# ABSTRACT KEY POLYNOMIALS AND MACLANE-VAQUIÉ CHAINS

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ABSTRACT. In this paper, for a valued field (K,v) of arbitrary rank and an extension w of v to K(X), a relation between induced complete sequences of abstract key polynomials and MacLane-Vaquié chains is given.

## 1. Introduction

Let (K, v) be a valued field. Starting with a valuation  $w_0$  of K(X), extending v, which admit key polynomials of degree one, Nart [6] introduced the notion of MacLane-Vaquié chains

$$w_0 \xrightarrow{\phi_1,\gamma_1} w_1 \xrightarrow{\phi_2,\gamma_2} \cdots \longrightarrow w_{n-1} \xrightarrow{\phi_n,\gamma_n} w_n \longrightarrow \cdots$$

consisting of a mixture of ordinary and limit augmentations satisfying some conditions (see Definition 2.18). The main result (Theorem 4.3 of [6]) says that all extensions w of v to K(X) fall exactly in one of the following categories:

- (i) It is the last valuation of a complete finite MacLane-Vaquié chain, i.e., after a finite number r of augmentation steps,  $w_r = w$ .
- (ii) After a finite number r of augmentation steps, it is the stable limit of a continuous family of augmentations of  $w_r$  defined by key polynomials of constant degree.
- (iii) It is the stable limit of a complete infinite MacLane-Vaquié chain.

It is known that [7, Theorem 1.1], every valuation w on K(X) admits a complete sequence of abstract key polynomials. Moreover, in [3] Mahboub et al. described a complete sequence of abstract key polynomials for w satisfying certain properties (see Remark 2.5) and we call this sequence an *induced complete sequence of abstract key polynomials* for w. In this paper, we prove that the concepts of MacLane-Vaquié chains and induced complete sequences of abstract key polynomials for w, are intimately connected.

To state the main result of the paper, we first recall some notation, definitions, and preliminary results.

## 2. Notation, Definitions, and Statements of Main Results

Throughout the paper, (K, v) denote a valued field of arbitrary rank with value group  $\Gamma_v$  and residue field  $k_v$ , and by  $\bar{v}$  we denote an extension of v to a fixed algebraic closure  $\overline{K}$  of K.

<sup>2020</sup> Mathematics Subject Classification. 12F20, 12J10, 13A18.

Key words and phrases. Abstract key polynomials, key polynomials, MacLane-Vaquié chains.

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An extension w of v to the simple transcendental extension K(X) of K such that  $k_w$  is algebraic over  $k_v$  is said to be valuation-algebraic if the quotient  $\Gamma_w/\Gamma_v$  is a torsion group and is said to be value-transcendental if  $\Gamma_w/\Gamma_v$  is a torsion-free group. The extension w of v to K(X) is called residually transcendental if the corresponding residue field extension  $k_w|k_v$  is transcendental. We call w valuation-transcendental if w is either value-transcendental or is residually transcendental.

An extension  $\overline{w}$  of w to  $\overline{K}(X)$  which is also an extension of  $\overline{v}$  is called a *common extension* of w and  $\overline{v}$ .

# 2.1. Abstract key polynomials.

**Definition 2.1.** Let w be a valuation of K(X) and  $\overline{w}$  a fixed common extension of w and  $\overline{v}$  to  $\overline{K}(X)$ . For any polynomial f in K[X], we define

$$\delta(f) \coloneqq \max\{\overline{w}(X - \alpha) \mid f(\alpha) = 0\}.$$

This value  $\delta(f)$  does not depend upon the choice of  $\overline{w}$  (see [8, Proposition 3.1]).

**Definition 2.2** (Abstract key polynomials). A monic polynomial Q in K[X] is said to be an abstract key polynomial (abbreviated as ABKP) for w if for each polynomial f in K[X] with deg  $f < \deg Q$  we have  $\delta(f) < \delta(Q)$ .

It is immediate from the definition that all monic linear polynomials are ABKPs for w. Also an ABKP for w is an irreducible polynomial (see [7, Proposition 2.4]).

**Definition 2.3.** For a polynomial Q in K[X] the Q-truncation of w is a map  $w_Q: K[X] \longrightarrow \Gamma_w$  defined by

$$w_Q(f) := \min_{i>0} \{w(f_i Q^i)\},$$

where  $f = \sum_{i \geq o} f_i Q^i$ , deg  $f_i < \deg Q$ , is the Q-expansion of f.

The Q-truncation  $w_Q$  of w need not be a valuation [7, Example 2.5]. However, if Q is an ABKP for w, then  $w_Q$  is a valuation on K(X) (see [7, Proposition 2.6]). Note that any ABKP, Q for w, is also an ABKP for the truncation valuation  $w_Q$ .

**Definition 2.4.** A family  $\Lambda = \{Q_i\}_{i \in \Delta}$  of ABKPs for w is said to be a complete sequence of ABKPs for w if the following conditions are satisfied:

- (i)  $\delta(Q_i) \neq \delta(Q_j)$  for every  $i \neq j \in \Delta$ .
- (ii)  $\Lambda$  is well-ordered with respect to the ordering given by  $Q_i < Q_j$  if and only if  $\delta(Q_i) < \delta(Q_j)$  for every  $i < j \in \Delta$ .
- (iii) For any  $f \in K[X]$ , there exists some  $Q_i \in \Lambda$  such that  $\deg Q_i \leq \deg f$  and  $w_{Q_i}(f) = w(f)$ .

It is known that [7, Theorem 1.1], every valuation w on K(X) admits a complete sequence of ABKPs. Moreover, there is a complete sequence  $\Lambda = \{Q_i\}_{i \in \Delta}$  of ABKPs for w having the following properties (see [3, Remark 4.6] and proof of [7, Theorem 1.1]).

- Remark 2.5. (i)  $\Delta = \bigcup_{j \in I} \Delta_j$  with  $I = \{0, 1, ..., N\}$  or  $\mathbb{N} \cup \{0\}$ , and for each  $j \in I$  we have  $\Delta_j = \{j\} \cup \vartheta_j$ , where  $\vartheta_j$  is an ordered set without a last element or is empty.
  - (ii)  $\deg Q_0 = 1$ .
- (iii) For all  $j \in I \setminus \{0\}$  we have j 1 < i < j, for all  $i \in \vartheta_{j-1}$ .

- (iv) All polynomials  $Q_i$  with  $i \in \Delta_j$  have the same degree and have degree strictly less than the degree of the polynomials  $Q_{i'}$  for every  $i' \in \Delta_{j+1}$ .
- (v) For each  $i < i' \in \Delta$  we have  $w(Q_i) < w(Q_{i'})$  and  $\delta(Q_i) < \delta(Q_{i'})$ .

The complete sequences of ABKPs satisfying the properties of Remark 2.5 will be called **induced complete sequences of ABKPs** for w.

Even though the set  $\{Q_i\}_{i\in\Delta}$  of ABKPs for w is not unique, the cardinality of I and the degree of an abstract key polynomial  $Q_i$  for each  $i\in I$  are uniquely determined by w.

The ordered set  $\Delta$  has a last element if and only if the following holds:

(2.1) 
$$I = \{0, 1, \dots, N\} \text{ is finite, and } \Delta_N = \{N\} \text{ (i.e., } \vartheta_N = \varnothing).$$

2.2. **MacLane-Vaquié chains.** We first recall the notion of key polynomials which was first introduced by MacLane in 1936 and later generalized by Vaquié in 2007 (see [2, 10]).

**Definition 2.6.** For a valuation w on K(X) and polynomials f, g in K[X], we say that

- (i) f and g are w-equivalent and write  $f \sim_w g$  if w(f-g) > w(f) = w(g).
- (ii) f w-divides g (denoted by  $f|_w g$ ) if there exists some polynomial  $h \in K[X]$  such that  $g \sim_w fh$ .
- (iii) f is w-irreducible if for any  $h, q \in K[X]$ , whenever  $f|_w hq$ , then either  $f|_w h$  or  $f|_w q$ .
- (iv) f is w-minimal if for every nonzero polynomial  $h \in K[X]$ , whenever  $f \mid_w h$ , then  $\deg h \ge \deg f$ .

**Definition 2.7** (Key polynomials). A monic polynomial f in K[X] is called a key polynomial for w, if f is w-irreducible and w-minimal.

In view of Proposition 2.10 of [1] any ABKP, Q for w is a key polynomial for  $w_Q$  of minimal degree. Let KP(w) denote the set of all key polynomials for w. Then for any  $\phi \in KP(w)$  we denote by  $[\phi]_w$  the set of all key polynomials which are w-equivalent to  $\phi$ . For any  $\phi$ ,  $\phi' \in KP(w)$ , we have

$$\phi \mid_w \phi'$$
 if and only if  $\phi \sim_w \phi'$ ,

and in this case,  $\deg \phi = \deg \phi'$  (see [5, Proposition 6.6]).

Let w be a valuation on K(X) which admits key polynomials. If  $\phi$  is a key polynomial for w of minimal degree, then we define

$$deg(w) := deg \phi$$
.

For any valuation w' on K(X) taking values in a subgroup of  $\Gamma_w$ , we say that

$$w' \le w$$
 if and only if  $w'(f) \le w(f)$ ,  $\forall f \in K[X]$ .

If w' < w, we denote by  $\Phi(w', w)$  the set of all monic polynomials  $g \in K[X]$  of minimal degree (say) d such that w'(g) < w(g). We denote

$$\deg(\Phi(w',w)) = d.$$

**Definition 2.8** (Ordinary augmentation). Let  $\phi$  be a key polynomial for a valuation w' on K(X) and  $\gamma > w'(\phi)$  be an element of a totally ordered abelian group  $\Gamma$  containing  $\Gamma_{w'}$  as an ordered subgroup. The map  $w: K[X] \longrightarrow \Gamma \cup \{\infty\}$  defined by

$$w(f) \coloneqq \min_{i>0} \{ w'(f_i) + i\gamma \},$$

where  $f = \sum_{i\geq 0} f_i \phi^i$ ,  $\deg f_i < \deg \phi$ , is the  $\phi$ -expansion of  $f \in K[X]$ , gives a valuation on K(X) (see [2, Theorem 4.2]) called the *ordinary augmentation of* w', and is denoted by  $w = [w'; \phi, \gamma]$ .

Note that  $w(\phi) = \gamma$ , i.e.,  $w'(\phi) < w(\phi)$  and the polynomial  $\phi$  is a key polynomial of minimal degree for the augmented valuation w (see [5, Corollary 7.3]).

**Theorem 2.9** (Theorem 1.15, [10]). Let w be a valuation on K(X) and w' < w. Then any  $\phi \in \Phi(w', w)$  is a key polynomial for w' and

$$w' < [w'; \phi, w(\phi)] \le w.$$

For any nonzero polynomial  $f \in K[X]$ , the equality w'(f) = w(f) holds if and only if  $\phi \downarrow_{w'} f$ .

Corollary 2.10 (Corollary 2.5, [6]). Let w' < w be as above. Then

- (i)  $\Phi(w', w) = [\phi]_{w'}$  for all  $\phi \in \Phi(w', w)$ .
- (ii) If  $w' < \nu \le w$  is a chain of valuations, then  $\Phi(w', w) = \Phi(w', \nu)$ . In particular,

$$w'(f) = w(f) \iff w'(f) = \nu(f), \forall f \in K[X].$$

Corollary 2.11. If  $w = [w'; \phi, \gamma]$  is an ordinary augmentation, then  $\Phi(w', w) = [\phi]_{w'}$ .

Consider a totally ordered family of valuations on K(X), taking values in a common ordered group

$$\mathcal{W} = (\rho_i)_{i \in \mathbf{A}},$$

and indexed by a totally ordered set **A**. We shall always assume that the assignment  $i \mapsto \rho_i$  is an isomorphism between totally ordered sets **A** and  $\mathcal{W}$ .

A polynomial f in K[X] is said to be W-stable if

$$\rho_i(f) = \rho_{i_0}(f), \ \forall \ i \ge i_0,$$

for some index  $i_0 \in \mathbf{A}$ . This stable value is denoted by  $\rho_{\mathcal{W}}(f)$ . We obtain in this way a *stability function*  $\rho_{\mathcal{W}}$  defined only on the set of stable polynomials which is a multiplicatively closed subset of K[X].

In view of Corollary 2.10 (ii), a polynomial  $f \in K[X]$  is W-unstable if and only if

$$\rho_i(f) < \rho_i(f), \ \forall \ i < j \in \mathbf{A}.$$

We denote

$$m_{\infty} = \min\{\deg f \mid f \in K[X], f \text{ is } W\text{-unstable}\}.$$

If all polynomials are W-stable, then we set  $m_{\infty} = \infty$ . We say that W has a *stable limit* if all polynomials in K[X] are W-stable. In this case,  $\rho_{W}$  is a valuation on K[X], and is called the *stable limit* of W.

**Definition 2.12.** Let w' be a valuation on K(X) admitting key polynomials. Then a continuous family of augmentations of w' is a family of ordinary augmentations of w'

$$\mathcal{W} = (\rho_i = [w'; \chi_i, \gamma_i])_{i \in \mathbf{A}},$$

indexed by a totally ordered set **A** such that  $\gamma_i < \gamma_j$  for all i < j in **A**, satisfying the following conditions:

- (i) The set **A** has no last element.
- (ii) All key polynomials  $\chi_i \in KP(w')$  have the same degree.
- (iii) For all i < j in  $\mathbf{A}$ ,  $\chi_j$  is a key polynomial for  $\rho_i$ ,  $\chi_j \not +_{\rho_i} \chi_i$  and  $\rho_j = [\rho_i; \chi_j, \gamma_j]$ .

The common degree  $\deg \chi_i$ , for all i, is called the *stable degree* of the family W and is denoted by  $\deg(W)$ .

Remark 2.13. The following properties hold for any continuous family  $W = (\rho_i)_{i \in \mathbf{A}}$  of augmentations (see p. 9, [6]):

- (i) The mapping defined by  $i \mapsto \gamma_i$  and  $i \mapsto \rho_i$  are isomorphisms of ordered sets between **A** and  $\{\gamma_i \mid i \in \mathbf{A}\}, \{\rho_i \mid i \in \mathbf{A}\}, \text{ respectively.}$
- (ii) For all  $i \in \mathbf{A}$ ,  $\chi_i$  is a key polynomial for  $\rho_i$  of minimal degree.
- (iii) For all  $i, j \in \mathbf{A}$ ,  $\rho_i(\chi_j) = \min\{\gamma_i, \gamma_j\}$ . Hence, all the polynomials  $\chi_i$  are stable.
- (iv)  $\Phi(\rho_i, \rho_j) = [\chi_j]_{\rho_i}, \forall i < j \in \mathbf{A}.$
- (v) All valuations  $\rho_i$  are residually transcendental.
- (vi) All the value groups  $\Gamma_{\rho_i}$  coincide and the common value group is denoted by  $\Gamma_{\mathcal{W}}$ .

Remark 2.14. Since a totally ordered set admits a well-ordered cofinal subset, so without loss of generality we can assume that A is well-ordered.

**Definition 2.15** (MacLane-Vaquié limit key polynomials). Let  $\mathcal{W}$  be a continuous family of augmentations of a valuation w'. Then a monic  $\mathcal{W}$ -unstable polynomial of minimal degree is called a MacLane-Vaquié limit key polynomial (abbreviated as MLV) for  $\mathcal{W}$ .

The set of all MLV limit key polynomials is denoted by  $KP_{\infty}(W)$ . Since the product of stable polynomials is stable, so all MLV limit key polynomials are irreducible in K[X].

Any continuous family W of augmentations of w' fall in one of the following three cases:

- (i) It has a stable limit, i.e.,  $\rho_{\mathcal{W}}$  is a valuation on K[X], if  $m_{\infty} = \infty$ .
- (ii) It is in-essential if  $m_{\infty} = \deg(\mathcal{W})$  (stable degree).
- (iii) It is essential if  $\deg(\mathcal{W}) < m_{\infty} < \infty$ .

Let W be an essential continuous family of augmentations of a valuation w'. Then W admit MLV limit key polynomials. If Q is an MLV limit key polynomial, then any polynomial f in K[X] with deg  $f < \deg Q$  is W-stable.

**Definition 2.16** (Limit augmentation). Let Q be any MLV limit key polynomial for an essential continuous family  $\mathcal{W} = (\rho_i)_{i \in \mathbf{A}}$  of augmentations of w' and  $\gamma > \rho_i(Q)$ , for all  $i \in \mathbf{A}$ , be an element of a totally ordered abelian group  $\Gamma$  containing  $\Gamma_{\mathcal{W}}$  as an ordered subgroup. Then the map  $w : K[X] \longrightarrow \Gamma \cup \{\infty\}$  defined by

$$w(f) \coloneqq \min_{i \ge 0} \{ \rho_{\mathcal{W}}(f_i) + i\gamma \},$$

where  $f = \sum_{i\geq 0} f_i Q^i$ , deg  $f_i < \deg Q$ , is the Q-expansion of  $f \in K[X]$ , gives a valuation on K(X) and is called the *limit augmentation* of  $\mathcal{W}$ , denoted by  $w = [\mathcal{W} = (\rho_i)_{i\in \mathbf{A}}; Q, \gamma]$ .

Note that  $w(Q) = \gamma$  and  $\rho_i < w$  for all  $i \in \mathbf{A}$ . Also, Q is a key polynomial for w of minimal degree [5, Corollary 7.13].

We now recall the definition of MacLane-Vaquié chains given by Nart in [6]. For this, we first consider a finite, or countably infinite, chain of mixed augmentations

$$(2.2) w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \longrightarrow w_n \xrightarrow{\phi_{n+1}, \gamma_{n+1}} w_{n+1} \longrightarrow \cdots$$

in which every valuation is an augmentation of the previous one and is of one of the following type:

- Ordinary augmentation:  $w_{n+1} = [w_n; \phi_{n+1}, \gamma_{n+1}]$ , for some  $\phi_{n+1} \in KP(w_n)$ .
- Limit augmentation:  $w_{n+1} = [\mathcal{W}_n; \phi_{n+1}, \gamma_{n+1}]$ , for some  $\phi_{n+1} \in \mathrm{KP}_{\infty}(\mathcal{W}_n)$ , where  $\mathcal{W}_n$  is an essential continuous family of augmentations of  $w_n$ .

Let  $\phi_0 \in KP(w_0)$  be a key polynomial of minimal degree and let  $\gamma_0 = w_0(\phi_0)$ . Then, in view of Theorem 2.9, Proposition 6.3 of [5], Proposition 2.1, 3.5 of [6] and Corollary 2.10, a chain (2.2) of augmentations have the following properties:

Remark 2.17. (i)  $\gamma_n = w_n(\phi_n) < \gamma_{n+1}$ .

(ii) For all  $n \ge 0$ , the polynomial  $\phi_n$  is a key polynomial for  $w_n$  of minimal degree and therefore

$$\deg(w_n) (= \deg \phi_n)$$
 divides  $\deg(\Phi(w_n, w_{n+1}))$ .

(iii)

$$\Phi(w_n, w_{n+1}) = \begin{cases} [\phi_{n+1}]_{w_n}, & \text{if } w_n \to w_{n+1} \text{ is an ordinary augmentation} \\ \Phi_{w_n, \mathcal{W}_n} = [\chi_i]_{w_n}, & \text{if } w_n \to w_{n+1} \text{ is a limit augmentation}. \end{cases}$$

(iv)

$$\deg(\Phi(w_n, w_{n+1})) = \begin{cases} \deg \phi_{n+1}, & \text{if } w_n \to w_{n+1} \text{ is an ordinary augmentation} \\ \deg(\mathcal{W}_n), & \text{if } w_n \to w_{n+1} \text{ is a limit augmentation.} \end{cases}$$

A valuation w on K(X) is called a *depth zero valuation* if  $w = w_{\alpha,\delta}$ , for some  $\alpha \in K$  and  $\delta \in \Gamma$ , where  $w_{\alpha,\delta}$  is a valuation on K[X] defined by

$$w_{\alpha,\delta}\left(\sum_{i\geq 0}c_i(X-\alpha)^i\right):=\min_{i\geq 0}\{v(c_i)+i\delta\}, c_i\in K.$$

**Definition 2.18** (MacLane-Vaquié chains). A finite, or countably infinite chain of mixed augmentations as in (2.2) is called a *MacLane-Vaquié chain* (abbreviated as MLV chain), if every augmentation step satisfies:

- (i) if  $w_n \to w_{n+1}$  is an ordinary augmentation, then  $\deg(w_n) < \deg(\Phi(w_n, w_{n+1}))$ .
- (ii) if  $w_n \to w_{n+1}$  is a limit augmentation, then  $\deg(w_n) = \deg(\Phi(w_n, w_{n+1}))$  and  $\phi_n \notin \Phi(w_n, w_{n+1})$ .

A MacLane-Vaquié chain is said to be *complete* if  $w_0$  is a depth zero valuation.

Remark 2.19. Every infinite MLV chain (2.2) has a stable limit. Since for any polynomial  $f \in K[X]$ , there exist some  $n \geq 0$  such that  $\deg f < \deg \phi_n$  and as  $\deg \phi_n = \deg(w_n) \leq \deg(\Phi(w_n, w_{n+1}))$ , so by Theorem 2.9,  $w_n(f) = w_{n+1}(f)$ . Therefore  $\rho_{\mathcal{W}}$  is the stable limit of the ordered family of valuations  $\mathcal{W} = (w_n)_{n \in I}$ , where  $I = \mathbb{N} \cup \{0\}$ .

In Theorem 3.1 of [4], given an induced complete sequence of ABKPs,  $\{Q_i\}_{i\in\Delta}$  for w such that  $\Delta$  has a last element a precise complete finite MLV chain of w is obtained, and conversely if w has a complete finite MLV chain, then Theorem 3.2 of [4] gives a construction of an induced complete sequence of ABKPs of the above type. Suppose now that w has an induced complete sequence of ABKPs such that  $\Delta$  has no last element. In the following result, using such a complete sequence we give an explicit construction of an MLV chain of w.

**Theorem 2.20.** Let (K, v) be a valued field and let w be an extension of v to K(X). If  $\{Q_i\}_{i\in\Delta}$  is an induced complete sequence of ABKPs for w such that  $\Delta$  has no last element, then w falls in exactly one of the following two cases.

(i) After a finite number, say, r of augmentation steps, it is the stable limit of a continuous family  $W_r$  of augmentations of  $w_r$ 

$$w_0 \xrightarrow{Q_1, \gamma_1} w_1 \xrightarrow{Q_2, \gamma_2} \cdots \longrightarrow w_{r-1} \xrightarrow{Q_r, \gamma_r} w_r \xrightarrow{W_r} \rho_{W_r} = w,$$

such that  $deg(\Phi(w_r, w)) = deg(w_r)$  and  $Q_r \notin \Phi(w_r, w)$ .

(ii) It is the stable limit of a complete infinite MLV chain.

$$w_0 \xrightarrow{Q_1,\gamma_1} w_1 \xrightarrow{Q_2,\gamma_2} \cdots \longrightarrow w_n \xrightarrow{Q_{n+1},\gamma_{n+1}} w_{n+1} \longrightarrow \cdots$$

In both cases,  $\gamma_j = w(Q_j)$  for all  $j \in I$ . Also, an augmentation  $w_j \longrightarrow w_{j+1}$  is ordinary if and only if  $\vartheta_j = \emptyset$ .

The converse of the above result also holds.

**Theorem 2.21.** Let (K, v) be a valued field and let w be an extension of v to K(X) such that w falls in exactly one of the following two cases.

(i) After a finite number, say, r of augmentation steps, it is the stable limit of a continuous family  $W_r = (\rho_i)_{i \in \mathbf{A}_r}$  of augmentations of  $w_r$ 

(2.3) 
$$w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \longrightarrow w_{r-1} \xrightarrow{\phi_r, \gamma_r} w_r \xrightarrow{(\rho_i)_{i \in \mathbf{A}_r}} \rho_{\mathcal{W}_r} = w$$

$$such that \deg(\Phi(w_r, w)) = \deg(w_r) \text{ and } \phi_r \notin \Phi(w_r, w).$$

(ii) It is the stable limit of a complete infinite MLV chain,

$$(2.4) w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \longrightarrow w_n \xrightarrow{\phi_{n+1}, \gamma_{n+1}} w_{n+1} \longrightarrow \cdots$$

Then, there is a totally ordered set  $\Delta$  containing no last element and an induced complete sequence  $\Lambda = \{Q_i\}_{i \in \Delta}$  of ABKPs for w, constructed as follows:

- (a)  $\Delta = \bigcup_{i \in I} \Delta_i$ , with  $I = \{0, 1, \dots, r\}$  in case (i) and  $I = \mathbb{N} \cup \{0\}$  in case (ii).
- (b)  $\Delta_j = \{j\} \cup \vartheta_j$ , for all  $j \in I$ , and  $\vartheta_j = \emptyset$  if and only if the augmentation  $w_j \longrightarrow w_{j+1}$  is ordinary. Moreover,  $Q_j = \phi_j$  for all  $j \in I$ .
- (c) Suppose that  $w_j oup w_{j+1}$  is a limit augmentation, or (in case (i)) j = r and  $w_j oup w$  is a stable limit step. Let  $W_j = (\rho_i)_{i \in \mathbf{A}_j}$  be the underlying totally ordered family. Then  $\vartheta_j = \mathbf{A}_j$  and  $Q_i = \chi_i$  for all  $i \in \mathbf{A}_j$ , where  $\chi_i$  is the key polynomial for  $w_j$  such that  $\rho_i = [w_j; \chi_i, w(\chi_i)]$ .

It is known that if  $\{Q_i\}_{i\in\Delta}$  is a complete sequence of ABKPs for w, then w is a valuation-transcendental extension of v to K(X) if and only if  $\Delta$  has a last element, say, N, and then  $w = w_{Q_N}$  (see [3, Theorem 5.6]). Therefore, as an immediate consequence of Theorems 2.20 and 2.21 we have the following result.

**Corollary 2.22.** Let (K, v) and (K(X), w) be as above. Then the following are equivalent:

- (i) The extension w is valuation-algebraic.
- (ii) There exists an induced complete sequence  $\{Q_i\}_{i\in\Delta}$  of ABKPs for w such that  $\Delta$  has no last element.
- (iii) The extension w has an MLV chain of type (2.3) or (2.4).

#### 3. Preliminaries

Let (K, v) be a valued field and  $(\overline{K}, \overline{v})$  be as before. Let w be an extension of v to K(X) and  $\overline{w}$  a common extension of w and  $\overline{v}$  to  $\overline{K}(X)$ . In this section we give some preliminary results which will be used to prove the main results.

We first recall some basic properties of ABKPs for w (see Lemma 2.11 of [7] and Proposition 3.8, Corollary 3.11, Theorem 6.1 of [9]).

**Proposition 3.1.** For ABKPs, Q and Q' for w the following holds:

- (i) If  $\delta(Q) < \delta(Q')$ , then  $w_Q(Q') < w(Q')$ .
- (ii) Suppose that  $\delta(Q) < \delta(Q')$ . For any polynomial  $f \in K[X]$ , we have

$$w_Q(f) = w(f) \implies w_{Q'}(f) = w(f),$$
  
 $w_{Q'}(f) < w(f) \implies w_Q(f) < w_{Q'}(f).$ 

- (iii) If  $Q' \in \Phi(w_Q, w)$ , then Q and Q' are key polynomials for  $w_Q$ . Moreover,  $w_{Q'} = [w_Q; Q', w(Q')]$ .
- (iv) Every  $F \in \Phi(w_Q, w)$  is an ABKP for w and  $\delta(Q) < \delta(F)$ .

The next two results gives a comparison between key polynomials and ABKPs.

**Theorem 3.2** (Theorem 2.17, [1]). Suppose that w' < w be valuations on K(X) and  $\phi$  a key polynomial for w'. Then  $\phi$  is an ABKP polynomial for w if and only if it satisfies one of the following two conditions:

- (i)  $\phi \in \Phi(w', w)$ .
- (ii)  $\phi \notin \Phi(w', w)$  and  $\deg \phi = \deg(w')$ .

In the first case  $w_{\phi} = [w'; \phi, w(\phi)]$ . In the second case  $w_{\phi} = w'$ .

**Theorem 3.3** (Theorem 2.18, [1]). Let  $\phi \in KP(w)$ . Then  $\phi$  is an ABKP for w if and only if deg  $\phi = \deg(w)$ . In this case,  $w_{\phi} = w$ .

As an application of the above two theorems, we have the following two results.

**Lemma 3.4.** Let  $w = [w'; \phi, \gamma]$  be an ordinary augmentation of a valuation w' of K(X). If  $\phi'$  is a minimal degree key polynomial for w', then both  $\phi'$  and  $\phi$  are ABKPs for w. Moreover, if  $\phi \downarrow_{w'} \phi'$ , then  $\delta(\phi') < \delta(\phi)$ .

*Proof.* Since  $\phi'$  is a minimal degree key polynomial for w', Theorem 3.2 shows that  $\phi'$  is an ABKP for w and as  $\phi$  is a minimal degree key polynomial for w, so by Theorem 3.3  $\phi$  is an ABKP for w.

Now, suppose that  $\phi \not\models_{w'} \phi'$ . By Corollary 2.11,  $\phi' \notin \Phi(w', w) = [\phi]_{w'}$ . Assume first that  $\deg \phi' = \deg \phi$ . Then by Theorem 3.2, we have  $w' = w_{\phi'}$ . Hence, Proposition 3.1 (iv) shows that  $\delta(\phi') < \delta(\phi)$ . Finally, if  $\deg \phi' < \deg \phi$ , then  $\delta(\phi') < \delta(\phi)$  because  $\phi$  is an ABKP for w.

**Lemma 3.5.** Let w' < w be valuations of K(X). If  $W = (\rho_i = [w'; \chi_i, \gamma_i])_{i \in \mathbf{A}}$  is a continuous family of augmentations of w' such that  $\rho_i < w$  for all  $i \in \mathbf{A}$ , then each  $\chi_i$  is an ABKP for w,  $\rho_i = w_{\chi_i}$  and  $\delta(\chi_i) < \delta(\chi_j)$  for all  $i < j \in \mathbf{A}$ . Moreover, if  $\phi'$  is a minimal degree key polynomial for w', then  $\phi'$  is also an ABKP for w, and if  $\chi_i \nmid_{w'} \phi'$ , then  $\delta(\phi') < \delta(\chi_i)$  for all  $i \in \mathbf{A}$ .

*Proof.* Since  $W = (\rho_i)_{i \in \mathbf{A}}$  is a continuous family of augmentations of w', so for all  $i < j \in \mathbf{A}$ , we have that  $\chi_j \not\vdash_{\rho_i} \chi_i$  and  $\rho_j = [\rho_i; \chi_j, \gamma_j]$  is an ordinary augmentation of  $\rho_i$ . Therefore, by Remark 2.13 (iv) and Corollary 2.10 (ii) we have that

$$\chi_i \notin \Phi(\rho_i, \rho_j) = [\chi_j]_{\rho_i} = \Phi(\rho_i, w),$$

i.e.,  $\rho_i(\chi_i) = \rho_j(\chi_i)$ , which in view of Corollary 2.10 (ii), implies that  $\rho_i(\chi_i) = w(\chi_i)$ , and as deg  $\chi_i = \deg \rho_i$ , so by Theorem 3.2,  $\chi_i$  is an ABKP for w and

(3.1) 
$$\rho_i = w_{\chi_i} \text{ for all } i \in \mathbf{A}.$$

Again from Corollary 2.10 (ii) and Remark 2.13 (iv), it follows that

$$\chi_i \in [\chi_i]_{\rho_i} = \Phi(\rho_i, \rho_i) = \Phi(\rho_i, w)$$
 for all  $i < j \in \mathbf{A}$ ,

i.e.,  $\rho_i(\chi_j) < w(\chi_j)$ , which on using (3.1), together with the fact that  $\deg \chi_i = \deg \chi_j$ , implies that  $\chi_j \in \Phi(w_{\chi_i}, w)$ . Hence from Proposition 3.1 (iv),  $\chi_j$  is an ABKP for w and

$$\delta(\chi_i) < \delta(\chi_j)$$
 for all  $i < j \in \mathbf{A}$ .

Since w' < w, so  $w'(\phi') \le w(\phi')$ . Now on using Theorem 3.2, together with the hypothesis that  $\phi'$  is a minimal degree key polynomial for w', i.e.,  $\deg \phi' = \deg(w')$ , we get that  $\phi'$  is an ABKP for w. Keeping in mind that  $\chi_i \nmid_{w'} \phi'$ , it immediately follows from Lemma 3.4, that  $\delta(\phi') < \delta(\chi_i)$  for all  $i \in \mathbf{A}$ .

Remark 3.6. In the above lemma, if  $w = [\mathcal{W}; \phi, \gamma]$  is a limit augmentation of an essential continuous family  $\mathcal{W} = (\rho_i = [w'; \chi_i, \gamma_i])_{i \in \mathbf{A}}$  of augmentations of w', then by Theorem 3.3,  $\phi$  is an ABKP for w, and as  $\phi$  is an MLV limit key polynomial for  $\mathcal{W}$ , so  $\deg \chi_i < \deg \phi$ . Consequently,

$$\delta(\chi_i) < \delta(\phi), \ \forall \ i \in \mathbf{A}.$$

In the next result we give some properties of an induced complete sequence of ABKPs.

**Lemma 3.7.** Let  $\{Q_i\}_{i\in\Delta}$  be an induced complete sequence of ABKPs for w such that  $\vartheta_j \neq \emptyset$  for some  $j \in \Delta$ . Then the following holds:

- (i)  $Q_{i'} \in \Phi(w_{Q_i}, w)$  for every  $i < i' \in \Delta_j = \{j\} \cup \vartheta_j$ .
- (ii)  $W_j = (w_{Q_i})_{i \in \vartheta_j}$  is a continuous family of augmentations of  $w_{Q_j}$ .

*Proof.* (i) Follows from Remark 2.5 and Proposition 3.1 (i).

(ii) Since for each  $i \in \Delta$ ,  $Q_i$  is an ABKP for w, so  $w_{Q_i}$  is a valuation on K(X). By hypothesis, as  $\vartheta_j \neq \emptyset$ , so by (i), for each  $i \in \vartheta_j$  we have that  $Q_i \in \Phi(w_{Q_j}, w)$  and  $\deg Q_i = \deg Q_j$ , where  $Q_j$  is the ABKP corresponding to  $\{j\}$ . From Proposition 3.1 (iii), it follows that each  $Q_i$  is a key polynomial for  $w_{Q_j}$  and

$$w_{Q_i} = [w_{Q_i}; Q_i, w(Q_i)].$$

Similarly, for each  $i < i' \in \vartheta_i$ , we get that  $Q_{i'}$  is a key polynomial for  $w_{Q_i}$  and

$$w_{Q_{i'}} = [w_{Q_i}; Q_{i'}, w(Q_{i'})].$$

Now by Corollary 2.10 (ii),  $\Phi(w_{O_i}, w) = \Phi(w_{O_i}, w_{O_{i,l}})$ , and as  $Q_i \notin \Phi(w_{O_i}, w)$ , so

$$w_{Q_i}(Q_i) = w_{Q_{i,l}}(Q_i), \ \forall \ i < i' \in \vartheta_i,$$

which in view of Theorem 2.9, implies that  $Q_{i'} \not \sim_{w_{Q_i}} Q_i$ . Hence  $\mathcal{W}_j = (w_{Q_i})_{i \in \vartheta_j}$  is a continuous family of augmentations of  $w_{Q_i}$ .

## 4. Proof of Main Results

Proof of Theorem 2.20. Since  $\{Q_i\}_{i\in\Delta}$  is an induced complete sequence of ABKPs for w, so by Remark 2.5 (i),  $\Delta=\bigcup_{j\in I}\Delta_j$ , where  $I=\{0,1,\ldots,N\}$  or  $I=\mathbb{N}\cup\{0\}$  and for each  $j\in I$ ,  $\Delta_j=\{j\}\cup\vartheta_j$ , where  $\vartheta_j$  is either empty or an ordered set without a last element. In view of hypothesis as  $\Delta$  has no last element, so by (2.1), either  $I=\mathbb{N}\cup\{0\}$  or  $I=\{0,1,\ldots,N\}$  and  $\Delta_N=\{N\}\cup\vartheta_N$ , with  $\vartheta_N\neq\varnothing$ . Since for each  $i\in\Delta$ ,  $Q_i$  is an ABKP for w, so  $w_{Q_i}$  is a valuation on K(X) and we denote it by  $w_i$ .

Arguing as in the proof of [4, Theorem 3.1], we get that each  $Q_i$ ,  $i \in \Delta$  is a key polynomial for  $w_{Q_i}$  of minimal degree and

- (1) if  $\vartheta_j = \emptyset$ , then  $w_j \longrightarrow w_{j+1}$  is an ordinary augmentation.
- (2) if  $\vartheta_j \neq \emptyset$ , then  $w_j \longrightarrow w_{j+1}$  is a limit augmentation of an essential continuous family  $\mathcal{W}_j = (\rho_i = w_{Q_i})_{i \in \vartheta_j}$  of augmentations of  $w_j$ .

In either case, denote

$$(4.1) \gamma_i = w(Q_i), \ \forall \ i \in \Delta.$$

Suppose first that  $I = \{0, 1, \dots, N\}$  and  $\vartheta_N \neq \emptyset$ . Then from above

$$(4.2) w_0 \xrightarrow{Q_1, \gamma_1} w_1 \xrightarrow{Q_2, \gamma_2} \cdots \longrightarrow w_{N-1} \xrightarrow{Q_N, \gamma_N} w_N < w,$$

is a finite MLV chain of  $w_N$ , and this chain is complete because  $w_0 = w_{Q_0}$ , where  $\deg Q_0 = 1$ , is a depth zero valuation. Since  $\vartheta_N \neq \emptyset$ , so by Lemma 3.7 (ii), we have that  $\mathcal{W}_N = (\rho_i = w_{Q_i})_{i \in \vartheta_N}$  is a continuous family of augmentations of  $w_N$ . We now claim the following:

- (a) w is the stable limit of  $\mathcal{W}_N$ ,
- (b)  $\deg(\Phi(w_N, w)) = \deg(w_N)$  and  $Q_N \notin \Phi(w_N, w)$ .

Let f in K[X] be any polynomial. As  $\Lambda$  is complete, so there exists some  $j_0 \in \Delta$  such that

$$(4.3) w_{j_0}(f) = w(f).$$

Take  $i \in \vartheta_N$  such that  $j_0 < i$ . Then,  $\delta(Q_{j_0}) < \delta(Q_i)$  and, in view of (4.3), Proposition 3.1 (ii) shows that  $w_i(f) = w(f)$ . Therefore,

$$\rho_{\mathcal{W}_N}(f) = w_{i_0}(f) = w_i(f) = w(f)$$
 for every  $i \in \vartheta_N$ .

Thus, every polynomial in K[X] is  $\mathcal{W}_N$ -stable, i.e.,  $\rho_{\mathcal{W}_N}$  is a valuation on K(X) and hence is a stable limit of  $\mathcal{W}_N$ . From the above argument it also follows that

$$\rho_{\mathcal{W}_N}(f) = w(f), \ \forall \ f \in K[X]$$

and this proves (a).

Since  $\vartheta_N \neq \emptyset$ , so by Lemma 3.7 (i)  $Q_i \in \Phi(w_{Q_N}, w)$  for every  $i \in \vartheta_N$ , which implies that

$$\deg(\Phi(w_N, w)) = \deg(w_N)$$

and  $Q_N \notin \Phi(w_N, w)$  because  $w_{Q_N}(Q_N) = w(Q_N)$ , proving (b). Thus in view of (4.2) and the claim, after N augmentation steps, we have that w is the stable limit of a continuous family  $\mathcal{W}_N = (\rho_i)_{i \in \partial_N}$ , of augmentations of  $w_N$ :

$$w_0 \xrightarrow{Q_1,\gamma_1} w_1 \xrightarrow{Q_2,\gamma_2} \cdots \longrightarrow w_{N-1} \xrightarrow{Q_N,\gamma_N} w_N \xrightarrow{(\rho_i)_{i \in \vartheta_N}} \rho_{\mathcal{W}_N} = w,$$

where  $w_j = w_{Q_j}$ ,  $\gamma_j = w(Q_j)$  for every  $0 \le j \le N$ , such that  $\deg(\Phi(w_N, w)) = \deg(w_N)$  and  $Q_N \notin \Phi(w_N, w)$ .

Assume now that  $I = \mathbb{N} \cup \{0\}$ . Then keeping in mind (1), (2) and (4.1), we have that

$$w_0 \xrightarrow{Q_1, \gamma_1} w_1 \xrightarrow{Q_2, \gamma_2} \cdots \longrightarrow w_i \xrightarrow{Q_{j+1}, \gamma_{j+1}} w_{j+1} \longrightarrow \cdots$$

is a complete infinite MLV chain of w. Since  $\{Q_i\}_{i\in\Delta}$  is an induced complete sequence of ABKPs for w, so  $\mathcal{W} = (w_i = w_{Q_i})_{i\in\Delta}$  is a totally ordered family of valuations, taking values in a common value group, such that the bijection  $i\mapsto w_{Q_i}$ , is an isomorphism between  $\Delta$  and  $\mathcal{W}$ . It only remains to prove that w is the stable limit of  $\mathcal{W}$ . For this it is enough to show that every polynomial in K[X] is  $\mathcal{W}$ -stable. Let f be any polynomial in K[X]. Since  $\Lambda$  is an induced complete sequence of ABKPs for w, so there exist some  $j_0 \in \Delta$  such that

$$w_{i_0}(f) = w(f).$$

For all  $i > j_0$  in  $\Delta$ , we have  $\delta(Q_{j_0}) < \delta(Q_i)$ , which together with the above equality, on using Proposition 3.1 (ii) implies that  $w_i(f) = w(f)$ . Therefore,

$$w_i(f) = w_{i_0}(f), \ \forall \ i > j_0 \in \Delta,$$

so every polynomial in K[X] is W-stable, i.e.,  $\rho_{\mathcal{W}}$  is a valuation on K(X), and

$$\rho_{\mathcal{W}}(f) = w_{Q_{j_0}}(f).$$

It also follows from the above argument that

$$\rho_{\mathcal{W}}(f) = w(f), \ \forall \ f \in K[X].$$

Thus w is the stable limit of  $\mathcal{W}$ .

Proof of Theorem 2.21. Since w has an MLV chain of type (2.3) or (2.4), so by Lemmas 3.4, 3.5 and Remark 3.6 we have that

- (1) if  $w_j oup w_{j+1}$  is an ordinary augmentation, i.e.,  $w_{j+1} = [w_j; \phi_{j+1}, \gamma_{j+1}]$  for some  $\phi_{j+1} \in KP(w_j)$ , then  $\phi_j$  and  $\phi_{j+1}$  are ABKPs for  $w_{j+1}$ .
- (2) if  $w_j oup w_{j+1}$  is a limit augmentation, i.e.,  $w_{j+1} = [\mathcal{W}_j; \phi_{j+1}, \gamma_{j+1}]$  for some  $\phi_{j+1} \in \mathrm{KP}_{\infty}(w_j)$ , where  $\mathcal{W}_j = (\rho_i = [w_i; \chi_i, \gamma_i])_{i \in \mathbf{A}_j}$  is an essential continuous family of augmentations of  $w_j$ , then  $\phi_j$ ,  $\phi_{j+1}$  and  $\chi_i$  for all  $i \in \mathbf{A}_j$  are ABKPs for  $w_{j+1}$ .

Now by Remark 2.17 and the definition of an MLV chain of w, if  $w_j \longrightarrow w_{j+1}$  is an ordinary augmentation, then

$$\phi_j \notin \Phi(w_j, w_{j+1}) = [\phi_{j+1}]_{w_i}, \text{ i.e., } \phi_{j+1} +_{w_i} \phi_j,$$

and if  $w_j \longrightarrow w_{j+1}$  is a limit augmentation, then

$$\phi_j \notin \Phi(w_j, w_{j+1}) = [\chi_i]_{w_j}$$
, i.e.,  $\chi_i \nmid_{w_i} \phi_j$ ,  $\forall i \in \mathbf{A}_j$ .

Since,  $w_i < w_{i+1} < w$  and  $\rho_i < \rho_{i'} < w$ , so by Corollary 2.10 (ii), we have that

$$\Phi(w_j, w_{j+1}) = \Phi(w_j, w)$$
 and  $\Phi(\rho_i, \rho_{i'}) = \Phi(\rho_i, w)$ , for all  $i < i' \in \mathbf{A}_j$ .

Therefore, keeping in mind the definition of an MLV chain of w together with Theorem 3.2, Lemmas 3.4, 3.5 and Remark 3.6 it immediately follows that

(a)  $\phi_j$ ,  $\phi_{j+1}$ ,  $Q_i := \chi_i$ ,  $i \in \mathbf{A}_j$  are ABKPs for w, and

$$w_j = w_{\phi_j}, \ \rho_i = w_{Q_i}, \ w_{j+1} = w_{\phi_{j+1}}, \ \text{for all } i \in \mathbf{A}_j.$$

- (b)  $\deg \phi_j = \deg Q_i < \deg \phi_{j+1}$  for all  $i \in \mathbf{A}_j$ .
- (c)  $\delta(\phi_i) < \delta(Q_i) < \delta(Q_{i'}) < \delta(\phi_{j+1})$  for all  $i < i' \in \mathbf{A}_j$ .
- (d) If  $w_j \longrightarrow w_{j+1}$  is an ordinary augmentation, then  $\phi_{j+1} \in \Phi(w_{\phi_j}, w)$ .

Suppose first that w is the stable limit of a continuous family of augmentations  $W_r = (\rho_i = [w_r; \chi_i, \gamma_i])_{i \in \mathbf{A}_r}$  of  $w_r$ 

$$w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \longrightarrow w_r \xrightarrow{(\rho_i)_{i \in \mathbf{A}_r}} \rho_{\mathcal{W}_r} = w.$$

Then from Lemma 3.5, we have that  $Q_i := \chi_i$ , for all  $i \in \mathbf{A}_r$ , is an ABKP for w,

$$(4.4) \rho_i = w_{Q_i}, \text{ and}$$

$$\delta(\phi_r) < \delta(Q_i) < \delta(Q_{i'}) \ \forall i < i' \in \mathbf{A}_r.$$

Let  $I = \{0, 1, ..., r\}$  and for every  $j \in I$ , let  $\Delta_j = \{j\} \cup \mathbf{A}_j$ , where in view of (1), (2)  $\mathbf{A}_j$  is either empty or an ordered set without a last element, for  $0 \le j \le r - 1$ . Let  $\Delta = \bigcup_{j=0}^r \Delta_j$ , keeping in mind (1), (2) and the continuous family  $\mathcal{W}_r$ , consider the set  $\Lambda = \{Q_i\}_{i \in \Delta}$ , where  $Q_j := \phi_j$  for every  $j \in I$ . Then in view of (c) and (4.5) we have that

$$\delta(Q_i) < \delta(Q_{i'}) \ \forall i < i' \in \Delta.$$

Moreover, the set  $\Lambda$  satisfies the properties of Remark 2.5. Let f in K[X] be any polynomial. Since w is the stable limit of  $W_r$ , so f is  $W_r$ -stable. Therefore, there exists some  $i_0 \in \mathbf{A}_r \subset \Delta$  such that

$$w(f) = \rho_{i_0}(f)$$
  
=  $w_{Q_{i_0}}(f)$  (by (4.4)).

If  $\deg Q_r = \deg Q_{i_0} \leq \deg f$ , then  $\Lambda$  fulfills the condition to be a complete sequence of ABKPs for w. Otherwise,  $\deg Q_i \leq \deg f < \deg Q_{j+1}$  for all  $i \in \Delta_j$  and some minimal j < r. Since  $\delta(c^{-1}f) = \delta(f)$ , where  $c \in K$  is the leading coefficient of f, so we can assume without loss of generality that f is monic. If  $\delta(f) \leq \delta(Q_i)$  for some  $i \in \Delta_j$ , then clearly  $w_{Q_i}(f) = w(f)$ . We now show that the case  $\delta(f) > \delta(Q_i)$  for every  $i \in \Delta_j$ , does not occur. If  $\mathbf{A}_j = \emptyset$ , then  $f \in \Phi(w_{Q_j}, w)$  and from (d) we also have that  $Q_{j+1} \in \Phi(w_{Q_j}, w)$ , which implies that  $\deg f = \deg Q_{j+1}$ . Therefore  $\mathbf{A}_j \neq \emptyset$ . If  $\deg f = \deg Q_i$ , then f is an ABKP for w and by Remark 2.17 and Corollary 2.10, f belongs to the set  $\{Q_i \mid i \in \mathbf{A}_j\}$ , which contradicts the definition of  $\mathbf{A}_j$ . So  $\deg Q_i < \deg f < \deg Q_{j+1}$ . Since for every  $i \in \Delta_j$ ,  $\delta(Q_i) < \delta(f)$ , so  $w_{Q_i}(f) < w(f)$ . In particular,  $w_{Q_i}(f) < w(f)$  for every  $i \in \mathbf{A}_j$ , which in view of Proposition 3.1 (ii), implies that  $w_{Q_i}(f) < w_{Q_{i'}}(f)$  for every  $i < i' \in \mathbf{A}_j$ . By definition of MLV chain, it now follows that  $\deg f \geq \deg Q_{j+1}$ , which is not the case. Hence  $\Lambda$  is an induced complete sequence of ABKPs for w such that  $\Delta$  has no last element.

Assume now that w is the stable limit of a complete infinite MLV chain,

$$w_0 \xrightarrow{\phi_1,\gamma_1} w_1 \xrightarrow{\phi_2,\gamma_2} \cdots \longrightarrow w_j \xrightarrow{\phi_{j+1},\gamma_{j+1}} w_{j+1} \longrightarrow \cdots$$

i.e., w is the stable limit of  $\mathcal{W} = (w_i)_{i \in I}$ , where  $I = \mathbb{N} \cup \{0\}$  (see Remark 2.19). For every  $j \in I$ , let  $\Delta_j = \{j\} \cup \mathbf{A}_j$ , where  $\mathbf{A}_j$  is an ordered set without a last element, whenever  $w_j \longrightarrow w_{j+1}$  is a limit augmentation, or is an empty set, if  $w_j \longrightarrow w_{j+1}$  is an ordinary augmentation. Let  $\Delta = \bigcup_{j \in I} \Delta_j$ , and, we denote  $Q_j := \phi_j$  for every  $j \in I$ . Keeping in mind (1), (2) consider the set  $\Lambda = \{Q_i\}_{i \in \Delta}$ . For any  $i < i' \in \Delta$  by (c) we have that

$$\delta(Q_i) < \delta(Q_{i'}),$$

and therefore the set  $\Lambda$  satisfies the properties of Remark 2.5. Now for any polynomial  $f \in K[X]$ , on using the fact that w is the stable limit of W, we get

$$w(f) = w_{Q_{i_0}}(f)$$
 for some  $i_0 \in \Delta$ .

Now, arguing similarly as in the previous case, we have that for every polynomial  $f \in K[X]$  there exists a polynomial  $Q_i \in \Lambda$  such that  $\deg Q_i \leq \deg f$  and  $w_{Q_i}(f) = w(f)$ . Hence,  $\{Q_i\}_{i \in \Delta}$  is an induced complete sequence of ABKPs for w such that  $\Delta$  has no last element.

Thus, in either case  $\Lambda$  =  $\{Q_i\}_{i\in\Delta}$  is an induced complete sequence of ABKPs for w such that

- $\Delta = \bigcup_{j \in I} \Delta_j$ , with  $I = \{0, 1, \dots, r\}$  or  $\mathbb{N} \cup \{0\}$  and  $\Delta_j = \{j\} \cup \mathbf{A}_j$  for all  $j \in I$ . Moreover,  $Q_j = \phi_j$  for all  $j \in I$ .
- $w_j \longrightarrow w_{j+1}$  is an ordinary augmentation if and only if  $\vartheta_j = \varnothing$ .
- if  $w_j \longrightarrow w_{j+1}$  is a limit augmentation (or  $w_j \longrightarrow w$  is a stable limit step) with respect to an essential continuous family (or continuous family)  $W_j = (\rho_i)_{i \in \mathbf{A}_j}$  of augmentations of  $w_j$ , then  $\vartheta_j = \mathbf{A}_j$  and  $Q_i = \chi_i$  for all  $i \in \mathbf{A}_j$ .

## Acknowledgement

We would like to thank the anonymous referee for a careful reading and providing useful suggestions which led to an improvement in the presentation of the paper. The research of first author is supported by CSIR (Grant No. 09/045(1747)/2019-EMR-I).

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