# THE ISOMORPHISM PROBLEM OF PROJECTIVE SCHEMES AND RELATED ALGORITHMIC PROBLEMS 

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#### Abstract

We discuss the isomorphism problem of projective schemes; given two projective schemes, can we algorithmically decide whether they are isomorphic? We give affirmative answers in the case of one-dimensional projective schemes, the case of smooth irreducible varieties with a big canonical sheaf or a big anti-canonical sheaf, and the case of K3 surfaces with a finite automorphism group. As related algorithmic problems, we also discuss decidability of positivity properties of invertible sheaves, and approximation of the nef cone and the pseudo-effective cone.


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## 1. Introduction

The main purpose of this paper is to discuss the isomorphism problem of projective schemes over the field $\overline{\mathbb{Q}}$ of algebraic numbers. Is it algorithmically decidable whether two given projective $\overline{\mathbb{Q}}$-schemes are isomorphic? What if we restrict ourselves to some class of projective schemes, for example, the class of smooth projective varieties having a prescribed invariant. Poonen [Poo11] writes that Totaro asked him about this problem in 2007. The case of smooth irreducible curves was treated earlier in the 2005 paper [BGJGP05 by Baker, González-Jiménez, González, and Poonen. The same problem was asked also by Arapura on MathOverflow ${ }^{11}$ in 2010.

To the best of the author's knowledge, the decidability of the isomorphism problem has been proved in the following two cases:
(1) Smooth irreducible curves ([BGJGP05, Lem. 5.1] for the case of genus $\neq 1$ and Poonen's comment in the MathOverflow thread mentioned above for the case of genus one).
(2) Varieties of general type (see Poo14, Rem. 12.3] for the proof due to Totaro).

In the same MathOverflow thread as above, there is also discussion about the cases of K3 surfaces and abelian surfaces, which has not reached a definite conclusion.

The main result of the paper is to prove that the isomorphism problem is decidable in the following cases:
(1) One-dimensional projective schemes (Theorem 7.31).
(2) Smooth projective varieties with a big canonical sheaf or a big anti-canonical sheaf (Theorem 8.11).
(3) K3 surfaces with a finite automorphism group (Theorem 11.3).

The first two cases slightly generalize ones mentioned above. As an application of the first case, we show the decidability also in the case of one-dimensional reduced quasi-projective schemes (Theorem 7.4). In birational geometry, varieties of Kodaira dimensions $-\infty$ and 0 as well as ones of general type have special importance, as they are considered as building blocks of all varieties. With case (2) above being solved, varieties of Kodaira dimension 0 would be the remaining most imporant case. Besides K3 surfaces, the isomorphism problem for abelian varieties should be important, but we do not discuss it in this paper.

Remark 1.1. It appears difficult to apply the global Torelli theorem for K3 surfaces [BHPV04, p. 332] to solve the isomorphism problem affirmatively. We can approximate the Hodge structure on cohomology groups with arbitrary precision Sim08. But the moduli space of (marked) K3 surfaces is not Hausdorff [BHPV04, p. 334]. This suggests that we cannot detect non-isomorphism of K3 surfaces by approximation.

Our strategy to prove these results is to compute the Iso schemes $\underline{\mathrm{Iso}}_{P_{i}}(X, Y)$ for the given projective schemes $X$ and $Y$ and for finitely many polynomials $P_{i}$. The entire Iso scheme $\underline{\text { Iso }}(X, Y)$ is the moduli scheme of isomorphisms $X \xrightarrow{\sim} Y$ and can be embedded into the Hilbert scheme $\operatorname{Hilb}(X \times Y)$ by sending an isomorphism $f: X \rightarrow Y$

[^1]to its graph $\Gamma_{f} \subset X \times Y$. The Hilbert scheme is decomposed as $\operatorname{Hilb}(X \times Y)=$ $\coprod_{P} \operatorname{Hilb}_{P}(X \times Y)$, where $P$ runs over countably many polynomials. This induces a decomposition $\underline{\operatorname{Iso}}(X, Y)=\coprod_{P} \underline{\operatorname{Iso}}(X, Y)$ of the Iso scheme. For each polynomial $P$, $\underline{\mathrm{Iso}}_{P}(X, Y)$ is of finite type, but the entire $\operatorname{Iso}(X, Y)$ is not generally so. We explain how to algorithmically compute $\underline{\operatorname{Iso}}_{P}(X, Y)$ for each $P$. Having the method of computing Iso schemes, we then construct an algorithm for each of the classes of projective schemes mentioned above that produces finitely many polynomials $P_{1}, \ldots, P_{n}$ from the given projective schemes $X$ and $Y$. These polynomials satisfy the condition that $X$ and $Y$ are isomorphic if and only if $\underline{\operatorname{Iso}}_{P_{i}}(X, Y) \neq \emptyset$ for some $i$. Then, whether $X$ and $Y$ are isomorphic or not is checked by computing these Iso schemes. In construction of finitely many polynomials as above, we use the Kodaira vanishing as a key ingredient in case (2) and use computation of the nef cone in case (3).

We also discuss several algorithmic problems related to the isomorphism problem. Firstly, we explicitly describe an algorithm to check whether two given projective schemes embedded in the same projective space are projectively equivalent (Section (5). If two projective schemes are projectively equivalent, then they are isomorphic, but the converse does not generally hold. Secondly, partly using computation of intersection numbers, we discuss positivity properties of invertible sheaves from the algorithmic viewpoint. We see that global generation of a coherent sheaf and very ampleness of an invertible sheave on a projective scheme is decidable. Using the Nakai-Moishezon criterion for ampleness and computation of intersection numbers, we see that, if the scheme is smooth and irreducible, then ampleness of an invertible sheaf is also decidable (Proposition 10.4). We do not know whether other positivities, bigness, nefness and pseudo-effectivity, are decidable. However, if we can compute the Picard number, then we can approximate the nef cone and the pseudo-effective cone with arbitrary precision (Proposition 10.14). Note that Poonen, Testa and van Luijk [PTL15] proved that the Picard number of a smooth irreducible projective variety is computable, if the Tate conjecture is true. This is the case for K3 surfaces. For a K3 surface with a finite automorphism group, we can compute its nef cone (not approximately but exactly), which is used to show the decidability in case (3) above.

To end this introduction, we mention a few more related works. Truong [Tru18] proved the decidability of the bounded birationality problem. Namely, he proved that for projective varieties $X \subset \mathbb{P}^{m}$ and $Y \subset \mathbb{P}^{n}$ and for a positive integer $d$, we can decide whether there exists a rational map $\mathbb{P}^{m} \longrightarrow \mathbb{P}^{n}$ of degree $\leq d$ that restricts to a birational map $X \rightarrow Y$. He also proved the decidability of the bounded isomorphism problem in the case where one of the two given varieties is smooth. To prove these results, he showed computability of a variety parametrizing rational maps $\mathbb{P}^{m} \rightarrow \mathbb{P}^{n}$ with this property, which is similar to our computability result regarding Iso schemes.

The isomorphism problem that we consider in this paper is a speical case of the problem regarding the existence of a morphism $X \rightarrow Y$ of $k$-schemes possibly imposed with some condition for a more general field or ring $k$. For example, the famous negative solution by Davis, Matiyasevich, Putnam, and Robinson to Hilbert's tenth problem says that the existence of a morphism $\operatorname{Spec} \mathbb{Z} \rightarrow Y$ with $Y$ an affine scheme of finite type over $\mathbb{Z}$ is undecidable. One of the other undecidability results in this direction is the
one of Kanel'-Belov and Chilikov KBC19 (see also Kol20) that the existence of an embedding $X \hookrightarrow Y$ of varieties over $\mathbb{R}$ or $\overline{\mathbb{Q}}$ is undecidable.

The rest of the paper is organized as follows. In Section 2, we set up our basic convention. In particular, we clarify what we mean by saying that some object (for example, a scheme or an invertible sheaf) is given. In Section 3, we show that the isomorphism problem of projective schemes is semi-decidable. In Section 4, we explain how to compute the Hilbert scheme for each polynomial. In Section 5, we apply computation of the Hilbert scheme to show that it is decidable whether two projective schemes embedded in the same projective space is projectively equivalent. Although eash result in sections 3 to 5 would be known to specialists, we include them for the sake of reader's convenience. The reader who knows these materials well may skip these sections. In Section 6. we explain how to compute the Hom scheme and the Iso scheme for each polynomial. In Section 7, we show the decidability of the isomorphism problem for one-dimensional projective schemes and the one for one-dimensional quasi-projective reduced schemes. In Section 8, we do the same for the case of smooth irreducible varieties with a big canonical sheaf or a big anti-canonical sheaf. In Section 9, we explain how to compute intersection numbers on a smooth irreducible projective variety. In Section 10, we discuss decidability of positivity properties of invertible sheaves. In particular, we show that ampleness of an invertible sheaf on a smooth variety is decidable and that the nef cone and the pseudo-effective cone are approximated by rational polyhedral cones with arbitrary precision. In Section 11, we discuss the isomorphism problem for K3 surfaces as well as smooth varieties with a rational polyhedral nef cone.

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## 2. Preliminaries

Throughout the paper, we work over the field of complex algebraic numbers, $\overline{\mathbb{Q}} \subset \mathbb{C}$, which is denoted by $k$. As explained in [Sim08, Section 2.1], elements of this field are expressed by finite data and four basic arithmetic operations on them, addition, subtraction, multiplication and division, are algorithmically computable. We can also algorithmically decide whether or not two expressions give the same number. It follows that we can also express polynomials with coefficients in $k$ by finite data and algorithmically compute their addition, subtraction and multiplication. We can also compute the Gröbner basis of an ideal in a polynomial ring $k\left[x_{1}, \ldots, x_{m}\right]$. Thus we can also make various computation based on the Gröbner basis. For example, we can algorithmically check whether or not an ideal is contained in another ideal in the same polynomial ring (this is an application of the ideal membership test; see [Eis95, 15.10.1]). We also note that the elements of $k$ as well as the elements of $k\left[x_{1}, \ldots, x_{m}\right]$ are enumerable.

When we say that a projective scheme $X$ is given, we mean that we are given finitely many homogeneous polynomials $f_{1}, \ldots, f_{l} \in k\left[x_{1}, \ldots, x_{m}\right]$ such that $X$ is the closed subscheme of $\mathbb{P}^{m-1}=\operatorname{Proj} k\left[x_{1}, \ldots, x_{m}\right]$ defined by the ideal $\left(f_{1}, \ldots, f_{l}\right)$. In particular, we are given an embedding $\iota: X \hookrightarrow \mathbb{P}^{m-1}$ into a projective space, the induced very ample invertible sheaf $\iota^{*} \mathcal{O}_{\mathbb{P}^{m-1}}(1)$ and the homogeneous coordinate ring
$R_{X}:=k\left[x_{1}, \ldots, x_{m}\right] /\left(f_{1}, \ldots, f_{l}\right)$, which is also denoted by $R$ omitting the subscript $X$. From these data, we can compute the standard affine charts $X_{i}:=X \cap\left\{x_{i} \neq 0\right\}$ for $1 \leq i \leq m$, which cover $X$. Indeed, if $k\left[x_{1}, \ldots, \check{x}_{i}, \ldots, x_{m}\right]$ is the polynomial ring with $x_{i}$ removed and if $f_{j}^{(i)}$ denotes the polynomial obtained from $f_{j}$ by substituting 1 for $x_{i}$, then $X_{i}$ is the closed subscheme of $\mathbb{A}^{m-1}=\operatorname{Spec} k\left[x_{1}, \ldots, \check{x}_{i}, \cdots, x_{m}\right]$ defined by $f_{1}^{(i)}, \ldots, f_{l}^{(i)}$.

For a projective scheme $X \subset \mathbb{P}^{m-1}$ defined by $f_{1}, \ldots, f_{l}$, we suppose that every coherent sheaf of $X$ (in particular, an invertible sheaf) is represented by a finitely generated module over $R=R_{X}$. In turn, every finitely generated $R$-module is represented by a matrix $A \in \mathrm{M}_{r \times s}(R)$ which defines a free presentation of $M$,

$$
\bigoplus_{j=1}^{s} R\left(b_{j}\right) \xrightarrow{A} \bigoplus_{i=1}^{r} R\left(a_{i}\right) \rightarrow M \rightarrow 0
$$

Here maps are supposed to be degree-preserving and $R(a)$ denotes the graded free $R$-module of rank one defined by $R(a)_{c}=R_{a+c}$.

When two projective schemes $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ are given, we can embed the product $X \times Y$ into $\mathbb{P}^{m n-1}$ via the Segre embedding $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{m n-1}$. When we say that a morphism $f: X \rightarrow Y$ is given, we mean that its graph $\Gamma_{f} \subset X \times Y$ is given as a closed subscheme of $\mathbb{P}^{m n-1}$.

## 3. SEMI-DECIDABILITY OF THE ISOMORPHISM PROBLEM OF PROJECTIVE SCHEMES

In this section, we show the probably well-known fact that the isomorphism problem of projective schemes is semi-decidable; there exists an algorithm such that, when two projective schemes are given as an input, then the algorithm stops after finitely many steps if and only if these schemes are isomorphic. The algorithm given in this section is a very naive one and would be very inefficient. An approach via Iso schemes would give a more efficient algorithm (see Remark 6.7).

Remark 3.1. Poonen pointed out to the author that the isomorphism problem of finitetype $k$-schemes is also semi-decidable and it appears well-known. Roughly, the proof is by checking whether the given schemes have the "same" affine open coverings. Our proof below for projective schemes is more along our basic strategy in terms of graphs. Arguments in it will be repeated in computation of Iso schemes in Section 6 ,

Let $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ be projective schemes. We first enumerate all the closed subschemes of $X \times Y$. To do so, we enumerate all the finite sequences $f_{1}, \ldots, f_{l}$ of homogeneous polynomials in $k\left[w_{i j} \mid 1 \leq i \leq m, 1 \leq j \leq n\right]$. For each positive integer $i$, let $I_{i}$ be the ideal generated by the $i$-th sequence and let $Z_{i} \subset \mathbb{P}^{n m-1}$ be the closed subscheme corresponding to $I_{i}$. Thus we obtain the sequence $Z_{i}, i>0$ of closed subschemes such that for every closed subscheme $Z \subset \mathbb{P}^{n m-1}$, there exists $i>0$ such that $Z=Z_{i}$. For each $i$, we can check whether or not $Z_{i}$ is included in $X \times Y$. Removing the ones not included in $X \times Y$, we can algorithmically produce every closed subscheme of $X \times Y$ one by one. If we prefer, we may remove redundancies to get a sequence where every closed subscheme of $X \times Y$ appears exactly once. We let $Z_{i}, i>0$ be thus obtained sequence of closed subschemes of $X \times Y$.

Proposition 3.2. The isomorphism problem of projective schemes is semi-decidable
Proof. For each integer $i>0$, from Lemma 3.3 below, we can algorithmically check whether $Z_{i}$ is the graph of an isomorphism $X \xrightarrow{\sim} Y$. As soon as one finds that this is the case, we stop this algorithm.

Lemma 3.3. We can algorithmically check whether or not a given closed subscheme $\Gamma \subset X \times Y$ is the graph of an isomorphism $X \xrightarrow{\sim} Y$.

Proof. We need to check whether the two projections $\Gamma \rightarrow X$ and $\Gamma \rightarrow Y$ are both isomorphisms. We discuss only the former projection, denoting it by $f$. Let $X=\bigcup X_{j}$ be the standard affine open covering and let $R_{j}$ be the coordinate ring of $X_{j}$. Let $\Gamma_{j}$ be the preimage of $X_{j}$ by the morphism $\Gamma \rightarrow X$, which is a closed subscheme of $\mathbb{P}_{R_{j}}^{n-1}=\operatorname{Proj} R_{j}\left[y_{1}, \ldots, y_{n}\right]$. The morphism $f: \Gamma \rightarrow X$ is an isomorphism if and only if every

$$
f_{j}:=\left.f\right|_{\Gamma_{j}}: \Gamma_{j} \rightarrow X_{j}
$$

is an isomorphism. We can compute the coherent sheaf $\Omega_{\Gamma_{j} / X_{j}}$ of differentials (see Remark (3.4) and check whether or not it is the zero sheaf. Thus we can algorithmically check whether or not $f_{j}$ is unramified. If it is not unramified, then it is not an isomorphism. Suppose that $f_{j}$ is unramified. Then it is also a finite morphism. We compute an $R_{j}$-module $M_{j}$ such that $\widetilde{M}_{j}:=\left(f_{j}\right)_{*} \mathcal{O}_{\Gamma_{j}}$ (see Remark 3.5). Consider the following three conditions:
(1) $\operatorname{Supp}\left(M_{j}\right)=X_{j}$.
(2) $V\left(\operatorname{Fitt}_{1}\left(M_{j}\right)\right)=\emptyset$, where $\operatorname{Fitt}_{1}\left(M_{j}\right)$ denote the first Fitting ideal of $M_{j}$.
(3) $\operatorname{Tor}^{R_{j}}\left(\left(R_{j}\right)_{\text {red }}, M_{j} \otimes\left(R_{j}\right)_{\mathrm{red}}\right)=0$.

The second condition means that for every point $x \in X_{j}$, the stalk $\left(\widetilde{M}_{j}\right)_{x}$ is generated by one element as an $\mathcal{O}_{X_{j}, x}$-module. From [Har77, II, Exercise 5.8], the first two conditions together are equivalent to that $M_{j} \otimes\left(R_{j}\right)_{\text {red }}$ is a flat $\left(R_{j}\right)_{\text {red }}$-module of constant rank one. Under these conditions, the third condition means that $M_{j}$ is a flat $R_{j}$-module (see [Mat89, Th. 22.3] or Sta21, tag 051C]). We conclude that these three conditions all hold if and only if the finite unramified morphism $f_{j}: \Gamma_{j} \rightarrow X_{j}$ is surjective and flat of constant rank one, that is, an isomorphism.

Remark 3.4 (cf. [Sti, Prop. 5.7]). Let $R$ be a commutative ring and let

$$
X=\operatorname{Proj} R\left[x_{1}, \ldots, x_{r}\right] /\left(f_{1}, \ldots, f_{l}\right)
$$

be a projective scheme over $R$, where $f_{1}, \ldots, f_{l}$ are homogeneous polynomials. Then the cotangent sheaf $\Omega_{X / R}$ is associated to the homology module of the sequence

$$
\bigoplus_{i=1}^{l} S\left(-\operatorname{deg} f_{i}\right) \xrightarrow{\left(\partial f_{i} / \partial x_{j}\right)_{i, j}} S(-1)^{\oplus r} \xrightarrow{\left(x_{1} \cdots x_{r}\right)} S
$$

Stillman's notes cited above treat the case where $R$ is a field, but this lemma holds for an arbitrary $R$. This is a straightforward consequence of Har77, Prop. 8.12 and Th. 8.13].

Remark 3.5. Let $X \subset \mathbb{P}_{R}^{n}$ be a projective scheme over a commutative ring $R$, let $\pi: X \rightarrow \operatorname{Spec} R$ be the structure morphism and let $\mathcal{M}$ be a coherent sheaf on $X$. Algorithms computing the pushfoward $\pi_{*} \mathcal{M}$ (and more generally, higher direct images $R^{i} \pi_{*} \mathcal{M}$ ) are explained in [Smi98, ES08].

## 4. Hilbert schemes

In this section, we discuss how to compute Hilbert schemes of general projective schemes and their universal families. Their computability, Proposition 4.4, has been already proved in PTL15, Lem. 8.23]. As this result is the core of our approach, we explain it in more details below.
4.1. The Hilbert scheme of a projective space. Bayer Bay82 explained how to compute equations defining the Hilbert scheme $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$ for each Hilbert polynomial $P$ as a closed subset of a Grassmaniann variety. It turned out that his equations also give the right scheme structure of the Hilbert scheme. We recall this description of the Hilbert scheme $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$, closely following the presentation by Iarrobino and Kleiman in [IK99, Appendix C] but with emphasis on algorithmic aspects.

Throughout this section, we fix a positive integer $r>0$. Let $R:=k\left[x_{1}, \ldots, x_{r}\right]=$ $\bigoplus_{i \geq 0} R_{i}$ with $R_{i}$ denoting the degree- $i$ part and let $\mathbb{P}^{r-1}=\operatorname{Proj} R$, the $(r-1)$ dimensional projective space. For a closed subscheme $Z \subset \mathbb{P}^{r-1}$ defined by a homogeneous ideal $I \subset R$, the Hilbert polynomial of $Z$ is a polynomial $P \in \mathbb{Q}[t]$ such that $P(i)=\operatorname{dim}_{k}(R / I)_{i}$ for $i \gg 0$, where $(R / I)_{i}$ denotes the degree $i$ part of the graded ring $R / I$. A polynomial is said to be a Hilbert polynomial if it is the Hilbert polynomial of some closed subscheme $Z \subset \mathbb{P}^{r-1}$. For each Hilbert polynomial $P$, the Hilbert scheme $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$ for P is the moduli scheme of closed subschemes $Z \subset \mathbb{P}^{r-1}$ with the Hilbert polynomial $P$.

For a Hilbert polynomial $P$, there exists a unique sequence of positive integers, $0<$ $a_{0} \leq \cdots \leq a_{k}$ such that $0 \leq k \leq r-2$ and

$$
\binom{r+t-1}{r-1}-P(t)=\binom{t-a_{0}+r-1}{r-1}+\cdots+\binom{t-a_{k}+r-1-k}{r-1-k} .
$$

We can algorithmically compute these integers $a_{0}, \ldots, a_{k}$ from the polynomial $P$. The Gotzmann number $\varphi(P)$ of $P$ is defined to be $a_{k}$.

We now fix a Hilbert polynomial $P$ and an integer $d \geq \varphi(P)$. Let $r_{d}:=\operatorname{dim}_{k} R_{d}$, $p:=P(d)$ and $p^{\vee}:=r_{d}-p$. Let $\operatorname{Grass}_{p} \vee\left(R_{d}\right)$ be the Grassmannian parameterizing $p^{\vee}$-dimensional subspaces of the $r_{d}$-dimensional vector space $R_{d}$. There exists a closed embedding

$$
\begin{aligned}
\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right) & \hookrightarrow \operatorname{Grass}_{p \vee}\left(R_{d}\right), \\
{[Z] } & \mapsto\left[\left(I_{Z}\right)_{d}\right]
\end{aligned}
$$

where $I_{Z} \subset R$ is the saturated ideal of $Z$ and $\left(I_{Z}\right)_{d}$ is its degree- $d$ part. In particular, the closed subscheme $Z$ is recovered from the subspace $\left(I_{Z}\right)_{d} \subset R_{d}$. Indeed $Z$ is defined by the ideal $\left(\left(I_{Z}\right)_{d}\right) \subset R$ generated by $\left(I_{Z}\right)_{d}$.

Let $M_{d}$ be the set of the monomials of degree $d$, which is a basis of $R_{d}$. The Grassmannian $\operatorname{Grass}_{p} \vee\left(R_{d}\right)$ has the standard affine open covering

$$
\operatorname{Grass}_{p^{\vee}}\left(R_{d}\right)=\bigcup_{\substack{K \subset M_{d} \\ \sharp K=p}} U_{K} .
$$

Each affine chart $U_{K}$ is isomorphic to the $p \cdot p^{\vee}$-dimensional affine space $\mathbb{A}^{p \cdot p^{\vee}}$. In what follows, we identify $M_{d}$ with $\left\{1,2, \ldots, r_{d}\right\}$ say by the lex order. Then, the affine chart $U_{K}=\mathbb{A}^{p \cdot p^{\vee}}$ is the space of $r_{d}$-by- $p^{\vee}$ matrices

$$
\begin{equation*}
A=\left(a_{i, j}\right)_{i \in M_{d}, 1 \leq j \leq p^{\vee}} \tag{4.1}
\end{equation*}
$$

such that the $p^{\vee}$-by- $p^{\vee}$ submatrix

$$
\left(a_{i, j}\right)_{i \in M_{d} \backslash K, 1 \leq j \leq p^{\vee}}
$$

is the identity matrix. Note that the $j$-th column of $A$ corresponds to the homogeneous polynomial $\sum_{i=1}^{r_{d}} a_{i j} m_{i}$, where $m_{i}$ denotes the $i$-th monomial in $M_{d}$. Thus we can write the coordinate ring of $U_{K}$ as

$$
k\left[U_{K}\right]=k\left[u_{i, j} \mid i \in K, 1 \leq j \leq p^{\vee}\right],
$$

where $u_{i, j}$ are indeterminates corresponding to entries $a_{i, j}$ above, respectively. For example, if $K$ consists of the last $p$ monomials in $M_{d}$, then a matrix $A$ as above is of the form:

$$
A=\left(\begin{array}{ccc}
1 & & 0 \\
& \ddots & \\
0 & & 1 \\
a_{p^{\vee}+1,1} & \cdots & a_{p^{\vee}+1, p^{\vee}} \\
\vdots & \ddots & \vdots \\
a_{r_{d}, 1} & \cdots & a_{r_{d}, p^{\vee}}
\end{array}\right)
$$

The $p \cdot p^{\vee}$ free entries in the last $p$ rows serve as coordinates of the affine space $U_{K}$. For general $K$, if we write $A=\left(\mathbf{a}_{1} \cdots \mathbf{a}_{p} \vee\right)$ with column vectors $\mathbf{a}_{i}$, then the matrix $A$ corresponds to the subspace $\left\langle\mathbf{a}_{1}, \ldots, \mathbf{a}_{p^{\vee}}\right\rangle \subset R_{d}$. A matrix as above is also regarded as a linear map

$$
k^{p^{\vee}} \rightarrow k^{r_{d}}=R_{d}
$$

For each $i$ with $1 \leq i \leq r$, let $B_{i}$ be the $r_{d+1}$-by- $p^{\vee}$ matrix corresponding to the composite map

$$
k^{p^{\vee}} \xrightarrow{A} R_{d} \xrightarrow{\times x_{i}} R_{d+1}=k^{r_{d+1}} .
$$

These matrices are easily computed from $A$. Indeed their nonzero entries are the ones of $A$ suitably arranged. Finally we define the $r_{d+1}$-by- $r p^{\vee}$ matrix

$$
B:=\left(B_{1}|\cdots| B_{r}\right)
$$

by lining $B_{i}$ 's horizontally. If $V_{A} \subset R_{d}$ is the subspace corresponding to $A$, then the image of $B$ regarded as the map $k^{r p^{\vee}} \rightarrow R_{d+1}$ is nothing but $R_{1} \cdot V_{A} \subset R_{d+1}$. We have that the point $\left[V_{A}\right] \in \operatorname{Grass}_{p \vee}\left(R_{d}\right)$ lies in $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$ if and only if the inequality

$$
\operatorname{dim}_{k} R_{1} \cdot V_{A} \leq r_{d+1}-P(d+1)=: q^{\vee}
$$

holds. Note that the inequality is equivalent to the equality, since the opposite inequality $\geq$ always holds. Thus, on the affine chart $U_{K} \subset \operatorname{Grass}_{p^{\vee}}\left(R_{d}\right)$, the Hilbert scheme $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$ is cut out, at least set-theoretically, by the $\left(q^{\vee}+1\right)$-by- $\left(q^{\vee}+1\right)$ minors of the matrix $B$; each such minor is a polynomial in coordinates $a_{i, j}$ of $U_{K}=\mathbb{A}^{p \cdot p^{\vee}}$. It turns out that this is the case also scheme-theoretically. Precisely:

Proposition 4.1. The closed subscheme $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right) \cap U_{K}$ of $U_{K}=\mathbb{A}^{p \cdot p^{\vee}}$ is defined by the $\left(q^{\vee}+1\right)-b y-\left(q^{\vee}+1\right)$ minors of the matrix $B$.
4.2. The universal family. We can also compute the universal family

$$
\mathcal{Z}_{K} \subset\left(\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right) \cap U_{K}\right) \times \mathbb{P}^{r-1}
$$

over $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right) \cap U_{K}$ as follows. Recall that $U_{K}=\mathbb{A}^{p \cdot p^{\vee}}$ has coordinates $u_{i, j}(i \in K$, $1 \leq j \leq p^{\vee}$ ) and consider the universal $r_{d}$-by- $p^{\vee}$ matrix

$$
\begin{equation*}
\mathcal{A}=\mathcal{A}_{K}:=\left(u_{i, j}\right)_{i \in M_{d}, 1 \leq j \leq p^{\vee}} \tag{4.2}
\end{equation*}
$$

Here, for $i \in K$, the entry $u_{i, j}$ is the indeterminate $u_{i, j} \in k\left[U_{K}\right]$ and, for $i \in M_{d} \backslash K$, $u_{i, j}$ is defined to be either 1 or 0 so that the $p^{\vee}$-by- $p^{\vee}$ matrix

$$
\left(u_{i, j}\right)_{i \in M_{d} \backslash K, 1 \leq j \leq p^{\vee}}
$$

is the identity matrix. At each point $\left(a_{i, j}\right) \in U_{K}$, the universal matrix to the matrix $A$ in (4.1). The $j$-th column of $\mathcal{A}$ defines the universal $j$-th polynomial

$$
h_{K, j}:=\sum_{i=1}^{r_{d}} u_{i, j} m_{i}=n_{i}+\sum_{i \in K} u_{i, j} m_{i} \in k\left[U_{K}\right]\left[x_{1}, \ldots, x_{r}\right],
$$

where $m_{i}$ is the $i$-th monimial in $M_{d}$ as before and $n_{i}$ is the $i$-th monomial in $M_{d} \backslash K$. They are homogeneous of degree $d$ with respect to variables $x_{1}, \ldots, x_{r}$.

Proposition 4.2. Let $I_{K} \subset k\left[U_{K}\right]$ be the defining ideal of $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right) \cap U_{K} \subset U_{K}$, which can be computed as in Proposition 4.1. Then the universal family $\mathcal{Z}_{K}$ is written as

$$
\mathcal{Z}_{K}=\operatorname{Proj} \frac{\left(k\left[U_{K}\right] / I_{K}\right)\left[x_{1}, \ldots, x_{r}\right]}{\left(h_{K, 1}, \ldots, h_{K, p^{v}}\right)}
$$

4.3. The Hilbert scheme of a general projective scheme. Next we consider the Hilbert scheme $\operatorname{Hilb}_{P}(X)$ of a projective scheme $X \subset \mathbb{P}_{k}^{r-1}$. This is the moduli scheme of those closed subschemes $Z \subset X$ that have the Hilbert polynomial $P$ as a closed subscheme of $\mathbb{P}_{k}^{r-1}$. Let $Q$ be the Hilbert polynomial of $X$. If $d \geq \varphi(Q)$, then $I_{X}$ is $d$-regular. In particular, the truncated ideal $\left(I_{X}\right)_{\geq d}$, which is the part of the saturated ideal $I_{X}$ with degree $\geq d$, is generated by the degree- $d$ part $\left(I_{X}\right)_{d}$. Now we choose the integer $d$ to be $\max \{\varphi(P), \varphi(Q)\}$. For a closed subscheme $Z \subset \mathbb{P}_{k}^{r-1}$ with the Hilbert polynomial $P$, we have $Z \subset X$ if and only if $\left(I_{Z}\right)_{d} \supset\left(I_{X}\right)_{d}$. Let

$$
f_{1}=\left(\begin{array}{c}
f_{1,1} \\
\vdots \\
f_{r_{d}, 1}
\end{array}\right), \ldots, f_{l}=\left(\begin{array}{c}
f_{1, l} \\
\vdots \\
f_{r_{d}, l}
\end{array}\right) \in k^{r_{d}}=R_{d}
$$

be a basis of $\left(I_{X}\right)_{d}$. Note that we can explicitly construct such a basis from the given finitely many defining polynomials of $X$. Firstly, there is an algorithm to compute the saturation $\left(I:\left(x_{1}, \ldots, x_{r}\right)^{\infty}\right)$ of an homogeneous ideal $I \subset k\left[x_{1}, \ldots, x_{r}\right]$, see [Eis95, page 360]. If $g_{1}, \ldots, g_{m}$ are thus computed generators of the saturated ideal $I_{X}$, which we may assume have degree $\leq d$ (since those of degree $>d$ are redundant), then $\left(I_{X}\right)_{d}$ is generated by elements of the form $x g_{i}$, where $x$ is a monomial of degree $d-\operatorname{deg} g_{i}$. We can choose a basis from them in a standard linear algebra procedure. As before, a point of $U_{K}=\mathbb{A}^{p \cdot p^{\vee}}$ is identified with a $r_{d}$-by- $p^{\vee}$ matrix $A$. To such a matrix, we associate the $r_{d}$-by- $\left(p^{\vee}+l\right)$ matrix $C_{A}=\left(A\left|f_{1}\right| \cdots \mid f_{l}\right)$. For example, if $K$ is the first $p$ monomials in $M_{d}$, then

$$
C_{A}=\left(\begin{array}{cccccc}
1 & & 0 & f_{1,1} & \cdots & f_{1, l} \\
& \ddots & & & & \\
0 & & 1 & \vdots & \ddots & \vdots \\
a_{p^{\vee}+1,1} & \cdots & a_{p^{\vee}+1, p^{\vee}} & \vdots & & \vdots \\
\vdots & \ddots & \vdots & & & \\
a_{r_{d}, 1} & \cdots & a_{r_{d}, p^{\vee}} & f_{r_{d}, 1} & \cdots & f_{r_{d}, l}
\end{array}\right) .
$$

For a closed subscheme $Z \subset \mathbb{P}^{r-1}$ with $[Z] \in U_{K}$, the following conditions are equivalent:
(1) $\left(I_{Z}\right)_{d} \supset\left(I_{X}\right)_{d}$.
(2) $\operatorname{rank} C_{A}=p^{\vee}$.
(3) all the $\left(p^{\vee}+1\right)$-by- $\left(p^{\vee}+1\right)$ minors of $C_{A}$ vanish.

The minors in the last condition are polynomials in the coordinates of $U_{K}$.
Proposition 4.3. The closed subscheme

$$
\begin{equation*}
\operatorname{Hilb}_{P}(X)_{K}:=\operatorname{Hilb}_{P}(X) \cap U_{K} \tag{4.3}
\end{equation*}
$$

of $U_{K}=\mathbb{A}^{p \cdot p^{\vee}}$ is defined by the defining polynomials of $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$ given in Proposition 4.1 and the above minors of $C_{A}$.

We can compute the universal family $\mathcal{Z}_{X, K} \subset \operatorname{Hilb}_{P}(X)_{K} \times X$ over $\operatorname{Hilb}_{P}(X)_{K}$ in a similar way as in the case of $\mathcal{Z}_{K}$.

If $I_{X, K} \subset k\left[U_{K}\right]$ denotes the defining ideal of $\operatorname{Hilb}_{P}(X) \cap U_{K}$, then

$$
\begin{align*}
\mathcal{Z}_{X, K} & =\operatorname{Proj} \frac{\left(k\left[U_{K}\right] / I_{X, K}\right)\left[x_{1}, \ldots, x_{r}\right]}{\left(f_{1}, \ldots, f_{l}, h_{K, 1}, \ldots, h_{K, p^{v}}\right)} \\
& \subset \operatorname{Proj} \frac{\left(k\left[U_{K}\right] / I_{X, K}\right)\left[x_{1}, \ldots, x_{r}\right]}{\left(f_{1}, \ldots, f_{l}\right)}=\operatorname{Hilb}_{P}(X)_{K} \times X . \tag{4.4}
\end{align*}
$$

As a conclusion of the above computation, we have:
Proposition 4.4 ([PTL15, Lem. 8.23]). There is an algorithm such that given a projective variety $X \subset \mathbb{P}^{r-1}$ and a Hilbert polynomial $P$, then it outputs a positive integer $d$ and computes the closed subscheme $\operatorname{Hilb}_{P}(X) \subset \operatorname{Grass}_{p^{\vee}}\left(R_{d}\right)$ in terms of defining equations on each affine chart $U_{K}$ together with defining equations of the universal families $\mathcal{Z}_{X, K} \subset \operatorname{Hilb}_{P}(X)_{K} \times X$.
4.4. The action of $\mathrm{GL}_{r}$. Since it will be used to prove the decidability of projective equivalence in Section 5, we describe the natural action of the general linear group $\mathrm{GL}_{r}$ on the Hilbert scheme $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$. The action of $\mathrm{GL}_{r}$ on $\mathbb{P}^{r-1}$ induces an action of $\mathrm{GL}_{r}$ on $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$. The Hilbert scheme has the open covering $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)=\bigcup_{K} U_{K}$ and we have an explicit presentation of $U_{K}$ for each $K$.
Definition 4.5. We define $V_{K, K^{\prime}}$ to be the preimage of $U_{K^{\prime}}$ by the morphism

$$
\mu: \mathrm{GL}_{r} \times U_{K} \rightarrow \operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)
$$

We explain how to compute an explicit presentation of the affine scheme $V_{K, K^{\prime}}$ as well as the morphism

$$
\begin{equation*}
\mu_{K, K^{\prime}}:=\left.\mu\right|_{V_{K, K^{\prime}}}: V_{K, K^{\prime}} \rightarrow U_{K^{\prime}} \tag{4.5}
\end{equation*}
$$

Let $S_{r}$ be the coordinate ring of $\mathrm{GL}_{r}$, that is, the localization $k[\underline{s}]_{D}$ of the polynomial ring $k[\underline{s}]=k\left[s_{i, j} \mid 1 \leq i, j \leq r\right]$ by the determinant $D=\operatorname{det}\left(s_{i, j}\right)$. We have the universal matrix

$$
\left(\begin{array}{ccc}
s_{1,1} & \cdots & s_{1, r} \\
\vdots & \ddots & \vdots \\
s_{r, 1} & \cdots & s_{r, r}
\end{array}\right) \in \operatorname{GL}_{r}\left(S_{r}\right)
$$

The action $\mathrm{GL}_{r} \times \mathbb{A}^{r} \rightarrow \mathbb{A}^{r}$ is given by the following ring map:

$$
\begin{aligned}
k\left[x_{1}, \ldots, x_{r}\right] & \rightarrow S_{r}\left[x_{1}, \ldots, x_{r}\right] \\
x_{i} & \mapsto \sum_{j=1}^{r} s_{i, j} x_{j}
\end{aligned}
$$

As before, we fix $d \geq \varphi(P)$. We write a monomial in $k\left[x_{1}, \ldots, x_{r}\right]$ as $x^{e}=x_{1}^{e_{1}} \cdots x_{r}^{e_{r}}$ with multi-index notation. The last map sends a monomial $x^{e}$ of degree $d$ to the polynomial,

$$
\prod_{i=1}^{d}\left(\sum_{j=1}^{r} s_{i, j} x_{j}\right)^{e_{i}}=: \sum_{e^{\prime}} \eta_{e, e^{\prime}} x^{e^{\prime}}
$$

which is homogeneous of degree $d$ both in $x_{1}, \ldots, x_{r}$ and in $s_{i, j}$. Here $\eta_{e^{\prime}, e}$ is a homogenous polynomial of degree $d$ in $k[\underline{s}]$. For each $e$, we can compute $\eta_{e, e^{\prime}}$ 's explicitly. We get the map

$$
\begin{aligned}
\mathrm{GL}_{r} & \rightarrow \mathrm{GL}_{r_{d}} \\
\left(a_{i, j}\right)_{1 \leq i, j \leq r} & \mapsto\left(\eta_{e, e^{\prime}}(\underline{a})\right)_{e, e^{\prime} \in M_{d}},
\end{aligned}
$$

which induces an action of $\mathrm{GL}_{r}$ on $\mathbb{A}^{r_{d}}$. In terms of coordinate rings, this is given by

$$
S_{r_{d}} \rightarrow S_{r}, s_{e, e^{\prime}} \mapsto \eta_{e, e^{\prime}}\left(s_{1,1}, \ldots, s_{r, r}\right)
$$

Let $K, K^{\prime} \subset M_{d}$ be two subsets with $\sharp K=\sharp K^{\prime}=p$. Let $\mathcal{A}_{K}=\left(u_{i, j}\right)$ be the universal $r_{d}$-by- $p^{\vee}$ matrix for $K$ (see (4.2)). Compute the matrix product

$$
\mathcal{B}:=\left(\eta_{e, e^{\prime}}\right) \cdot \mathcal{A}_{K} \in \mathrm{M}_{r_{d} \times p^{\vee}}(k(\underline{s})[\underline{u}])
$$

with $k(\underline{s})$ the fraction field of $k[\underline{s}]$ and $k(\underline{s})[\underline{u}]=k(\underline{s})\left[u_{i, j} \mid i \in K, 1 \leq j \leq p^{\vee}\right]$. By Gaussian elimination in the field $k(\underline{s}, \underline{u})$, we can algorithmically and uniquely transform
$\mathcal{B}$ to a matrix belonging to $U_{K^{\prime}}(k(\underline{s}, \underline{u}))$ by applying elementary column operations finitely many times; let $\theta=\left(\theta_{i, j}\right) \in \mathrm{M}_{r_{d} \times p^{\vee}}(k(\underline{s}, \underline{u}))$ be the resulting matrix. Here the $p^{\vee}$-by- $p^{\vee}$ matrix $\left(\theta_{i, j}\right)_{i \in M_{d} \backslash K^{\prime}, 1 \leq j \leq r_{d}}$ is the identity matrix. We get $p \cdot p^{\vee}$ rational functions $\theta_{i, j} \in k(\underline{s}, \underline{u})$ for $i \in K^{\prime}$ and $1 \leq j \leq p^{\vee}$. We see that the rational map $\mathrm{GL}_{r} \times U_{K} \rightarrow U_{K^{\prime}}$ is given by:

$$
\begin{aligned}
k[\underline{u}] & \rightarrow k(\underline{s}, \underline{u}) \\
u_{i, j} & \mapsto \theta_{i, j}
\end{aligned}
$$

Let $V_{K, K^{\prime}} \subset \mathrm{GL}_{r} \times U_{K}$ be the domain of this rational map, that is, the preimage of $U_{K^{\prime}}$ by the map $\mathrm{GL}_{r} \times U_{K} \rightarrow \operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$. This is an affine scheme, since $\mathrm{GL}_{r} \times U_{K}$ and $U_{K^{\prime}}$ are affine and $\operatorname{Hilb}_{P}\left(\mathbb{P}^{r-1}\right)$ is separated. The coordinate ring $T_{K, K^{\prime}}$ of $V_{K, K^{\prime}}$ is obtained by adjoining $\theta_{i, j}$ 's to $S_{r}[\underline{u}]$.

## 5. Projective equivalence

Two projective schemes $X$ and $Y$ embedded in the same projective space $\mathbb{P}^{r-1}$ are said to be projectively equivalent if there exists an invertible matrix $g \in \mathrm{GL}_{r}(k)$ such that $g(X)=Y$.

Proposition 5.1. Let $X$ and $Y$ be projective schemes embedded in the same projective space $\mathbb{P}^{r-1}$. Then we can algorithmically check whether $X$ and $Y$ are projectively equivalent.

Proof. We follow the notation of Section 4. We can compute the Hilbert polynomials of $X$ and $Y$ (see [Eis95, Sections 15.1.1 and 15.10.2]) and check whether they are the same. If they are different, then there is no $g \in \mathrm{GL}_{r}(k)$ as in the proposition. Suppose that they are the same and denote it by $P$. Let $d:=\varphi(P)$, the Gotzmann number of $P$, let $I_{X}$ and $I_{Y}$ be the saturated ideals of $X$ and $Y$ and let $\left(I_{X}\right)_{d}$ and $\left(I_{Y}\right)_{d}$ be their degree- $d$ parts. We compute bases of $\left(I_{X}\right)_{d}$ and $\left(I_{Y}\right)_{d}$, which are represented by $r_{d}$-by$p^{\vee}$ matrices $A_{X}$ and $A_{Y}$ respectively. We then compute their reduced column echelon forms and denote them by $B_{X}$ and $B_{Y}$. Let $K$ and $K^{\prime}$ be the set of indices such that the corresponding rows of $B_{X}$ and $B_{Y}$ have pivots. Then $B_{X}$ and $B_{Y}$ define the points $[X] \in U_{K}$ and $[Y] \in U_{K^{\prime}}$ respectively. Consider the morphism $\mu_{K, K^{\prime}}: V_{K, K^{\prime}} \rightarrow U_{K^{\prime}}$ (see (4.5)) and the morphism

$$
p: V_{K, K^{\prime}} \hookrightarrow \mathrm{GL}_{r} \times U_{K} \xrightarrow{\text { projection }} U_{K}
$$

of affine schemes. We compute the closed subset

$$
\left(\mu_{K, K^{\prime}}\right)^{-1}([Y]) \cap p^{-1}([X]) \subset V_{K, K^{\prime}},
$$

which is the set of pairs $(g,[X])$ such that $g \in \mathrm{GL}_{r}(k)$ and $g(X)=Y$. Thus, there exists $g \in \mathrm{GL}_{r}(k)$ as in the proposition if and only if this closed subset is not empty.

## 6. Hom schemes and Iso schemes

For projective schemes $X$ and $Y$, the Hom scheme, denoted by $\underline{\operatorname{Hom}}(X, Y)$, and Iso scheme, denoted by Iso $(X, Y)$, are the moduli schemes of morphisms $X \rightarrow Y$ and
isomorphisms $X \rightarrow Y$ respectively. From [Kol96, p. 16], we have an open immersion

$$
\begin{aligned}
& \underline{\operatorname{Hom}}(X, Y) \rightarrow \operatorname{Hilb}(X \times Y), \\
& {[f: X \rightarrow Y] \mapsto\left[\Gamma_{f}\right]}
\end{aligned}
$$

where $\Gamma_{f} \subset X \times Y$ is the graph of $f$. Namely, we can identify $\underline{\operatorname{Hom}}(X, Y)$ with the locus of points $[Z] \in \operatorname{Hilb}(X \times Y)$ such that the first projection $Z \rightarrow X$ is an isomorphism. Similarly we can identify $\underline{\operatorname{Iso}}(X, Y)$ with the locus of points $[Z]$ where both the projections $Z \rightarrow X$ and $Z \rightarrow Y$ are isomorphisms. Thus, if we embed also $\underline{\operatorname{Hom}}(Y, X)$ into $\operatorname{Hilb}(X \times Y)$ via the obvious isomorphism $\operatorname{Hilb}(X \times Y) \cong \operatorname{Hilb}(Y \times X)$, then we have

$$
\underline{\operatorname{Iso}}(X, Y)=\underline{\operatorname{Hom}}(X, Y) \cap \underline{\operatorname{Hom}}(Y, X) .
$$

We now fix embeddings $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ and embed the product $X \times Y$ into $\mathbb{P}^{m n-1}$ by the Segre embedding. Then the Hilbert scheme Hilb $(X \times Y)$ decomposes into the disjoint union of countably many open and closed subschemes as

$$
\operatorname{Hilb}(X \times Y)=\coprod_{P} \operatorname{Hilb}_{P}(X \times Y) .
$$

Here $P$ runs over Hilbert polynomials. Recall that $\operatorname{Hilb}_{P}(X \times Y)_{K}$ denotes $\operatorname{Hilb}_{P}(X \times$ $Y) \cap U_{K}$, see (4.3).

Definition 6.1. We define

$$
\begin{aligned}
\underline{\operatorname{Hom}}_{P}(X, Y) & :=\underline{\operatorname{Hom}}(X, Y) \cap \operatorname{Hilb}_{P}(X \times Y), \\
\underline{\operatorname{Hom}}_{P}(X, Y)_{K} & : \underline{\operatorname{Hom}}(X, Y) \cap \operatorname{Hilb}_{P}(X \times Y)_{K}, \\
\underline{\operatorname{Iso}_{P}(X, Y)} & : \underline{\mathrm{Iso}}(X, Y) \cap \operatorname{Hilb}_{P}(X \times Y), \\
\underline{\operatorname{Iso}}_{P}(X, Y)_{K} & : \underline{\mathrm{Iso}}(X, Y) \cap \operatorname{Hilb}_{P}(X \times Y)_{K} .
\end{aligned}
$$

We have the open coverings,

$$
\begin{gathered}
\underline{\operatorname{Hom}}_{P}(X, Y)=\bigcup_{K} \underline{\operatorname{Hom}}_{P}(X, Y)_{K}, \\
\underline{\operatorname{Iso}}_{P}(X, Y)=\bigcup_{K} \underline{\operatorname{Iso}}_{P}(X, Y)_{K} .
\end{gathered}
$$

Definition 6.2. We define the Hilbert polynomial of an isomorphism $f: X \rightarrow Y$ to be the Hilbert polynomial of its graph $\Gamma_{f} \subset X \times Y$ as a closed subscheme of $\mathbb{P}^{m n-1}$. Namely the Hilbert polynomial of $f$ is the polynomial $P$ such that $[f] \in \underline{\operatorname{Iso}}_{P}(X, Y)$.

When we show the decidability of the isomorphism problem for several classes of projective schemes, our strategy will be to construct finitely many polynomials $P_{1}, \ldots, P_{l}$ from given projective schemes $X$ and $Y$ that satisfy the following property; if $X$ and $Y$ are isomorphic, then there exists an isomorphism $f: X \rightarrow Y$ having the Hilbert polynomial $P_{i}$ for some $i \in\{1, \ldots, l\}$. Namely $X$ and $Y$ are isomorphic if and only if $\bigcup_{i=1}^{l} \underline{\operatorname{Iso}}_{P_{i}}(X, Y) \neq \emptyset$. Then what remains to do is to compute $\underline{\mathrm{IsO}}_{P_{i}}(X, Y)$ for every $i \in\{1, \ldots, l\}$.

For each $P$ and $K$, we have

$$
\underline{\operatorname{Iso}}_{P}(X, Y)_{K}=\underline{\operatorname{Hom}}_{P}(X, Y)_{K} \cap \underline{\operatorname{Hom}}_{P}(Y, X)_{K} .
$$

Thus, computation of ${\underline{\operatorname{Iso}_{P}}}_{P}(X, Y)_{K}$ is reduced to the one of $\underline{\operatorname{Hom}}_{P}(X, Y)_{K}$. We focus on the latter computation in what follows. Replacing $X$ in (4.4) with $X \times Y$, we get explicit presentation of the universal family

$$
\mathcal{Z}_{X \times Y, K} \subset \operatorname{Hilb}_{P}(X \times Y)_{K} \times X \times Y
$$

From discussion above, the open subscheme $\underline{\operatorname{Hom}}_{P}(X, Y)_{K}$ of $\operatorname{Hilb}_{P}(X \times Y)_{K}$ is the largest open subscheme $U$ such that the composite morphism

$$
g: \mathcal{Z}_{X \times Y, K} \hookrightarrow \operatorname{Hilb}_{P}(X \times Y)_{K} \times X \times Y \xrightarrow{\text { projection }} \operatorname{Hilb}_{P}(X \times Y)_{K} \times X
$$

is an isomorphism over $U \times X$. Since we are given the embedding $X \subset \mathbb{P}^{m-1}$, we have the standard affine open covering $X=\bigcup_{i=1}^{m} X_{i}$. Let

$$
g_{i}: g^{-1}\left(X_{i}\right) \rightarrow \operatorname{Hilb}_{P}(X \times Y)_{K} \times X_{i}
$$

be restriction of $g$. This is a projective morphism with the target being affine. Let $U_{i} \subset \operatorname{Hilb}_{P}(X \times Y)_{K}$ be the largest open subset such that $g_{i}$ is an isomorphism over $U_{i} \times X_{i}$. Then

$$
U=\underline{\operatorname{Hom}}_{P}(X, Y)_{K}=\bigcap_{i=1}^{m} U_{i} .
$$

There exists an algorithm to compute each $U_{i}$ :
Proposition 6.3. Let $H$ and $X$ be affine schemes and let $f: Z \rightarrow H \times X$ be a projective morphism. Then there exists an algorithm to compute the largest open subset $U \subset H$ such that $f$ is an isomorphism over $U \times X$.

Proof. We first compute the closed subset $C_{1}:=\operatorname{Supp}\left(f_{*} \Omega_{Z / H \times X}\right)$ of $H \times X$ (see Remarks 3.4 and 3.5). Next we compute the closed subset $C_{2}:=(H \times X) \backslash V$, where $V$ is the invertible locus of $f_{*} \mathcal{O}_{Z}$ (see Lemma 6.4). The desired open subset $U \subset H$ is $H \backslash p_{H}\left(C_{1} \cup C_{2}\right)$, where $p_{H}$ denotes the projection $H \times X \rightarrow H$. Indeed, obviously $H \backslash p_{H}\left(C_{1} \cup C_{2}\right)$ is contained in the desired subset. On the other hand, $f$ is unramified (in particular, finite and affine) over $H \backslash p_{H}\left(C_{1}\right)$. That $f_{*} \mathcal{O}_{Z}$ is invertible on $H \backslash p_{H}\left(C_{1} \cup C_{2}\right)$ means that $f$ is an isomorphism over this open subset.

Lemma 6.4. There is an algorithm to compute the invertible locus of a coherent sheaf on an affine scheme, that is, the largest open subset on which the sheaf is invertible.

Proof. Let $X$ be an affine scheme and let $\mathcal{M}$ be a coherent sheaf on $X$. We first compute the closed subset

$$
C_{1}:=\overline{X \backslash \operatorname{Supp}(\mathcal{M})} \subset X
$$

(see Remark 6.5). Its complement $X \backslash C_{1}$ is the largest open subset of $X$ that is included in $\operatorname{Supp}(\mathcal{M})$. We then put $\mathcal{M}^{\prime}:=\left.\mathcal{M}\right|_{X_{\text {red }}}$ and compute the closed subset

$$
C_{2}:=V\left(\operatorname{Fitt}_{1}\left(\mathcal{M}^{\prime}\right)\right),
$$

where Fitt ${ }_{1}$ denotes the first Fitting ideal. Its complement $X \backslash C_{2}$ is the locus of points $x \in X$ where $\mathcal{M}_{x}^{\prime}$ is generated by one element as an $\mathcal{O}_{X, x}$-module. Finally we compute

$$
C_{3}:=\operatorname{Supp}\left(\mathcal{T}_{\text {or }}{ }^{\mathcal{O}_{X}}\left(\mathcal{O}_{X_{\text {red }}}, \mathcal{M}\right)\right)
$$

From Mat89, Th. 22.3], its complement $X \backslash C_{3}$ is the locus where $\left.\mathcal{M}\right|_{X}$ is flat. Now the open subset

$$
U:=X \backslash\left(C_{1} \cup C_{2} \cup C_{3}\right)
$$

is the largest open subset such that

- $\operatorname{Supp}\left(\left.\mathcal{M}\right|_{U}\right)=U$,
- for every $x \in U, \mathcal{M}_{x}$ is generated by one element as an $\mathcal{O}_{X, x}$-module,
- $\left.\mathcal{M}\right|_{U}$ is a flat $\mathcal{O}_{U}$-module.

Therefore, $U$ is the desired open subset. We can compute $C_{1}, C_{2}$ and $C_{3}$ by some algorithms, for example, ones implemented to Macaulay2 [GS].

Remark 6.5. Let $X=\operatorname{Spec} R$ and let $C=V\left(f_{1}, \ldots, f_{n}\right) \subset X$ be a closed subset. Then the closed subset $\overline{X \backslash C}$ is defined by the ideal

$$
\bigcap_{i=1}^{n} \operatorname{Ker}\left(R \rightarrow R_{f_{i}}\right) .
$$

Indeed, $X \backslash C$ is covered by the affine open subsets $U_{i}=\left\{f_{i} \neq 0\right\}$ and we have $\overline{X \backslash C}=\bigcup \overline{U_{i}}$. Each $\overline{U_{i}}$ is defined by the ideal $\operatorname{Ker}\left(R \rightarrow R_{f_{i}}\right)$.

We conclude:
Proposition 6.6. For each Hilbert polynomial $P$, we can explicitly compute open subschemes $\underline{\operatorname{Hom}}_{P}(X, Y)$ and $\underline{\operatorname{Iso}}_{P}(X, Y)$ of $\operatorname{Hilb}_{P}(X \times Y)$ by means of explicit presentation of open subsets $\operatorname{Hom}_{P}(X, Y)_{K}$ and $\underline{\operatorname{Iso}}_{P}(X, Y)_{K}$ of $\operatorname{Hilb}_{P}(X \times Y)_{K}$ for each $K$.

Remark 6.7. Using Iso schemes, we can give an alternative proof of the semi-decidability of the isomorphism, which was proved in Section 3. We enumerate all the Hilbert polynomials as $P_{i}, i \in \mathbb{Z}_{>0}$. For each $i>0$, we compute the Iso scheme $\underline{\operatorname{Iso}}_{P_{i}}(X, Y)$. We stop if we get a non-empty Iso scheme.

## 7. One-dimensional schemes

In this section, we show that the isomorphism problem for one-dimensional projective schemes and the one for one-dimensional reduced quasi-projective schemes are decidable. This generalizes the known case of smooth irreducible curves (BGJGP05, Lem. 5.1] for the case of genus $\neq 1$ and the MathOverflow thread mentioned in Introduction for elliptic curves). We need the following version of the Riemann-Roch formula for one-dimensional projective schemes.

Proposition 7.1 ( $\backslash$ Vak15, Exercise 18.4.S]). Let $X$ be a one-dimensional projective scheme and let $X_{i}, 1 \leq i \leq l$, be its one-dimensional irreducible components given with reduced structure and let $\eta_{i}$ be the generic point of $X_{i}$. Let $\mathcal{L}$ be an invertible sheaf on $X$ and let $\mathcal{F}$ be a coherent sheaf on $X$. Then

$$
\begin{equation*}
\chi(\mathcal{F} \otimes \mathcal{L})-\chi(\mathcal{F})=\sum_{i=1}^{l} \operatorname{length}\left(\mathcal{F}_{\eta_{i}}\right) \operatorname{deg}\left(\left.\mathcal{L}\right|_{X_{i}}\right) \tag{7.1}
\end{equation*}
$$

Here length $\left(\mathcal{F}_{\eta_{i}}\right)$ is the length of $\mathcal{F}_{\eta_{i}}$ as an $\mathcal{O}_{X, \eta_{i}}$-module. In particular,

$$
\begin{equation*}
\chi(\mathcal{L})-\chi(\mathcal{O})=\sum_{i=1}^{l} \operatorname{length}\left(\mathcal{O}_{X, \eta_{i}}\right) \operatorname{deg}\left(\left.\mathcal{L}\right|_{X_{i}}\right) \tag{7.2}
\end{equation*}
$$

Proof. The outline of the proof is written in Vak15. For the sake of completeness, we write it down in more details. We first observe that zero-dimensional connected components of $X$ do not contribute to either side of (7.1). Therefore we may suppose that $X$ has only one-dimensional irreducible components, that is, $X=\bigcup_{i=1}^{l} X_{i}$. Note also that both sides of (7.1) are also additive for short exact sequences; if $v(\mathcal{F})$ denotes either side of the equality, for a short exact sequence of coherent $\mathcal{O}_{X}$-modules,

$$
0 \rightarrow \mathcal{F}_{1} \rightarrow \mathcal{F}_{2} \rightarrow \mathcal{F}_{3} \rightarrow 0
$$

we have $v\left(\mathcal{F}_{2}\right)=v\left(\mathcal{F}_{1}\right)+v\left(\mathcal{F}_{3}\right)$. This implies that for a filtration of coherent sheaves,

$$
\mathcal{F}=\mathcal{F}_{0} \supset \mathcal{F}_{1} \supset \cdots \supset \mathcal{F}_{n}=0
$$

we have

$$
\begin{equation*}
v(\mathcal{F})=\sum_{i} v\left(\mathcal{F}_{i} / \mathcal{F}_{i+1}\right) \tag{7.3}
\end{equation*}
$$

Let $\mathcal{I} \subset \mathcal{O}_{X}$ be the defining ideal sheaf of the associated reduced scheme $X_{\text {red }}$ of $X$, which is necessarily nilpotent. We apply equality (7.3) to the filtration

$$
\mathcal{F} \supset \mathcal{I F} \supset \mathcal{I}^{2} \mathcal{F} \supset \cdots \supset \mathcal{I}^{n} \mathcal{F}=0
$$

Thus it suffices to show (7.1) for sheaves $\mathcal{I}^{i} \mathcal{F} / \mathcal{I}^{i+1} \mathcal{F}$. Since they are $\mathcal{O}_{X_{\text {red }}}$-modules, in turn, it suffices to show the proposition in the case where $X$ is reduced. Let us now write $\mathcal{L}=\mathcal{O}_{X}\left(\sum_{j=1}^{m} n_{j} p_{j}\right)$, where $n_{i}$ are integers and $p_{i}$ are closed points of $X$ at which $X$ is smooth and $\mathcal{F}$ is locally free. We prove (7.1) in this situation by induction on $n:=\sum_{j=1}^{m}\left|n_{j}\right|$. If $n=0$, this is obvious. If $n=1$, then $\mathcal{L}$ is either $\mathcal{O}_{X}(p)$ or $\mathcal{O}_{X}(-p)$. In the latter case, from the exact sequence

$$
\left.0 \rightarrow \mathcal{F} \otimes \mathcal{L} \rightarrow \mathcal{F} \rightarrow \mathcal{F}\right|_{p} \rightarrow 0
$$

we have

$$
\chi(\mathcal{F} \otimes \mathcal{L})-\chi(\mathcal{F})=-\chi\left(\left.\mathcal{F}\right|_{p}\right)=- \text { length }\left(\mathcal{F}_{\eta_{i}}\right)
$$

where $i$ is such that $p \in X_{i}$. If $\mathcal{L}=\mathcal{O}_{X}(p)$ and if we put $\mathcal{F}^{\prime}:=\mathcal{F} \otimes \mathcal{L}$, then

$$
\chi(\mathcal{F} \otimes \mathcal{L})-\chi(\mathcal{F})=-\left(\chi\left(\mathcal{F}^{\prime} \otimes \mathcal{O}_{X}(-p)\right)-\chi\left(\mathcal{F}^{\prime}\right)\right)=\operatorname{length}\left(\mathcal{F}_{\eta_{i}}\right)
$$

Thus (7.1) holds when $n=1$. For general $n \geq 1$, if we write $\mathcal{L}=\mathcal{L}^{\prime} \otimes \mathcal{L}^{\prime \prime}$ with $\mathcal{L}^{\prime}$ and $\mathcal{L}^{\prime \prime}$ having smaller $n$, we have

$$
\begin{aligned}
\chi(\mathcal{F} \otimes \mathcal{L})-\chi(\mathcal{F}) & =\left(\chi\left(\left(\mathcal{F} \otimes \mathcal{L}^{\prime}\right) \otimes \mathcal{L}^{\prime \prime}\right)-\chi\left(\mathcal{F} \otimes \mathcal{L}^{\prime}\right)\right)+\left(\chi\left(\mathcal{F} \otimes \mathcal{L}^{\prime}\right)-\chi(\mathcal{F})\right) \\
& =\sum_{i=1}^{l} \operatorname{length}\left(\mathcal{F}_{\eta_{i}}\right) \operatorname{deg}\left(\left.\mathcal{L}^{\prime}\right|_{X_{i}}\right)+\sum_{i=1}^{l} \operatorname{length}\left(\mathcal{F}_{\eta_{i}}\right) \operatorname{deg}\left(\left.\mathcal{L}^{\prime \prime}\right|_{X_{i}}\right) \\
& =\sum_{i=1}^{l} \operatorname{length}\left(\mathcal{F}_{\eta_{i}}\right) \operatorname{deg}\left(\left.\mathcal{L}\right|_{X_{i}}\right) .
\end{aligned}
$$

Consider two one-dimensional projective schemes $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$, which have very ample invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ corresponding to the given embeddings to projective spaces respectively. Suppose that there exists an isomorphism $f: X \rightarrow Y$. We will bound possibilities for the Euler characteristic of $\mathcal{L} \otimes f^{*} \mathcal{M}$ without using data of $f$.

Corollary 7.2. Let $X_{i}, 1 \leq i \leq l$ and $Y_{j}, 1 \leq j \leq m$ be the one-dimensional irreducible components of $X$ and $Y$ respectively. We give them with reduced structure. Let $\eta_{i}$ be the generic point of $X_{i}$. Let $d:=\sum_{j=1}^{m} \operatorname{deg}\left(\left.\mathcal{M}\right|_{Y_{j}}\right)$. Then there exists a partition of $d$ into positive integers, $d=\sum_{i=1}^{l} d_{i}$, such that

$$
\chi\left(\mathcal{L} \otimes f^{*} \mathcal{M}\right)=\chi\left(\mathcal{O}_{X}\right)+\sum_{i=1}^{l} \operatorname{length}\left(\mathcal{O}_{X, \eta_{i}}\right)\left(\operatorname{deg}\left(\left.\mathcal{L}\right|_{X_{i}}\right)+d_{i}\right)
$$

Proof. We put $d_{i}:=\operatorname{deg}\left(\left.f^{*} \mathcal{M}\right|_{X_{i}}\right)$ and apply the second equality in Proposition 7.1 with $\mathcal{L} \otimes f^{*} \mathcal{M}$ in place of $\mathcal{L}$.

The Hilbert polynomial of $f$ (see Definition 6.2) is equal to the Hilbert polynomial of $X$ with respect to the very ample sheaf $\mathcal{L} \otimes f^{*} \mathcal{M}$. It is the polynomial $P(t)$ of degree at most one such that

$$
P(0)=\chi\left(\mathcal{O}_{X}\right) \text { and } P(1)=\chi\left(\mathcal{L} \otimes f^{*} \mathcal{M}\right)
$$

Theorem 7.3. The isomorphism problem for one-dimensional projective schemes is decidable.

Proof. Let $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ be one-dimensional projective schemes. We compute their one-dimensional irreducible components with reduced structure $X_{i}, 1 \leq$ $i \leq l$ and $Y_{j}, 1 \leq j \leq m$ respectively; there exist algorithms to compute associated reduced schemes and (geometric) irreducible components (see EHV92, Chi86]). Then we compute $d=\sum_{j=1}^{m} \operatorname{deg}\left(\left.\mathcal{M}\right|_{Y_{j}}\right)$.

For each partition

$$
\lambda: d=d_{1}+\cdots+d_{l}
$$

of $d$ into $l$ positive integers, we compute

$$
e_{\lambda}:=\chi\left(\mathcal{O}_{X}\right)+\sum_{i=1}^{l} \operatorname{length}\left(\mathcal{O}_{X, \eta_{i}}\right)\left(\operatorname{deg}\left(\left.\mathcal{L}\right|_{X_{i}}\right)+d_{i}\right)
$$

and define the polynomial

$$
P_{\lambda}(t):=\left(e_{\lambda}-\chi\left(\mathcal{O}_{X}\right)\right) t+\chi\left(\mathcal{O}_{X}\right)
$$

The polynomials $P_{\lambda}$ are the only potential Hilbert polynomials for an isomorphism $f: X \rightarrow Y$ if any. For each $\lambda$, we compute $\underline{\operatorname{Iso}}_{P_{\lambda}}(X, Y)$. If one of them is non-empty, then $X$ and $Y$ are isomorphic. Otherwise, they are not isomorphic.

Theorem 7.4. The isomorphism problem for one-dimensional reduced quasi-projective schemes is decidable. Here we suppose that each quasi-projective scheme $X$ is given an embedding $X \hookrightarrow \mathbb{P}^{m-1}$ and represented by two projective schemes $\bar{X} \subset \mathbb{P}^{m-1}$, the closure of $X$ in $\mathbb{P}^{m-1}$, and $\bar{X} \backslash X \subset \mathbb{P}^{m-1}$.

Proof. Let $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ be quasi-projective one-dimensional reduced schemes and let $\bar{X} \subset \mathbb{P}^{m-1}$ and $\bar{Y} \subset \mathbb{P}^{n-1}$ be their closures respectively. If $\bar{X}$ is singular at some point of $\bar{X} \backslash X$, then we resolve this singularity by repeating blowups. Note that a blowup of $\mathbb{P}^{m-1}$ at a point is a closed subvariety of $\mathbb{P}^{m-1} \times \mathbb{P}^{m-2}$ and hence one of $\mathbb{P}^{m(m-1)-1}$ by the Segre embedding. Thus a blowup of $\bar{X}$ at a point has an embedding into $\mathbb{P}^{m(m-1)-1}$, which can be explicitly computed. Therefore we can replace the embedding $X \subset \mathbb{P}^{m-1}$ so that $\bar{X}$ becomes smooth at every point of $\bar{X} \backslash X$. Similarly for $Y$. If $\bar{X} \backslash X$ and $\bar{Y} \backslash Y$ have different numbers of points, then $X$ and $Y$ are not isomorphic. Thus, we may suppose that they have the same number of points. Now $X$ and $Y$ are isomorphic if and only if there exists an isomorphism $f: \bar{X} \rightarrow \bar{Y}$ such that $\bar{Y} \backslash Y \subset f(\bar{X} \backslash X)$. From the assumption which we just put, $\bar{Y} \backslash Y \subset f(\bar{X} \backslash X)$ implies $\bar{Y} \backslash Y=f(\bar{X} \backslash X)$. Following the algorithm described in the proof of Theorem 7.3, we can compute the Iso scheme

$$
\underline{\mathrm{Iso}}(\bar{X}, \bar{Y})\left(=\bigcup_{\lambda} \underline{\operatorname{Iso}}_{P_{\lambda}}(\bar{X}, \bar{Y})\right)
$$

where $\lambda$ runs over partitions of a positive integer $d$ as in the proof of Theorem 7.3. In particular, since there are only finitely many partitions, we can algorithmically compute this Iso scheme. Recall that the Iso scheme is by definition a subscheme of the Hilbert scheme $\operatorname{Hilb}(\bar{X} \times \bar{Y})$. Let

$$
\mathcal{U} \subset \underline{\operatorname{Iso}}(\bar{X}, \bar{Y}) \times \bar{X} \times \bar{Y}
$$

be the universal family. Let $A \subset \mathcal{U}$ be the preimage of $\bar{X} \backslash X$ by the projection $\mathcal{U} \rightarrow \bar{X}$. Let us write $\bar{Y} \backslash Y=\left\{y_{1}, \ldots, y_{n}\right\}$ and let $B_{1}, \ldots, B_{n} \subset \mathcal{U}$ be the preimages of $y_{1}, \ldots, y_{n}$ by the projection $\mathcal{U} \rightarrow \bar{Y}$ respectively. Let $\pi: \mathcal{U} \rightarrow \underline{\mathrm{Iso}}(\bar{X}, \bar{Y})$ be the projection. For each $i$, we claim that

$$
\pi\left(A \cap B_{i}\right)=\left\{[f] \in \underline{\operatorname{Iso}}(\bar{X}, \bar{Y}) \mid y_{i} \in f(\bar{X} \backslash X)\right\}
$$

Indeed, $\pi^{-1}([f])$ is identical to the graph $\Gamma_{f} \subset \bar{X} \times \bar{Y}$ and we have

$$
\pi^{-1}([f]) \cap A \cap B_{i}=\left\{\left(x, y_{i}\right) \in \bar{X} \times \bar{Y} \mid x \in \bar{X} \backslash X\right\}
$$

The last set is non-empty if and only if $y_{i} \in f(\bar{X} \backslash X)$. This shows the above claim.
Thus $\bigcap_{i=1}^{n} \pi\left(A \cap B_{i}\right)$ is exactly the locus of isomorphisms $f: \bar{X} \rightarrow \bar{Y}$ with $\bar{Y} \backslash Y \subset$ $f(\bar{X} \backslash X)$. We compute this closed subset $\bigcap_{i=1}^{n} \pi\left(A \cap B_{i}\right)$ and check whether this is empty. The given quasi-projective schemes $X$ and $Y$ are isomorphic if and only if this is not empty.

## 8. Varieties with a big canonical sheaf or a big anti-canonical sheaf

In this section, we show the decidability of the isomorphism problem for varieties as in the title, generalizing the case of general type solved by Totaro (see Poo14, Rem.
12.3]). The key ingredients are computation of Iso schemes and the Kodaira vanishing theorem.

Theorem 8.1. Let $X$ and $Y$ be smooth irreducible projective varieties. Suppose that either $\omega_{X}$ or $\omega_{X}^{-1}$ is big. Then we can algorithmically decide whether $X$ and $Y$ are isomorphic.

Proof. It is enough to consider the case where $X$ and $Y$ have the same dimension $d>0$. We may also suppose that either both $\omega_{X}$ and $\omega_{Y}$ are big or both $\omega_{X}^{-1}$ and $\omega_{Y}^{-1}$ are big. For, otherwise, $X$ and $Y$ are not isomorphic. We denote these big sheaves by $\mathcal{B}_{X}$ and $\mathcal{B}_{Y}$ respectively. Note that we can algorithmically decide which of $\omega_{X}$ and $\omega_{X}^{-1}$ is big by checking the birationality of maps $\Phi_{\omega_{X}^{\otimes n}}, n \in \mathbb{Z}$ in turn (see Section 10.4). Let $\mathcal{L}$ and $\mathcal{M}$ be the very ample invertible sheaves on $X$ and $Y$ respectively corresponding to the given embeddings into projective spaces. We compute the least positive integer $e$ such that $\mathcal{L}^{\otimes e} \otimes \omega_{X}^{-1}$ is ample (see Section 10.3). We replace $\mathcal{L}$ with $\mathcal{L}^{\otimes e}$, which amounts to replacing the embedding $X \hookrightarrow \mathbb{P}^{m-1}$ with the one obtained by the $e$-uple Veronese embedding $\mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{\binom{m-1+e}{m-1}-1}$. Now $\mathcal{L} \otimes \omega_{X}^{-1}$ is ample. In this situation, we will algorithmically output finitely many polynomials $Q_{1}, \ldots, Q_{c}$ such that the Hilbert polynomial of every isomorphism $f: X \rightarrow Y$ (if any) is one of them. To do so, we first note that for every positive integer $l,\left(\mathcal{L} \otimes f^{*} \mathcal{M}\right)^{\otimes l} \otimes \omega_{X}^{-1}$ is ample. From the Kodaira vanishing, we have

$$
\mathrm{H}^{i}\left(X,\left(\mathcal{L} \otimes f^{*} \mathcal{M}\right)^{\otimes l}\right)=0 \quad(i>0)
$$

Hence the Hilbert polynomial $P_{f}$ of $f$ satisfies

$$
P_{f}(l)=\chi\left(X,\left(\mathcal{L} \otimes f^{*} \mathcal{M}\right)^{\otimes l}\right)=h^{0}\left(X,\left(\mathcal{L} \otimes f^{*} \mathcal{M}\right)^{\otimes l}\right)
$$

Here $h^{0}$ means the dimension of $\mathrm{H}^{0}$. Then we compute the least positive integer $q$ such that $\mathcal{B}_{X}^{\otimes q} \otimes \mathcal{L}^{-1}$ are $\mathcal{B}_{Y}^{\otimes q} \otimes \mathcal{M}^{-1}$ are both effective. Then, there exists an injection

$$
0 \rightarrow\left(\mathcal{L} \otimes f^{*} \mathcal{M}\right)^{\otimes l} \rightarrow \mathcal{B}_{X}^{\otimes 2 q l}
$$

which implies

$$
P_{f}(l)=h^{0}\left(X,\left(\mathcal{L} \otimes f^{*} \mathcal{M}\right)^{\otimes l}\right) \leq h^{0}\left(X, \mathcal{B}_{X}^{\otimes 2 q l}\right) .
$$

Note that the obtained upper bound $h^{0}\left(X, \mathcal{B}_{X}^{\otimes 2 q l}\right)$ of $P_{f}(l)$ is independent of the isomorphism $f: Y \rightarrow X$. In general, a polynomial

$$
h(t)=a_{d} t^{d}+\cdots+a_{0}
$$

of degree $d$ is determined by its values at $d+1$ distinct points $t_{0}, \ldots, t_{d}$, by Lagrange interpolation. We compute $h^{0}\left(X, \mathcal{B}_{X}^{\otimes 2 q l}\right)$ for $1 \leq l \leq d+1$. For each tuple $\lambda=\left(\lambda_{1}, \ldots, \lambda_{d+1}\right)$ of nonnegative integers with $\lambda_{i} \leq h^{0}\left(X, \mathcal{B}_{X}^{\otimes 2 q i}\right)$, we compute the polynomial $Q_{\lambda}(t)$ such that $Q_{\lambda}(i)=\lambda_{i}$. Thus obtained finitely many polynomials $Q_{\lambda}$ are the desired ones. For each $\lambda$, we compute $\underline{\mathrm{Iso}}_{Q_{\lambda}}(X, Y)$ and check whether it is empty or not. If one of them is not empty, then $X$ and $Y$ are isomorphic. Otherwise, they are not isomorphic.

## 9. Computing intersection numbers

The aim of this section is to prove the following proposition.
Proposition 9.1 ([PTL15, Lem. 8.7]). For a smooth irreducible projective variety $X$, an irreducible closed subset $Z \subset X$ of codimension $c$ and an invertible sheaf $\mathcal{L}$ on $X$, we can algorithmically compute the intersection number $Z \cdot \mathcal{L}^{\operatorname{dim} Z} \in \mathbb{Z}$.

This will be used in Section 10 to discuss decidability of various positivity properties of invertible sheaves. The proof in PTL15] uses étale cohomology (in fact, its authors considered, more generally, the intersection number of cycles of complementary dimensions). We give an alternative proof using Simpson's algorithm Sim08, Section 2.5] to compute singular cohomology. We only consider the case where the ambient variety $X$ is smooth, as Simpson's algorithm is valid only for smooth varieties. According to his algorithm, for a smooth projective variety $X \subset \mathbb{P}^{r-1}$, we can compute a finite simplicial complex $\mathcal{H}(X)$ in $\mathbb{P}^{r-1}(\mathbb{C})$ which is homotopy equivalent to $X(\mathbb{C})$. In particular, we can compute the singular homology $\mathrm{H}_{i}(X(\mathbb{C}), \mathbb{Z})$ and cohomology $\mathrm{H}^{i}(X(\mathbb{C}), \mathbb{Z})$ using $\mathcal{H}(X)$. Their elements are represented by simplicial $i$-chains and $i$-cochains on $\mathcal{H}(X)$ respectively.

When we have a morphism $f: Z \rightarrow X$ of smooth projective varieties, denoting its graph by $\Gamma_{f}$, we can compute maps of simplicial complexes

$$
\mathcal{H}(Z) \leftarrow \mathcal{H}\left(\Gamma_{f}\right) \rightarrow \mathcal{H}(X)
$$

and the induced map

$$
\mathrm{H}_{i}(Z(\mathbb{C}), \mathbb{Z}) \cong \mathrm{H}_{i}\left(\Gamma_{f}(\mathbb{C}), \mathbb{Z}\right) \xrightarrow{f_{*}} \mathrm{H}_{i}(X(\mathbb{C}), \mathbb{Z})
$$

If $X$ and $Z$ are irreducible and have dimensions $d$ and $p$ respectively and if $f$ is generically finite onto the image, then the cycle class $[f] \in \mathrm{H}^{2 d-2 p}(X(\mathbb{C}), \mathbb{Z})$ of $f$ is computed to be the element corresponding to $f_{*}([Z]) \in \mathrm{H}_{2 p}(X(\mathbb{C}), \mathbb{Z})$ via the Poincaré duality

$$
\mathrm{H}^{2 d-2 p}(X(\mathbb{C}), \mathbb{Z}) \cong \mathrm{H}_{2 p}(X(\mathbb{C}), \mathbb{Z})
$$

Here $[Z] \in \mathrm{H}_{2 p}(Z(\mathbb{C}), \mathbb{Z})$ denotes the fundamental class of $Z$. When $Z \subset X$ is a (possibly singular) irreducible closed subvariety of dimension $p$, then we can algorithmically construct a resolution of singularities $f: \widetilde{Z} \rightarrow Z \subset X$ (see Vil89, VU92, BM91, Bie97, BS00]) and define the cycle class $[Z] \in \mathrm{H}^{2 d-2 p}(X(\mathbb{C}), \mathbb{Z})$ to be $[f]$.

For an invertible sheaf $\mathcal{L}$ on $X$, we can compute a divisor $D$ on $X$ such that $\mathcal{L} \cong$ $\mathcal{O}_{X}(D)$, for example, by an algorithm given in [SY18, Section 3]. If we write $D=$ $\sum_{i=1}^{n} a_{i} D_{i}$ with $D_{i}$ prime divisors and $a_{i}$ integers, then the cohomology class $[\mathcal{L}]$ of $\mathcal{L}$ is defined to be $[D]=\sum_{i=1}^{n} a_{i}\left[D_{i}\right]$.

We can also compute the cup product

$$
\mathrm{H}^{i}(X(\mathbb{C}), \mathbb{Z}) \times \mathrm{H}^{j}(X(\mathbb{C}), \mathbb{Z}) \rightarrow \mathrm{H}^{i+j}(X(\mathbb{C}), \mathbb{Z})
$$

again by using the representation of $\mathrm{H}^{i}(X(\mathbb{C}), \mathbb{Z})$ in terms of the simplicial complex $\mathcal{H}(X)$. In summary, in the situation of Proposition 9.1, we can algorithmically compute elements $[Z],[\mathcal{L}] \in \mathrm{H}^{2 d-2 c}(X(\mathbb{C}), \mathbb{Z})$ as represented by explicit cochains on $\mathcal{H}(X)$ and compute the product $[Z][\mathcal{L}]^{\operatorname{dim} Z} \in \mathrm{H}^{2 d}(X(\mathbb{C}), \mathbb{Z})$ with respect to the cup product. The desired intersection number $Z \cdot \mathcal{L}^{\operatorname{dim} Z} \in \mathbb{Z}$ is then computed as the integer $n$ such that
$[Z][\mathcal{L}]^{\operatorname{dim} Z}=n[\mathbf{p t}]$, where $[\mathbf{p t}]$ is the cycle class of a point of $X(\mathbb{C})$. This completes the proof of Proposition 9.1 .

## 10. Positivity of invertible sheaves

Positivity properties of invertible sheaves, such as ample, big, and nef, are closely related to the isomorphism problem. In this section, we discuss the decidability problem of these properties. We also show that, for a smooth variety whose Picard number can be computable, we can approximate its nef cone and pseudo-effective cone with arbitrary precision.

### 10.1. Global generation.

Proposition 10.1. For a projective variety $X$ and a coherent sheaf $\mathcal{L}$ on it, we can algorithmically check whether it is globally generated. (We do not assume that $\mathcal{L}$ is invertible, although it is the case of our main interest.)

Proof. Let $R$ be the homogeneous coordinate ring of $X$ and let $L$ be the given finitely generated graded $R$-module, which defines $\mathcal{L}$. We can compute the graded $R$-module $L^{\prime}:=\bigoplus_{v \geq 0} \mathrm{H}^{0}(X, \widetilde{L}(v))$ as the Hom module $\operatorname{Hom}_{R}\left(R_{\geq r}, L\right)_{\geq 0}$ for some sufficiently large $r$, see Theorem 8.2 of Chapter 8 by Eisenbud in the book [Vas98]. Let

$$
F_{1} \rightarrow F_{0} \rightarrow L^{\prime} \rightarrow 0
$$

be the obtained minimal free presentation of $L^{\prime}$. Here the arrows are degree-preserving $R$-linear maps and the free module $F_{0}$ is written as

$$
F_{0}=\bigoplus_{i=1}^{c} R y_{i}
$$

with homogeneous generators $y_{i}$ with

$$
0 \leq \operatorname{deg}\left(y_{1}\right) \leq \cdots \leq \operatorname{deg}\left(y_{c}\right)
$$

Let $y_{1}, \ldots, y_{n}(n \leq c)$ be the ones of degree 0 , which are regarded as a basis of $\mathrm{H}^{0}(X, \widetilde{L})$, and let

$$
F_{0}^{\prime}:=\bigoplus_{i=1}^{n} R y_{i} .
$$

The derived map $F_{0}^{\prime} \rightarrow L^{\prime}$ induces the map of sheaves,

$$
\mathcal{O}_{X} \otimes \mathrm{H}^{0}(X, \widetilde{L}) \rightarrow \widetilde{L}
$$

The sheaf $\widetilde{L}$ is globally generated if and only if this map is surjective. We can check the latter condition, for example, by computing the support of the $\mathcal{O}_{X}$-module corresponding to the graded $R$-module $\operatorname{Coker}\left(F_{0}^{\prime} \rightarrow L^{\prime}\right)$ and see whether it is empty.

### 10.2. Very ampleness.

Proposition 10.2. For a projective variety $X$ and an invertible sheaf $\mathcal{L}$ on it, we can algorithmically check whether it is very ample.

Proof. We first check the global generation of $\mathcal{L}$. If $\mathcal{L}$ is not globally generated, then $\mathcal{L}$ is not very ample. Suppose that $\mathcal{L}$ is globally generated. We then compute the morphism $\Phi_{\mathcal{L}}: X \rightarrow \mathbb{P}^{n-1}$ associated to $\mathcal{L}$, where $n=\operatorname{dim} \mathrm{H}^{0}(X, \mathcal{L})$. Let $L$ be the given graded $R$-module defining $\mathcal{L}$ and we construct a map of $R$-modules, $F_{0}^{\prime} \rightarrow L^{\prime}$, as in the proof of Proposition 10.1. Let $M$ be the image of this map, which defines the same sheaf on $X$ as $L^{\prime}$ and $L$ do. We have a free presentation

$$
F_{1}^{\prime} \rightarrow F_{0}^{\prime} \rightarrow M \rightarrow 0
$$

From [Eis95, Prop. A2.2], this induces the exact sequence

$$
\operatorname{Sym}_{R}\left(F_{0}^{\prime}\right) \otimes_{R} F_{1}^{\prime} \rightarrow \operatorname{Sym}_{R}\left(F_{0}^{\prime}\right) \rightarrow \operatorname{Sym}_{R}(M) \rightarrow 0
$$

If the basis of $F_{1}^{\prime}$ maps to

$$
g_{i}=\sum_{j=1}^{n} g_{i j} y_{j} \quad(i=1, \ldots, r)
$$

with $r$ denoting the rank of $F_{1}^{\prime}$, then

$$
\begin{aligned}
\operatorname{Sym}_{R}(M) & =R\left[y_{1}, \ldots, y_{n}\right] /\left(g_{1}, \ldots, g_{r}\right) \\
& =k\left[x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n}\right] /\left(f_{1}, \ldots, f_{l}, g_{1}, \ldots, g_{r}\right),
\end{aligned}
$$

which is bi-graded. This defines a closed subscheme $\Gamma \subset \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$. The projection $\Gamma \rightarrow \mathbb{P}^{m-1}$ is an isomorphism onto $X$. In other words, $\Gamma$ is the graph of a morphism $X \rightarrow \mathbb{P}^{n-1}$. The last morphism is the morphism $\Phi_{\widetilde{L}}$ associated to the globally generated invertible sheaf $\widetilde{L}$. Now we can compute the image $Y:=\Phi_{\widetilde{L}}(X)$ of $\Phi_{\widetilde{L}}$ by projective elimination (see Remark (10.3). From Lemma 3.3, we can check whether the morphism $X \rightarrow Y$, which corresponds to $\Gamma \subset \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$, is an isomorphism. Our invertible sheaf $\mathcal{L}$ is very ample if and only if the last morphism is an isomorphism.

Remark 10.3 (Projective elimination). Suppose that a closed subscheme $V \subset \mathbb{P}^{m-1} \times$ $\mathbb{P}^{n-1}$ is defined by a bi-homogeneous ideal $I \subset k\left[x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n}\right]$. Then the scheme-theoretic image $p_{2}(V)$ by the second projection is defined by the ideal

$$
\left(I:\left(x_{1}, \ldots, x_{m}\right)^{\infty}\right) \cap k\left[y_{1}, \ldots, y_{n}\right] .
$$

At the set-theoretic level, this is written [GP08, page 503]; the closed subset $p_{2}(V)$ is the zero set of the last homogeneous ideal. The scheme-theoretic version follows from [GP08, Lemma A.7.9].

### 10.3. Ampleness.

Proposition 10.4. We can algorithmically decide whether an invertible sheaf on a smooth irreducible projective variety is ample.

Proof. Let $X$ be a projective scheme and let $\mathcal{L}$ be an invertible sheaf on $X$. From an effective version of Matsusaka's Big Theorem [Siu93], there exists a positive integer $m(\mathcal{L})$ explicitly determined by $\mathcal{L}^{\operatorname{dim} X}, \omega_{X} \cdot \mathcal{L}^{\operatorname{dim} X-1}$, and $\operatorname{dim} X$ such that $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes m(\mathcal{L})}$ is very ample. Thus, we only need to compute the number $m(\mathcal{L})$ and check whether $\mathcal{L}^{\otimes m(\mathcal{L})}$ is very ample.
10.4. Nefness, bigness and pseudo-effectivity. These properties in the title of invertible sheaves are all positivity properties in some sense, which are weaker than ampleness, and play important roles in birational geometry. In what follows, we restrict ourselves to the case where the ambient scheme is irreducible and smooth, unless otherwise noted.

Definition 10.5. Let $X$ be a smooth irreducible projective variety and let $\mathcal{L}$ be an invertible sheaf on $X$. The $\mathcal{L}$ is big if for some integer $n>0$, the rational map associated to $\mathcal{L}^{n}$,

$$
\Phi_{\mathcal{L}^{n}}: X \rightarrow \mathbb{P}^{m}
$$

is birational onto the image. The $\mathcal{L}$ is nef if for every irreducible curve $C \subset X$, we have $C \cdot \mathcal{L} \geq 0$. The $\mathcal{L}$ is pseudo-effective if its class in $\operatorname{NS}(X) \otimes \mathbb{R}$ is the limit of classes of effective divisors.

Note that it is easy to check whether $\mathcal{L}$ is effective (that is, isomorphic to $\mathcal{O}_{X}(D)$ for some effective divisor $D$ ) by computing the cohomology group $\mathrm{H}^{0}(X, \mathcal{L})$. For each $n$, we can algorithmically check whether the map $\Phi_{\mathcal{L}^{n}}$ is birational, see [Sim04, DHS12. See also [BHSS19] for implementation of such an algorithm. Thus, bigness of an invertible sheaf is semi-decidable. On the other hand, not being nef is a semi-decidable property. Indeed, we enumerate irreducible curves in $X$ as $C_{1}, C_{2}, \ldots$ and for each $i$, we compute the intersection number $C_{i} \cdot \mathcal{L}$, until we get a negative intersection number. As for pseudo-effectivity, we have the following theorem [BDPP13, 0.2 Theorem]: $\mathcal{L}$ is pseudoeffective if and only if $\mathcal{L} \cdot C \geq 0$ for every irreducible curve $C \subset X$ which moves in a family covering $X$. Using this, we can prove:

Proposition 10.6. Not being pseudo-effective is a semi-decidable property.
Proof. We enumerate all irreducible curves on $X$ as $C_{1}, C_{2}, \ldots$ Consider the following algorithm:
(1) Put $n=1$.
(2) Check whether $C_{n}$ moves in a family covering $X$ as follows. We first compute the Hilbert polynomial $P_{n}$ of $C_{n}$ and then compute the connected component $W$ of $\operatorname{Hilb}_{P_{i}}(X)$ containing $\left[C_{n}\right]$. We then check whether the universal family $\mathcal{U}_{W}$ on $W$ maps onto $X ; C_{n}$ is a movable curve if and only if this is the case. When $C_{n}$ is movable, we compute the intersection number $C_{n} \cdot D$ and stop the algorithm if $C_{n} \cdot D<0$.
(3) Put $n=n+1$ and go back to (2).

From BDPP13, 0.2 Theorem], this algorithm stops after finitely many steps if and only if $D$ is not pseudo-effective. The proposition follows.

In summary, the following properties of invertible sheaves on a smooth irreducible projective variety are semi-decidable:
(1) Being big.
(2) Not being nef.
(3) Not being pseudo-effective.

It is now quite natural to ask:
Problem 10.7. Are the three properties, big, nef and pseudo-effective, decidable?
Remark 10.8. Note that when the given invertible sheaf is known to be nef, then we only need to compute the intersection number $\mathcal{L}^{\operatorname{dim} X}$ to check whether $\mathcal{L}$ is big (see [Laz04, Theorem 2.2.16]).

In the case of the canonical sheaf $\omega_{X}$, the most important invetible sheaf, we may take advantage of the following conjectures:

Conjecture 10.9 (The abundance conjecture). For a smooth projective variety $X$, the canonical sheaf $\omega_{X}$ is nef if and only if it is semi-ample (that is, $\mathcal{L}^{n}$ is globally generated for some $n>0$ ).
Conjecture 10.10 (The non-vanishing conjecture). For a smooth projective variety $X$, the canonical sheaf $\omega_{X}$ is pseudo-effective if and only if it is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor.

The abundunce conjecture is recognized as one of the most important conjectures in the minimal model program. The importance of the non-vanishing conjecture was pinned down by Birkar [Bir11]. The above form of the non-vanishing conjecture is slightly different from the one considered by Birkar. However Hashizume [Has18] proved that they are equivalent.
Proposition 10.11. Let $X$ be a smooth irreducible projective variety.
(1) If the abundance conjecture holds for $X$, then the nefness of $\omega_{X}$ is decidable.
(2) If the weak nonvanishing conjecture holds for $X$, then the pseudo-effectivity of $\omega_{X}$ is decidable.

Proof. (1) We first note that we can compute the canonical sheaf [Sti, Section 5.6]. We enumerate all irreducible curves in $X$ as $C_{1}, C_{2}, \ldots$ Consider the following algorithm:
(1) Put $n=1$.
(2) We check whether $\omega_{X}^{\otimes n}$ is globally generated. If this is the case, then stop the algorithm and output True.
(3) We check whether $C_{n} \cdot \omega_{X}<0$. If this is the case, then stop the algorithm and output False.
(4) Put $n=n+1$ and go to (2).

If the abundance conjecture holds, then this algorithm always stops after finitely many steps and outputs True if $\omega_{X}$ is nef and False if $\omega_{X}$ is not nef.
(2) Effective $\mathbb{Q}$-divisors on $X$ are enumerable. For each of them, we can compute its class in $\mathrm{H}^{2}(X(\mathbb{C}), \mathbb{Q})$ by the method explained in Section 9 and check whether it coincides with the class of $\omega_{X}$. From the non-vanishing conjecture, we see that the
pseudo-effectivity of $\omega_{X}$ is semi-decidable. Combining this with Proposition 10.6 shows the assertion.

Remark 10.12. If $\operatorname{dim} X \leq 3$, then the bigness of $\omega_{X}$ is also decidable. Indeed, from [HM06, Tak06, Tsu06], for each dimension $d$, there exists a positive integer $n_{d}$ such that for every smooth variety $X$ of general type and of dimension $d$ and for every integer $n \geq n_{d}$, the rational map $\Phi_{\omega_{X}^{\otimes n}}$ is birational onto the image. Moreover, for $d \leq 3$, we can take $n_{1}=3, n_{2}=5, n_{3}=126$ (see Bom73, CC10]); we only need to check whether $\Phi_{\omega_{X}^{\otimes n_{d}}}$ is birational onto the image. To generalize this argument to dimensions $\geq 4$, we need to compute $n_{d}$.

If $\omega_{X}$ is nef, then we can check its bigness in any dimension, see Remark 10.8. If $\omega_{X}$ is not nef, then we may run the minimal model program. As an output of the program, we would get a Mori fiber space or a minimal model birational to the given variety $X$. In the former case, $\omega_{X}$ is not pseudo-effective, in particular, not big. In the latter case, we can check the bigness of $\omega_{X}$ by computing the intersection number $\left(\omega_{X}\right)^{\operatorname{dim} X}$. This strategy provides motivation for studying the following problem, which would be important also on its own right:

Problem 10.13. Describe each step of the minimal model program as a strict algorithm, starting from algorithmically finding a $\omega_{X}$-negative ray of the cone of curves.
10.5. Approximating nef and pseudo-effective cones. Let $\mathrm{NS}(X)_{\mathbb{R}}:=\mathrm{NS}(X) \otimes \mathbb{R}$ denote the Néron-Severi group tensored with $\mathbb{R}$, This is a finite-dimensional $\mathbb{R}$-vector space and its dimension $\rho(X)$ is called the Picard number of $X$. The nef cone of $X$, denoted by $\operatorname{Nef}(X)$, is the smallest closed convex cone in $\operatorname{NS}(X)_{\mathbb{R}}$ such that, for an invertible sheaf $\mathcal{L}$, the class $[\mathcal{L}]$ belongs to it if and only if $\mathcal{L}$ is nef. The pseudo-effective cone $\operatorname{PEff}(X)$ is similarly defined. The ample cone and the big cone are the interiors of the nef cone and the pseudo-effective cone respectively.

As we do not have an algorithm to decide whether a given invertible sheaf is big/nef, we can not compute the cones $\operatorname{PEff}(X)$ and $\operatorname{Nef}(X)$ at least for now. However, if we know the value of the Picard number $\rho(X)$, then we can approximate these cones with arbitrary precision. Note that, if we know the value of $\rho(X)$, then we can compute the subspace $\operatorname{NS}(X)_{\mathbb{R}} \subset \mathrm{H}^{2}(X(\mathbb{C}), \mathbb{R})$ by giving a basis of it. To do so, we only need to compute classes $[D] \in \mathrm{H}^{2}(X(\mathbb{C}), \mathbb{R})$ of divisors $D \subset X$, until we have enough to span a subspace of dimension $\rho(X)$. Poonen, Testa and van Luijk PTL15 gave an algorithm to compute $\rho(X)$, assuming the Tate conjecture. In particular, we can compute $\rho(X)$ if $X$ is a K3 surface.

Proposition 10.14. Let $X$ be a smooth irreducible projective variety. Suppose that we know the value of $\rho(X)$. We fix a metric on $\mathrm{NS}(X)_{\mathbb{R}}$. Let $S \subset \mathrm{NS}(X)_{\mathbb{R}}$ be the unit sphere with center at the origin. Then, for any positive real number $\epsilon>0$, we can algorithmically construct rational polyhedral convex cones $A_{\epsilon}$ and $B_{\epsilon}$ such that $A_{\epsilon} \subset$ $\operatorname{Nef}(X) \subset B_{\epsilon}$ and $B_{\epsilon} \cap S$ is contained in the $\epsilon$-neighborhood of $A_{\epsilon} \cap S$. Similarly for $\operatorname{PEff}(X)$.
Proof. Let $\rho$ denote the Picard number of $X$. From Proposition 10.4, we can enumerate all the ample divisors on $X$ as $D_{1}, D_{2}, \ldots$ Let $A_{n}=\sum_{i=1}^{n} \mathbb{R}_{\geq 0}\left[D_{i}\right]$ be the convex
cone generated by $\left[D_{1}\right], \ldots,\left[D_{n}\right]$. The closure of $\bigcup_{n \geq 0} A_{n}$ is the nef cone $\operatorname{Nef}(X)$. In particular, each $A_{n}$ is a rational convex polyhedral cone contained in $\operatorname{Nef}(X)$. We can also enumerate the irreducible curves in $X$ as $C_{1}, C_{2}, \ldots$. Let

$$
\begin{aligned}
B_{n}^{c} & :=\bigcup_{i=1}^{n}\left\{x \in \operatorname{NS}(X)_{\mathbb{R}} \mid x \cdot C_{i}<0\right\} \text { and } \\
B_{n} & :=\operatorname{NS}(X)_{\mathbb{R}} \backslash B_{n}^{c}=\bigcap_{i=1}^{n}\left\{x \in \operatorname{NS}(X)_{\mathbb{R}} \mid x \cdot C_{i} \geq 0\right\} .
\end{aligned}
$$

We see that $\bigcup_{n \geq 0} B_{n}^{c}=\operatorname{NS}(X)_{\mathbb{R}} \backslash \operatorname{Nef}(X)$. Thus each $B_{n}$ is a rational convex polyhedral cone containing $\operatorname{Nef}(X)$. It is also strongly convex (that is, it has a vertex at the origin) for $n \gg 0$. We have got two sequences $\left(A_{n}\right)_{n}$ and $\left(B_{n}\right)_{n}$ of rational convex polyhedral cones approximating $\operatorname{Nef}(X)$ from inside and outside respectively. Therefore, for $n \gg 0$, $A_{n}$ and $B_{n}$ satisfy the desired condition. To see for which value of $n$ this is the case, we first check whether $B_{n}$ is strongly convex. If this is the case, then for each vertex $w \in B_{n} \cap S$, we check whether every vertex $w \in B_{n} \cap S$ is contained in the $\epsilon$-neighborhood of $A_{n}$. If this is the case, $B_{n}$ is contained in the $\epsilon$-neighborhood of $A_{n}$. This completes the proof for the nef cone $\operatorname{Nef}(X)$.

As for the pseudo-effective cone $\operatorname{PEff}(X)$, we only need to replace ample divisors with big divisors and irreducible curves with movable irreducible curves. To enumerate movable irreducible curves, we can use the algorithm in the proof of 10.6. To enumerate big divisors, we can use the following algorithm: We first enumerate all the divisors on $X$ as $D_{1}, D_{2}, \ldots$.
(1) Put $n=1$ and put $b=()$, the empty ordered tuple.
(2) For each $i, j \leq n$, if $\Phi_{i \cdot D_{j}}$ is a birational map onto the image and if $D_{j} \notin b$, then append $D_{j}$ to $b$.
(3) Put $n=n+1$ and go to (2).

For every big divisor $D$ on $X$, the above algorithm appends $D$ to $b$ after finitely many steps. Thus, for every positive integer $n$, we can algorithmically construct the $n$-th big divisor. (Thus, big divisors on $X$ are listable. But this does not mean that the bigness of each divisor is decidable.)

## 11. K3 Surfaces

In this section, we discuss the isomorphism problem for K3 surfaces, which would be natural as the next case to study after the one-dimensional case and the case with $\omega_{X}$ or $\omega_{X}^{-1}$ big were treated in Sections 7 and 8 respectively. The main result of this section is the decidability of the isomorphism problem for K3 surfaces with an automorphism group finite.

Proposition 11.1. Let $X$ be a K3 surface. If $\operatorname{Aut}(X)$ is finite, then we can compute the nef cone $\operatorname{Nef}(X)$ by giving finitely many effective divisors $D_{1}, \ldots, D_{n}$ such that $\operatorname{Nef}(X)=\sum_{i=1}^{n} \mathbb{R}_{\geq 0}\left[D_{i}\right]$.
Proof. From PTL15, we can compute the Picard number $\rho(X)$ and compute $\mathrm{NS}(X)_{\mathbb{R}}$ as explained in Section 10.5, For an effective divisor $D$ on $X$, we can check whether it
is nef; we check whether $C \cdot D \geq 0$ for every prime divisor $C$ contained in the support of $D$. Therefore we can enumerate all the effective divisors as $D_{1}, D_{2}, \ldots$ and all the nef and effective divisors as $N_{1}, N_{2}, \ldots$. For each $n$, let

$$
\begin{aligned}
A_{n} & :=\sum_{i=1}^{n} \mathbb{R}_{\geq 0}\left[N_{i}\right] \\
B_{n} & :=\bigcap_{i=1}^{n}\left\{x \in \operatorname{NS}(X)_{\mathbb{R}} \mid x \cdot D_{i} \geq 0\right\} .
\end{aligned}
$$

These are rational polyhedral convex cones satisfying

$$
\begin{equation*}
A_{n} \subset \operatorname{Nef}(X) \subset B_{n} \tag{11.1}
\end{equation*}
$$

If $\operatorname{Aut}(X)$ is finite, then $\operatorname{PEff}(X)$ is a rational polyhedral cone spanned by effective classes Kov94. It follows that every point of $\operatorname{NS}(X)_{\mathbb{Q}} \cap \operatorname{PEff}(X)$ is represented by an effective $\mathbb{Q}$-divisor. Since $\operatorname{Nef}(X)$ is the dual cone of $\operatorname{PEff}(X)$, it is also rational polyhedral and spanned by finitely many points of $\operatorname{NS}(X)_{\mathbb{Q}}$. Since $\operatorname{Nef}(X) \subset \operatorname{PEff}(X)$, these points are represented by nef and effective divisors after multiplied with some positive integer. We conclude that $\operatorname{Nef}(X)$ is a rational polyhedral cone spanned by nef and effective classes. Therefore, for $n \gg 0$, inclusions on this page are equalities. For each $n$, we compute $A_{n}$ and $B_{n}$ and check whether $A_{n}=B_{n}$. If this equality holds, then the cone $A_{n}=B_{n}$ is the nef cone.

Proposition 11.2. For a K3 surface $X$, we can algorithmically decide whether $\operatorname{Aut}(X)$ is finite.

Proof. We define cones $A_{n}$ and $B_{n}$ in $\mathrm{NS}(X)_{\mathbb{R}}$ as in the proof of Proposition 11.1. From [Kov94], $A_{n}=B_{n}$ for $n \gg 0$ if and only if $\operatorname{Aut}(X)$ is finite. Therefore we have an algorithm which stops after finitely many steps exactly when $\operatorname{Aut}(X)$ is finite. We denote this algorithm by $\Theta$.

For an automorphism $f: X \rightarrow X$, the tangent space of $\underline{\operatorname{Aut}}(X)=\underline{\operatorname{Iso}}(X, X)$ at $[f]$ is isomorphic to $\mathrm{H}^{0}\left(X, \mathcal{T}_{X}\right)$ with $\mathcal{T}_{X}$ denoting the tangent sheaf. Since

$$
\mathrm{H}^{0}\left(X, \mathcal{T}_{X}\right)^{\vee}=\mathrm{H}^{2}\left(X, \omega_{X} \otimes \Omega_{X}\right)=\mathrm{H}^{2}\left(X, \Omega_{X}\right)=0
$$

the Aut scheme $\underline{\text { Aut }}(X)$ has only isolated points. From Kon99], any finite subgroup of $\operatorname{Aut}(X)$ has order at most 3840. The following algorithm stops after finitely many steps exactly when $\operatorname{Aut}(X)$ is infinite: We enumerate all the Hilbert polynomials as $P_{1}, P_{2}, \ldots$.
(1) Put $n=1$ and numAuts $=0$.
(2) Put numAuts $=$ numAuts $+\sharp \underline{\text { Iso }}_{P_{n}}(X, X)$.
(3) If numAuts $>3840$, then stop.
(4) Put $n=n+1$ and go to (2).

We denote this algorithm by $\Theta^{\prime}$. Now the following algorithm is the desired one:
(1) Put $n=1$.
(2) If $\Theta$ stops after $n$ steps, then stop and output Finite.
(3) If $\Theta^{\prime}$ stops after $n$ steps, then stop and output Infinite.
(4) Put $n=n+1$ and go to (2).

Theorem 11.3. For $K 3$ surfaces $X$ and $Y$ with finite automorphism groups, we can algorithmically decide whether they are isomorphic.
Proof. We compute the nef cones $\operatorname{Nef}(X)$ and $\operatorname{Nef}(Y)$. There exist at most finitely many isomorphisms $g: \mathrm{NS}(Y) \rightarrow \mathrm{NS}(X)$ such that $g(\operatorname{Nef}(Y))=\operatorname{Nef}(X)$. If there is no such isomorphism, then $X$ and $Y$ are not isomorphic. Suppose that this is not the case and let $g_{1}, \ldots, g_{n}$ be all the isomorphisms with this property. Let $\mathcal{L}$ and $\mathcal{M}$ be the given very ample sheaves on $X$ and $Y$. We compute the Hilbert polynomial for each $[\mathcal{L}]+g_{i}[\mathcal{M}]$ and call it by $P_{i}$. Note that the Hilbert polynomial of an ample invertible sheaf depends only on its numerical class. Indeed, the numerical class $[\mathcal{N}]$ of an ample invertible sheaf determines the Euler characteristics $\chi(\mathcal{N})$ and $\chi\left(\mathcal{N}^{2}\right)$ from the Riemann-Roch formula for surfaces. These values together with the one of $\chi\left(\mathcal{O}_{X}\right)$ determines the Hilbert polynomial of $\mathcal{N}$. If there is an isomorphism $f: X \rightarrow Y$, then the induced isomorphism $\mathrm{NS}(Y) \rightarrow \mathrm{NS}(X)$ is one of the $g_{i}$ 's and the Hilbert polynomial of $f$ is one of the $P_{i}$ 's. Thus $[f]$ is a point of $\bigcup_{i=1}^{n} \underline{\operatorname{Iso}}_{P_{i}}(X, Y)$. Thus, $X$ and $Y$ are isomorphic if and only if $\bigcup_{i=1}^{n} \underline{\operatorname{Iso}}_{P_{i}}(X, Y) \neq \emptyset$. From Section 6, the last condition can be algorithmically checked.

Remark 11.4. For a general K3 surface $X$, there are only finitely many very ample class $x$ with $x^{2}$ being the prescribed number modulo the action of $\operatorname{Aut}(X)$ Ste85, 2.6]. Therefore, if we replace the given very ample sheaf of $X$ by a suitable automorphism of $X$, we can find an isomorphism $X \rightarrow Y$ (if any) with the Hilbert polynomial in a finite set of potential candidates. But there is a priori no way to know which automorphism does this job.
Remark 11.5. Let $X$ and $Y$ be K3 surfaces and let $\mathcal{L}$ be the given very ample sheaf of $Y$. There is no intrinsic invariants of $X$ and $Y$ to determine the place of $f^{*}[\mathcal{L}]$ in the ample cone of $X$ for a potential isomorphism $f: X \rightarrow Y$. Indeed, when $\rho=2$ and they have infinite automorphisms, then a very ample class $l$ with $l^{2}=d$ is sent to infinitely many distinct lattice points on the curve $x^{2}=d$. For two lattice points $l_{1}$ and $l_{2}$ on the curve, the sum $l_{1}+l_{2}$ can have arbitrarily large Euler characteristic.
Remark 11.6. Discussion in this section indicates that the isomorphism problem is closely related to complexity of the automorphism group. Recently, Lesieutre [Les18] showed that there exists a projective variety $X$ whose automorphism group is discrete, but not finitely generated (see also (DO19]). This result may be considered to suggest that the isomorphism problem for general projective schemes is not decidable.

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[^1]:    ${ }^{1}$ https://mathoverflow.net/questions/21883/isomorphism-problem-for-commutative-algebras-andschemes

