# RELATIVE MALTSEV DEFINABILITY OF SOME COMMUTATOR PROPERTIES

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ABSTRACT. We show that, when restricted to the class of varieties that have a Taylor term, several commutator properties are definable by Maltsev conditions.

# 1. INTRODUCTION

A strong Maltsev condition is a positive primitive sentence in the language of clones. That is, it is a sentence expressing the existence of some clone elements satisfying some equalities. The name derives from the original example in [16]: A. I. Maltsev proved that the class of varieties whose members have permuting congruences is exactly the class of varieties whose clones satisfy the p.p. clone sentence

$$\sigma: \qquad (\exists p)((p(x, x, y) \approx y) \& (p(x, y, y) \approx x)).$$

A variety is said to satisfy a strong Maltsev condition if its clone does. In this article I will say that a class of varieties is *definable by a strong Maltsev condition* if it is exactly the class of all varieties that satisfy the strong Maltsev condition.

A Maltsev condition (without the word strong) is a sequence  $\Sigma = (\sigma_n)_{n \in \omega}$  of successively weaker strong Maltsev conditions ( $\sigma_n \vdash \sigma_{n+1}$  for all n). A variety  $\mathcal{V}$  satisfies  $\Sigma$  if its clone satisfies  $\sigma_n$  for some n. A class of varieties is definable by a Maltsev condition, or is Maltsev definable, if it is exactly the class of all varieties that satisfy some Maltsev condition  $\Sigma$ . A class of varieties definable by a strong Maltsev condition  $\sigma$  is also definable by the ordinary Maltsev condition  $\Sigma = (\sigma, \sigma, \ldots)$  that is a constant sequence.

In this article, I investigate classes of varieties that are not Maltsev definable, but which become Maltsev definable relative to some weak 'ground' Maltsev condition. That is, suppose that  $\mathscr{P}$  is a property of varieties. Let  $\Gamma$  be a Maltsev condition. I will investigate some instances where the class of varieties satisfying  $\mathscr{P}$  is not Maltsev definable, but the class of varieties satisfying both  $\mathscr{P}$  and  $\Gamma$  is Maltsev definable. In symbols, I might write  $\mathscr{P} + \Gamma = \Sigma$  to mean that, restricted to varieties satisfying the ground condition  $\Gamma$ , the class of varieties satisfying the condition  $\mathscr{P}$  is definable by the Maltsev condition  $\Sigma$ . I then say that the class of varieties satisfying  $\mathscr{P}$  is *Maltsev definable relative to*  $\Gamma$ . In this article, the 'ground' Maltsev condition will always be 'the existence of a Taylor term'. It is known,

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through Corollaries 5.2 and 5.3 of [17], that an idempotent variety has a Taylor term if and only if it contains no algebra with at least 2 elements in which every operation interprets as a projection operation. It is easy to see that this means exactly that 'the existence of a Taylor term' is the weakest nontrivial idempotent Maltsev condition. It is known that 'the existence of a Taylor term' is expressible as a strong Maltsev condition, see [15].

We investigate relative Maltsev definability for the ten commutator properties  $\mathscr{P}$  from the following list. To understand these statements completely, it is necessary to know the definitions of  $\mathbf{C}(x, y; z)$  (Definition 3.3), of [x, y] (Definition 3.5), and of (relative) right or left annihilators (Definition 3.6). For intuition about these statements, it may help to remember that for the variety of groups "the commutator operation" coincides with the usual commutator operation of group theory ( $[M, N] = [M, N]_{\text{group}}$ ) while for the variety of commutative rings "the commutator operation" coincides with ideal product ( $[I, J] = I \cdot J$ ). "The centralizer relation",  $\mathbf{C}(x, y; z)$ , coincides with the relation  $[x, y] \leq z$  in both cases. For commutative rings, "the relative (right or left) annihilator of J modulo I" is the colon ideal (I: J) = { $r \in R \mid rJ \subseteq I$ }, while "the annihilator of J" is special case (0: J).

- [x, y] = [y, x] (Commutativity of the commutator.)
- [x + y, z] = [x, z] + [y, z] (Left distributivity of the commutator.)
- [x, y + z] = [x, y] + [x, z] (Right distributivity of the commutator.)
- $[x, y] = [x, y'] \implies [x, y] = [x, y + y']$  (Right semidistributivity of the commutator.)
- Given x, there exists a largest y such that [x, y] = 0 (Right annihilators exist.)
- Given x, z, there exists a largest y such that  $\mathbf{C}(x, y; z)$  (Relative right annihilators exist.) We write  $(z : x)_R$  for the relative right annihilator of x modulo z, when it exists.
- $\mathbf{C}(x, y; z) \iff \mathbf{C}(y, x; z)$  (Symmetry of the centralizer relation in its first two places.)
- $\mathbf{C}(x, y; z) \iff [x, y] \le z$  (The centralizer relation is determined by the commutator.)
- $\mathbf{C}(x, y; z)$  &  $(z \le z') \Longrightarrow \mathbf{C}(x, y; z')$  (Stability of the centralizer relation under lifting in its third place.)
- $\mathbf{C}(x, y; z)$  &  $(z \le z' \le x \cap y) \Longrightarrow \mathbf{C}(x, y; z')$  (Weak stability of the centralizer relation under lifting in its third place.)

The main results of this article may be summarized as follows. First, I explain why no one of the ten commutator properties listed above is Maltsev definable [Section 2]. Then I explain why the following are equivalent for varieties  $\mathcal{V}$  with a Taylor term:

- $\mathcal{V}$  is congruence modular.
- The commutator is left distributive throughout  $\mathcal{V}$ .
- The commutator is right distributive throughout  $\mathcal{V}$ .
- The centralizer relation is symmetric in its first two places throughout  $\mathcal{V}$ .
- Relative right annihilators exist throughout  $\mathcal{V}$ .
- The centralizer relation is determined by the commutator.
- The centralizer relation is stable under lifting in its third place.

See Theorems 4.32 and 4.43. Also, for varieties  $\mathcal{V}$  with a Taylor term, the following are equivalent:

- $\mathcal{V}$  has a difference term.
- The commutator is commutative throughout  $\mathcal{V}$ .
- Right annihilators exist throughout  $\mathcal{V}$ .
- The commutator is right semidistributive throughout  $\mathcal{V}$ .
- The centralizer relation is weakly stable under lifting in its third place.

See Theorems 4.29, 4.33, 4.44.

A specific Maltsev condition defining the class of congruence modular varieties may be found in [1, Section 2]. A specific Maltsev condition defining the class of varieties with a difference term may be found in [12, Section 4]. Thus, Theorems 4.29, 4.32, 4.33, 4.43, and 4.44 establish the Maltsev definability of all ten commutator properties relative to the existence of a Taylor term.

The proofs of relative Maltsev definability for the ten commutator properties identified will be called the "primary" results of this article, and will be identified as such when we prove them. All other results are considered "secondary", although some secondary results are as interesting as the primary results. For example, some nontrivial commutator-theoretic facts are proved in Section 3 whose proofs do not require the existence of a Taylor term. In addition to this, a commutator-theoretic characterization of the class of varieties that have a weak difference term is established in Theorem 4.5.

For background, I direct the reader to Section 2.4 of [8] for a discussion of Maltsev conditions and Section 2.5 of [8] for a discussion of the properties of the centralizer relation  $\mathbf{C}(x, y; z)$ . The most important elements from this source will be reproduced below when needed. In particular, it will be necessary to know the definitions of  $\mathbf{C}(x, y; z)$  (Definition 3.3), of [x, y] (Definition 3.5), of (relative) right or left annihilators (Definition 3.6), of a difference term (Definition 4.1), and of a Taylor term (see the opening paragraph of Section 4).

#### 2. The ten commutator properties are not Maltsev definable

The variety  $\mathcal{V}$  of sets has the properties that  $\mathbf{C}(\alpha, \beta; \delta)$  and  $[\alpha, \beta] = 0$  hold for any  $\alpha, \beta, \delta \in \text{Con}(\mathbf{A}), \mathbf{A} \in \mathcal{V}$ . This implies that each of the following are true in the variety of sets:

- [x,y] = [y,x]
- [x + y, z] = [x, z] + [y, z]
- [x, y + z] = [x, y] + [x, z]
- $[x, y] = [x, y'] \Longrightarrow [x, y] = [x, y + y']$
- Given x, there exists a largest y such that [x, y] = 0
- Given x, z, there exists a largest y such that  $\mathbf{C}(x, y; z)$
- $\mathbf{C}(x, y; z) \iff \mathbf{C}(y, x; z)$

- $\mathbf{C}(x, y; z) \iff [x, y] \le z$
- $\mathbf{C}(x,y;z)$  &  $(z \leq z') \Longrightarrow \mathbf{C}(x,y;z')$
- $\mathbf{C}(x, y; z) \& (z \le z' \le x \cap y) \Longrightarrow \mathbf{C}(x, y; z')$

If one of these properties  $\mathscr{P}$  were Maltsev definable, then, since the variety of sets is interpretable in any variety, every variety would satisfy  $\mathscr{P}$ . To prove that no one of these properties is Maltsev definable it suffices to exhibit varieties where the properties fail.

All the properties fail in the variety of semigroups  $\mathcal{V} = \mathsf{HSP}(\mathbb{Z}_2 \times \mathbb{S}_2)$  where  $\mathbb{Z}_2$  is the 2element group considered as a semigroup and  $\mathbb{S}_2$  is the 2-element semilattice. This is a variety of commutative semigroups satisfying  $x^3 \approx x$ . In this variety, the term T(x, y, z) = xyz is a Taylor term for  $\mathcal{V}$  (see [8, Definition 2.15] or the opening paragraph of Section 4 below). One can conclude this by noting that T is idempotent in  $\mathcal{V}$  (since  $T(x, x, x) \approx x^3 \approx x$ ) and satisfies *i*-th place Taylor identities in  $\mathcal{V}$  for every *i* (since  $T(x, y, z) \approx xyz \approx zxy \approx T(z, x, y)$ ).

From the main results of this article, the fact that  $\mathcal{V}$  has a Taylor term implies that, if  $\mathcal{V}$  had one of the commutator properties listed above, then  $\mathcal{V}$  would have a difference term. Then, from Theorem 4.25 below, any pentagon in a congruence lattice of a member of  $\mathcal{V}$  would have a 'neutral' critical interval. This is not the case, since  $\operatorname{Con}(\mathbb{Z}_2 \times \mathbb{S}_2)$  is a pentagon and its critical interval is abelian. The lattice  $\operatorname{Con}(\mathbb{Z}_2 \times \mathbb{S}_2)$  is indicated in Figure 1 with some congruences identified using the notation "partition : congruence" or "congruence : partition". By hand computations, or by UACalc [3], it can be shown that  $\mathbf{C}(\theta, \theta; \delta)$ . This



means that the critical interval of this copy of  $N_5$  is abelian, hence is *not* neutral.

#### 3. Commutator theoretic results true for every variety

In this section we prove some new facts about the commutator which we will need later in the paper. They have been extracted from their rightful places in the next section and recorded here solely because the proofs require no ground Maltsev condition among their hypotheses.

Our notation follows that of [2], and we direct the reader to that source for fuller explanations. For example,  $\operatorname{Con}(\mathbf{A})$  is the congruence lattice of  $\mathbf{A}$ . The meet (= intersection) and join of congruences  $\alpha, \beta \in \operatorname{Con}(\mathbf{A})$  will be denoted  $\alpha \cap \beta$  and  $\alpha + \beta$ . We might write

 $u \stackrel{\alpha}{=} v$  as an alternative to  $(u, v) \in \alpha$ . When  $\delta \leq \theta$ ,  $I[\delta, \theta]$  denotes the interval in Con(A) consisting of all congruences between  $\delta$  and  $\theta$ , namely  $I[\delta, \theta] = \{x \in \text{Con}(\mathbf{A}) \mid \delta \leq x \leq \theta\}$ . A five-element sublattice of Con(A) is called a pentagon if it is isomorphic to the lattice depicted in Figure 1. The critical interval of a pentagon is the interval that corresponds to  $I[\delta, \theta]$  in Figure 1. If  $\alpha \in \text{Con}(\mathbf{A})$ , then  $\mathbf{A}(\alpha)$  denotes the subalgebra of  $\mathbf{A} \times \mathbf{A}$  whose universe is  $\alpha$  (reference for notation: page 37 of [2]). We use product notation for congruences of  $\mathbf{A}(\alpha)$ , so for  $\beta, \gamma \in \text{Con}(\mathbf{A})$  we let  $\beta_1 = \{((x,y), (z,w)) \in A(\alpha)^2 \mid (x,z) \in \beta\}$  and  $\gamma_2 = \{((x,y), (z,w)) \in A(\alpha)^2 \mid (y,w) \in \gamma\}$  (reference: page 85 of [2]).<sup>1</sup> Following [2], we deviate from this convention by using  $\eta_1$  and  $\eta_2$  in place of  $0_1$  and  $0_2$ . E.g.,  $\eta_1 = \{((x,y), (z,w)) \in A(\alpha)^2 \mid (x,z) \in \beta\}$  (reference: on  $\mathbf{A}(\alpha)$  generated by the  $\beta$ -diagonal relation  $\{((x,x), (z,z)) \in A(\alpha)^2 \mid (x,z) \in \beta\}$  (reference: page 37 of [2, Definition 4.7]). A fact we use when necessary is that

$$(3.1) \qquad \qquad \Delta_{\alpha,\beta} \le \beta_1 \times \beta_2$$

always holds, since the generators of the congruence  $\Delta_{\alpha,\beta}$  lie in the congruence  $\beta_1 \times \beta_2$ .

Next we define S, T-matrices and the centralizer relation. The definitions are made for *tolerances* of an algebra **A**. A tolerance on **A** is a reflexive, symmetric, compatible binary relation. (A *congruence* is a transitive tolerance.)

**Definition 3.2.** If S and T are tolerances on an algebra A, then an S, T-matrix is a  $2 \times 2$  matrix of elements of A of the form

$$\left[\begin{array}{cc} p & q \\ r & s \end{array}\right] = \left[\begin{array}{cc} t(\mathbf{a}, \mathbf{u}) & t(\mathbf{a}, \mathbf{v}) \\ t(\mathbf{b}, \mathbf{u}) & t(\mathbf{b}, \mathbf{v}) \end{array}\right]$$

where  $t(\mathbf{x}, \mathbf{y})$  is an (m + n)-ary term operation of  $\mathbf{A}$ ,  $\mathbf{a} \ S \ \mathbf{b}$ , and  $\mathbf{u} \ T \ \mathbf{v}$ . The set of all S, T-matrices of  $\mathbf{A}$  is denoted M(S, T).

The symmetry of tolerances guarantees that the set M(S,T) is invariant under the operations of interchanging rows or columns.

**Definition 3.3.** Let *S* and *T* be tolerances of an algebra **A** and let  $\delta$  be a congruence on **A**. If  $p \stackrel{\delta}{\equiv} q$  implies that  $r \stackrel{\delta}{\equiv} s$  whenever

$$\left[\begin{array}{cc} p & q \\ r & s \end{array}\right] \in M(S,T),$$

then we say that  $C(S, T; \delta)$  holds, or S centralizes T modulo  $\delta$ .

Many of the basic properties of the centralizer relation are proved in Theorem 2.19 of [8]. I copy the statement of that theorem here because its many items will be referenced repeatedly throughout this article.

<sup>&</sup>lt;sup>1</sup>Observe that the definitions of the relations  $\beta_1$  and  $\gamma_2$  depend on the choice of  $\alpha$ .

**Theorem 3.4.** Let **A** be an algebra with tolerances S, S', T, T' and congruences  $\alpha, \alpha_i, \beta, \delta, \delta', \delta_j$ . The following are true.

- (1) (Monotonicity in the first two variables) If  $\mathbf{C}(S,T;\delta)$  holds and  $S' \subseteq S, T' \subseteq T$ , then  $\mathbf{C}(S',T';\delta)$  holds.
- (2)  $\mathbf{C}(S,T;\delta)$  holds if and only if  $\mathbf{C}(\mathrm{Cg}^{\mathbf{A}}(S),T;\delta)$  holds.
- (3)  $\mathbf{C}(S,T;\delta)$  holds if and only if  $\mathbf{C}(S,\delta \circ T \circ \delta;\delta)$  holds.
- (4) If  $T \cap \delta = T \cap \delta'$ , then  $\mathbf{C}(S, T; \delta)$  holds if and only if  $\mathbf{C}(S, T; \delta')$  holds.
- (5) (Semidistributivity in the first variable) If  $\mathbf{C}(\alpha_i, T; \delta)$  holds for all  $i \in I$ , then  $\mathbf{C}(\bigvee_{i \in I} \alpha_i, T; \delta)$  holds.
- (6) If  $\mathbf{C}(\bar{S}, T; \delta_j)$  holds for all  $j \in J$ , then  $\mathbf{C}(S, T; \bigwedge_{i \in J} \delta_j)$  holds.
- (7) If  $T \cap (S \circ (T \cap \delta) \circ S) \subseteq \delta$ , then  $\mathbf{C}(S, T; \delta)$  holds.
- (8) If  $\beta \cap (\alpha + (\beta \cap \delta)) \leq \delta$ , then  $\mathbf{C}(\alpha, \beta; \delta)$  holds.
- (9) Let **B** be a subalgebra of **A**. If  $\mathbf{C}(S, T; \delta)$  holds in **A**, then  $\mathbf{C}(S|_{\mathbf{B}}, T|_{\mathbf{B}}; \delta|_{\mathbf{B}})$  holds in **B**.
- (10) If  $\delta' \leq \delta$ , then the relation  $\mathbf{C}(S,T;\delta)$  holds in  $\mathbf{A}$  if and only if  $\mathbf{C}(S/\delta',T/\delta';\delta/\delta')$  holds in  $\mathbf{A}/\delta'$ .

The commutator operation is defined in terms of the centralizer relation.

**Definition 3.5.** Let S, T be tolerances on an algebra **A**. The commutator [S, T] equals the least congruence  $\delta$  on **A** for which  $\mathbf{C}(S, T; \delta)$  holds.

According to Definition 3.5, [S, T] = 0 holds if and only if  $\mathbf{C}(S, T; 0)$  holds. By Theorem 3.4 (5), if T is a tolerance on some algebra, then the join  $\alpha$  of all congruences  $\alpha_i$ satisfying  $\mathbf{C}(\alpha_i, T; 0)$  satisfies  $\mathbf{C}(\alpha, T; 0)$ . Using Theorem 3.4 (2) we see that this join  $\alpha$  is a congruence and it is the largest congruence x such that  $\mathbf{C}(x, T; 0)$  or equivalently the largest x such that [x, T] = 0. We denote this largest x by (0:T) and call it the annihilator of T. If we want to emphasize that the annihilator x = (0:T) appears in the left variable of the commutator in the equation [x, T] = 0 we will add a subscript L to write  $x = (0:T)_L$ and say that  $(0:T)_L$  the left annihilator of T. For the same reasons, given a tolerance T and a congruence  $\delta \in \text{Con}(\mathbf{A})$  there exists a largest tolerance  $\alpha$  such that  $\mathbf{C}(\alpha, T; \delta)$  which we denote  $(\delta:T)$  or  $(\delta:T)_L$ . We call  $(\delta:T)_L$  the <u>relative</u> left annihilator of T <u>modulo  $\delta$ . We record the notation we have just introduced in Definition 3.6. Although the definitions from [8] of the centralizer relation and the commutator operation involve tolerance relations (reflexive, symmetric, compatible binary relations) rather than congruence relations (transitive tolerances), in this paper we henceforth concentrate on the centralizer, commutator, and annihilators of congruences only.</u>

**Definition 3.6.** Let A be an algebra and let  $\delta, \beta \in \text{Con}(A)$  be congruences on A.

(1) The largest congruence  $\alpha \in \text{Con}(\mathbf{A})$  satisfying  $\mathbf{C}(\alpha, \beta; 0)$  is called the *left annihilator* of  $\beta$  and it is denoted  $(0:\beta)_L$ . If there is a largest congruence  $\alpha \in \text{Con}(\mathbf{A})$  satisfying  $\mathbf{C}(\beta, \alpha; 0)$  it is called the *right annihilator of*  $\beta$  and it is denoted  $(0:\beta)_R$ . (2) The relative left annihilator of  $\beta$  modulo  $\delta$ , denoted  $(\delta : \beta)_L$ , is the largest congruence  $\alpha \in \text{Con}(\mathbf{A})$  satisfying  $\mathbf{C}(\alpha, \beta; \delta)$ . If there is a largest congruence  $\alpha \in \text{Con}(\mathbf{A})$  satisfying  $\mathbf{C}(\beta, \alpha; \delta)$  it is called the *relative right annihilator of*  $\beta$  modulo  $\delta$ , and it is denoted  $(\delta : \beta)_R$ .

The first new result in this section shows that if a variety contains an algebra whose congruence lattice contains a certain kind of pentagon with an abelian critical interval, then the variety contains an algebra with a pentagon satisfying other (usually stronger) abelianness conditions.



FIGURE 2.  $CON(\mathbf{A})$  OR  $CON(\mathbf{B})$ .

**Theorem 3.7.** (Better pentagons) Let  $\mathcal{V}$  be an arbitrary variety and assume that  $\mathcal{V}$  contains an algebra  $\mathbf{A}$  with congruences  $\beta, \theta, \delta$  generating a pentagon, as shown in Figure 2. Assume that  $\mathbf{C}(\theta, \theta; \delta)$  holds and  $\mathbf{C}(\beta, \theta; \delta)$  fails.<sup>2</sup> There exists an algebra  $\mathbf{B} \in \mathcal{V}$  with congruences ordered as in Figure 2 and satisfying  $\mathbf{C}(\alpha, \alpha; \beta)$  and  $\mathbf{C}(\theta, \theta; 0)$ .

*Proof.* Let **A** have congruences  $\beta$ ,  $\theta$ ,  $\delta$  with the properties described. We will find a pentagon of the desired type in the congruence lattice of the algebra  $\mathbf{A}(\beta)$ .

The desired pentagon will be the one depicted in Figure 3. The congruence  $\gamma$  that appears in this figure will be defined in the course of the proof.

To establish that the congruences indicated form a pentagon with the required properties we will use the fact that  $\mathbf{C}(\beta, \theta; \delta)$  fails. According to Definition 3.3, this assumption yields a  $\beta, \theta$ -matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} t(\mathbf{a}, \mathbf{c}) & t(\mathbf{a}, \mathbf{d}) \\ t(\mathbf{b}, \mathbf{c}) & t(\mathbf{b}, \mathbf{d}) \end{bmatrix}$$

where  $(a_i, b_i) \in \beta$ ,  $(c_j, d_j) \in \theta$ ,  $(p, q) \in \delta$ , and  $(r, s) \in \theta - \delta$ . This implies that  $((r, p), (s, q)) \in \Delta_{\beta,\theta} \cap \delta_2$ ,  $((r, p), (s, q)) \in \Delta_{\beta,\theta} \cap \theta_1$ , but  $((r, p), (s, q)) \notin \delta_1$ .

The comparabilities indicated in the chain on the left side in Figure 3, namely

(3.8)  $\delta_1 \cap \Delta_{\beta,\theta} \leq \delta_1 \leq \delta_1 + (\delta_2 \cap \Delta_{\beta,\theta}),$ 

<sup>&</sup>lt;sup>2</sup>The assumption that " $\mathbf{C}(\beta, \theta; \delta)$  fails" will always hold if  $\mathcal{V}$  has a Taylor term – see Theorem 4.2.



FIGURE 3. A SUBLATTICE OF  $CON(\mathbf{A}(\beta))$ .

are obvious: the middle congruence  $\delta_1$  is a meetand on the left of (3.8) and a join on the right of (3.8). To show that the middlemost and the rightmost elements of chain (3.8) are distinct, observe that ((r, p), (s, q)) lies in the rightmost congruence of (3.8) but not in the middlemost. To show that the middlemost and leftmost elements of chain (3.8) are distinct, choose  $(u, v) \in \beta - \theta$ . This is possible since  $\beta \not\leq \theta$  (see Figure 2). Both pairs (u, u), (u, v) belong to  $\mathbf{A}(\beta)$  and  $((u, u), (u, v)) \in \delta_1$ . We have  $\Delta_{\beta,\theta} \leq \theta_1 \times \theta_2$  by (3.1), so  $\delta_1 \cap \Delta_{\beta,\theta} \leq \delta_1 \times \theta_2$ . However the pair ((u, u), (u, v)) does not belong to  $\delta_1 \times \theta_2$ , since  $(u, v) \notin \theta$ . This shows that ((u, u), (u, v)) is contained in the middlemost element of the chain in (3.8), but not the leftmost.

Let us use " $\gamma_1$ " to denote the top congruence  $\delta_1 + \delta_2 \cap \Delta_{\beta,\theta}$  in Figure 3. Here, note that since  $\delta_1 + \delta_2 \cap \Delta_{\beta,\theta}$  is strictly above  $\delta_1$ , it is indeed of the form  $\gamma_1$  for some  $\gamma \in \text{Con}(\mathbf{A})$  satisfying  $\gamma > \delta$ . Moreover, since  $\gamma_1 = \delta_1 + (\delta_2 \cap \Delta_{\beta,\theta}) \stackrel{(3.1)}{\leq} \delta_1 + (\delta_2 \cap (\theta_1 \times \theta_2)) = \delta_1 + (\theta_1 \times \delta_2) = \theta_1$  we have that

$$(3.9) \qquad \qquad \delta < \gamma \le \theta$$

in  $\operatorname{Con}(\mathbf{A})$ . Note also that

$$(3.10) \qquad \qquad \Delta_{\beta,\gamma} \le \gamma_1 \times \gamma_2 \le \gamma_1$$

in  $\operatorname{Con}(\mathbf{A}(\beta))$ .

We have a chain on the right side of Figure 3, namely

$$\begin{aligned} \delta_1 \cap \Delta_{\beta,\theta} &\leq (\delta_1 \cap \Delta_{\beta,\theta}) + (\delta_2 \cap \Delta_{\beta,\theta}) & (\delta_2 \cap \Delta_{\beta,\theta}) \\ &\leq \Delta_{\beta,\gamma} + (\delta_1 \cap \Delta_{\beta,\theta}) + (\delta_2 \cap \Delta_{\beta,\theta}) & \Delta_{\beta,\gamma} \\ &\leq \delta_1 + \delta_2 \cap \Delta_{\beta,\theta} = \gamma_1 & \delta_1. \end{aligned}$$

One sees that, on each of these lines, the congruence immediately to the right of the " $\leq$ " is obtained from the preceding congruence in the chain by joining the additional congruence indicated on the same line at the far right. The claim just made involves three assertions, the first two of which are formally true. For the third claim, which is the claim that we

obtain  $\gamma_1$  when we join  $\delta_1$  to  $\Omega = \Delta_{\beta,\gamma} + (\delta_1 \cap \Delta_{\beta,\theta}) + (\delta_2 \cap \Delta_{\beta,\theta})$ , we use (3.10) in the last step of the following computation.

$$\delta_{1} + \Omega = \delta_{1} + (\Delta_{\beta,\gamma} + (\delta_{1} \cap \Delta_{\beta,\theta}) + (\delta_{2} \cap \Delta_{\beta,\theta}))$$
  
$$= \Delta_{\beta,\gamma} + (\delta_{1} + (\delta_{1} \cap \Delta_{\beta,\theta})) + (\delta_{2} \cap \Delta_{\beta,\theta}))$$
  
$$= \Delta_{\beta,\gamma} + (\delta_{1} + \delta_{2} \cap \Delta_{\beta,\theta})$$
  
$$= \Delta_{\beta,\gamma} + \gamma_{1}$$
  
$$= \gamma_{1}.$$

Next, we argue that the comparability

(3.11)  $\Psi := (\delta_1 \cap \Delta_{\beta,\theta}) + (\delta_2 \cap \Delta_{\beta,\theta}) \le \Delta_{\beta,\gamma} + (\delta_1 \cap \Delta_{\beta,\theta}) + (\delta_2 \cap \Delta_{\beta,\theta}) =: \Omega,$  is strict.

# Claim 3.12. $\Psi < \Omega$ .

Proof of Claim 3.12. If  $(u, v) \in \gamma - \delta$ , then the pair  $((u, u), (v, v)) \in \Delta_{\beta,\gamma}$  will belong to  $\Omega$ , since  $\Delta_{\beta,\gamma}$  is a summand of  $\Omega$ . We shall argue that  $((u, u), (v, v)) \notin \Psi$ . Our path will be to show that the  $\Psi$ -class of (u, u) is contained in the  $\delta_1 \times \delta_2$ -class of (u, u). This will suffice, since we have  $(u, v) \notin \delta$  and therefore  $((u, u), (v, v)) \notin \delta_1 \times \delta_2$ , so we will be able to derive that  $((u, u), (v, v)) \notin \Psi$ .

Since Figure 2 is a pentagon in which  $\beta \cap (\theta + (\beta \cap \delta)) \leq \delta$  holds, Theorem 3.4 (8) guarantees that the relation  $\mathbf{C}(\theta, \beta; \delta)$  holds. This property can be restated in this way: if D is the set of pairs  $(x, y) \in \mathbf{A}(\beta)$  that satisfy  $(x, y) \in \delta$ , then the subset  $D \subseteq \mathbf{A}(\beta)$  is a union of  $\Delta_{\beta,\theta}$ -classes. In other words,  $(e, f) \in \delta$  and  $(e, f) \stackrel{\Delta_{\beta,\theta}}{\equiv} (g, h)$  jointly imply  $(g, h) \in \delta$ . We can apply this information to the intersection congruence  $\delta_1 \cap \Delta_{\beta,\theta}$  to derive that if

- $(e, f) \stackrel{\Delta_{\beta, \theta}}{\equiv} (g, h),$
- $(e, f) \in \delta$ , and we also have
- $(e,g) \in \delta$ ,

then  $(g,h), (f,h) \in \delta$ . (Here  $(f,h) \in \delta$  follows from  $f \stackrel{\delta}{\equiv} e \stackrel{\delta}{\equiv} g \stackrel{\delta}{\equiv} h$ .) In conclusion, if  $(e,f) \in \delta$  and  $((e,f), (g,h)) \in \delta_1 \cap \Delta_{\beta,\theta}$ , then necessarily  $(g,h) \in \delta$  and  $((e,f), (g,h)) \in \delta_2 \cap \Delta_{\beta,\theta}$ . This establishes that, when  $(e,f) \in \delta$ , the  $\delta_1 \cap \Delta_{\beta,\theta}$ -class of (e,f) agrees with the  $\delta_2 \cap \Delta_{\beta,\theta}$ -class of (e,f), and hence agrees with the  $\Psi$ -class of (e,f) ( $\Psi$  is the join of  $\delta_1 \cap \Delta_{\beta,\theta}$  and  $\delta_2 \cap \Delta_{\beta,\theta}$ ). Therefore, when  $(e,f) \in \delta$ , the  $\Psi$ -class of (e,f) is contained in the  $\delta_1 \times \delta_2$ -class of (e,f). By applying this reasoning to (e,f) = (u,u) we see that  $((u,u), (v,v)) \notin \Psi$ , since the  $\delta_1 \times \delta_2$ -class of (u,u) does not contain (v,v).

What remains to do to establish that our congruences form a pentagon is to show that (i)  $\delta_1 + \Psi = \gamma_1$  and (ii)  $\delta_1 \cap \Omega = \delta_1 \cap \Delta_{\beta,\theta}$ . For the first of these we calculate that

$$\delta_1 + \Psi = (\delta_1 + (\delta_1 \cap \Delta_{\beta,\theta})) + (\delta_2 \cap \Delta_{\beta,\theta}) = \delta_1 + (\delta_2 \cap \Delta_{\beta,\theta}) = \gamma_1.$$

For (ii), we use the fact  $\gamma \leq \theta$  from (3.9) to derive that  $\Delta_{\beta,\gamma} \leq \Delta_{\beta,\theta}$ . Therefore all summands in

$$\Omega = \Delta_{\beta,\gamma} + (\delta_1 \cap \Delta_{\beta,\theta}) + (\delta_2 \cap \Delta_{\beta,\theta})$$

lie below  $\Delta_{\beta,\theta}$ . Since one of the summands is  $\delta_1 \cap \Delta_{\beta,\theta}$ , we obtain

$$\delta_1 \cap \Delta_{\beta,\theta} \le \Omega \le \Delta_{\beta,\theta}.$$

If we meet this chain throughout with  $\delta_1$  we obtain

$$\delta_1 \cap \Delta_{\beta,\theta} \le \delta_1 \cap \Omega \le \delta_1 \cap \Delta_{\beta,\theta},$$

or  $\delta_1 \cap \Omega = \delta_1 \cap \Delta_{\beta,\theta}$ , which is what (ii) asserts.

We have established the pentagon shape, so what is left is to establish that the asserted centralities hold. In (3.9) we showed above that  $\delta < \gamma \leq \theta$  in Con(**A**). Since  $\mathbf{C}(\theta, \theta; \delta)$  holds in Con(**A**), we get  $\mathbf{C}(\theta_1, \theta_1; \delta_1)$  in Con( $\mathbf{A}(\beta)$ ) by Theorem 3.4 (10) and the Correspondence Theorem. We then get  $\mathbf{C}(\gamma_1, \gamma_1; \delta_1)$  by monotonicity (Theorem 3.4 (1)). This shows that the interval  $I[\delta_1, \gamma_1]$  between  $\delta_1$  and the top of the pentagon,  $\gamma_1$ , is abelian. Using this, we can derive that the interval  $I[\delta_1 \cap \Delta_{\beta,\theta}, \Omega]$  between  $\Omega = \Delta_{\beta,\gamma} + (\delta_1 \cap \Delta_{\beta,\theta}) + (\delta_2 \cap \Delta_{\beta,\theta})$  and the bottom of the pentagon is also abelian, as follows: By the facts that  $\mathbf{C}(\gamma_1, \gamma_1; \delta_1)$  and  $\Omega \leq \gamma_1$ , we can get  $\mathbf{C}(\Omega, \Omega; \delta_1)$ . We always have  $\mathbf{C}(\Omega, \Omega; \Omega)$  according to Theorem 3.4 (8). Then, by Theorem 3.4 (6), we get  $\mathbf{C}(\Omega, \Omega; \delta_1 \cap \Omega)$ , which is the claim that the interval between  $\Omega$  and the bottom of the pentagon is abelian. This completes the proof of the theorem up to relabeling the congruences in Figure 3. (In particular, since the bottom element of Figure 3 is labeled 0, we should factor our algebra and take  $\mathbf{B} = \mathbf{A}(\beta)/(\delta_1 \cap \Delta_{\beta,\theta})$ .)

**Lemma 3.13.** Assume that **A** is an algebra whose commutator operation is not commutative. Some quotient **B** of **A** will have congruences  $\alpha, \beta \in \text{Con}(\mathbf{B})$  such that  $[\beta, \alpha] = 0 < [\alpha, \beta]$ .

*Proof.* For this proof (and later proofs) we will adopt "relative commutator" notation first introduced above [5, Theorem 4.22]. This notation is useful for discussing the relationship between the commutator operation in  $\mathbf{A}$  and the commutator operations on quotients of  $\mathbf{A}$ . Define

 $[\alpha,\beta]_{\varepsilon}:=\bigcap\{\gamma\mid (\gamma\geq\varepsilon) \text{ and } \mathbf{C}(\alpha,\beta;\gamma)\}.$ 

It is easy to see from Theorem 3.4 (10) that this notation has the property that if  $\varepsilon \leq \alpha, \beta$ , then  $[\alpha/\varepsilon, \beta/\varepsilon] = [\alpha, \beta]_{\varepsilon}/\varepsilon$ , so the ordinary (= unsubscripted) commutator operation [-, -]on  $\operatorname{Con}(\mathbf{A}/\varepsilon)$  is reflected by the operation  $[-, -]_{\varepsilon}$  on the interval  $I[\varepsilon, 1]$  of  $\operatorname{Con}(\mathbf{A})$ .

If  $\mathbf{A} \in \mathcal{V}$  has noncommutative commutator, then it has congruences  $\gamma, \delta \in \operatorname{Con}(\mathbf{A})$  such that  $[\gamma, \delta] \not\leq [\delta, \gamma]$ . Set  $\varepsilon = [\delta, \gamma]$ . This is a congruence which lies below both  $\gamma$  and  $\delta$ . The fact that  $[\gamma, \delta] \not\leq [\delta, \gamma] = \varepsilon$  implies that  $\mathbf{C}(\gamma, \delta; \varepsilon)$  fails, hence  $[\gamma, \delta]_{\varepsilon} \neq \varepsilon = [\delta, \gamma]$ . But  $[\gamma, \delta]_{\varepsilon} \geq \varepsilon$  from the definition of the relative commutator notation. Hence we have  $[\delta, \gamma]_{\varepsilon} = [\delta, \gamma] < [\gamma, \delta]_{\varepsilon}$ . This means that the algebra  $\mathbf{A}/\varepsilon$  has congruences  $\delta/\varepsilon, \gamma/\varepsilon$  satisfying  $[\delta/\varepsilon, \gamma/\varepsilon] = 0 < [\gamma/\varepsilon, \delta/\varepsilon]$ . By changing notation to work modulo  $\varepsilon$  we have a quotient of  $\mathbf{A}$  with congruences  $\alpha = \gamma/\varepsilon, \beta = \delta/\varepsilon$  satisfying  $[\beta, \alpha] = 0 < [\alpha, \beta]$ .

We will use Lemma 3.13 in the next result where we connect left and right distributivity of the commutator with commutativity of the commutator.

**Theorem 3.14.** Let  $\mathcal{V}$  be an arbitrary variety.

(1) If the commutator is left distributive throughout  $\mathcal{V}$ ,

$$(\forall x, y, z) \ [x + y, z] = [x, z] + [y, z],$$

then it is also commutative throughout  $\mathcal{V}$ 

$$(\forall x, y) \ [x, y] = [y, x].$$

(2) If the commutator is right distributive throughout  $\mathcal{V}$ ,

$$(\forall x, y, z) \ [x, y + z] = [x, y] + [x, z],$$

then the commutator satisfies the following "partial commutativity" on comparable pairs of congruences.

$$(\forall x, y) \ (y \le x) \Rightarrow [x, y] \le [y, x].$$

*Proof.* We start by proving the contrapositive form of Item (1), so assume that the commutator fails to be commutative throughout  $\mathcal{V}$ . There must be some  $\mathbf{A} \in \mathcal{V}$  that has congruences  $\alpha, \beta \in \text{Con}(\mathbf{A})$  such that  $[\alpha, \beta] \not\leq [\beta, \alpha]$ . By Lemma 3.13 we may assume that  $[\beta, \alpha] = 0 < [\alpha, \beta]$ .

Recall that  $\eta_1 = 0_1 = \{((w, x), (y, z)) \in A(\alpha)^2 \mid w = y\}$  and  $\eta_2 = 0_2 = \{((w, x), (y, z)) \in A(\alpha)^2 \mid x = z\}$  are the restrictions of the coordinate projection kernels of  $\mathbf{A}^2$  to the subalgebra  $\mathbf{A}(\alpha)$ .

Let  $\delta: \mathbf{A} \to \mathbf{A}(\alpha): x \mapsto (x, x)$  be the diagonal embedding. We will write D for the settheoretic image  $\delta(A)$  and  $\mathbf{D}$  for the algebra-theoretic image  $\delta(\mathbf{A})$  (the diagonal subuniverse of  $\mathbf{A}(\alpha)$ ). For a congruence  $\theta \in \operatorname{Con}(\mathbf{A})$ , we write  $\delta(\theta)$  for  $\{((x, x), (y, y)) \in A(\alpha)^2 \mid (x, y) \in \theta\}$ , which is a congruence on  $\mathbf{D}$ .

**Claim 3.15.** The diagonal subuniverse  $D \leq \mathbf{A}(\alpha)$  is a union of singleton  $([\eta_1, \Delta_{\alpha,\beta}] + [\eta_2, \Delta_{\alpha,\beta}])$ -classes.

Proof of Claim 3.15. Since  $[\beta, \alpha] = 0$ , the set D is a union of  $\Delta_{\alpha,\beta}$ -classes. No two elements of D are related by  $\eta_1$ , so every element of D is a singleton  $(\eta_1 \cap \Delta_{\alpha,\beta})$ -class. Since  $[\eta_1, \Delta_{\alpha,\beta}]$ is contained in  $\eta_1 \cap \Delta_{\alpha,\beta}$ , every element of D is a singleton  $[\eta_1, \Delta_{\alpha,\beta}]$ -class. Similarly every element of D is a singleton  $[\eta_2, \Delta_{\alpha,\beta}]$ -class. Therefore every element of D is a singleton  $([\eta_1, \Delta_{\alpha,\beta}] + [\eta_2, \Delta_{\alpha,\beta}])$ -class.

Claim 3.16. The diagonal subuniverse  $D \leq \mathbf{A}(\alpha)$  is not a union of singleton  $([\eta_1 + \eta_2, \Delta_{\alpha,\beta}])$ classes.

Proof of Claim 3.16. We will show that the restriction of the congruence  $[\eta_1 + \eta_2, \Delta_{\alpha,\beta}]$  to D is not the equality relation, and this will prove that D is not a union of singleton  $([\eta_1 + \eta_2, \Delta_{\alpha,\beta}])$ -classes. For this, notice that

$$[\eta_1 + \eta_2, \Delta_{\alpha,\beta}]|_{\mathbf{D}} \ge [(\eta_1 + \eta_2)|_{\mathbf{D}}, \Delta_{\alpha,\beta}|_{\mathbf{D}}] = [\delta(\alpha), \delta(\beta)] = \delta([\alpha, \beta]) > 0.$$

The leftmost inequality is derived from Theorem 3.4 (9).

Claims 3.15 and 3.16 show that  $[\eta_1 + \eta_2, \Delta_{\alpha,\beta}] \neq [\eta_1, \Delta_{\alpha,\beta}] + [\eta_2, \Delta_{\alpha,\beta}]$ , so the commutator is not left distributive on  $\mathbf{A}(\alpha)$ .

Next we argue the contrapositive form of Item (2) of the theorem. Assume that there is some  $\mathbf{A} \in \mathcal{V}$  that has congruences  $\alpha, \beta \in \text{Con}(\mathbf{A})$  such that (i)  $\beta \leq \alpha$  but (ii)  $[\alpha, \beta] \not\leq [\beta, \alpha]$ . Consulting the proof of Lemma 3.13, we see that we may refine these assumptions to (i)  $\beta \leq \alpha$  and (ii)  $[\beta, \alpha] = 0 < [\alpha, \beta]$ .

Claim 3.17. The diagonal subuniverse  $D \leq \mathbf{A}(\alpha)$  is a union of singleton  $([\eta_1, \eta_2] + [\eta_1, \Delta_{\alpha,\beta}])$ classes.

Proof of Claim 3.17. We have  $[\eta_1, \eta_2] \leq \eta_1 \cap \eta_2 = 0$ , so  $[\eta_1, \eta_2]$  is the equality relation on  $\mathbf{A}(\alpha)$ and all  $[\eta_1, \eta_2]$ -classes are singletons. Hence the subuniverse D consists of singleton  $[\eta_1, \eta_2]$ classes. We may copy the proof of Claim 3.15 to establish that D is a union of singleton  $[\eta_1, \Delta_{\alpha,\beta}]$ -classes. (The situation here is the same as the one there, except here we have the extra property that  $\beta \leq \alpha$ .) It follows that D is a union of singleton  $([\eta_1, \eta_2] + [\eta_1, \Delta_{\alpha,\beta}])$ classes.

**Claim 3.18.** *D* is not a union of singleton  $[\eta_1, \eta_2 + \Delta_{\alpha,\beta}]$ -classes.

Proof of Claim 3.18. For this proof, note that  $\eta_2 + \Delta_{\alpha,\beta} = \beta_2$ .

We started the proof by arranging that  $[\alpha, \beta] > 0$ . This means that there is an  $\alpha, \beta$ -matrix

$$\begin{bmatrix} t(\mathbf{a}, \mathbf{u}) & t(\mathbf{a}, \mathbf{v}) \\ t(\mathbf{b}, \mathbf{u}) & t(\mathbf{b}, \mathbf{v}) \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad \mathbf{a} \alpha \mathbf{b}, \quad \mathbf{u} \beta \mathbf{v},$$

with p = q but  $r \neq s$ . Consider the  $\eta_1, \beta_2$ -matrix of  $\mathbf{A}(\alpha)$ 

(M) 
$$\begin{bmatrix} t\left((\mathbf{b},\mathbf{a}),(\mathbf{u},\mathbf{u})\right) & t\left((\mathbf{b},\mathbf{a}),(\mathbf{u},\mathbf{v})\right) \\ t\left((\mathbf{b},\mathbf{b}),(\mathbf{u},\mathbf{u})\right) & t\left((\mathbf{b},\mathbf{b}),(\mathbf{u},\mathbf{v})\right) \end{bmatrix} = \begin{bmatrix} (r,p) & (r,q) \\ (r,r) & (r,s). \end{bmatrix}$$

The fact that this truly is an  $\eta_1, \beta_2$ -matrix of  $\mathbf{A}(\alpha)$  follows from our assumption that  $\beta \leq \alpha$ (and this is the only place in the argument where this assumption is needed). Namely, to know that (M) is an  $\eta_1, \beta_2$ -matrix of  $\mathbf{A}(\alpha)$  we need to know that  $\mathbf{A}(\alpha)$  contains all pairs of the form  $(b_i, a_i), (u_i, u_i), (b_i, b_i)$ , and  $(u_i, v_i)$ . It is easy to see that  $\mathbf{A}(\alpha)$  contains all pairs of these types except possibly those of the type  $(u_i, v_i)$ . Such pairs lie in  $\beta$ , so if  $\beta \leq \alpha$  they will also lie in  $A(\alpha) = \alpha$ . We have (r, p) = (r, q), so ((r, r), (r, s)) belongs to  $[\eta_1, \beta_2]$ . Since  $(r, r) \in D$  and  $(r, s) \notin D$ , the  $[\eta_1, \beta_2]$ -class of  $(r, r) \in D$  is not a singleton, hence D is not a union of singleton  $[\eta_1, \beta_2]$ -classes.

Claims 3.17 and 3.18 show that  $[\eta_1, \eta_2 + \Delta_{\alpha,\beta}] \neq [\eta_1, \eta_2] + [\eta_1, \Delta_{\alpha,\beta}]$ , so the commutator is not right distributive on  $\mathbf{A}(\alpha)$ .

#### 4. Main results

Now we prove results which seem to require a ground Maltsev condition. The most important theorems of this section will be proved under the assumption of "existence of a Taylor term", [8, Definition 2.15]. A Taylor term for a variety  $\mathcal{V}$  is a term  $T(x_1, \ldots, x_n)$ such that  $\mathcal{V}$  satisfies the identity  $T(x, \ldots, x) \approx x$  and, for each *i* between 1 and  $n, \mathcal{V}$  satisfies some identity of the form  $T(\mathbf{w}) \approx T(\mathbf{z})$  with  $w_i \neq z_i$ . Any identity of the form  $T(\mathbf{w}) \approx T(\mathbf{z})$ with  $w_i \neq z_i$  is called an "*i*-th place Taylor identity" of T.

Some of the results of this section will be proved under the stronger assumptions "existence of a difference term" or "existence of a weak difference term", Definition 4.1. The class of varieties with a difference term is definable by a Maltsev condition. The same is true for the class of varieties with a weak difference term. The Maltsev conditions were identified in principle in [10] in Theorem 4.8 and the paragraph following the proof of that theorem. We define weak and ordinary difference terms next.

**Definition 4.1.** Let  $\mathcal{V}$  be a variety. A ternary  $\mathcal{V}$ -term t(x, y, z) shall be called

- (1) a right Maltsev term for  $\mathcal{V}$  if  $\mathcal{V} \models t(x, x, y) \approx y$ .
- (2) a left Maltsev term for  $\mathcal{V}$  if  $\mathcal{V} \models t(x, y, y) \approx x$ .
- (3) a *Maltsev term* for  $\mathcal{V}$  if it is both a right and left Maltsev term.
- (4) a right difference term for  $\mathcal{V}$  if, for any  $\mathbf{B} \in \mathcal{V}$ ,  $t^{\mathbf{B}}(a, a, b) = b$  holds whenever the pair (a, b) is contained in an abelian congruence.
- (5) a left difference term for  $\mathcal{V}$  if, for any  $\mathbf{B} \in \mathcal{V}$ ,  $t^{\mathbf{B}}(a, b, b) = a$  holds whenever the pair (a, b) is contained in an abelian congruence.
- (6) a weak difference term for  $\mathcal{V}$  if it is both a right and left difference term.
- (7) a difference term for  $\mathcal{V}$  if it is a right Maltsev term and a left difference term.

This left/right terminology is not standard, but it is introduced here because the new concept "right difference term" will play a role in the proof of Theorem 4.5 (e.g., in Claim 4.7).

As we noted in the Introduction, there is a weakest nontrivial idempotent Maltsev condition. The class of varieties defined by this condition is the the class of varieties with a Taylor term. As noted before Definition 4.1, the classes of varieties with (i) a weak difference term or (ii) an ordinary difference term are also definable by idempotent Maltsev conditions. Ordinary difference terms are formally stronger than weak difference terms, which are stronger than Taylor terms. These differences in strength are strict. The algebra I of Example 4.4 generates a variety that has a Taylor term, but does not have a weak difference term. (The justification for this claim is given in that example.) The semigroup  $\mathbb{Z}_2 \times \mathbb{S}_2$  described in Section 2 generates a variety with a weak difference term, but with no difference term. (The justification for this claim is given in that section.)

Throughout this section, the assumption that the variety under consideration has a Taylor term will be used to invoke the following theorem, which expresses limits on the behavior of the centralizer relation.

**Theorem 4.2.** ([8, Theorem 4.16(2)(i)]) Let  $\mathcal{V}$  be a variety that has a Taylor term and let **A** be a member of  $\mathcal{V}$ . There is no pentagon sublattice of Con(**A**), labeled as in Figure 4, such that  $\mathbf{C}(\beta, \theta; \delta)$  holds.



FIGURE 4. (THEOREM 4.2) A PENTAGON IN  $CON(\mathbf{A})$  WILL NOT SATISFY  $C(\beta, \theta; \delta)$ .

We restate this theorem in a positive and formally stronger way.

**Theorem 4.3.** Let  $\mathcal{V}$  be a variety that has a Taylor term and let  $\mathbf{A}$  be a member of  $\mathcal{V}$ . Given any pentagon sublattice of  $\operatorname{Con}(\mathbf{A})$ , labeled as in Figure 5,  $[\alpha, x]_{\delta} = x$  holds for every  $x \in I[\delta, \theta]$ . (Equivalently, if  $\delta \leq y < x \leq \theta$ , then  $\mathbf{C}(\alpha, x; y)$  fails.)



FIGURE 5. IF  $\delta \leq x \leq \theta$ , THEN  $[\alpha, x]_{\delta} = x$ .

There are two observations to make to see that Theorem 4.2 and Theorem 4.3 have the same content. The first observation is: (i)  $\mathbf{C}(\beta, \theta; \delta)$  holds in the pentagon of Figure 4 if and only if (ii)  $\mathbf{C}(\alpha, \theta; \delta)$  holds in that pentagon. One obtains (ii) $\Rightarrow$ (i) by monotonicity of the centralizer in its first place (Theorem 3.4 (1)). One obtains (i) $\Rightarrow$ (ii) by assuming  $\mathbf{C}(\beta, \theta; \delta)$ ,

deriving  $\mathbf{C}(\delta, \theta; \delta)$  from Theorem 3.4 (8), then deriving  $\mathbf{C}(\beta + \delta, \theta; \delta)$  from Theorem 3.4 (5). Since  $\alpha = \beta + \delta$ , we get that (ii) holds.

The second observation is that if  $\delta \leq y < x \leq \theta$ , then  $\{\alpha, \beta, x, y, \beta \cap y\}$  is another pentagon in Con(**A**). Applying Theorem 4.2 to this new pentagon we get that  $\mathbf{C}(\beta, x; y)$  fails. Using the idea of the preceding paragraph, this is equivalent to the assertion that  $\mathbf{C}(\alpha, x; y)$  fails whenever  $\delta \leq y < x \leq \theta$ . In particular, since  $\delta \leq [\alpha, x]_{\delta} \leq x \leq \theta$  and  $\mathbf{C}(\alpha, x; [\alpha, x]_{\delta})$  holds, we cannot have  $[\alpha, x]_{\delta} < x$ . The alternative is that  $[\alpha, x]_{\delta} = x$  whenever  $\delta \leq x \leq \theta$ . On the other hand, if we have  $[\alpha, x]_{\delta} = x$  for  $x = \theta$ , then we recover the conclusion of Theorem 4.2.

The first main theorem of this section gives a commutator-theoretic characterization of varieties with a weak difference term (Theorem 4.5). Using this characterization, we shall deduce that any variety with a Taylor term and commutative commutator must have a weak difference term (Theorem 4.22). The proofs of these two results were developed from an analysis of a simple example, which we describe first. The inclusion of this example is meant to help guide the reader through the lengthy proof of Theorem 4.5.

**Example 4.4.** Let  $\mathbb{R}$  be the real line considered as a 1-dimensional real vector space. Let  $\mathbb{R}^{\circ}$  be the reduct of  $\mathbb{R}$  to the idempotent linear operations of the form  $f_r(x, y) = rx + (1 - r)y$ , 0 < r < 1. Let  $\mathbf{I}$  be the subalgebra of  $\mathbb{R}^{\circ}$  whose universe is the unit interval I = [0, 1]. Thus,  $\mathbf{I} = \langle [0, 1]; \{f_r(x, y) \mid 0 < r < 1\} \rangle$  is a subalgebra of a reduct of an abelian algebra  $\mathbb{R}$ , which makes  $\mathbf{I}$  an abelian algebra. From Definition 4.1, the concepts of "Maltsev term", "difference term", and "weak difference term" all coincide for abelian algebras. The fact that I is closed under all of the  $f_r$ -operations and is not closed under the unique Maltsev operation x - y + z of  $\mathbb{R}$  shows that neither  $\mathbb{R}^{\circ}$  nor  $\mathbf{I}$  have Maltsev operations, and therefore neither  $\mathbb{R}^{\circ}$  nor  $\mathbf{I}$  has a weak difference term.  $\mathcal{V}(\mathbf{I})$  does have a Taylor term, namely  $T(x, y) = f_{\frac{1}{2}}(x, y) = \frac{1}{2}x + \frac{1}{2}y$ . This is a Taylor operation for  $\mathcal{V}(\mathbf{I})$ , since  $T(x, x) \approx x$  and  $T(x, y) \approx T(y, x)$  hold in  $\mathbf{I}$ , and the latter is both a first-place and a second-place Taylor identity for T(x, y) in  $\mathcal{V}(\mathbf{I})$ .

The algebra  $\mathbf{I}$  is free in  $\mathcal{V}(\mathbf{I})$  over the 2-element generating set  $\{0, 1\}$ . We identify some congruences in  $\text{Con}(\mathbf{I} \times \mathbf{I})$  and indicate the sublattice they generate in Figure 6.



FIGURE 6. A SUBLATTICE OF  $\operatorname{Con}(\mathbf{I} \times \mathbf{I})$ .

The projection kernels are the congruences  $\eta_1 = Cg(((0,0), (0,1)), ((1,0), (1,1)))$  and  $\eta_2 = Cg(((0,0), (1,0)), ((0,1), (1,1)))$ . The congruence  $\eta_1$  partitions the "square"  $\mathbf{I} \times \mathbf{I}$  into



FIGURE 7. THE PARTITION OF  $\mathbf{I} \times \mathbf{I}$  INDUCED BY  $\delta = Cg(((0,0),(1,0)))$ .

congruence classes that are "vertical lines" and  $\eta_2$  partitions  $\mathbf{I} \times \mathbf{I}$  into congruence classes that are "horizontal lines". The interesting congruence is  $\delta = \text{Cg}(((0,0),(1,0)))$ . The partition of  $\mathbf{I} \times \mathbf{I}$  it yields is depicted in Figure 7. In the partition depicted in Figure 7, all congruence classes of  $\delta$  agree with those of  $\eta_2$  except the class that is the "top line",  $X := \mathbf{I} \times \{1\}$ . The top line X is a single  $\eta_2$ -class, and hence a union of  $\delta$ -classes, but  $\delta$  restricts to be the equality relation on the top line X while  $\eta_2$  restricts to be the total relation.

This unusual structure for  $\delta$  can be exploited the following way. The operation  $T(x, y) = f_{\frac{1}{2}}(x, y) = \frac{1}{2}x + \frac{1}{2}y$  may be used to create a 1,  $\eta_2$ -matrix

$$\begin{bmatrix} T\left((1,0),(0,1)\right) & T\left((1,0),(1,1)\right) \\ T\left((1,1),(0,1)\right) & T\left((1,1),(1,1)\right) \end{bmatrix} = \begin{bmatrix} (.5,.5) & (1,.5) \\ (.5,1) & (1,1) \end{bmatrix}$$

where the elements on the top row are  $\delta$ -related while the elements on the bottom row are not. This matrix witnesses that  $\neg \mathbf{C}(1, \eta_2; \delta)$ . In the quotient  $(\mathbf{I} \times \mathbf{I})/\delta$  we must have  $[\overline{1}, \overline{\eta}_2] > 0$ where  $\overline{\eta}_2 := \eta_2/\delta$  and  $\overline{1} := 1/\delta$ . It is possible to argue that  $[\overline{\eta}_2, \overline{1}] = 0$ , and therefore that  $[\overline{\eta}_2, \overline{1}] \neq [\overline{1}, \overline{\eta}_2]$ . Although this example is special, the location of noncommutativity in  $\mathcal{V}$  is general as we shall see in the proofs of the next two results.

**Theorem 4.5.** The following are equivalent for a variety  $\mathcal{V}$ .

- (1)  $\mathcal{V}$  has a weak difference term.
- (2) Whenever  $\mathbf{A} \in \mathcal{V}$  and  $\alpha \in \operatorname{Con}(\mathbf{A})$  is abelian, the interval  $I[0, \alpha]$  consists of permuting equivalence relations.
- (3) Whenever  $\mathbf{A} \in \mathcal{V}$  and  $\alpha \in \operatorname{Con}(\mathbf{A})$  is abelian, the interval  $I[0, \alpha]$  is modular.
- (4) Whenever  $\mathbf{A} \in \mathcal{V}$  and  $\alpha \in \operatorname{Con}(\mathbf{A})$ , there is no pentagon labeled as in Figure 8 with  $[\alpha, \alpha] = 0$ . (No "spanning pentagon" in  $I[0, \alpha]$  if  $\alpha$  is abelian.)
- (5) Whenever  $\mathbf{A} \in \mathcal{V}$  and  $\alpha \in \text{Con}(\mathbf{A})$ , there is no pentagon labeled as in Figure 8 where  $[\alpha, \alpha] = 0$  and  $\mathbf{C}(\theta, \alpha; \delta)$ .

*Proof.* [(1)  $\Rightarrow$  (2)] Assume that t(x, y, z) is a weak difference term for  $\mathcal{V}$ ,  $\mathbf{A} \in \mathcal{V}$ , and  $\alpha \in \operatorname{Con}(\mathbf{A})$  is abelian. If  $\sigma, \tau \in I[0, \alpha]$  and  $a \stackrel{\sigma}{\equiv} b \stackrel{\tau}{\equiv} c$ , then  $a = t^{\mathbf{A}}(a, b, b) \stackrel{\tau}{\equiv} t^{\mathbf{A}}(a, b, c) \stackrel{\sigma}{\equiv} t^{\mathbf{A}}(b, b, c) = c$ . This is all that is needed to verify that  $\sigma \circ \tau \subseteq \tau \circ \sigma$  ( $\subseteq \sigma \circ \tau$ ).



FIGURE 8. FORBIDDEN SUBLATTICE IF BOTH  $[\alpha, \alpha] = 0$  and  $\mathbf{C}(\theta, \alpha; \delta)$  hold.

 $[(2) \Rightarrow (3)]$  Every lattice of permuting equivalence relations is modular.

 $[(3) \Rightarrow (4)]$  A lattice is modular if and only if it has no pentagon sublattice. Notice that Item (3) asserts that when  $\alpha$  is abelian, then  $I[0, \alpha]$  contains no pentagon sublattice. Item (4) asserts something slightly more: when  $\alpha$  is abelian, then  $I[0,\alpha]$  contains no "spanning pentagon" sublattice, by which we mean a pentagon whose bottom is 0 and whose top is  $\alpha$ .

 $[(4) \Rightarrow (5)]$  Item (5) is identical to Item (4) except the pentagons in Item (5) are more restricted. In Item (5) we assert no abelian interval contains a spanning pentagon which also satisfies  $\mathbf{C}(\theta, \alpha; \delta)$ .

 $[(5) \Rightarrow (1)]$  This implication is the only nontrivial claim of the theorem. We prove it in the contrapositive form: if  $\mathcal{V}$  does not have a weak difference term, then it will contain an algebra with a pentagon like the one described in Item (5).

Let

- $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$  be the free  $\mathcal{V}$ -algebra over the set  $\{x, y\}$ .
- $\theta = \mathrm{Cg}^{\mathbf{F}}(x, y).$
- $\overline{\mathbf{F}} = \mathbf{F}/[\theta, \theta].$   $\overline{x} = x/[\theta, \theta], \overline{y} = \underline{y}/[\theta, \theta].$
- $\overline{\theta} = \theta / [\theta, \theta] = \mathrm{Cg}^{\overline{\mathbf{F}}}(\overline{x}, \overline{y}).$

It follows from properties of the commutator (Theorem 3.4 (10)) that  $[\overline{\theta}, \overline{\theta}] = 0$ .

**Claim 4.6.** ( $\overline{\theta}$  is the "free principal abelian congruence" in  $\mathcal{V}$ ) Suppose that  $\mathbf{B} \in \mathcal{V}$  and  $\beta \in \text{Con}(\mathbf{B})$  satisfies  $[\beta, \beta] = 0$ . For any  $(a, b) \in \beta$  there is a unique homomorphism  $\overline{\varphi} \colon \overline{\mathbf{F}} \to \mathbf{B} \text{ satisfying } \overline{x} \mapsto a, \overline{y} \mapsto b.$ 

Proof of Claim 4.6. Given  $(a,b) \in \beta$ , let  $\varphi \colon \mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y) \to \mathbf{B}$  be the homomorphism determined by  $x \mapsto a, y \mapsto b$ . Since  $\mathbf{C}(\beta, \beta; 0)$  holds in **B**,  $\mathbf{C}(\varphi^{-1}(\beta), \varphi^{-1}(\beta); \ker(\varphi))$  holds in  $\overline{\mathbf{F}}$ . Since  $(x,y) \in \varphi^{-1}(\beta)$ , we have  $\mathbf{C}(\theta,\theta;\ker(\varphi))$  for  $\theta = \mathrm{Cg}(x,y)$  by monotonicity. Therefore  $[\theta, \theta] \leq \ker(\varphi)$  holds. We may factor  $\varphi$  modulo  $[\theta, \theta]$  as

$$\varphi \colon \mathbf{F} \to \mathbf{F} / [\theta, \theta] = \overline{\mathbf{F}} \stackrel{\varphi}{\to} \mathbf{B}$$

where the first map is the natural map. This yields the desired map  $\overline{\varphi} \colon \overline{\mathbf{F}} \to \mathbf{B} \colon \overline{x} \mapsto a, \overline{y} \mapsto b$ . The uniqueness is a consequence of the fact that  $\overline{\mathbf{F}}$  is generated by  $\{\overline{x}, \overline{y}\}$ , so any homomorphism with domain  $\overline{\mathbf{F}}$  is uniquely determined by its values on this set.

Let **A** be the subalgebra of  $\overline{\mathbf{F}} \times \overline{\mathbf{F}}$  that is generated by the set

$$G = \{ (\overline{x}, \overline{x}), (\overline{y}, \overline{x}), (\overline{x}, \overline{y}), (\overline{y}, \overline{y}) \}.$$

Since  $\overline{\mathbf{F}}$  is generated by  $\{\overline{x}, \overline{y}\}$ , the universe of  $\mathbf{A}$  is the reflexive, symmetric, compatible, binary relation (or "tolerance") generated by the pair  $(\overline{x}, \overline{y})$  on the algebra  $\overline{\mathbf{F}}$ . Since  $G \subseteq \overline{\theta}$ we have  $A \subseteq \overline{\theta}$ , so in fact  $\mathbf{A} \leq \overline{\mathbf{F}}(\overline{\theta}) \leq \overline{\mathbf{F}} \times \overline{\mathbf{F}}$ . This is enough to draw some conclusions about  $\mathbf{A}$ . The most tautological conclusion that follows from  $A \subseteq \overline{\theta}$  is that if  $(\overline{u}, \overline{v}) \in A$ , then  $(\overline{u}, \overline{v}) \in \overline{\theta}$ , so any pair in  $\mathbf{A}$  generates an abelian congruence in  $\overline{\mathbf{F}}$ . This fact will be used without reminders. A less obvious conclusion is that, since  $\overline{\theta}$  is an abelian congruence that relates  $\overline{x}$  to  $\overline{y}$  in  $\overline{\mathbf{F}}$ , the congruence  $\overline{\theta}_1 \times \overline{\theta}_2$  is an abelian congruence on  $\overline{\mathbf{F}} \times \overline{\mathbf{F}}$  whose restriction to  $\mathbf{A}$  is an abelian congruence that relates any two elements of G.

Let  $\eta_1$  and  $\eta_2$  be the coordinate projection kernels of  $\overline{\mathbf{F}} \times \overline{\mathbf{F}}$  restricted to  $\mathbf{A}$ . Following Example 4.4, let  $\delta = \operatorname{Cg}^{\mathbf{A}}(((\overline{x}, \overline{x}), (\overline{y}, \overline{x})))$ . Let  $X = (\overline{y}, \overline{y})/\eta_2$  be the  $\eta_2$ -class of  $(\overline{y}, \overline{y})$ . The set X plays the role of the "top edge" in Example 4.4. Elements of X have the form  $(\overline{P}, \overline{y})$ for certain elements  $\overline{P} \in \overline{\mathbf{F}}$  satisfying  $\overline{P} \stackrel{\overline{\theta}}{=} \overline{y} \stackrel{\overline{\theta}}{=} \overline{x}$  in  $\overline{\mathbf{F}}$ .

**Claim 4.7.** (Characterization of  $\delta|_X$ ) Two pairs  $(\overline{P}, \overline{y})$  and  $(\overline{Q}, \overline{y})$  lying in X are  $\delta$ -related if and only if there is a ternary  $\mathcal{V}$ -term  $t_{\overline{P},\overline{Q}}(x, y, z)$  such that

(†)  $t_{\overline{P},\overline{Q}}$  is a right difference term for  $\mathcal{V}$  (cf. Definition 4.1), and (‡)  $t_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{P}) = \overline{Q}$ .

Proof of Claim 4.7. Recall that  $\delta$  is the principal congruence on  $\mathbf{A}$  that is generated by the pair  $((\overline{x}, \overline{x}), (\overline{y}, \overline{x}))$ . Since X is an  $\eta_2$ -class and  $\delta \leq \eta_2$ , if  $(\overline{P}, \overline{y})$  and  $(\overline{Q}, \overline{y})$  lying in X are  $\delta$ -related, then they are connected by a Maltsev chain that lies entirely inside X. A link of such a Maltsev chain has the form  $(f((\overline{x}, \overline{x})), f((\overline{y}, \overline{x})))$  or  $(f((\overline{y}, \overline{x})), f((\overline{x}, \overline{x})))$  for some polynomial  $f \in \text{Pol}_1(\mathbf{A})$ . Since  $\mathbf{A}$  is generated by the set  $G = \{(\overline{x}, \overline{x}), (\overline{y}, \overline{x}), (\overline{x}, \overline{y}), (\overline{y}, \overline{y})\}$ , we may assume that the parameters of the polynomial f lie in G, and hence we may write

$$f((z,w)) = s^{\mathbf{A}}((z,w), (\overline{x},\overline{x}), (\overline{y},\overline{x}), (\overline{x},\overline{y}), (\overline{y},\overline{y}))$$

for some 5-ary  $\mathcal{V}$ -term s. If  $(f((\overline{x}, \overline{x})), f((\overline{y}, \overline{x}))) = ((\overline{R}, \overline{y}), (\overline{S}, \overline{y}))$  is a link in a Maltsev chain in X, then there must exist such a 5-ary term s such that

$$s^{\mathbf{A}}((\overline{x},\overline{x}),(\overline{x},\overline{x}),(\overline{y},\overline{x}),(\overline{x},\overline{y}),(\overline{y},\overline{y})) = (R,\overline{y})$$
  
$$s^{\mathbf{A}}((\overline{y},\overline{x}),(\overline{x},\overline{x}),(\overline{y},\overline{x}),(\overline{x},\overline{y}),(\overline{y},\overline{y})) = (\overline{S},\overline{y}),$$

which simplifies to the coordinate equations

$$(4.8) \qquad s^{\overline{\mathbf{F}}}(\overline{x}, \overline{x}, \overline{y}, \overline{x}, \overline{y}) = \overline{R} \\ s^{\overline{\mathbf{F}}}(\overline{y}, \overline{x}, \overline{y}, \overline{x}, \overline{y}) = \overline{S} \\ s^{\overline{\mathbf{F}}}(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}) = \overline{y}.$$

Let  $t_{\overline{R},\overline{S}}(x,y,z) = s(x,y,y,z,z).$ 

**Subclaim 4.9.** (The (†)-part of "only if" when  $((\overline{R}, \overline{y}), (\overline{S}, \overline{y}))$  is a Maltsev link)

 $t_{\overline{R},\overline{S}}$  is a right difference term for  $\mathcal{V}$ .

Proof of Subclaim 4.9. We derive from the definition of  $t_{\overline{R},\overline{S}}$  and the third equation in (4.8) that  $t_{\overline{R},\overline{S}}^{\overline{\mathbf{F}}}(\overline{x},\overline{x},\overline{y},\overline{y}) = s^{\overline{\mathbf{F}}}(\overline{x},\overline{x},\overline{x},\overline{y},\overline{y},\overline{y}) = \overline{y}$ . The claim then follows from the fact that  $\overline{\theta} = \operatorname{Cg}^{\overline{\mathbf{F}}}(\overline{x},\overline{y})$  is the "free principal abelian congruence" in  $\mathcal{V}$ . Specifically, given any  $\mathbf{B} \in \mathcal{V}$  and any pair (a,b) from an abelian congruence of  $\mathbf{B}$ , Claim 4.6 guarantees a unique homomorphism  $\overline{\varphi} \colon \overline{\mathbf{F}} \to \mathbf{B}$  satisfying  $\overline{x} \mapsto a, \overline{y} \mapsto b$ . Applying this homomorphism to the equality  $t_{\overline{R},\overline{S}}(\overline{x},\overline{x},\overline{y}) = \overline{y}$  yields  $t_{\overline{R},\overline{S}}^{\mathbf{B}}(a,a,b) = b$ . This is all that is required to prove that  $t_{\overline{R},\overline{S}}$  is right difference term for  $\mathcal{V}$ . (Cf. Definition 4.1 (4).)

**Subclaim 4.10.** (The (‡)-part of "only if" when  $((\overline{R}, \overline{y}), (\overline{S}, \overline{y}))$  is a Maltsev link)

$$t_{\overline{R},\overline{S}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{R}) = \overline{S}.$$

*Proof of Subclaim* 4.10. The following matrix is a  $\overline{\theta}, \overline{\theta}$ -matrix in  $\overline{\mathbf{F}}$ :

$$(4.11) \qquad \begin{bmatrix} s^{\overline{\mathbf{F}}}(\overline{x}, \overline{x}, \overline{y}, \overline{x}, \overline{y}) & s^{\overline{\mathbf{F}}}(\overline{x}, \overline{x}, \overline{x}, \overline{R}, \overline{R}) \\ s^{\overline{\mathbf{F}}}(\overline{y}, \overline{x}, \overline{y}, \overline{x}, \overline{y}) & s^{\overline{\mathbf{F}}}(\overline{y}, \overline{x}, \overline{x}, \overline{R}, \overline{R}) \end{bmatrix} = \begin{bmatrix} \overline{R} & t^{\overline{\mathbf{F}}}_{\overline{R}, \overline{S}}(\overline{x}, \overline{x}, \overline{R}) \\ \overline{S} & t^{\overline{\mathbf{F}}}_{\overline{R}, \overline{S}}(\overline{y}, \overline{x}, \overline{R}) \end{bmatrix} = \begin{bmatrix} \overline{R} & \overline{R} \\ \overline{S} & t^{\overline{\mathbf{F}}}_{\overline{R}, \overline{S}}(\overline{y}, \overline{x}, \overline{R}) \end{bmatrix}$$

The justifications for the claims that these matrices are equal and that the leftmost is an  $\overline{\theta}, \overline{\theta}$ -matrix follow from the facts that (i)  $(\overline{x}, \overline{y}), (\overline{y}, \overline{y}), (\overline{R}, \overline{y}), (\overline{S}, \overline{y})$  belong to  $X \subseteq A$ , hence  $\overline{x}, \overline{y}, \overline{R}, \overline{S}$  belong to the same  $\overline{\theta}$ -class, (ii) equations (4.8) hold, (iii)  $t_{\overline{R},\overline{S}}(x, y, z) :=$ s(x, y, y, z, z), and (iv)  $t_{\overline{R},\overline{S}}$  is a right difference term for  $\mathcal{V}$  (Subclaim 4.9). Item (i) is enough to show that the leftmost matrix is a  $\overline{\theta}, \overline{\theta}$ -matrix, Items (ii) and (iii) are enough to show that the leftmost matrix reduces to the middlemost, and Item (iv) is enough to show that the middlemost matrix reduces to the rightmost one.

Since  $\theta$  is abelian and the top row of the matrix in (4.11) is constant, the bottom row must also be constant.

Subclaim 4.12. The "only if" part of Claim 4.7 holds.

Proof of Subclaim 4.12. Subclaims 4.9 and 4.10 prove the "only if" part of Claim 4.7 when  $((\overline{R}, \overline{y}), (\overline{S}, \overline{y}))$  is a Maltsev link. Maltsev links are Maltsev chains of length 1. Here we prove that the "only if" part holds for Maltsev chains of any length. For this, let  $\Omega$  be the relation on X consisting of all pairs  $((\overline{P}, \overline{y}), (\overline{Q}, \overline{y}))$  which satisfy (†) and (‡) of Claim 4.7.  $\Omega$  will contain all pairs  $((\overline{R}, \overline{y}), (\overline{S}, \overline{y}))$  that are Maltsev links, so to prove this subclaim it suffices to prove that  $\Omega$  is an equivalence relation on X.

( $\Omega$  is **reflexive**) Given  $((\overline{P}, \overline{y}), (\overline{P}, \overline{y}))$ , the third projection term  $t_{\overline{P},\overline{P}}(x, y, z) := z$  witnesses membership in  $\Omega$ . To see this, note that both ( $\dagger$ ) and ( $\ddagger$ ) are trivial when  $\overline{P} = \overline{Q}$  and  $t_{\overline{P},\overline{P}}(x, y, z) = z$ :

- (†)  $t_{\overline{P},\overline{P}}(x, y, z) := z$  is a right difference term for  $\mathcal{V}$ . (It is even right Maltsev, which is formally stronger.)
- $(\ddagger) \ t_{\overline{P},\overline{P}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{P}) = \overline{P}.$

( $\Omega$  is **symmetric**) Assume that the ternary term  $s_{\overline{P},\overline{Q}}(x, y, z)$  witnesses membership in  $\Omega$  for the pair  $((\overline{P}, \overline{y}), (\overline{Q}, \overline{y}))$ . We argue that the ternary term  $t_{\overline{P},\overline{Q}}(x, y, z) := s_{\overline{P},\overline{Q}}(y, x, z)$  obtained by swapping the first two variables in  $s_{\overline{P},\overline{Q}}(x, y, z)$  witnesses membership in  $\Omega$  for  $((\overline{Q}, \overline{y}), (\overline{P}, \overline{y}))$ :

(†)  $t_{\overline{P},\overline{O}}$  is a right difference term for  $\mathcal{V}$ .

Reason: Choose (a, b) generating an abelian congruence in some  $\mathbf{B} \in \mathcal{V}$ .  $t^{\mathbf{B}}_{\overline{P},\overline{Q}}(a, a, b) = s^{\mathbf{B}}_{\overline{P},\overline{Q}}(a, a, b) = b.$ 

$$(\ddagger) \ t^{\overline{\mathbf{F}}}_{\overline{P},\overline{Q}}(\overline{y},\overline{x},\overline{Q}) = \overline{P}.$$

Reason: We know from (†) for  $s_{\overline{P},\overline{Q}}$  and from the fact that  $s_{\overline{P},\overline{Q}}(x, y, z)$  witnesses membership in  $\Omega$  for  $((\overline{P}, \overline{y}), (\overline{Q}, \overline{y}))$  that

$$s_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{P}) = \overline{Q} = s_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{x},\overline{x},\overline{Q}).$$

The following is a  $\overline{\theta}, \overline{\theta}$ -matrix in  $\overline{\mathbf{F}}$ .

$$(4.13) \qquad \begin{bmatrix} s_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{P}) & s_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{x},\overline{x},\overline{Q}) \\ s_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{y},\overline{y},\overline{P}) & s_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{x},\overline{y},\overline{Q}) \end{bmatrix} = \begin{bmatrix} \overline{Q} & \overline{Q} \\ \overline{P} & s_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{x},\overline{y},\overline{Q}) \end{bmatrix} = \begin{bmatrix} \overline{Q} & \overline{Q} \\ \overline{P} & t_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{Q}) \end{bmatrix}.$$

Since  $\overline{\theta}$  is abelian and this matrix is constant on the first row we must have  $t_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{Q}) = \overline{P}$ , which is the statement to be proved.

( $\Omega$  is **transitive**) Assume that the ternary term  $r_{\overline{P},\overline{Q}}(x, y, z)$  witnesses membership in  $\Omega$  for the pair  $((\overline{P}, \overline{y}), (\overline{Q}, \overline{y}))$  and that  $s_{\overline{Q},\overline{W}}(x, y, z)$  witnesses membership in  $\Omega$  for the pair

 $((\overline{Q}, \overline{y}), (\overline{W}, \overline{y}))$ . We shall argue that the ternary term  $t_{\overline{P}, \overline{W}}(x, y, z) := s_{\overline{Q}, \overline{W}}(x, y, r_{\overline{P}, \overline{Q}}(x, y, z))$  witnesses that  $((\overline{P}, \overline{y}), (\overline{W}, \overline{y}))$  belongs to  $\Omega$ .

(†)  $t_{\overline{P},\overline{W}}$  is a right difference term for  $\mathcal{V}$ .

Reason: Choose (a, b) generating an abelian congruence in some  $\mathbf{B} \in \mathcal{V}$ . We have  $t_{\overline{P},\overline{W}}^{\mathbf{B}}(a, a, b) = s_{\overline{Q},\overline{W}}^{\mathbf{B}}(a, a, r_{\overline{P},\overline{Q}}^{\mathbf{B}}(a, a, b)) = s_{\overline{Q},\overline{W}}^{\mathbf{B}}(a, a, b) = b.$ 

 $(\ddagger) \ t^{\overline{\mathbf{F}}}(\overline{y}, \overline{x}, \overline{P}) = \overline{W}.$ 

Reason: 
$$t_{\overline{P},\overline{W}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{P}) = s_{\overline{Q},\overline{W}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},r_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{P})) = s_{\overline{Q},\overline{W}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{Q}) = \overline{W}.$$

**Subclaim 4.14.** ("if" statement in Claim 4.7) Assume that  $(\overline{P}, \overline{y})$  and  $(\overline{Q}, \overline{y})$  lie in X and there is a ternary  $\mathcal{V}$ -term t(x, y, z) such that

(†)  $t_{\overline{P},\overline{\mathcal{Q}}}$  is a right difference term for  $\mathcal{V}$ , and

$$(\ddagger) \ t_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{y},\overline{x},\overline{P}) = \overline{Q}.$$

Then  $(\overline{P}, \overline{y})$  and  $(\overline{Q}, \overline{y})$  are  $\delta$ -related.

Proof of Subclaim 4.14. Since  $t_{\overline{P},\overline{Q}}$  is a right difference term and  $\overline{x}, \overline{y}, \overline{P}, \overline{Q}$  belong to a single class of the abelian congruence  $\overline{\theta}$  we have both  $t_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{x},\overline{x},\overline{P}) = \overline{P}$  and  $t_{\overline{P},\overline{Q}}^{\overline{\mathbf{F}}}(\overline{x},\overline{x},\overline{y}) = \overline{y}$ . Hence, working with pairs in  $\mathbf{A}$ ,

$$\begin{split} (\overline{P},\overline{y}) &= t^{\mathbf{A}}_{\overline{P},\overline{Q}}((\overline{x},\overline{x}),(\overline{x},\overline{x}),(\overline{P},\overline{y})) \\ &\stackrel{\delta}{\equiv} t^{\mathbf{A}}_{\overline{P},\overline{Q}}((\overline{y},\overline{x}),(\overline{x},\overline{x}),(\overline{P},\overline{y})) \\ &= (\overline{Q},\overline{y}). \end{split}$$

In moving from the first line to the second we have underlined the only change, indicating where we replaced  $(\overline{x}, \overline{x})$  with the  $\delta$ -related pair  $(\overline{y}, \overline{x})$ . In moving from the second line to the third we made coordinatewise use of  $(\dagger)$  and  $(\ddagger)$  for  $t_{\overline{P},\overline{Q}}$ .

This completes the proof of Claim 4.7.

## Claim 4.15. $((\overline{x}, \overline{y}), (\overline{y}, \overline{y})) \notin \delta$ .

Proof of Claim 4.15. Assume instead that  $((\overline{x}, \overline{y}), (\overline{y}, \overline{y})) \in \delta$ . Then, for  $\overline{P} = \overline{x}$  and  $\overline{Q} = \overline{y}$ , we have  $((\overline{P}, \overline{y}), (\overline{Q}, \overline{y})) \in \delta|_X$ . Claim 4.7 guarantees the existence of a ternary term  $t_{\overline{x},\overline{y}}(x, y, z)$  such that  $(\dagger) t_{\overline{x},\overline{y}}$  is a right difference term for  $\mathcal{V}$  and  $(\ddagger)$ 

$$(4.16) t_{\overline{x},\overline{y}}(\overline{y},\overline{x},\overline{x}) = \overline{y}$$

In Claim 4.6 we showed that  $\overline{x}$  and  $\overline{y}$  are generators of the "free principal abelian congruence" in  $\mathcal{V}$ . We used this information in Subclaim 4.9 to prove that the equality  $t_{\overline{R},\overline{S}}(\overline{x},\overline{x},\overline{y}) = \overline{y}$ 

in **F** suffices to prove that  $t_{\overline{R},\overline{S}}$  is a right difference term for  $\mathcal{V}$ . The same argument can be applied here to prove that (4.16) suffices to prove that  $t_{\overline{x},\overline{y}}$  is a left difference term for  $\mathcal{V}$ . But now  $t_{\overline{x},\overline{y}}$  is both a right and a left difference term, which contradicts our initial assumption that  $\mathcal{V}$  has no weak difference term (= left and right difference term).

At present we have no understanding of how  $\delta$  behaves off of the set X. To deal with this, we enlarge  $\delta$  to the largest congruence below  $\eta_2$  that "behaves like  $\delta$  on X". Since X is an  $\eta_2$ -class, all congruences  $\gamma$  satisfying  $\gamma \leq \eta_2$  have the property that X is a union of  $\gamma$ -classes. This implies that the set of those  $\gamma$  satisfying

- (i)  $\gamma \leq \eta_2$  and
- (ii)  $\gamma|_X \subseteq \delta|_X$

contains  $\delta$  and is closed under complete join. Let  $\varepsilon \in \text{Con}(\mathbf{A})$  be the join of all congruences satisfying (i) and (ii). Since  $\delta$  is a joinand, we get that  $\delta \leq \varepsilon \leq \eta_2$  and  $\varepsilon|_X = \delta|_X$ . Since  $((\overline{x}, \overline{y}), (\overline{y}, \overline{y})) \in \eta_2$  and, by Claim 4.15,  $((\overline{x}, \overline{y}), (\overline{y}, \overline{y})) \notin \delta|_X$ , we get that  $\eta_2|_X \neq \delta|_X = \varepsilon|_X$ , and therefore  $\delta \leq \varepsilon < \eta_2$ . This is enough to imply that  $\{\eta_1, \eta_2, \varepsilon\}$  generates a pentagon in  $\text{Con}(\mathbf{A})$  that is labeled like the one in Figure 9. In this figure, it might happen that  $\delta = \varepsilon$ , but no other pair of differently-labeled congruences in the figure could be equal.



FIGURE 9.  $\delta \leq \varepsilon < \eta_2$ ,  $\delta|_X = \varepsilon|_X$ .

Claim 4.17. (Characterization of  $\varepsilon$ )

$$\varepsilon = \{ (a,b) \in \eta_2 \mid \forall f \in \operatorname{Pol}_1(\mathbf{A}) \ (f(a), f(b)) \in X \Rightarrow (f(a), f(b)) \in \delta ) \}.$$

Proof of Claim 4.17. It is not difficult to see that the relation on the right hand side of the equality symbol is (i) an equivalence relation contained in  $\eta_2$  (ii) that is closed under the application of unary polynomials and (iii) whose restriction to X is contained in  $\delta$ . It is also clear that the relation on the right hand side of the equality symbol contains all other relations with these three properties, hence it is the largest congruence  $\gamma \leq \eta_2$  satisfying  $\gamma|_X \subseteq \delta|_X$ . This is enough to conclude that the relation on the right hand side of the equality symbol is  $\varepsilon$ . **Claim 4.18.** (i)  $\eta_1 + \eta_2$  is abelian and (ii)  $\mathbf{C}(\eta_2, \eta_1 + \eta_2; \varepsilon)$  holds.

Proof of Claim 4.18. For (i), we already noted in the paragraph preceding Claim 4.7,  $\overline{\theta}_1 \times \overline{\theta}_2$  is an abelian congruence of **A**. Since  $\mathbf{A} \leq \overline{\mathbf{F}}(\overline{\theta})$  we have  $\eta_1, \eta_2 \leq \overline{\theta}_1 \times \overline{\theta}_2$ . From this we derive that  $\eta_1 + \eta_2 \leq \overline{\theta}_1 \times \overline{\theta}_2$  and then that  $\eta_1 + \eta_2$  is an abelian congruence of **A**.

For (ii), we must show that, given an  $\eta_2$ ,  $(\eta_1 + \eta_2)$ -matrix

(4.19) 
$$\begin{bmatrix} t(\mathbf{a}, \mathbf{u}) & t(\mathbf{a}, \mathbf{v}) \\ t(\mathbf{b}, \mathbf{u}) & t(\mathbf{b}, \mathbf{v}) \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

if  $p \stackrel{\varepsilon}{\equiv} q$ , then  $r \stackrel{\varepsilon}{\equiv} s$ . We shall assume that  $p \stackrel{\varepsilon}{\equiv} q$  and  $r \stackrel{\varepsilon}{\neq} s$  and argue to a contradiction.

We do have  $r \stackrel{\eta_2}{\equiv} p \stackrel{\varepsilon}{\equiv} q \stackrel{\eta_2}{\equiv} s$ . Since  $\varepsilon \leq \eta_2$ , all elements p, q, r, s are  $\eta_2$ -related. If indeed  $r \stackrel{\varepsilon}{\neq} s$ , then by Claim 4.17 there is a unary polynomial f(x) of **A** such that  $f(r), f(s) \in X$  and  $f(r) \stackrel{\delta}{\neq} f(s)$ . We may apply f to the matrix in (4.19) to obtain another  $\eta_2, (\eta_1 + \eta_2)$ -matrix (4.20)  $\begin{bmatrix} f(p) & f(q) \\ f(r) & f(s) \end{bmatrix}.$ 

The new matrix has the same properties as the old one, except now we also have all entries in  $X = (\overline{y}, \overline{y})/\eta_2$ . Let us write  $(\overline{P}, \overline{y}), (\overline{Q}, \overline{y}), (\overline{R}, \overline{y}), (\overline{S}, \overline{y})$  for f(p), f(q), f(r), f(s).

By Claim 4.7, the fact that  $(\overline{P}, \overline{y}) \stackrel{\delta|_X}{\equiv} (\overline{Q}, \overline{y})$  holds yields a right difference term  $t_{\overline{P},\overline{Q}}$  such that  $t_{\overline{P},\overline{Q}}(\overline{y}, \overline{x}, \overline{P}) = \overline{Q}$ . We must have  $t_{\overline{P},\overline{Q}}(\overline{y}, \overline{x}, \overline{R}) \neq \overline{S}$ , or the same right difference term would yield  $(\overline{R}, \overline{y}) \stackrel{\delta|_X}{\equiv} (\overline{S}, \overline{y})$ , which is false.

Now consider the  $\eta_2$ ,  $(\eta_1 + \eta_2)$ -matrix (4.21)

$$t_{\overline{P},\overline{Q}}\left(\begin{bmatrix} (\overline{y},\overline{y}) & (\overline{x},\overline{y}) \\ (\overline{y},\overline{y}) & (\overline{x},\overline{y}) \end{bmatrix}, \begin{bmatrix} (\overline{x},\overline{y}) & (\overline{x},\overline{y}) \\ (\overline{x},\overline{y}) & (\overline{x},\overline{y}) \end{bmatrix}, \begin{bmatrix} (\overline{P},\overline{y}) & (\overline{Q},\overline{y}) \\ (\overline{R},\overline{y}) & (\overline{S},\overline{y}) \end{bmatrix}\right) = \begin{bmatrix} (\overline{Q},\overline{y}) & (\overline{Q},\overline{y}) \\ (t_{\overline{P},\overline{Q}}(\overline{y},\overline{x},\overline{R}),\overline{y}) & (\overline{S},\overline{y}) \end{bmatrix}$$

The rightmost matrix witnesses that  $\mathbf{C}(\eta_2, \eta_1 + \eta_2; 0)$  fails, since the top row is constant while the bottom row is not, since  $t_{\overline{P},\overline{Q}}(\overline{y}, \overline{x}, \overline{R}) \neq \overline{S}$ . But the failure of  $\mathbf{C}(\eta_2, \eta_1 + \eta_2; 0)$  contradicts  $\mathbf{C}(\eta_1 + \eta_2, \eta_1 + \eta_2; 0)$ , which we established in part (i) of this claim. This completes the proof of (ii).

We have constructed the desired pentagon, so to complete the proof of  $\neg(1) \Rightarrow \neg(5)$  of Theorem 4.5 we just have to explain how to relabel the elements of the pentagon. Relabel each congruence in the sequence  $(\eta_1 + \eta_2, \eta_1, \eta_2, \varepsilon, 0)$  of the pentagon of in Figure 9 with the corresponding label in  $(\alpha, \beta, \theta, \delta, 0)$  to obtain the pentagon in Figure 8. From Claim 4.18 we have that  $\alpha$  is abelian and  $\mathbf{C}(\theta, \alpha; \delta)$ , as desired.

The next result represents a half-step toward proving that a variety with a Taylor term and commutative commutator has a difference term. It turns out that we only need to assume that commutativity of the commutator on comparable pairs of congruences to prove this result.

**Theorem 4.22.** If  $\mathcal{V}$  has a Taylor term and, whenever  $x \leq y$  in Con(A) for  $\mathbf{A} \in \mathcal{V}$ , the equation [x, y] = [y, x] is satisfied, then  $\mathcal{V}$  must have a weak difference term.

*Proof.* We shall assume that  $\mathcal{V}$  has a Taylor term and does not have a weak difference term and argue that  $\mathcal{V}$  contains an algebra with a noncommutative commutator. Our construction produces an algebra in  $\mathcal{V}$  which has a pair of comparable congruences  $x \leq y$  such that  $[x, y] \neq [y, x]$ .

Since we have assumed that  $\mathcal{V}$  does not have a weak difference term, Theorem 4.5 guarantees that there is some algebra  $\mathbf{A} \in \mathcal{V}$  whose congruence lattice contains a pentagon where



FIGURE 10. BOTH  $[\alpha, \alpha] = 0$  and  $\mathbf{C}(\theta, \alpha; \delta)$  hold.

 $\alpha$  is abelian and  $\mathbf{C}(\theta, \alpha; \delta)$  holds. We claim that the algebra  $\mathbf{A}/\delta \in \mathcal{V}$  has noncommutative commutator. Specifically, we claim that

(4.23) 
$$[\theta/\delta, \alpha/\delta] = 0 \neq [\alpha/\delta, \theta/\delta].$$

To see this, first observe that since  $\mathbf{C}(\theta, \alpha; \delta)$  holds we have  $[\theta/\delta, \alpha/\delta] = 0$  from Theorem 3.4 (10), and this is the equality in (4.23).

It remains to prove the inequality  $[\alpha/\delta, \theta/\delta] \neq 0$  from (4.23). If instead we had equality, then again by Theorem 3.4 (10) we would have that  $\mathbf{C}(\alpha, \theta; \delta)$  holds. By monotonicity (Theorem 3.4 (1)), we would have  $\mathbf{C}(\beta, \theta; \delta)$  holds. This contradicts Theorem 4.2. Thus, for  $x = \theta/\delta$  and  $y = \alpha/\delta$  we have  $x \leq y$  and  $[x, y] \neq [y, x]$ .

Next we begin a sequence of results to strengthen the conclusion of Theorem 4.22 from "weak difference term" to "difference term". The argument is completed in Theorem 4.29 below.

**Theorem 4.24.** Assume that  $\mathcal{V}$  has a weak difference term. If  $\mathbf{A}$  in  $\mathcal{V}$  has congruences satisfying

(1)  $\alpha \ge \theta \ge \delta$ , and (2)  $[\alpha, \theta] = 0$ , then  $\mathbf{C}(\theta, \alpha; \delta)$  holds. *Proof.* Let d(x, y, z) be some fixed weak difference term for  $\mathcal{V}$ .

This is a proof by contradiction, so assume that the hypotheses hold and that the conclusion  $\mathbf{C}(\theta, \alpha; \delta)$  fails. This failure is witnessed by a  $\theta, \alpha$ -matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} t(\mathbf{a}, \mathbf{c}) & t(\mathbf{a}, \mathbf{d}) \\ t(\mathbf{b}, \mathbf{c}) & t(\mathbf{b}, \mathbf{d}) \end{bmatrix}$$

where  $(a_i, b_i) \in \theta$ ,  $(c_j, d_j) \in \alpha$ ,  $(p, q) \in \delta$ , and  $(r, s) \notin \delta$ . Since  $(r, p) \in \theta$ ,  $(q, s) \in \theta$ , and  $(p, q) \in \delta \leq \theta$  we have  $r \stackrel{\theta}{\equiv} p \stackrel{\delta}{\equiv} q \stackrel{\theta}{\equiv} s$ , so p, q, r and s are all  $\theta$ -related. From the hypotheses (1)  $\alpha \geq \theta$ , and (2)  $[\alpha, \theta] = 0$ , we derive that  $[\theta, \theta] = 0$  by monotonicity. This implies that d(x, y, z) acts like a Maltsev operation on the  $\theta$ -class containing p, q, r, s. Let  $t'(\mathbf{x}, \mathbf{y}) = d(t(\mathbf{x}, \mathbf{y}), t(\mathbf{x}, \mathbf{c}), t(\mathbf{b}, \mathbf{c}))$ . We have an  $\alpha, \theta$ -matrix

$$\begin{bmatrix} t'(\mathbf{a}, \mathbf{c}) & t'(\mathbf{b}, \mathbf{c}) \\ t'(\mathbf{a}, \mathbf{d}) & t'(\mathbf{b}, \mathbf{d}) \end{bmatrix} = \begin{bmatrix} d(p, p, r) & d(r, r, r) \\ d(q, p, r) & d(s, r, r) \end{bmatrix} = \begin{bmatrix} r & r \\ d(q, p, r) & s \end{bmatrix}$$

In moving from the middlemost matrix to the rightmost matrix we use the fact that d acts like a Maltsev operation on the  $\theta$ -class containing p, q, r, s. Since this matrix is an  $\alpha, \theta$ -matrix, the top row is constant, and  $[\alpha, \theta] = 0$ , we derive that the bottom row is constant, i.e. d(q, p, r) = s. This proves that  $s = d(q, p, r) \stackrel{\delta}{=} d(q, q, r) = r$ , which contradicts our earlier assumptions that  $(p, q) \in \delta$  and  $(r, s) \notin \overline{\delta}$ .

**Theorem 4.25.** ([6, Theorem 3.3(a) $\Leftrightarrow$ (b)]) If  $\mathcal{V}$  is a variety, then the following conditions are equivalent.

- (a)  $\mathcal{V}$  has a difference term.
- (b) For each  $\mathbf{A} \in \mathcal{V}$  the solvability relation is a congruence on  $\operatorname{Con}(\mathbf{A})$  which is preserved by homomorphisms. Furthermore, whenever the pentagon  $\mathbf{N}_5$  is a sublattice of  $\operatorname{Con}(\mathbf{A})$ , then the critical interval is neutral.

(A congruence interval  $I[\delta, \theta]$  is called *neutral* if it contains no nontrivial abelian subinterval I[x, y], equivalently if  $[y, y]_x = y$  whenever  $\delta \leq x < y \leq \theta$ .)

**Corollary 4.26.** A variety has a difference term if and only if it has a weak difference term and whenever  $N_5$  is a sublattice of Con(A), then the critical interval is neutral.

*Proof.* This result will be derived from Theorem 4.25. Assume first that  $\mathcal{V}$  has a difference term. This term is also a weak difference term for  $\mathcal{V}$ . Also, whenever  $\mathbf{N}_5$  is a sublattice of Con(**A**), then the critical interval must be neutral by Theorem 4.25 (a) $\Rightarrow$ (b).

Conversely, assume that  $\mathcal{V}$  has a weak difference term and whenever  $\mathbf{N}_5$  is a sublattice of  $\operatorname{Con}(\mathbf{A})$ , then the critical interval is neutral. Our goal is to prove that  $\mathcal{V}$  has a difference term. According to Theorem 4.25, what remains to show is that for each  $\mathbf{A} \in \mathcal{V}$  the solvability relation is a congruence on  $\operatorname{Con}(\mathbf{A})$  which is preserved by homomorphisms. This property always holds for varieties with a weak difference term, as we now explain.

Following the notation of Chapter 6 of [8], write  $\alpha \triangleleft \beta$  to mean that  $\alpha \leq \beta$  and that  $\mathbf{C}(\beta, \beta; \alpha)$  holds ( $\beta$  is abelian over  $\alpha$ ). The solvability relation is defined so that  $\gamma \stackrel{s}{\sim} \delta$  holds exactly when there is a finite chain

$$\gamma \cap \delta = \varepsilon_0 \lhd \cdots \lhd \varepsilon_n = \gamma + \delta.$$

A related notion,  $\infty$ -solvability, is defined in [8, Definition 6.5] the same way, but with finite chains replaced by continuous well-ordered chains. That is,  $\alpha \ll \beta$  means  $\alpha \leq \beta$  and there is continuous well-ordered chain  $(\varepsilon_{\mu})_{\mu < \kappa+1}$  such that  $\alpha = \varepsilon_0, \varepsilon_{\mu} < \varepsilon_{\mu+1}$  for  $\mu < \kappa + 1$ and  $\varepsilon_{\lambda} = \bigcup_{\mu < \lambda} \varepsilon_{\mu}$  for limit  $\lambda \leq \kappa$ , and  $\bigcup_{\mu < \kappa} \varepsilon_{\mu} = \beta$ . Then  $\gamma$  is  $\infty$ -solvably related to  $\delta$  if  $\gamma \cap \delta \ll \gamma + \delta$ . The claims that we need to prove here for solvability were proved for  $\infty$ -solvability in [8], and the proofs given there work for our purposes. In particular, Lemma 6.10 of [8] proves that, for any  $\gamma$ ,

• if  $\alpha \triangleleft \beta$ , then  $\alpha \cap \gamma \triangleleft \beta \cap \gamma$  and  $\alpha + \gamma \triangleleft \beta + \gamma$ .

This is the technical lemma which is used in Theorem 6.11 of [8] to prove that the  $\infty$ -solvability relation is compatible with finite meets and arbitrary joins. The same arguments show that the ordinary solvability relation is compatible with finite meets and finite joins.

The fact that the  $\infty$ -solvability relation is preserved by homomorphisms is proved in Theorem 6.19 (1) of [8]. The same proof works here for the ordinary solvability relation. This completes the proof (sketch) for the converse.

The next lemma refines the statement of Theorem 4.3 for varieties that do *not* have a difference term. This lemma will be used in the proofs of Theorem 4.28, Theorem 4.33, and Theorem 4.44.

**Lemma 4.27.** If  $\mathcal{V}$  has a Taylor term and does not have a difference term, then  $\mathcal{V}$  contains an algebra  $\mathbf{A}$  with congruences labeled as in Figure 11 satisfying the following commutator conditions:

- (1)  $[\alpha, \theta] = 0,$
- (2)  $\mathbf{C}(\theta, \alpha; \delta)$ , and
- (3)  $[\alpha, x]_{\delta} = x \text{ for all } x \in I[\delta, \theta].$



Figure 11.  $\mathcal{V}$  has a Taylor term but not a difference term.

*Proof.* We split the proof into two cases depending on whether  $\mathcal{V}$  has a weak difference term.

For the first case, assume that  $\mathcal{V}$  does not have a weak difference term. Theorem 4.5 (5) guarantees that some  $\mathbf{A} \in \mathcal{V}$  has a pentagon in its congruence lattice, labeled as in Figure 11, where (i)  $[\alpha, \alpha] = 0$  and (ii)  $\mathbf{C}(\theta, \alpha; \delta)$ . Since (i) is stronger than Item (1) of this lemma statement (by monotonicity of the commutator), and (ii) is the same as (2), we only have to verify Item (3) of the lemma statement. That follows from Theorem 4.3, since  $\mathcal{V}$  has a Taylor term.

For the second case, assume that  $\mathcal{V}$  does have a weak difference term. We still assume that  $\mathcal{V}$  does not have a difference term. By Corollary 4.26, the fact that  $\mathcal{V}$  does not have a difference term means that some  $\mathbf{A} \in \mathcal{V}$  has a pentagon in its congruence lattice, labeled as in Figure 11, where the critical interval  $I[\delta, \theta]$  is not neutral. The nonneutrality means that  $[x, x]_{\delta} < x$  for some  $x \in I[\delta, \theta]$ . We have

$$\delta \le [x, x]_{\delta} < x \le \theta,$$

so we can alter the pentagon by shrinking  $I[\delta, \theta]$  to  $I[[x, x]_{\delta}, x]$ . This produces a new pentagon with abelian critical interval. Change to this pentagon and change notation. That is, assume that  $\{\beta, \delta, \theta\}$  generates a pentagon in Con(**A**), labeled as in Figure 4, with critical interval  $I[\delta, \theta]$  where  $\mathbf{C}(\theta, \theta; \delta)$  holds.

By Theorem 4.2,  $\mathbf{C}(\beta, \theta; \delta)$  fails, since  $\mathcal{V}$  has a Taylor term. Now, citing Theorem 3.7 (Better pentagons), we may further adjust our pentagon so that  $\mathbf{C}(\theta, \theta; 0)$  holds. We have  $\mathbf{C}(\beta, \theta; 0)$  by Theorem 3.4 (8), so for  $\alpha = \beta + \theta$  we have  $\mathbf{C}(\alpha, \theta; 0)$  by Theorem 3.4 (5). This may be written as  $[\alpha, \theta] = 0$ , which is Item (1) of the lemma statement. Since  $\alpha \ge \theta \ge \delta$ , we may invoke Theorem 4.24 to derive that  $\mathbf{C}(\theta, \alpha; \delta)$  holds. This is Item (2) of the lemma statement. We derive Item (3) from Theorem 4.3, as we did in the first case of this proof.  $\Box$ 

**Theorem 4.28.** If  $\mathcal{V}$  has a weak difference term and the commutator is commutative on pairs of comparable congruences, then  $\mathcal{V}$  has a difference term.

Proof. Assume that  $\mathcal{V}$  has a weak difference term and does not have a difference term. The hypotheses of Lemma 4.27 hold, so some  $\mathbf{A} \in \mathcal{V}$  has a pentagon in its congruence lattice with congruences labeled as in Figure 11, and which satisfies the commutator conditions (1), (2), and (3) of Lemma 4.27. In  $\mathbf{A}/\delta$  the congruences  $x = \alpha/\delta$  and  $y = \theta/\delta$  satisfy 0 < y < x, [y, x] = 0 (by Item (2) of that lemma) and [x, y] = y (by Item (3) of that lemma).

The next theorem is one of the primary results of this article.

**Theorem 4.29.** The following are equivalent for a variety  $\mathcal{V}$ .

- (1)  $\mathcal{V}$  has a difference term
- (2)  $\mathcal{V}$  has a Taylor term and has commutative commutator.
- (3)  $\mathcal{V}$  has a Taylor term and the commutator operation is commutative on pairs of comparable congruences.

*Proof.* The class of varieties that have a difference term is definable by a nontrivial idempotent Maltsev condition. (The reason that the class of varieties with a difference term is

definable by an idempotent Maltsev condition is explained in the paragraph following the proof of Theorem 4.8 of [10]. A specific Maltsev condition defining the class of varieties with a difference term is in [12, Section 4].) The weakest nontrivial idempotent Maltsev condition is the one that asserts the existence of a Taylor term. Thus, the implication  $(1) \Rightarrow (2)$  follows from Lemma 2.2 of [6], which proves that a variety with a difference term has commutative commutator. Item (2) is formally stronger than Item (3), so it remains to prove that Item (3) implies Item (1). For this, combine Theorems 4.22 and 4.28.

Now we turn to an examination of distributivity of the commutator. You will recall that we proved some results about the left or right distributivity of the commutator in Theorem 3.14. The results obtained there were left/right-asymmetric, but that asymmetry disappears when a Taylor term is present, as we establish with the next two results.

**Theorem 4.30.** If  $\mathcal{V}$  has a Taylor term, then for any  $\mathbf{A} \in \mathcal{V}$  and any congruences  $\alpha, \beta \in \text{Con}(\mathbf{A})$  the following are equivalent:

(a)  $[\beta, \alpha] = 0$  and  $[\alpha, \alpha \cap \beta] = 0$ .

(b)  $\Delta_{\alpha,\beta}$  is disjoint from the coordinate projection kernels of  $\mathbf{A}(\alpha)$ .

- (c)  $[\alpha, \beta] = 0$  and  $[\beta, \alpha \cap \beta] = 0$ .
- (d)  $\Delta_{\beta,\alpha}$  is disjoint from the coordinate projection kernels of  $\mathbf{A}(\beta)$ .

In particular, if  $\mathcal{V}$  has a Taylor term, then  $\mathcal{V}$  satisfies the commutator congruence quasiidentity

$$[\beta, \alpha] = 0 \quad \& \quad [\alpha, \alpha \cap \beta] = 0 \implies \quad [\alpha, \beta] = 0.$$

*Proof.* This proof is a refinement of the proof of Lemma 4.4 of [10].

To prove that (a) implies (b), it suffices to prove that (a) implies that  $\eta_2 \cap \Delta_{\alpha,\beta} = 0$ . For then, by interchanging the two coordinates of  $\mathbf{A}(\alpha)$ , the same argument will show that  $\eta_1 \cap \Delta_{\beta,\beta} = 0$  also. Let  $\theta = \eta_2 \cap \Delta_{\alpha,\beta} \in \operatorname{Con}(\mathbf{A}(\alpha))$ . Assuming (a), that  $[\beta, \alpha] = 0$  holds, the diagonal of  $\mathbf{A}(\alpha)$  is a union of  $\Delta_{\alpha,\beta}$ -classes. No two distinct diagonal elements are related by  $\eta_2$ , and  $\theta = \eta_2 \cap \Delta_{\alpha,\beta}$ , so it follows that each element of the diagonal of  $\mathbf{A}(\alpha)$  is a singleton  $\theta$ -class. Choose an arbitrary pair  $((a, c), (b, c)) \in \theta$ . Let  $T(x_1, \ldots, x_n)$  be a Taylor term for  $\mathcal{V}$ . Consider a first-place Taylor identity  $T(x, \mathbf{w}) \approx T(y, \mathbf{z})$  where  $\mathbf{w}, \mathbf{z} \in \{x, y\}^{n-1}$ . Substitute b for all occurrences of x and c for all occurrences of y. This yields  $T(b, \bar{u}) = T(c, \bar{v})$  where all  $u_i$  and  $v_i$  are in  $\{b, c\}$ . Since  $(b, c) \in \mathbf{A}(\alpha)$  we have  $b \stackrel{\alpha}{\equiv} c$ , hence  $(u_i, v_i) \in \alpha$  for all i. This implies that  $p((x,y)) = (T(x,\bar{u}), T(y,\bar{v}))$  is a unary polynomial of  $\mathbf{A}(\alpha)$ . The equation  $T(x, \mathbf{w}) \approx T(y, \mathbf{z})$  implies that p((b, c)) lies on the diagonal of  $\mathbf{A}(\alpha)$ . The element p((a, c))is  $\theta$ -related to p((b,c)), and each element of the diagonal is a singleton  $\theta$ -class, therefore p((a,c)) = p((b,c)). This has the consequence that  $T(a,\bar{u}) = T(b,\bar{u})$  where each  $u_i \in \{b,c\}$ . Now, since  $((a, c), (b, c)) \in \theta \leq \Delta_{\alpha,\beta} \leq \beta_1 \times \beta_2$  we get that  $(a, b) \in \beta$ . Since (a, c) and (b, c)are elements of our algebra we have  $a \stackrel{\alpha}{\equiv} c \stackrel{\alpha}{\equiv} b$ , so  $(a, b) \in \alpha$ . Together, the last two sentences show that  $(a, b) \in \alpha \cap \beta$ . Now, applying  $[\alpha, \alpha \cap \beta] = 0$  to the equality  $T(a, \bar{u}) = T(b, \bar{u})$ , we deduce that  $T(a, \bar{y}) = T(b, \bar{y})$  for any  $\bar{y}$  whose entries lie in the  $\alpha$ -class containing a, b and c. The argument we just gave concerning a, b and T, which showed that  $T(a, \bar{y}) = T(b, \bar{y})$ whenever each  $y_i$  is in the  $\alpha$ -class containing a, b, and c is an argument which works in each of the n variables of T. That is,

$$T(y_1, \ldots, y_{i-1}, a, y_{i+1}, \ldots, y_n) = T(y_1, \ldots, y_{i-1}, b, y_{i+1}, \ldots, y_n)$$

for each *i* and any choice of values for  $y_1, \ldots, y_n$  in the  $\alpha$ -class of a, b, and c. Therefore, using the fact that T is idempotent, we have

$$a = T(a, a, \dots, a) = T(b, a, \dots, a) = \dots = T(b, b, \dots, b) = b.$$

This proves that (a, c) = (b, c). Since  $((a, c), (b, c)) \in \theta$  was arbitrarily chosen,  $\theta = \eta_2 \cap \Delta_{\alpha,\beta} = 0$ , completing the proof that (a) implies (b).

Now we argue by contraposition that (b) implies (c). Assume that (c) fails because  $[\alpha, \beta] > 0$ . There is an  $\alpha, \beta$ -matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} t(\mathbf{e}, \mathbf{u}) & t(\mathbf{e}, \mathbf{v}) \\ t(\mathbf{f}, \mathbf{u}) & t(\mathbf{f}, \mathbf{v}) \end{bmatrix}$$

where  $(e_i, f_i) \in \alpha$ ,  $(u_i, v_i) \in \beta$ , p = q, and  $r \neq s$ . The pair

$$((r, p), (s, q)) = (t((\mathbf{f}, \mathbf{e}), (\mathbf{u}, \mathbf{u})), t((\mathbf{f}, \mathbf{e}), (\mathbf{v}, \mathbf{v})))$$

belongs to  $\eta_2$  (since p = q) and  $\Delta_{\alpha,\beta}$  (since  $((u_i, u_i), (v_i, v_i)) \in \Delta_{\alpha,\beta}$ ), but not to  $\eta_1$  (since  $r \neq s$ ). Therefore,  $((r, p), (s, q)) \in (\eta_2 \cap \Delta_{\alpha,\beta}) - 0$ , establishing that  $\eta_2 \cap \Delta_{\alpha,\beta} \neq 0$ . This shows that if (c) fails because  $[\alpha, \beta] > 0$ , then (b) fails because  $\eta_2 \cap \Delta_{\alpha,\beta} > 0$ .

Now assume that (c) fails because  $[\beta, \alpha \cap \beta] > 0$ . There is a  $\beta, \alpha \cap \beta$ -matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} t(\mathbf{e}, \mathbf{u}) & t(\mathbf{e}, \mathbf{v}) \\ t(\mathbf{f}, \mathbf{u}) & t(\mathbf{f}, \mathbf{v}) \end{bmatrix}$$

where  $(e_i, f_i) \in \beta$ ,  $(u_j, v_j) \in \alpha \cap \beta$ , p = q, and  $r \neq s$ . The pair

$$((p,q),(r,s)) = (t((\mathbf{e},\mathbf{e}),(\mathbf{u},\mathbf{v})),t((\mathbf{f},\mathbf{f}),(\mathbf{u},\mathbf{v})))$$

belongs to  $\Delta_{\alpha,\beta}$  (since  $((e_i, e_i), (f_i, f_i)) \in \Delta_{\alpha,\beta}$ ). The pair (r, p) belongs to  $\beta$  (since  $(p, r) = (t(\mathbf{e}, \mathbf{u}), t(\mathbf{f}, \mathbf{u}))$  and  $(e_i, f_i) \in \beta$ ). Hence

$$(r,r) \stackrel{\Delta_{\alpha,\beta}}{\equiv} (p,p) = (p,q) \stackrel{\Delta_{\alpha,\beta}}{\equiv} (r,s).$$

Therefore,  $((r,r), (r,s)) \in \eta_1 \cap \Delta_{\alpha,\beta}$ , but  $((r,r), (r,s)) \notin \eta_2$  since  $r \neq s$ . This shows that if (c) fails because  $[\beta, \alpha \cap \beta] > 0$ , then (b) fails because  $\eta_1 \cap \Delta_{\alpha,\beta} > 0$ .

At this point we have shown that  $(a) \Rightarrow (b) \Rightarrow (c)$ . Interchanging the roles of  $\alpha$  and  $\beta$  in these arguments we deduce that  $(c) \Rightarrow (d) \Rightarrow (a)$ . This shows that all four properties are equivalent. The commutator congruence quasi-identity of the theorem statement follows from the equivalence of (a) and (c).

**Theorem 4.31.** If  $\mathcal{V}$  has a Taylor term, then the commutator operation in  $\mathcal{V}$  is left distributive if and only if it is right distributive. If either distributivity condition holds, then the commutator is commutative in  $\mathcal{V}$ .

*Proof.* By Theorem 3.14 (1), if  $\mathcal{V}$  has left distributive commutator, then it has commutative commutator, hence it has right distributive commutator. This part of the theorem did not require the assumption that  $\mathcal{V}$  has a Taylor term.

Now assume that  $\mathcal{V}$  has a Taylor term and has right distributive commutator. We shall argue that the commutator operation in  $\mathcal{V}$  is commutative, hence left distributive. This is a proof by contradiction, so assume also that some algebra in  $\mathcal{V}$  has noncommutative commutator. By Lemma 3.13, we may assume that some  $\mathbf{A} \in \mathcal{V}$  has congruences  $\alpha$  and  $\beta$ such that  $0 = [\beta, \alpha] < [\alpha, \beta]$ . Theorem 4.30 implies that  $0 < [\alpha, \alpha \cap \beta]$ . This means that for  $x := \alpha$  and  $y := \alpha \cap \beta$  we have  $y \leq x, 0 < [x, y]$ , and  $[y, x] = [\alpha \cap \beta, \alpha] \leq [\beta, \alpha] = 0$ , so  $[x, y] \not\leq [y, x]$ , in contradiction to Theorem 3.14 (2). 

We strengthen the previous theorem with the following result, which is one of the primary results of this article.

**Theorem 4.32.** If  $\mathcal{V}$  has a Taylor term, then the following are equivalent:

- (1)  $\mathcal{V}$  is congruence modular.
- (2) The commutator is left distributive in  $\mathcal{V}$ .
- (3) The commutator is right distributive in  $\mathcal{V}$ .

*Proof.* For any congruence modular variety, the commutator is both left and right distributive. (See [2, Proposition 4.3] for the fact that the modular commutator is right distributive and commutative.) This shows that Item (1) implies both Item (2) and Item (3).

By Theorem 4.31, Items (2) and (3) are equivalent in the presence of a Taylor term, and they imply that the commutator is commutative in  $\mathcal{V}$ . From commutativity, we derive the existence of a difference term from Theorem 4.29. Thus, it remains to prove that if  $\mathcal{V}$  has (left) distributive commutator and a difference term, then  $\mathcal{V}$  is congruence modular. This fact follows from Theorem 3.2 (i) of [13], but we give an argument for this based on the results of this paper.

We are going to argue by contradiction, so assume that the commutator is left distributive throughout  $\mathcal{V}$ , but there is some algebra in  $\mathcal{V}$  that does not have a modular congruence lattice. We can find such an algebra  $\mathbf{A} \in \mathcal{V}$  with congruences  $\beta, \theta, \delta \in \text{Con}(\mathbf{A})$  generating a pentagon satisfying  $\delta < \theta$ ,  $\beta \cap \theta = 0 < \delta$ , and  $\alpha := \beta + \delta > \theta$  (Figure 12). By the left distributivity of the commutator,

$$[\alpha, \theta] = [\beta + \delta, \theta] = [\beta, \theta] + [\delta, \theta] = 0 + [\delta, \theta] \le \delta.$$

In this line we are using that  $[\beta, \theta] \leq \beta \cap \theta = 0$  to obtain the last equality and last inequality.

Let  $\sigma$  denote  $[\alpha, \theta] = [\delta, \theta]$ , a congruence which satisfies  $0 \leq \sigma \leq \delta$ . Working modulo  $\sigma$ , and writing  $\overline{x}$  for  $x/\sigma$  for any congruence  $x \geq \sigma$ , we have

- (1)  $\overline{\alpha} \geq \overline{\theta} \geq \overline{\delta}$ , and (2)  $[\overline{\alpha}, \overline{\theta}] = 0$ .

Since  $\mathcal{V}$  has a difference term, Theorem 4.24 guarantees that  $\mathbf{C}(\overline{\theta}, \overline{\alpha}; \overline{\delta})$  holds in  $\mathbf{A}/\sigma$ . Since  $\overline{\alpha} > \overline{\theta} > \overline{\delta}$ , Theorem 3.4 (10) guarantees that in  $\mathbf{A}/\delta$  we have  $\mathbf{C}(\overline{\overline{\theta}}, \overline{\alpha}; 0)$ , where  $\overline{\overline{x}}$  denotes



FIGURE 12.  $CON(\mathbf{A})$ .

 $x/\delta$ . Because  $\mathbf{C}(\overline{\overline{\theta}}, \overline{\alpha}; 0)$  holds and the commutator is commutative in  $\mathcal{V}$  we derive that

$$0 = [\overline{\overline{\theta}}, \overline{\overline{\alpha}}] = [\overline{\overline{\alpha}}, \overline{\overline{\theta}}].$$

Hence  $\mathbf{C}(\overline{\alpha}, \overline{\theta}; 0)$  holds in  $\mathbf{A}/\delta$ . Theorem 3.4 (10) guarantees that  $\mathbf{C}(\alpha, \theta; \delta)$  holds in  $\mathbf{A}$ , and so by monotonicity  $\mathbf{C}(\beta, \theta; \delta)$  holds in  $\mathbf{A}$ . This instance of centrality in a pentagon contradicts Theorem 4.2.

Next we consider the existence of right annihilators and the right semidistributivity of the commutator.

**Theorem 4.33.** If  $\mathcal{V}$  has a Taylor term, then the following are equivalent:

- (1)  $\mathcal{V}$  has a difference term.
- (2) Right annihilators exist throughout  $\mathcal{V}$ .
- (3) The commutator is right semidistributive throughout  $\mathcal{V}$ .

*Proof.* We shall argue that  $(1) \Rightarrow (2) \Rightarrow (3)$  and  $\neg(1) \Rightarrow \neg(3)$ .

Assume (1), that  $\mathcal{V}$  has a difference term. According to Theorem 4.29,  $\mathcal{V}$  has commutative commutator. Since the left annihilator of any congruence on any algebra exists, right annihilators will also exist in any variety with a commutative commutator (and  $(0 : \beta)_R = (0 : \beta)_L$  will hold). This proves that (2) holds.

Now assume that (2) holds. Assume that  $\mathbf{A} \in \mathcal{V}$  and that  $\alpha, \beta, \beta' \in \operatorname{Con}(\mathbf{A})$  satisfy  $[\alpha, \beta] = [\alpha, \beta']$ . The congruence  $\delta := [\alpha, \beta]$  is below each of  $\alpha, \beta, \beta'$  according to Theorem 3.4 (7), and both  $\mathbf{C}(\alpha, \beta; \delta)$  and  $\mathbf{C}(\alpha, \beta'; \delta)$  hold since  $[\alpha, \beta] = \delta = [\alpha, \beta']$ . Let's factor by  $\delta = [\alpha, \beta]$  to obtain  $\overline{\mathbf{A}} = \mathbf{A}/\delta \in \mathcal{V}$ ,  $\overline{\alpha} = \alpha/\delta$ ,  $\overline{\beta} = \beta/\delta$ ,  $\overline{\beta}' = \beta'/\delta$ . Both  $\mathbf{C}(\overline{\alpha}, \overline{\beta}; 0)$  and  $\mathbf{C}(\overline{\alpha}, \overline{\beta}; 0)$  hold in  $\overline{\mathbf{A}}$  by Theorem 3.4 (10). This implies that  $\overline{\beta}, \overline{\beta}' \leq (0:\overline{\alpha})_R$  in  $\operatorname{Con}(\overline{\mathbf{A}})$ . Hence  $\overline{\beta} + \overline{\beta}' \leq (0:\overline{\alpha})_R$ . Hence  $\mathbf{C}(\alpha, \overline{\beta} + \overline{\beta}'; 0)$  in  $\overline{\mathbf{A}}$  by the definition of  $(0:\overline{\alpha})_R$ . Hence  $\mathbf{C}(\alpha, \beta + \beta'; \delta)$  in  $\mathbf{A}$  by Theorem 3.4 (10).

(4.34) 
$$[\alpha,\beta] \le [\alpha,\beta+\beta'] \le \delta = [\alpha,\beta].$$

Here the first  $\leq$  is an instance of the monotonicity of the commutator in its second variable, while the second  $\leq$  follows from  $\mathbf{C}(\alpha, \beta + \beta'; \delta)$  and the definition of the commutator. Altogether, the line (4.34) yields that  $[\alpha, \beta] = [\alpha, \beta + \beta']$ , completing the proof of the right semidistributivity of the commutator.

The rest of the argument is devoted to establishing the difficult implication  $\neg(1) \Rightarrow \neg(3)$ . We start with the assumptions that  $\mathcal{V}$  has a Taylor term but does not have a difference term and construct a failure of right semidistributivity in the congruence lattice of some algebra in  $\mathcal{V}$ . Since  $\mathcal{V}$  has a Taylor term but does not have a difference term, Lemma 4.27 guarantees the existence of an algebra  $\mathbf{A} \in \mathcal{V}$  with a pentagon in its congruence lattice, labeled as in Figure 11, satisfying the commutator conditions (1), (2), and (3) of that lemma. We shall start our construction with the quotient  $\mathbf{B} = \mathbf{A}/\delta$ . Writing  $\overline{x}$  for  $x/\delta$ , Lemma 4.27 guarantees that  $\mathbf{B}$  has congruences  $0 < \overline{\theta} < \overline{\alpha}$  such that (i)  $[\overline{\theta}, \overline{\alpha}] = 0$  (from Item (2) of that lemma) and (ii)  $[\overline{\alpha}, \overline{x}] = \overline{x}$  for any congruence  $\overline{x}$  satisfying  $0 \leq \overline{x} \leq \overline{\theta}$  (from Item (3) of that lemma). In particular, we shall use the part of (ii) that guarantees that  $[\overline{\alpha}, \overline{\theta}] = \overline{\theta}$ . Let  $\mathbf{D} = \mathbf{B}(\overline{\alpha})$ . Write  $\Delta$  for  $\Delta_{\overline{\alpha},\overline{\theta}}$  ( $\in \operatorname{Con}(\mathbf{D})$ ). Write  $\Gamma$  for  $\overline{\theta}_1 \times \eta_2$  ( $\in \operatorname{Con}(\mathbf{D})$ ). It is clear that  $\Delta, \Gamma, \leq \overline{\theta}_1 \times \overline{\theta}_2$ . As usual, the coordinate projection kernels on  $\mathbf{D} = \mathbf{B}(\overline{\alpha})$  will be denoted  $\eta_1$  and  $\eta_2$ , but to minimize subscripts below we shall use  $\eta$  as a duplicate name for  $\eta_1$  (that is,  $\eta := \eta_1$ ). Define sequences of congruences



FIGURE 13. A HERRINGBONE-LIKE PORTION OF CON(D).

(4.35) 
$$\Delta^{1} = \Delta, \qquad \Delta^{2n+1} = \Delta + \eta^{2n}$$
$$\Gamma^{0} = \Gamma, \qquad \Gamma^{2n+2} = \Gamma + \eta^{2n+1}$$
$$\eta^{0} = 0, \qquad \eta^{2n+1} = \eta \cap \Delta^{2n+1}, \qquad \eta^{2n} = \eta \cap \Gamma^{2n},$$
$$\Delta^{\infty} = \bigcup \Delta^{2n+1}, \qquad \Gamma^{\infty} = \bigcup \Gamma^{2n}, \qquad \eta^{\infty} = \bigcup \eta^{n}$$

See Figure 13 for a depiction of the ordering of these congruences in  $Con(\mathbf{D})$ . This figure need not be a sublattice of  $Con(\mathbf{D})$  and the congruences in the figure need not be distinct,

but the indicated (non-strict) comparabilities hold and the meet or join of any element in the central chain with any element in a side chain is depicted correctly, as we explain in the next claim.

## Claim 4.36.

- (1)  $\eta^0 \leq \eta^1 \leq \eta^2 \leq \cdots \leq \eta^\infty \leq \eta \cap (\Delta + \Gamma).$
- (2)  $\Delta = \Delta^1 \leq \Delta^3 \leq \cdots \leq \Delta^\infty \leq \Delta + \Gamma$  and  $\Gamma = \Gamma^0 \leq \Gamma^2 \leq \cdots \leq \Gamma^\infty \leq \Delta + \Gamma$ .
- (3)  $\eta \cap \Delta^{\infty} = \eta^{\infty} = \eta \cap \overline{\Gamma}^{\infty}$ .
- $\begin{array}{l} (4) \ \Delta^{\infty} + \Gamma^{\infty} = \Delta + \Gamma \leq \overline{\theta}_{1} \times \overline{\theta}_{2}. \\ (5) \ The \ sets \ \{\eta^{2m+1}, \eta^{2m+2}, \eta^{2m+3}, \Delta^{2m+1}, \Delta^{2m+3}\} \ and \ \{\eta^{2n}, \eta^{2n+1}, \eta^{2n+2}, \Gamma^{2n}, \Gamma^{2n+2}\} \ are \end{array}$ sublattices of  $\operatorname{Con}(\mathbf{D})$  for every m, n > 0.
- (6) The meet or join of any element in the central chain with any element in a side chain is depicted correctly in Figure 13.

Proof of Claim 4.36. For Item (1), observe that  $\eta^0 = 0 = \eta \cap \Gamma^0 \leq \eta$ . If  $\eta^{2n} \leq \eta$  for some n, then  $\eta^{2n} \leq \eta \cap (\Delta + \eta^{2n}) = \eta^{2n+1}$  and  $\eta^{2n+1} = \eta \cap (\Delta + \eta^{2n}) \leq \eta$ . Similarly, if  $\eta^{2n+1} \leq \eta$  for some n, then  $\eta^{2n+1} \leq \eta \cap (\Gamma + \eta^{2n+1}) = \eta^{2n+2}$  and  $\eta^{2n+2} = \eta \cap (\Gamma + \eta^{2n+1}) \leq \eta$ . Inductively we get that  $\eta^0 < \eta^1 < \eta^2 < \cdots$  and that all of these elements lie below  $\eta$ .

To complete the proof of (1) it will suffice to show that  $\eta^k \leq \Delta + \Gamma$  for all k. This is true of k = 0 since that  $\eta^0 = 0$ . If  $\eta^{2n} \leq \Delta + \Gamma$ , then  $\Delta + \eta^{2n} \leq \Delta + \Gamma$ , so  $\eta^{2n+1} = \eta \cap (\Delta + \eta^{2n}) \leq \Delta + \Gamma$ . Similarly, if  $\eta^{2n+1} \leq \Delta + \Gamma$ , then  $\Gamma + \eta^{2n+1} \leq \Delta + \Gamma$ , so  $\eta^{2n+2} = \eta \cap (\Gamma + \eta^{2n+1}) \leq \Delta + \Gamma$ . By induction,  $\eta^k \leq \Delta + \Gamma$  for all k (hence  $\eta^{\infty} \leq \Delta + \Gamma$ , too).

For Item (2), the facts that (i) the  $\eta$ -sequence is increasing and bounded above by  $\Delta + \Gamma$ and (ii)  $\Delta^{2n+1} = \Delta + \eta^{2n}$  are jointly sufficient to imply that the  $\Delta$ -sequence is increasing and bounded above by  $\Delta + \Gamma$ . A similar argument proves that the  $\Gamma$ -sequence is increasing and bounded above by  $\Delta + \Gamma$ .

For Item (3),  $\eta \cap \Delta^{\infty} = \eta \cap (\bigcup \Delta^{2n+1}) = \bigcup (\eta \cap \Delta^{2n+1}) = \bigcup \eta^{2n+1} = \eta^{\infty}$ . Also  $\eta \cap \Gamma^{\infty} =$  $\eta \cap (\lfloor \rfloor \Gamma^{2n}) = \lfloor \rfloor (\eta \cap \Gamma^{2n}) = \lfloor \rfloor \eta^{2n} = \eta^{\infty}.$ 

For the equality in Item (4), we have  $\Delta \leq \Delta^{\infty} \leq \Delta + \Gamma$  and  $\Gamma \leq \Gamma^{\infty} \leq \Delta + \Gamma$ . Joining these yields  $\Delta + \Gamma \leq \Delta^{\infty} + \Gamma^{\infty} \leq \Delta + \Gamma$ , so  $\Delta^{\infty} + \Gamma^{\infty} = \Delta + \Gamma$ . For the inequality in Item (4), we have  $\Delta = \Delta_{\overline{\alpha},\overline{\theta}} \leq \overline{\theta}_1 \times \overline{\theta}_2$  and  $\Gamma = \overline{\theta}_1 \times \eta_2 \leq \overline{\theta}_1 \times \overline{\theta}_2$ , so  $\Delta + \Gamma \leq \overline{\theta}_1 \times \overline{\theta}_2$ . We have already established in Items (1) and (2) that  $\eta^{2m+1} \leq \eta^{2m+2} \leq \eta^{2m+3}$  and

 $\Delta^{2m+1} \leq \Delta^{2m+3}$ . For the first part of Claim 4.36 (5), it remains to show that (i)  $\Delta^{2m+1} +$  $\overline{\eta^{2m+2}} = \Delta^{2m+3} \text{ and (ii) } \Delta^{2m+1} \cap \eta^{2m+3} = \eta^{2m+1}. \text{ For (i), } \Delta^{2m+1} + \eta^{2m+2} = (\Delta + \eta^{2m}) + (\Delta + \eta^{2m}) + \eta^{2m+2} = (\Delta + \eta^{2m}) + (\Delta + \eta^{2m$  $\Delta + (\eta^{2m} + \eta^{2m+2}) = \Delta + \eta^{2m+2} = \Delta^{2m+3}.$  For (ii), recall that  $\eta^{2m+1} = \eta \cap \Delta^{2m+1} \le \Delta^{2m+1}.$ Intersect the inequalities  $\eta^{2m+1} \leq \eta^{2m+3} \leq \eta$  throughout with  $\Delta^{2m+1}$  to obtain  $\eta^{2m+1} = \Delta^{2m+1} \cap \eta^{2m+1} \leq \Delta^{2m+1} \cap \eta^{2m+3} \leq \Delta^{2m+1} \cap \eta = \eta^{2m+1}$ . The middle value must equal the outer value, so  $\overline{\Delta}^{2m+1} \cap \eta^{2m+3} = \eta^{\overline{2m+1}}$ .

Item (6) is a consequence of Items (1), (2), and (5). For example, the fact that  $\eta^8 + \Gamma^0 = \Gamma^8$  may be argued:

$$\eta^{8} + \Gamma^{0} = (\eta^{8} + \eta^{2}) + \Gamma^{0} \quad (1)$$

$$= \eta^{8} + (\eta^{2} + \Gamma^{0})$$

$$= \eta^{8} + \Gamma^{2} \qquad (5)$$

$$= \eta^{8} + \Gamma^{4} \qquad (1) + (5)$$

$$= \eta^{8} + \Gamma^{6} \qquad (1) + (5)$$

)

while the fact that  $\eta^8 \cap \Gamma^0 = 0$  may be argued:

$$\eta^{8} \cap \Gamma^{0} = \eta^{8} \cap (\Gamma^{6} \cap \Gamma^{0}) \quad (2) \\
= (\eta^{8} \cap \Gamma^{6}) \cap \Gamma^{0} \\
= \eta^{6} \cap \Gamma^{0} \quad (5) \\
= \eta^{4} + \Gamma^{0} \quad (2) + (5) \\
= \eta^{2} + \Gamma^{0} \quad (2) + (5) \\
= 0 \quad (5)$$

From Claim 4.36 (4) we have  $\eta \cap \Delta^{\infty} = \eta^{\infty} = \eta \cap \Gamma^{\infty}$ . It follows from this and Theorem 3.4 (8) that  $\mathbf{C}(\eta, \Delta^{\infty}; \eta^{\infty})$  and  $\mathbf{C}(\eta, \Gamma^{\infty}; \eta^{\infty})$  hold. The rest of the proof is devoted to proving that  $\mathbf{C}(\eta, \Delta^{\infty} + \Gamma^{\infty}; \eta^{\infty})$  does *not* hold. If we do this, then, factoring by  $\eta^{\infty}$  (which is below all congruences involved), we get that in  $\mathbf{D}/\eta^{\infty}$  we have the following failure of the right semidistributive law:

$$[\eta/\eta^{\infty}, \Delta^{\infty}/\eta^{\infty}] = 0 = [\eta/\eta^{\infty}, \Gamma^{\infty}/\eta^{\infty}], \text{ but } [\eta/\eta^{\infty}, \Delta^{\infty}/\eta^{\infty} + \Gamma^{\infty}/\eta^{\infty}] \neq 0.$$

Since  $\Delta^{\infty} + \Gamma^{\infty} = \Delta + \Gamma$ , we can write our remaining goal as:

**Goal 4.37.** Show that  $\mathbf{C}(\eta, \Delta + \Gamma; \eta^{\infty})$  fails.

Recall from the fourth paragraph of this proof (i.e. of the proof of Theorem 4.33) that  $0 < \overline{\theta} < \overline{\alpha}$  and  $[\overline{\alpha}, \overline{\theta}] = \overline{\theta}$ . This puts us in a position to mimic the construction in Theorem 3.14 (2). As was the case there (with  $\alpha, \beta$  there replaced by  $\overline{\alpha}, \overline{\theta}$  here), there is an  $\overline{\alpha}, \overline{\theta}$ -matrix

$$\begin{bmatrix} t(\mathbf{a}, \mathbf{u}) & t(\mathbf{a}, \mathbf{v}) \\ t(\mathbf{b}, \mathbf{u}) & t(\mathbf{b}, \mathbf{v}) \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad \mathbf{a} \ \overline{\alpha} \ \mathbf{b}, \quad \mathbf{u} \ \overline{\theta} \ \mathbf{v},$$

with p = q but  $r \neq s$ . The fact that this is an  $\overline{\alpha}, \overline{\theta}$ -matrix implies, in particular, that  $(p, r), (q, s) \in \overline{\alpha}$  and  $(p, q), (r, s) \in \overline{\theta}$ .

# Claim 4.38.

(MM) 
$$\begin{bmatrix} t ((\mathbf{b}, \mathbf{a}), (\mathbf{u}, \mathbf{u})) & t ((\mathbf{b}, \mathbf{a}), (\mathbf{u}, \mathbf{v})) \\ t ((\mathbf{b}, \mathbf{b}), (\mathbf{u}, \mathbf{u})) & t ((\mathbf{b}, \mathbf{b}), (\mathbf{u}, \mathbf{v})) \end{bmatrix} = \begin{bmatrix} (r, p) & (r, q) \\ (r, r) & (r, s) \end{bmatrix}$$

is an  $\eta$ ,  $(\Delta + \Gamma)$ -matrix of  $\mathbf{D} = \mathbf{B}(\overline{\alpha})$  that is constant on the first row and not constant on the second row.

Proof of Claim 4.38. To show that the matrix given is truly an  $\eta$ ,  $(\Delta + \Gamma)$ -matrix, we first argue that the elements of the form  $(b_i, a_i)$ ,  $(u_i, u_i)$ ,  $(b_i, b_i)$ , and  $(u_i, v_i)$  belong to the algebra  $\mathbf{D} = \mathbf{B}(\overline{\alpha})$ . This is so, because  $\mathbf{a} \ \overline{\alpha} \ \mathbf{b}$ ,  $\mathbf{u} \ \overline{\theta} \ \mathbf{v}$ ,  $\overline{\theta} \subseteq \overline{\alpha}$ , and the universe of  $\mathbf{D}$  is  $\overline{\alpha}$ .

Next, we need to argue that  $((b_i, a_i), (b_i, b_i)) \in \eta = \eta_1$ , and  $((u_i, u_i), (u_i, v_i)) \in \Delta + \Gamma$ . The former is clear, since  $(b_i, a_i)$  and  $(b_i, b_i)$  have the same first coordinate. The latter is clear, since  $\Delta = \Delta_{\overline{\alpha},\overline{\theta}}$  and  $\Gamma = \overline{\theta}_1 \times \eta_2$  and for each subscript *i* we have  $(u_i, v_i) \in \overline{\theta}$ , so

$$(u_i, u_i) \stackrel{\Delta}{\equiv} (v_i, v_i) \stackrel{\Gamma}{\equiv} (u_i, v_i).$$

We have shown that the matrix in (MM) is truly an  $\eta_1$ ,  $(\Delta + \Gamma)$ -matrix. The first row is constant and the second is not (since p = q and  $r \neq s$  as one sees in the lines before the statement of Claim 4.38).

Establishing Goal 4.37, that  $\mathbf{C}(\eta, \Delta + \Gamma; \eta^{\infty})$  fails, is equivalent to establishing that there exists some  $\eta$ ,  $(\Delta + \Gamma)$ -matrix whose first row lies in  $\eta^{\infty}$  and whose second row does not. We shall argue that the matrix in (MM) is such a matrix. Already we know from Claim 4.38 that this matrix is an  $\eta$ ,  $(\Delta + \Gamma)$ -matrix. We also know that the first row lies in  $\eta^{\infty}$ , since the first row is constant. The rest of the proof is devoted to showing that the second row does not belong to  $\eta^{\infty}$ . For this, let  $V = r/\overline{\theta}$  be the  $\overline{\theta}$ -class of r in **B** and let  $U = V \times V =$  $(r,r)/(\overline{\theta}_1 \times \overline{\theta}_2)$  be the  $\overline{\theta}_1 \times \overline{\theta}_2$ -class of (r,r) in  $\mathbf{B}(\overline{\alpha})$  and let  $0_U$  be the equality relation on U. Since  $(r,s) \in \overline{\theta}$ , we have  $(r,r), (r,s) \in U$ . Since  $r \neq s$ , we also have  $(r,r) \neq (r,s)$ . We shall accomplish Goal 4.37 by showing that  $\eta^{\infty}|_U = 0_U$ , so  $((r,r), (r,s)) \notin \eta^{\infty}$ .

**Claim 4.39.** U is a union of  $\Delta$ -classes and a union of  $\Gamma$ -classes. In fact, U is a union of congruence classes for each of the congruences  $\Delta^{2n+1}, \Gamma^{2n}, \eta^k$  and  $\Delta^{\infty}, \Gamma^{\infty}, \eta^{\infty}$ .

Proof of Claim 4.39. Since U is a single class of the congruence  $\overline{\theta}_1 \times \overline{\theta}_2$ , it is a union of congruence classes of any smaller congruence. According to Claim 4.36, all of the congruences  $\Delta = \Delta^1$ ,  $\Gamma = \Gamma^0$ ,  $\Delta^{2n+1}$ ,  $\Gamma^{2n}$ ,  $\eta^k$ ,  $\Delta^{\infty}$ ,  $\Gamma^{\infty}$ ,  $\eta^{\infty}$  are contained in  $\Delta + \Gamma$ , which is contained in  $\overline{\theta}_1 \times \overline{\theta}_2$ .

Claim 4.40.  $\eta^1|_U = 0_U$ .

*Proof of Claim 4.40.* This claim is proved by localizing the proof of  $(a) \Rightarrow (b)$  of Theorem 4.30 to the congruence class U.

Since  $0 < \overline{\theta} < \overline{\alpha}$  in Con(**B**) and  $0 = [\overline{\theta}, \overline{\alpha}] (\geq [\overline{\theta}, \overline{\theta}])$  we get that  $\overline{\theta}$  is an abelian congruence of **B** and hence  $\overline{\theta}_1 \times \overline{\theta}_2$  is an abelian congruence of  $\mathbf{B}(\overline{\alpha})$ . The  $\overline{\theta}_1 \times \overline{\theta}_2$ -class U is therefore a class of an abelian congruence.

Recall that  $\eta = \eta_1$  is the first projection kernel of  $\mathbf{D} = \mathbf{B}(\overline{\alpha})$ . If  $((c, a), (c, b)) \in \eta^1|_U = \eta|_U \cap \Delta|_U$ , then since  $(c, a), (c, b) \in U$  it must be that  $a \stackrel{\overline{\theta}}{\equiv} c \stackrel{\overline{\theta}}{\equiv} b$ . Let  $T(x_1, \ldots, x_n)$ 

be a Taylor term for  $\mathcal{V}$ . Consider a first-place Taylor identity  $T(x, \mathbf{w}) \approx T(y, \mathbf{z})$  where  $\mathbf{w}, \mathbf{z} \in \{x, y\}^{n-1}$ . Substitute *b* for all occurrences of *x* and *c* for all occurrences of *y*. This yields  $T(b, \bar{u}) = T(c, \bar{v})$  where all  $u_i$  and  $v_i$  are in  $\{b, c\}$ . The facts that  $\bar{\theta} \leq \bar{\alpha}, [\bar{\theta}, \bar{\alpha}] = 0$ , and U is a  $\bar{\theta}_1 \times \bar{\theta}_2$ -class imply that the diagonal of *U* is a single  $\Delta = \Delta_{\bar{\alpha},\bar{\theta}}$ -class of  $\mathbf{D} = \mathbf{B}(\bar{\alpha})$ . As in the proof of Theorem 4.30, the fact that  $(T(b, \mathbf{u}), T(c, \mathbf{v}))$  lies on the diagonal of *U* implies that  $(T(a, \mathbf{u}), T(c, \mathbf{v})) = (T(b, \mathbf{u}), T(c, \mathbf{v}))$ , so  $T(a, \mathbf{u}) = T(b, \mathbf{u})$  where each  $u_i \in \{b, c\}$ . By the  $\bar{\theta}, \bar{\theta}$ -term condition,  $T(a, \bar{y}) = T(b, \bar{y})$  for any  $\bar{y}$  whose entries lie in the  $\bar{\theta}$ -class containing *a*, *b* and *c*. As in the proof of Theorem 4.30, this conclusion holds in every place of *T*. That is,

$$T(y_1, \ldots, y_{i-1}, a, y_{i+1}, \ldots, y_n) = T(y_1, \ldots, y_{i-1}, b, y_{i+1}, \ldots, y_n)$$

for each *i* and any choice of values for  $y_1, \ldots, y_n$  in the  $\overline{\theta}$ -class of *a*, *b*, and *c*. Using the fact that *T* is idempotent, we have

$$a = T(a, a, \dots, a) = T(b, a, \dots, a) = \dots = T(b, b, \dots, b) = b.$$

This proves that (c, a) = (c, b). Since  $((c, a), (c, b)) \in \eta^1|_U$  was arbitrarily chosen,  $\eta^1|_U = \eta|_U \cap \Delta|_U = 0_U$ .

# Claim 4.41. $\eta^{\infty}|_U = 0_U$ .

Proof of Claim 4.41. Since U is a single  $\overline{\theta}_1 \times \overline{\theta}_2$ -class, the restriction map from the congruence interval  $I[0, \overline{\theta}_1 \times \overline{\theta}_2]$  in Con(**D**) to the lattice of equivalence relations on the set U is a complete lattice homomorphism. (Under this map a congruence x maps to  $x|_U = x \cap (U \times U)$ .) If you apply this restriction map to the congruences in Figure 13 (excluding  $\eta$ , which need not be in the interval  $I[0, \overline{\theta}_1 \times \overline{\theta}_2]$ ) you will get a similarly-ordered set of equivalence relations on U. If you replace each congruence x in Figure 13 with  $x|_U$ , then all the claims of Claim 4.36 remain true.

In particular, the set  $\{\eta^0|_U, \eta^1|_U, \eta^2|_U, \Gamma^0|_U, \Gamma^2|_U\}$  is a sublattice that is a quotient of a pentagon. By Claim 4.40,  $\eta^0|_U = 0_U = \eta^1|_U$ . Since we are dealing with a quotient of a pentagon, we derive that  $\Gamma^0|_U = \Gamma^2|_U$ , since  $\Gamma^0|_U = \Gamma^0|_U + \eta^0|_U = \Gamma^0|_U + \eta^1|_U = \Gamma^2|_U$ . Then we have  $\eta^2|_U = \eta^1|_U = \eta^0|_U$ , since  $\eta^2|_U = \eta^2|_U \cap \Gamma^2|_U = \eta^2|_U \cap \Gamma^0|_U = \eta^0|_U$ . In summary, from  $\eta^1|_U = \eta^0|_U$  we derive  $\Gamma^2|_U = \Gamma^0|_U$  from which we derive  $\eta^2|_U = \eta^1|_U$ . A similar argument now allows us to derive from  $\eta^2|_U = \eta^1|_U$  that  $\Delta^3|_U = \Delta^1|_U$  from which we derive  $\eta^3|_U = \eta^2|_U$ . This may be continued to derive  $\eta^0|_U = \eta^1|_U = \eta^2|_U = \cdots$ ,  $\Gamma^0|_U = \Gamma^2|_U = \Gamma^4|_U = \cdots$ , and  $\Delta^1|_U = \Delta^3|_U = \Delta^5|_U = \cdots$ . Taking the complete joins of these constant sequences we get  $\eta^\infty|_U = \eta^0|_U = 0$ ,  $\Gamma^\infty|_U = \Gamma^0|_U$ , and  $\Delta^\infty|_U = \Delta^0|_U$ .

# Claim 4.42. $C(\eta, \Delta + \Gamma; \eta^{\infty})$ fails.

Proof of Claim 4.42. From Claim 4.38 we know that matrix (MM) is an  $\eta$ ,  $(\Delta + \Gamma)$ -matrix whose first row is constant, hence the elements in the first row are congruent modulo  $\eta^{\infty}$ . The second row is not constant but lies in U. Since  $\eta^{\infty}|_{U} = 0_{U}$ , the elements in the second row

are not congruent modulo  $\eta^{\infty}$ . Thus matrix (MM) witnesses the failure of  $\mathbf{C}(\eta, \Delta + \Gamma; \eta^{\infty})$ .

We complete the proof of Theorem 4.33 by reiterating ideas mentioned before the statement of Goal 4.37. By Claim 4.36 (3),  $\eta \cap \Delta^{\infty} = \eta \cap \Gamma^{\infty} = \eta^{\infty}$ , so all of the congruences  $\eta, \Delta^{\infty}, \Gamma^{\infty}$  lie above  $\eta^{\infty}$ . In  $\operatorname{Con}(\mathbf{D}/\eta^{\infty})$ , let  $x = \eta/\eta^{\infty}, y = \Delta^{\infty}/\eta^{\infty}, z = \Gamma^{\infty}/\eta^{\infty}$ . We have  $x \cap y = 0 = x \cap z$ , so  $\mathbf{C}(x, y; 0)$  and  $\mathbf{C}(x, z; 0)$  hold in  $\mathbf{D}/\eta^{\infty}$ . But we do not have  $\mathbf{C}(x, y+z; 0)$ in  $\mathbf{D}/\eta^{\infty}$ , since this translates to  $\mathbf{C}(\eta, \Delta^{\infty} + \Gamma^{\infty}; \eta^{\infty})$  in  $\mathbf{D}$ , which is the same statement as  $\mathbf{C}(\eta, \Delta + \Gamma; \eta^{\infty})$ . We proved in Claim 4.42 that  $\mathbf{C}(\eta, \Delta + \Gamma; \eta^{\infty})$  fails in  $\mathbf{D}$ .

Next we show that four of the centralizer properties from the Introduction are Maltsev definable relative to the existence of a Taylor term. The following is another of the primary results of this article.

**Theorem 4.43.** Let  $\mathcal{V}$  be a variety that has a Taylor term. The following are equivalent properties for  $\mathcal{V}$ :

- (1)  $\mathcal{V}$  is congruence modular.
- (2) The centralizer relation is symmetric in its first two places throughout  $\mathcal{V}$ .  $(\mathbf{C}(x, y; z) \iff \mathbf{C}(y, x; z).)$
- (3) Relative right annihilators exist. (Given x, z, there is a largest y such that  $\mathbf{C}(x, y; z)$ . Write  $y = (z : x)_R$ .)
- (4) The centralizer relation is determined by the commutator throughout  $\mathcal{V}$ .  $(\mathbf{C}(x, y; z) \iff [x, y] \le z.)$
- (5) The centralizer relation is stable under lifting in its third place throughout  $\mathcal{V}$ . ( $\mathbf{C}(x, y; z) \& (z \leq z') \Longrightarrow \mathbf{C}(x, y; z').$ )

Proof. The fact that the bi-implication in Item (2) holds in every congruence modular variety is proved in [2, Proposition 4.2]. (The bi-implication is  $(1)(iii) \Leftrightarrow (1)(iv)$  of Proposition 4.2 of the reference.) Hence  $(1) \Rightarrow (2)$ . We next explain how to derive (3) from (2): It follows from Theorem 3.4 (5) that the relative *left* annihilator,  $(\delta : \theta)_L$ , exists for any  $\delta, \theta \in \text{Con}(\mathbf{A})$  on any algebra  $\mathbf{A}$ . From (2), which asserts the symmetry of the centralizer relation in its first two places, it follows that relative right annihilators must also exist and that  $(\delta : \theta)_R = (\delta : \theta)_L$ for every  $\delta$  and  $\theta$ .

The fact that the bi-implication in Item (4) holds in every congruence modular variety is (1)(iii) $\Leftrightarrow$ (1)(v) of [2, Proposition 4.2]. Hence (1) $\Rightarrow$ (4). We also have (4) $\Rightarrow$ (5), for the following reason: (4) asserts that the centralizer relation is equivalent to the relation  $[x, y] \leq z$ , which is a relation that is stable under lifting in z. Thus, (4) implies that the centralizer is stable under lifting in its third place, which is Item (5).

So far we have  $(1) \Rightarrow (2) \Rightarrow (3)$  and  $(1) \Rightarrow (4) \Rightarrow (5)$ . To finish the proof it will suffice to establish  $(3) \Rightarrow (1)$  and  $(5) \Rightarrow (1)$ .

In this paragraph we prove  $(3) \Rightarrow (1)$  by contradiction. Therefore, assume that (3) holds (relative right annihilators exist) and (1) fails ( $\mathcal{V}$  is not congruence modular). We also assume throughout the proof that the global hypotheses of the theorem hold ( $\mathcal{V}$  is a variety

that has a Taylor term). If relative right annihilators (those of the form  $(\delta : \theta)_R$ ) always exist, then ordinary right annihilators (those of the form  $(0 : \theta)_R$ ) must also exist. Since  $\mathcal{V}$  has a Taylor term and the property that ordinary right annihilators exist throughout  $\mathcal{V}$ , it follows from Theorem 4.33 that  $\mathcal{V}$  has a difference term. Since we have assumed that  $\mathcal{V}$  is not congruence modular, there will exist an algebra  $\mathbf{A} \in \mathcal{V}$  with a pentagon in Con( $\mathbf{A}$ ). We label it as in Figure 4. For this algebra we have  $\mathbf{C}(\theta, \beta; \delta)$  and  $\mathbf{C}(\theta, \delta; \delta)$  by Theorem 3.4 (7). Hence  $\beta, \delta \leq (\delta : \theta)_R$ , which forces  $\alpha = \beta + \delta \leq (\delta : \theta)_R$ , or equivalently  $\mathbf{C}(\theta, \alpha; \delta)$ . By monotonicity in the middle place we derive  $\mathbf{C}(\theta, \theta; \delta)$ , which implies that the critical interval of the pentagon,  $I[\delta, \theta]$ , is abelian. But according to Theorem 4.25, critical intervals of pentagons are neutral in varieties with a difference term. We have arrived at a contradiction, since nontrivial congruence intervals like  $I[\delta, \theta]$  cannot be both abelian  $([\theta, \theta]_{\delta} = \delta)$  and neutral  $([x, x]_y = x \text{ for } \delta \leq y \leq x \leq \theta)$ . This completes the proof that (1), (2), and (3) are equivalent.

In this paragraph we prove  $(5) \Rightarrow (1)$  by contradiction. Therefore, assume that (5) holds (the centralizer is stable under lifting in its third place) and (1) fails ( $\mathcal{V}$  is not congruence modular). We proceed as above: since we have assumed that  $\mathcal{V}$  is not congruence modular, there will exist an algebra  $\mathbf{A} \in \mathcal{V}$  with a pentagon in Con( $\mathbf{A}$ ), which we label as in Figure 4. According to Theorem 3.4 (8), we have that  $\mathbf{C}(\beta, \theta; \beta \cap \theta)$  holds. Since we have assumed that the centralizer relation is stable under lifting in its third place,  $\mathbf{C}(\beta, \theta; \delta)$  holds. This contradicts Theorem 4.2. This completes the argument that Items (1), (4), and (5) are equivalent.

Finally we show that the property of weak stability of the centralizer relation in its third place is Maltsev definable relative to the existence of a Taylor term. This is our last primary result.

**Theorem 4.44.** Let  $\mathcal{V}$  be a variety that has a Taylor term. The following are equivalent properties for  $\mathcal{V}$ :

- (1)  $\mathcal{V}$  has a difference term.
- (2) The centralizer relation is weakly stable under lifting in its third place throughout  $\mathcal{V}$ .  $(\mathbf{C}(x, y; z) \& (z \leq z' \leq x \cap y) \Longrightarrow \mathbf{C}(x, y; z').)$

Proof. For  $(1) \Rightarrow (2)$ , assume that  $\mathcal{V}$  has a difference term and that some  $\mathbf{A} \in \mathcal{V}$  has congruences  $x = \alpha, y = \beta, z = \delta, z' = \gamma$  such that  $\mathbf{C}(\alpha, \beta; \delta)$  &  $(\delta \leq \gamma \leq \alpha \cap \beta)$ . By [6, Lemma 2.3 (i) $\Rightarrow$ (ii)],  $\mathbf{C}(\alpha, \beta; \delta)$  implies  $[\alpha, \beta]_{\delta} = \delta$ . It now follows from [6, Lemma 2.4] that  $[\alpha, \beta]_{\gamma} = [\alpha, \beta] + \gamma = \gamma$ . By [6, Lemma 2.3 (ii) $\Rightarrow$ (i)],  $\mathbf{C}(\alpha, \beta; \gamma)$  holds. This establishes the weak stability property.

Now we argue that if  $\mathcal{V}$  has a Taylor term and does not have a difference term, then the centralizer will not be weakly stable in its third place in some instances. By Lemma 4.27,  $\mathcal{V}$  has an algebra **A** with a pentagon in its congruence lattice, which satisfies the commutator conditions (1), (2), and (3) of that lemma. Take  $x = \alpha, y = \theta, z = 0, z' = \delta$ . From the

lemma,  $[\alpha, \theta] = 0$ , so  $\mathbf{C}(x, y; z)$  holds. By our choices,  $z \leq z' \leq x \cap y$ . Since  $[\alpha, \theta]_{\delta} = \theta$  we have  $\neg \mathbf{C}(x, y; z')$ . This shows that weak stability fails.  $\Box$ 

### 5. Outro

5.1. The intended applications. The results of this paper help to decide some cases of the following question: Given a finite algebra  $\mathbf{A}$  of finite type, does the variety  $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$  have commutative commutator? If  $\mathcal{V}$  has a Taylor term, then the answer is affirmative if and only if  $\mathcal{V}$  also has a difference term. There are known algorithms to decide whether  $\mathcal{V}$  has a Taylor term and whether  $\mathcal{V}$  has a difference term whenever  $\mathcal{V}$  is generated by a finite algebra of finite type. (These algorithms are implemented in UACalc, [3]). This gives a path to answer the question algorithmically for finite algebras of finite type that have a Taylor term.

Even in the case where  $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$  does not have a Taylor term, the results of this paper might apply. Suppose that A is a finite algebra of finite type and  $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$  does not have a Taylor term. It is possible that some  $\mathbf{B} \in \mathcal{V}$  generates a subvariety  $\mathcal{U} = \mathsf{HSP}(\mathbf{B})$  that has a Taylor term but does not have a difference term. In this case, the subvariety will not have commutative commutator, so  $\mathcal{V}$  cannot have commutative commutator. If there is such a  $\mathbf{B} \in \mathcal{V}$ , then there must exist such a **B** that is free on three generators in the subvariety it generates, hence will be a quotient of the finite, relatively free algebra  $\mathbf{F}_{\mathcal{V}}(3)$ . Determining whether such a **B** exists is a matter of a finite amount of computation. A concrete example where this happens is when **A** is the semigroup  $\mathbb{Z}_2 \times \mathbb{S}_2 \times \mathbb{L}_2$  and  $\mathbf{B} = \mathbb{Z}_2 \times \mathbb{S}_2$ . (Here  $\mathbb{Z}_2$ ) is the 2-element group considered as a semigroup,  $\mathbb{S}_2$  is the 2-element semilattice, and  $\mathbb{L}_2$  is the 2-element left zero semigroup.) In this example,  $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$  does not have a Taylor term, but one may still apply the results of this paper to derive that the commutator is not commutative in  $\mathcal{V}$  since the subvariety  $\mathcal{U} = \mathsf{HSP}(\mathbf{B})$  has a Taylor term and does not have a difference term. (Contrast with this example: the algebra  $\mathbf{C} = \mathbb{S}_2 \times \mathbb{L}_2$  generates a variety with noncommutative commutator, but the results of this paper do not help to establish this since every subvariety of  $HSP(\mathbf{C})$  that has a Taylor term also has a difference term.)

Another intended application of the results of this paper is to help understand whether some theorems are expressed with optimal hypotheses. For example, in [11], Ágnes Szendrei, Ross Willard and I proved Park's Conjecture for varieties with a difference term. Park's Conjecture is the conjecture that a finitely generated variety of finite type is finitely based whenever it has a finite residual bound. One question received after the publication of that paper was: How hard would it be to generalize the proof in [11], which assumes the existence of a difference term, to establish Park's Conjecture for varieties with a *weak* difference term?<sup>3</sup> Our proof in [11] depends on the commutativity of the commutator in some places. Thus one may ask: if one were to refine the proof in [11] so that it proves Park's Conjecture for varieties that have a *weak* difference term and commutative commutator, would this

<sup>&</sup>lt;sup>3</sup>Note: A finitely generated variety has a weak difference term if and only if it has a Taylor term.

refinement constitute a proper generalization of the result in [11]? The answer is negative, according to Theorem 4.28 of this paper. That is, the class of varieties which have a weak difference term and commutative commutator is exactly the same as the class of varieties with a difference term. Any proper generalization of the result in [11] must apply to some varieties  $\mathcal{V}$  in which either (i)  $\mathcal{V}$  has no Taylor term or (ii) the commutator operation in  $\mathcal{V}$  is not commutative.

5.2. Some problems from [13]. The results of this article partially solve some problems posed by Paolo Lipparini in [13]. The problems I refer to are:

## Problems 1.7 of [13].

- (a) Find conditions implying (if possible, equivalent to) left join distributivity, right join distributivity or commutativity of the commutator.
- (b) In particular, is there a (weak) Mal'cev condition strictly weaker than modularity and implying left join distributivity of the commutator?
- (c) Does right join distributivity always imply left join distributivity?
- (e) Answer the above questions at least in the particular cases of varieties with a (weak) difference term, *M*-permutable varieties, locally finite varieties (omitting type 1 or some other type).

## Partial solutions.

In this article we work at the level of varieties. At this level, we can say the following.

Regarding Problem 1.7(a), we have characterized those varieties with a Taylor term that have left distributive, right distributive, or commutative commutator.

Regarding Problem 1.7(b), Theorem 4.32 implies that there is <u>no</u> idempotent Maltsev condition strictly weaker than modularity that implies left distributivity of the commutator.

Regarding Problem 1.7(c), Theorem 3.14 (1) shows that left distributivity of the commutator throughout a variety implies commutativity of the commutator throughout the variety. Hence left distributivity implies right distributivity in any variety. We do not know if, conversely, right distributivity implies left distributivity in every variety. Nevertheless we have shown that left and right distributivity are equivalent for varieties with a Taylor term in Theorem 4.31.

Regarding Problem 1.7(e), if some variety  $\mathcal{V}$  has a difference term, a weak difference term, is *M*-permutable, or is a locally finite omitting type 1, then  $\mathcal{V}$  has a Taylor term. In these settings we have classified the varieties that have left distributive, right distributive, or commutative commutator.

5.3. **Two problems from** [9]. The results of this article solve two problems from the list of 64 open problems posed at the Workshop on Tame Congruence Theory which was held at the Paul Erdős Summer Research Center of Mathematics in 2001.

**Problem 10.6 of** [9]. Let  $\mathcal{V}$  be a locally finite variety that omits type **1**. Is it true that if  $[\alpha, \beta] = [\beta, \alpha]$  for all congruences  $\alpha, \beta$  of algebras in  $\mathcal{V}$ , then  $\mathcal{V}$  has a difference term?

A locally finite variety omits type **1** if and only if it has a Taylor term, according to Lemma 9.4 and Theorem 9.6 of [4]. Thus, Problem 10.6 of [9] asks about the truth of Theorem 4.29 of this paper in the restricted setting of locally finite varieties. Theorem 4.29 provides an affirmative answer.

**Problem 10.7 of** [9]. Are there natural conditions on a variety  $\mathcal{V}$  under which the implications

 $[\alpha,\beta] = [\alpha,\gamma] \Longrightarrow [\alpha,\beta] = [\alpha,\beta+\gamma] \text{ and } [\beta,\alpha] = [\gamma,\alpha] \Longrightarrow [\beta,\alpha] = [\beta+\gamma,\alpha]$ 

hold throughout the variety  $\mathcal{V}$ ? (Consider, e.g., the condition ' $\mathcal{V}$  has a difference term'.)

Problem 10.7 of [9] asks for natural conditions guaranteeing the right or left semidistributivity of the commutator for varieties, and suggests that having a difference term might be such a condition. Every variety has left semidistributive commutator by Theorem 3.4 (5) and the definition of the commutator, so the nontrivial part of this problem is the question about right semidistributivity. Theorem 4.33 proves that, for varieties with a Taylor term, the condition proposed in Problem 10.7 of [9] (that  $\mathcal{V}$  has a difference term) is a necessary and sufficient condition guaranteeing that  $\mathcal{V}$  has right (and left) semidistributive commutator.

5.4. A problem from [7]. The results of this article solve a problem I posed at the 90th Arbeitstagung Allgemeine Algebra held at the University of Novi Sad in 2015. There I gave a talk entitled *Problems on the frontier of commutator theory*. These twenty-five problems were not published formally, but the slides for the talk are posted at [7]. The thirteenth problem asks

**Problem.** Does  $\exists$  weak difference term + symmetric commutator imply  $\exists$  difference term?

This problem is answered affirmatively in Theorem 4.28 of this paper. The affirmative answer is strengthened in two ways to

 $\exists$ Taylor term + symmetric commutator  $\iff \exists$  difference term.

in Theorem 4.29. The conclusion is: If you want to prove some case of some conjecture about varieties, and you need (i) a Taylor term and (ii) a commutative commutator throughout your variety for the proof, then the assumption that the variety has a difference term guarantees both (i) and (ii) and it is the optimal hypothesis that guarantees both.

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