$ be the canonical model for T. Let $V_1 \subset V(G_\circ)$ denote the subset of all leaf ends. If $\#V_1$ is* *even, then there exists a pairing of vertices in V_1 such that the union of unique paths connecting paired vertices covers T.*

Proof. Since $\#V_1$ is even, there exists a pairing of vertices in V_1. If it is not desired, then there exists an edge e of $G_\circ$ which is not contained in the union of unique paths connecting paired vertices. Since T is a tree, there exist two vertices v, w such that the unique path connecting them contains e. By pairing again v, w and the two other vertices v', w' originally paired with v, w respectively, the number of covered edges increases. In fact, the union of the path from v to w and the path from v' to w' contains e and both the path from v to v' and the path from w to w'. Hence, by repeating this process, we have the conclusion. $\square$

Proposition 3.11. *Let T be a tree. Let $(G_\circ, l_\circ)$ be the canonical model for T and $V_1 \subset V(G_\circ)$ the subset of all leaf ends. Then there exists a generating set of $\mathrm{Rat}(T)$ consisting of at most $\frac{[\#V_1+1]}{2}$ elements, where $[x] = \max\{n \in \boldsymbol{Z} \mid n \leq x\}$.*

Proof. By Lemma 3.10, there exists a pairing of vertices in V_1 except at most one vertex v_0 such that the union of unique paths connecting paired vertices covers T except at most one edge e_0 incident v_0. Let v, w be any paired vertices and P the unique path connecting them. Let f be a rational function on T which has slope one on P in the direction from v to w and constant on other points. Let g be a rational function on T which has slope one on e_0 and constant on other points. Then such f and g generates $\mathrm{Rat}(T)$ as a semifield over $\boldsymbol{T}$. In fact, for a tree T' which is a metric graph obtained from T by contracting edges of length ∞, the restrictions of such f and g on T' generate $f_{e'}, g_{e'}, h_{e'}$ and $\mathrm{CF}(\{v\}, \infty)$ for each edge e' and each vertex v of the underlying graph of the canonical model for T' and chip firing moves of the form of $\mathrm{CF}(\overline{\Gamma \setminus L_i}, \infty)$ in the proof of Lemma 1.4. Hence we have the conclusion. $\square$

Third, we show that except the singleton case, rational function semifields of tropical curves are not finitely generated as a $\boldsymbol{T}$-*algebra* by the following two lemmas.

The following lemma holds by the definitions of two operators $\odot$, $\oplus$.

Lemma 3.12. *Let Γ be a tropical curve. For any rational functions $f, g \in \mathrm{Rat}(\Gamma) \setminus \{-\infty\}$, $f \odot g$ and $f \oplus g$ may have as these poles only points that are poles of f or g.*

Lemma 3.13. *Let Γ be a metric graph. Then,* $\mathrm{Rat}(\Gamma)$ *is finitely generated as a $\boldsymbol{T}$-algebra if and only if Γ is a singleton.*

Proof. The if part is clear. We shall show the only if part. If $\mathrm{Rat}(\Gamma)$ is finitely generated as a $\boldsymbol{T}$-algebra, then by Lemma 3.12, Γ must be a singleton. $\square$

Proposition 3.14. *Let $\varGamma$ be a tropical curve. Then,* $\mathrm{Rat}(\varGamma)$ *is finitely generated as a* $\boldsymbol{T}$*-algebra if and only if* $\varGamma$ *is a singleton.*

Proof. By Lemma 3.13, it is enough to show that with *any* metric graph $\varGamma'$ obtained from $\varGamma$ by contracting edges of length ∞, if $\mathrm{Rat}(\varGamma)$ is finitely generated as a $\boldsymbol{T}$-algebra, then so is $\mathrm{Rat}(\varGamma')$. Assume that $\mathrm{Rat}(\varGamma)$ is finitely generated as a $\boldsymbol{T}$-algebra. Let $\{f_1, \ldots, f_n\}$ be a finite generating set of $\mathrm{Rat}(\varGamma)$. Then the set of restrictions $\{f_1|_{\varGamma'}, \ldots, f_n|_{\varGamma'}\}$ is a finite generating set of $\mathrm{Rat}(\varGamma')$ with the natural inclusion $\varGamma' \hookrightarrow \varGamma$. In fact, the restriction map $\mathrm{Rat}(\varGamma) \to \mathrm{Rat}(\varGamma')$ is surjective since the contraction $\varGamma \twoheadrightarrow \varGamma'$ contracts only trees. Hence, we have the assertion. $\square$

Finally, we shall show Proposition 1.5:

Proof of Proposition 1.5. Fix loopless models (G, l), (G', l') for $\varGamma$, $\varGamma'$, respectively, such that $\varphi(V(G)) = V(G')$. For any edge e of G, if e is not incident to a point at infinity, then let $F_e := \mathrm{CF}(\varGamma \setminus e^\circ, l(e)/2)$; otherwise, let $F_e := \mathrm{CF}(\overline{\varGamma \setminus e}, \infty)$.

Assume that $l(e) < \infty$. Let v be one of the vertices incident to e. Let $G_{v,e}$ be a rational function on $\varGamma$ which has slope one from v to the midpoint of e; has a sufficiently large positive integer to be its slope from v to a point of each edge incident to v other than e; is the constant zero function on other points; has v as its unique point where attains the minimum value $-\frac{l(e)}{2}$. Let $x \in e^\circ$. Assume that $\mathrm{dist}(x, v) = l_x \leq \frac{l(e)}{2}$. If $\mathrm{dist}(x, v) = \frac{l(e)}{2}$, then we have

$$\begin{aligned} & \mathrm{CF}(\{x\}, l_x) \odot \{-(\deg_e(\varphi) - 1)l_x\} \\ = \; & \varphi^*(\mathrm{CF}(\{\varphi(x)\}, l_{\varphi(x)})) \odot F_e^{\odot(\deg_e(\varphi)-1)} \odot \bigodot_{\substack{e_1 \in E(G)\setminus\{e\}: \\ e_1 \subset \varphi^{-1}(\varphi(e))}} F_{e_1}^{\odot \deg_{e_1}(\varphi)}. \end{aligned}$$

Suppose $\mathrm{dist}(x, v) \leq \frac{l(e)}{4}$. Let f' be a rational function on $\varGamma'$ which coincides with $\mathrm{CF}(\{\varphi(x)\}, l_{\varphi(x)})$ on $U' := \{x' \in \varGamma' \mid \mathrm{dist}(\varphi(x), x') \leq l_{\varphi(x)}\}$; has a sufficiently small negative slope s from U' on the ε-neighborhood of U' with a sufficiently small positive real number ε enough to be the restriction of $G_{v,e}$ on the inverse image of the ε-neighborhood of U' does not take zero; is the constant $-l_{\varphi(x)} + s\varepsilon$ on other points. Then, there exists a positive integer b such that

$$\begin{aligned}
&\mathrm{CF}(\{x\}, l_x) \odot \{-(\deg_e(\varphi)-1)l_x\} \\
= \ &\varphi^*(f') \odot \left\{F_e \oplus G_{v,e} \odot \left(\frac{l(e)}{2} - 2l_x\right)\right\}^{\odot(\deg_e(\varphi)-1)} \\
&\odot \bigodot_{\substack{e_1 \in E(G)\setminus\{e\}: \\ e_1 \subset \varphi^{-1}(\varphi(e)), v \in e_1}} F_{e_1}^{\odot b} \odot \bigodot_{\substack{e_2 \in E(G)\setminus\{e\}, v_2 \in V(G)\setminus\{v\}: \\ e_2 \subset \varphi^{-1}(\varphi(e)), v_2 \in \varphi^{-1}(\varphi(v)), \\ v \notin e_2, v_2 \in e_2}} G_{v_2,e_2}^{\odot b} \\
&\oplus (-\deg_e(\varphi) l_x).
\end{aligned}$$

Suppose $\frac{l(e)}{4} < \mathrm{dist}(x, v) < \frac{l(e)}{2}$. Then, we have

$$\begin{aligned}
&\mathrm{CF}\left(\{x\}, \frac{l(e)}{2} - l_x\right) \odot \{-(\deg_e(\varphi)-1)l_x\} \\
= \ &\varphi^*(\mathrm{CF}(\{\varphi(x)\}, l_{\varphi(x)}) \odot \left\{F_e \oplus G_{v,e} \odot \left(\frac{l(e)}{2} - 2l_x\right)\right\}^{\odot(\deg_e(\varphi)-1)} \\
&\odot \bigodot_{\substack{e_1 \in E(G)\setminus\{e\}: \\ e_1 \subset \varphi^{-1}(\varphi(e))}} F_{e_1}^{\odot \deg_{e_1}(\varphi)} \oplus \left\{-(\deg_e(\varphi)-1)l_x - \left(\frac{l(e)}{2} - l_x\right)\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
&\mathrm{CF}\left(\left\{x_1 \in \Gamma \mid \mathrm{dist}(x, x_1) \leq \frac{l(e)}{4} - \frac{l_x}{2}\right\}, \frac{l(e)}{4} - \frac{l_x}{2}\right) \\
= \ &\mathrm{CF}\left(\{x\}, \frac{l(e)}{4} - \frac{l_x}{2}\right) \odot \bigodot_{\substack{x_1 \in e: \\ \mathrm{dist}(x, x_1) = \frac{l(e)}{4} - \frac{l_x}{2}}} \left\{\mathrm{CF}\left(\{x_1\}, \frac{l(e)}{4} - \frac{l_x}{2}\right) \odot \left(\frac{l(e)}{4} - \frac{l_x}{2}\right)\right\},
\end{aligned}$$

$$\begin{aligned}
&\mathrm{CF}\left(\left\{x_1 \in \Gamma \mid \mathrm{dist}(x, x_1) \leq \frac{l(e)}{2} - l_x\right\}, \frac{l(e)}{4} - \frac{l_x}{2}\right) \\
= \ &\mathrm{CF}\left(\left\{x_1 \in \Gamma \mid \mathrm{dist}(x, x_1) \leq \frac{l(e)}{4} - \frac{l_x}{2}\right\}, \frac{l(e)}{4} - \frac{l_x}{2}\right) \\
&\odot \bigodot_{\substack{x_1 \in e: \\ \mathrm{dist}(x, x_1) = \frac{l(e)}{2} - l_x}} \left\{\mathrm{CF}\left(\{x_1\}, \frac{l(e)}{4} - \frac{l_x}{2}\right) \odot \left(\frac{l(e)}{4} - \frac{l_x}{2}\right)\right\},
\end{aligned}$$

and $0 < 3\left(\frac{l(e)}{4} - \frac{l_x}{2}\right) < \frac{l(e)}{2}$, with inputs $l = \frac{l(e)}{4} - \frac{l_x}{2}$, $S_1 = \{x_1 \in \Gamma \mid \mathrm{dist}(x, x_1) \leq l(e)/2 - l_x\}$ in Remark 3.1, we can show that $\mathrm{CF}(\{x\}, l_x)$ is generated by $F_e, G_{v,e}$ as a $\varphi^*(\mathrm{Rat}(\Gamma'))$-algebra.

Assume that $l(e) = \infty$. Identify $e = [0, \infty]$. Let $x, y \in e$ be any distinct points such that $x < y$. Let $g_{[x,y]}$ be the rational function on Γ which has slope one in the direction from x to y, is constant on any other points, and whose minimum value is zero. Identify $\varphi(e) = [0, \infty]$.

Let $g'_{[\varphi(x),\varphi(y)]}$ be the rational function on Γ' which has slope one in the direction from $\varphi(x)$ to $\varphi(y)$, is constant on any other points, and whose minimum value is zero. Let v be the endpoint 0 of e. Let l_x (resp. l_y) be the length of $[v, x]$ (resp. $[v, y]$). Then, we have

$$g_{[x,y]} = g_{[v,y]} \odot (-l_x) \oplus 0$$

and

$$g_{[v,y]} = \varphi^* \left(g'_{[\varphi(v),\varphi(y)]} \right) \odot \mathrm{CF}(\{v\}, l_y)^{\odot(\deg_e(\varphi)-1)} \odot \bigodot_{\substack{e_1 \in E(G)\setminus\{e\}: \\ e_1 \subset \varphi^{-1}(\varphi(e))}} F_{e_1}^{\deg_{e_1}(\varphi)} \oplus 0.$$

Let w be the endpoint ∞ of e. Then, we have

$$g_{[x,y]}^{\odot(-1)} = g_{[x,w]}^{\odot(-1)} \oplus (-\operatorname{dist}(x, y))$$

and

$$g_{[x,w]}^{\odot(-1)} = F_e \odot g_{[v,x]}.$$

Let Γ_1 be the metric graph obtained from Γ by contracting all edges of G of length ∞ to the finite endpoints. We regard that Γ_1 is a subgraph of Γ. By the proof of Theorem 1.1, for any rational function $f \in \mathrm{Rat}(\Gamma)$, the restriction $f|_{\Gamma_1}$ is generated by (the restrictions on Γ_1 of elements of) $\{\mathrm{CF}(\{x\}, l_x), \mathrm{CF}(\{v\}, \infty) \,|\, x \in \Gamma_1 \setminus V(G), v \in V(G) \setminus \Gamma_\infty\}$ as a $\boldsymbol{T}$-algebra. Let g be the generated rational function such that $f|_{\Gamma_1} = g|_{\Gamma_1}$. By (tropical) multiplying rational functions of the forms of $g_{[x,y]}^{\odot(\pm 1)}$ above to g, f is made of g. Hence $\{F_e, G_{v_1,e_1}, \mathrm{CF}(\{v\}, \infty) \,|\, e \in E(G), v, v_1 \in V(G) \setminus \Gamma_\infty, e_1 \in E(G) \setminus E_\infty, v_1 \in e_1\}$ generates $\mathrm{Rat}(\Gamma)$ as a $\varphi^*(\mathrm{Rat}(\Gamma'))$-algebra, where E_∞ denotes the subset of $E(G)$ consisting of all edges of length ∞. $\square$

By Example 3.15, we know that $\mathrm{Rat}(\Gamma)$ may not be finitely generated as a $\varphi^*(\mathrm{Rat}(\Gamma'))$-module.

Example 3.15. Let $\Gamma := [0, 2]$ and $\Gamma' := [0, 1]$. The map $\varphi : \Gamma \to \Gamma'; x \mapsto x$ when $0 \leq x \leq 1; x \mapsto 2 - x$ when $1 < x \leq 2$ is a finite harmonic morphism of degree two. Assume that $\mathrm{Rat}(\Gamma)$ is finitely generated as a $\varphi^*(\mathrm{Rat}(\Gamma'))$-module, i.e., there exist $f_1, \ldots, f_n \in \mathrm{Rat}(\Gamma) \setminus \{-\infty\}$ such that $\mathrm{Rat}(\Gamma) = \bigoplus_{i=1}^n \varphi^*(\mathrm{Rat}(\Gamma')) \odot f_i$. Let x' be a point of $[0, 1) \subset \Gamma'$. Then, for any pair of values $a, b \in \boldsymbol{R}_{>0}$, $\mathrm{Rat}(\Gamma)$ contains a rational function which has a, b as values at each element x_1, x_2 of $\varphi^{-1}(x')$. For example, consider $(\mathrm{CF}(\{x_1\}, \varepsilon)^{\odot p} \odot a \oplus 0) \odot (\mathrm{CF}(\{x_2\}, \varepsilon)^{\odot q} \odot b \oplus 0) \in \mathrm{Rat}(\Gamma)$ with a small positive number $\varepsilon > 0$ and some $p, q \in \boldsymbol{Z}_{>0}$ such that $p\varepsilon > a$ and $q\varepsilon > b$. On the other hand, for any i and $f' \in \mathrm{Rat}(\Gamma')$ such that $(\varphi^*(f') \odot f_i)(x_1) = a$, since $a - f_i(x_1) = \varphi^*(f')(x_1) = \varphi^*(f')(x_2)$ hold, $(\varphi^*(f') \odot f_i)(x_2) = a - f_i(x_1) + f_i(x_2)$ holds. Hence, if for any j, $b \neq a - f_j(x_1) + f_j(x_2)$ holds, then $\bigoplus_{i=1}^n \varphi^*(\mathrm{Rat}(\Gamma')) \odot f_i$ contains no rational functions which take a, b at x_1, x_2 respectively. It is a contradiction. Therefore, $\mathrm{Rat}(\Gamma)$ is not finitely generated as a $\varphi^*(\mathrm{Rat}(\Gamma'))$-module.

References

[1] Omid Amini, Matthew Baker, Erwan Brugallé and Joseph Rabinoff, *Lifting harmonic morphisms II: tropical curves and metrized complexes*, Algebra Number Theory **9**(2)(2015), 267–315.

[2] Song JuAe, *Generators of linear systems on tropical curves*, Hokkaido Mathematical journal **50**(1):55–76, 2021.

[3] Song JuAe, *Semiring isomorphisms between rational function semifields of tropical curves induce isomorphisms between tropical curves*, arXiv:2110.08091.

[4] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, Vol. 161. American Mathematical Soc., Providence, RI, 2015.

Tokyo Metropolitan University 1-1 Minami-Ohsawa, Hachioji, Tokyo, 192-0397, Japan.

Email address: `song-juae@ed.tmu.ac.jp`