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ABSTRACT. Let D be a weighted oriented graph and I(D) be its edge ideal. We provide one method to find all the minimal generators of $I_{\subseteq C}$, where C is a maximal strong vertex cover of D and $I_{\subseteq C}$ is the intersections of irreducible ideals associated to the strong vertex covers contained in C. If D' is an induced digraph of D, under a certain condition on the strong vertex covers of D' and D, we show that $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$ implies $I(D)^{(s)} \neq I(D)^s$. We provide the necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideal of the union of two naturally oriented paths with a common sink vertex. We characterize all the maximal strong vertex covers of D such that at most one edge is oriented into each of its vertices and $w(x) \ge 2$ if $\deg_D(x) \ge 2$ for all $x \in V(D)$. Finally, if D is a weighted rooted tree with the degree of root is 1 and $w(x) \ge 2$ when $\deg_D(x) \ge 2$ for all $x \in V(D)$, we show that $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$.

Keywords: Weighted oriented graph, induced digraph, edge ideal, symbolic power, tree, path.

1. INTRODUCTION

Let k be a field and $R = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables. Let I be a homogeneous ideal of R. Then for $s \ge 1$, the s-th symbolic power of I is defined as $I^{(s)} = \bigcap_{P \in Ass I} (I^s R_P \cap R)$. We refer [5] to the reader to survey some known results of symbolic powers of ideals.

A weighted oriented graph D with underlying graph G, is a triplet (V(D), E(D), w)whose vertex set is V(D) = V(G), edge set is $E(D) = \{(x_i, x_j) \mid \{x_i, x_j\} \in E(G)\}$ and the weight function $w : V(D) \longrightarrow \mathbb{N}$. If $e = (x, y) \in E(D)$, then x is the initial vertex and y is the terminal vertex of the directed edge e. The weight of a vertex $x_i \in V(D)$ is $w(x_i)$ denoted by w_i or w_{x_i} . If a vertex x_i of D is a source (i.e., has only arrows leaving x_i), we fix $w_i = 1$. The edge ideal of D is denoted by I(D) and is defined as $I(D) = (x_i x_j^{w_j} | (x_i, x_j) \in E(D))$. This ideal was first studied in [6, 12]. Recently in [7], the authors give the classification of some normally torsion-free edge ideals of weighted

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oriented graphs, where the s-th symbolic power of I defined using the associated minimal primes of I.

The difficulty in the study of symbolic powers of edge ideals of weighted oriented graphs depends upon the structures of irreducible ideals associated to the strong vertex covers and the number of strong vertex covers. In general, the number of strong vertex covers of a weighted oriented graph is greater than the number of minimal vertex covers of its underlying graph and it occurs due to the weights on its vertices and orientation of its edges. So the study of symbolic powers of edge ideals of weighted oriented graphs is harder than simple graphs. In this paper, we provide some methods to study the symbolic powers of their edge ideals. In [9], we see that, if the set of all vertices of a weighted oriented graph forms a strong vertex cover, all the ordinary and symbolic powers of its edge ideal coincide. Comparision of ordinary and symbolic powers has been done for several classes of weighted oriented graphs in [1], [2] and [10]. In all those papers, to compute the symbolic powers, the authors always find the minimal generators of the intersections of irreducible ideals associated to the strong vertex covers contained in a maximal strong vertex cover. In this paper, we give a direct formula for that in Theorem 3.5 and it works for any weighted oriented graph. We compare the ordinary and symbolic powers of edge ideals of weighted oriented graphs by studying the ordinary and symbolic powers of edge ideals of their induced digraphs. In [11], if H is an induced subgraph of G, it is known that $I(H)^{(s)} \neq I(H)^s$ for some $s \ge 2$ implies $I(G)^{(s)} \neq I(G)^s$. We extend this result to weighted oriented graphs. If D' is an induced digraph of D, under a certain condition on the strong vertex covers of D' and D, we show that $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$ implies $I(D)^{(s)} \ne I(D)^s$ (see Theorem 4.3). We apply this result to compare the ordinary and symbolic powers of edge ideals of weighted oriented paths (see Theorem 4.5). In Theorem 5.12, we give the necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideal of union of two naturally oriented paths with a common sink.

The main problem in the computation of symbolic power is to find all the maximal strong vertex covers. In [12, Lemma 47], Pitones et al. proved that $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of D if $I(D) \subseteq (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s})$. We identify that, if $a_j = w(x_{i_j})$ and s is the least possible value, then $\{x_{i_1}, \ldots, x_{i_s}\}$ is a maximal strong vertex cover of D (see Lemma 6.2). In Theorem 6.5, we prove the converse of Lemma 6.2 is also true under the assumption " at most one edge is oriented into each vertex of D and $w(x) \ge 2$ if $\deg_D(x) \ge 2$ for all $x \in V(D)$ ". Recently in [1], Banerjee et al. prove the equality of ordinary and symbolic powers of a certain class of weighted rooted trees. In Theorem 6.6, we extend this result to prove that " if D is a weighted rooted tree with degree of root is 1 and $w(x) \ge 2$ whenever $\deg_D(x) \ge 2$ for all $x \in V(D)$, then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2^n$.

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2. Preliminaries

In this section, we provide some definitions and results for the weighted oriented graphs. By the result of Ha, Nguyen, Trung, and Trung in [8], we may assume that the underlying graph G of the weighted oriented graph D is connected.

Definition 2.1. A vertex cover C of D is a subset of V(D) such that if $(x, y) \in E(D)$, then $x \in C$ or $y \in C$. A vertex cover C of D is minimal if each proper subset of C is not a vertex cover of D. We set (C) to be the ideal generated by the vertices of C.

Definition 2.2. Let x be a vertex of a weighted oriented graph D, then the sets $N_D^+(x) = \{y \mid (x,y) \in E(D)\}$ and $N_D^-(x) = \{y \mid (y,x) \in E(D)\}$ are called the out-neighbourhood and the in-neighbourhood of x, respectively. Moreover, the neighbourhood of x is the set $N_D(x) = N_D^+(x) \cup N_D^-(x)$ and denote $N_D[x] = N_D(x) \cup \{x\}$. Define $\deg_D(x) = |N_D(x)|$ for $x \in V(D)$ and $\deg_D(x)$ is known as the degree of the vertex x in D. A vertex $x \in V(D)$ is called a source vertex if $N_D(x) = N_D^+(x)$. A vertex $x \in V(D)$ is called a sink vertex if $N_D(x) = N_D^-(x)$. We assume that the weights of source vertices are trivial. We set $V^+(D)$ as the set of vertices of D with non-trivial weights.

Definition 2.3. For $T \,\subset V(D)$, we define the induced digraph $D_T = (V(D_T), E(D_T), w)$ of D on T to be the weighted oriented graph such that $V(D_T) = T$ and for any $u, v \in$ $V(D_T), (u, v) \in E(D_T)$ if and only if $(u, v) \in E(D)$. Here $D_T = (V(D_T), E(D_T), w_T)$ is a weighted oriented graph with the same orientation as in D and for any $u \in V(D_T)$, if u is not a source in D_T , then its weight equals to the weight of u in D, otherwise, its weight in D_T is 1. For a subset $W \subset V(D)$, we define $D \setminus W$ to be the induced digraph of D on $V(D) \setminus W$.

Definition 2.4. [12, Definition 4] Let C be a vertex cover of a weighted oriented graph D. We define the following three sets that form a partition of C

$$L_1^D(C) = \{ x \in C \mid N_D^+(x) \cap C^c \neq \phi \},\$$

$$L_2^D(C) = \{ x \in C \mid x \notin L_1^D(C) \text{ and } N_D^-(x) \cap C^c \neq \phi \} \text{ and}\$$

$$L_3^D(C) = C \smallsetminus (L_1^D(C) \cup L_2^D(C)),\$$

where C^c is the complement of C, i.e., $C^c = V(D) \setminus C$.

Lemma 2.5. [12, Proposition 5] If C is a vertex cover of D, then $L_3^D(C) = \{x \in C \mid N_D(x) \subset C\}$.

Definition 2.6. [12, Definition 7] A vertex cover C of D is strong if for each $x \in L_3^D(C)$ there is $(y, x) \in E(D)$ such that $y \in L_2^D(C) \cup L_3^D(C)$ with $y \in V^+(D)$ (i.e., $w(y) \neq 1$). **Remark 2.7.** [12, Remark 8, Proposition 5] A vertex cover C of D is strong if and only if for each $x \in L_3^D(C)$, we have $N_D^-(x) \cap V^+(D) \cap [C \setminus L_1^D(C)] \neq \phi$.

Definition 2.8. A strong vertex cover C of D is said to be a maximal strong vertex cover of D if it is not contained in any other strong vertex cover of D.

Definition 2.9. [12, Definition 19] Let C be a vertex cover of D. The irreducible ideal associated to C is the ideal $I_C := (L_1^D(C) \cup \{x_j^{w(x_j)} | x_j \in L_2^D(C) \cup L_3^D(C)\}).$

Lemma 2.10. [12, Lemma 20] Let D be a weighted oriented graph. Then $I(D) \subseteq I_C$, for each vertex cover C of D.

The next lemma gives us the irreducible decomposition of the edge ideal of a weighted oriented graph D in terms of irreducible ideals associated with the strong vertex covers of D.

Lemma 2.11. [12, Theorem 25, Remark 26] Let D be a weighted oriented graph and C_1, \ldots, C_s are the strong vertex covers of D, then the irredundant irreducible decomposition of I(D) is

 $I(D) = I_{C_1} \cap \dots \cap I_{C_s}$ where each $I_{C_i} = (L_1^D(C_i) \cup \{x_j^{w(x_j)} \mid x_j \in L_2^D(C_i) \cup L_3^D(C_i)\}), \operatorname{rad}(I_{C_i}) = P_i = (C_i).$

Corollary 2.12. [12, Remark 26] Let D be a weighted oriented graph. Then P is an associated prime of I(D) if and only if P = (C) for some strong vertex cover C of D.

Let $I \,\subset R$ and $I = Q_1 \cap \cdots \cap Q_m$ be a primary decomposition of ideal I. For $P \in \operatorname{Ass}(R/I)$, we denote $Q_{\subseteq P}$ to be the intersection of all Q_i with $\sqrt{Q_i} \subseteq P$. If C is a strong vertex cover of a weighted oriented graph D, then $(C) \in \operatorname{Ass}(R/I(D))$. We denote $I_{\subseteq C}$ as $I_{\subseteq(C)}$. In the following lemma, we express the [4, Theorem 3.7] for edge ideals of weighted oriented graphs.

Lemma 2.13. [4, Theorem 3.7] Let I be the edge ideal of a weighted oriented graph D and C_1, \ldots, C_r be the maximal strong vertex covers of D. Then

$$I^{(s)} = (I_{\subseteq C_1})^s \cap \dots \cap (I_{\subseteq C_r})^s.$$

The following three lemmas are useful to get the necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideals of weighted oriented paths.

Definition 2.14. A path is said to be naturally oriented, if all of its edges are oriented in one direction.

Lemma 2.15. [10, Lemma 3.8] Let D be a weighted oriented graph such that at most one edge is oriented into each vertex. Let D' be an induced weighted naturally oriented path of length 3 of D with $V(D') = \{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}, E(D') = \{(x_j, x_{j+1}) \mid i-1 \le j \le i+1\},$ $w(x_i) \ge 2$ and $w(x_{i+1}) = 1$. Then $I(D)^{(3)} \ne I(D)^3$.

Theorem 2.16. [1, Theorem 3.6] Let D be a weighted naturally oriented path with $V(D) = \{x_1, x_2, x_3, \ldots, x_n\}$ and $E(D) = \{(x_i, x_{i+1}) \mid 1 \le i \le n-1\}$. Then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$ if and only if it satisfies the condition "if $w(x_j) \ge 2$ for some 1 < j < n then $w(x_i) \ge 2$ for some $j \le i \le n-1$ ".

Lemma 2.17. [10, Corollary 4.6] Let D be a weighted oriented graph. Let D' be the weighted oriented graph obtained from D after replacing the non-trivial weights of sink vertices by trivial weights. Then $I(D)^{(s)} = I(D)^s$ if and only if $I(D')^{(s)} = I(D')^s$ for each $s \ge 1$.

Notation 2.18. Let $I \subset k[x_1, \ldots, x_n]$ be a monomial ideal. We set $\mathcal{G}(I)$ be the set of minimal generators of the ideal I. If J is a set of some elements of I, then $\langle J \rangle$ denoted as the ideal generated by the elements of J.

Notation 2.19. Let $g \in k[x_1, ..., x_n]$ be a monomial. We define support of $g = \{x_i : x_i \mid g\}$ and we denote it by supp(g).

3. Symbolic powers of weighted oriented graphs

In this section, we give one method to find all the minimal generators of $I_{\subseteq C}$, where C is any maximal strong vertex cover of a weighted oriented graph D.

Let C be a vertex cover of D. We call that a vertex $x \in L_3^D(C)$ satisfies the SVC condition on C if and only if $N_D^-(x) \cap V^+(D) \cap [L_2^D(C) \cup L_3^D(C)] \neq \phi$. By Remark 2.7, C is strong if and only if each $x \in L_3^D(C)$ satisfies the SVC condition on C.

Next we give some lemmas, which are useful to prove our main results.

Lemma 3.1. Let D be a weighted oriented graph. Let C_1 and C_2 be two vertex covers of D such that $C_1 \,\subset C_2$. If $x \in L_3^D(C_1)$ satisfies the SVC condition on C_1 , then $x \in L_3^D(C_2)$ and it satisfies the SVC condition on C_2 .

Proof. Since $x \in L_3^D(C_1)$, by Lemma 2.5, we have $x \in L_3^D(C_2)$. Given that $x \in L_3^D(C_1)$ satisfies the SVC condition on C_1 . That means there exists $y \in N_D^-(x) \cap V^+(D) \cap [L_2^D(C_1) \cup L_3^D(C_1)]$. Here $y \notin L_1^D(C_1)$ implies $N_D^+(y) \cap C_1^c = \phi$. Since $C_1 \subset C_2$, we have $N_D^+(y) \cap C_2^c = \phi$ and so $y \notin L_1^D(C_2)$. Then $y \in N_D^-(x) \cap V^+(D) \cap [L_2^D(C_2) \cup L_3^D(C_2)]$. Hence x satisfies the SVC condition on C_2 . **Lemma 3.2.** Let D be a weighted oriented graph. Let C be a vertex cover of D. Then there exists a strong vertex cover $C' \subseteq C$ of D such that there is no strong vertex cover $C'' \subseteq C$ of D with $C' \subsetneq C''$.

Proof. Let C be a vertex cover of D. Let $J \subseteq L_3^D(C)$ be the set of vertices which does not satisfy the SVC condition on C. If $J = \phi$, then we take C' = C. Now we assume $J \neq \phi$. Let $J = \{x_{l_1}, \ldots, x_{l_r}\}$. Choose one element $x_{j_1} \in J \cap L_3^D(C)$ and set $C_1 = C \setminus \{x_{j_1}\}$. By Remark 2.5, $N_D[x_{j_1}] \subseteq C$ and so C_1 is a vertex cover of D. Now, we suppose that there are vertex covers C_0, \ldots, C_k such that $C_i = C_{i-1} \setminus \{x_{j_i}\}$ and $x_{j_i} \in J \cap L_3^D(C_{i-1})$ for $i = 1, \ldots, k$, where $C_0 = C$ and we give the following recursively process: If $J \cap L_3^D(C_{i-1}) = \phi$, then we take $C' = C_{i-1}$. Since |V(D)| is finite, the process is finite. Hence there exists $m \leq r$ such that $J \cap L_3^D(C_m) = \phi$. Then we take $C' = C_m = C \setminus \{x_{j_1}, \ldots, x_{j_m}\}$, where $\{x_{j_1}, \ldots, x_{j_m}\} \subseteq J$.

Now we claim that C' is strong. Let $x_p \in L_3^D(C')$. By Lemma 2.5, $x_p \in L_3^D(C)$ because $C' \subseteq C$. Since $J \cap L_3^D(C') = \phi$, we have $x_p \notin J$. Then x_p satisfies the SVC condition on C. Thus there is some $y_p \in N_D^-(x_p) \cap V^+(D) \cap [L_2^D(C) \cup L_3^D(C)]$. Note that $y_p \notin L_1^D(C)$. Suppose $y_p \in L_1^D(C')$. That means $x_{j_i} \in N_D^+(y_p) \cap C'^c$ for some $i \in [m]$. Now $y_p \in N_D^-(x_{j_i}) \cap V^+(D) \cap [L_2^D(C) \cup L_3^D(C)]$ and so $x_{j_i} \in L_3^D(C)$ satisfies the SVC condition on C, which is a contradiction. Therefore $y_p \notin L_1^D(C')$ and $y_p \in N_D^-(x_p) \cap V^+(D) \cap [L_2^D(C') \cup L_3^D(C')]$, i.e., $x_p \in L_3^D(C')$ satisfies the SVC condition on C'. Hence C' is strong.

Suppose there exists a strong vertex cover $C'' \subseteq C$ such that $C' \not\subseteq C''$. This implies $x_{j_i} \in C''$ for some $i \in [m]$. Since $x_{j_i} \notin C'$, we have $N_D(x_{j_i}) \subset C'$. By Lemma 2.5, $x_{j_i} \in L_3^D(C'')$ and it satisfies the SVC condition on C'' because C'' is strong. By Lemma 3.1, $x_{j_i} \in L_3^D(C)$ and it satisfies the SVC condition on C, which is a contradiction. Hence the proof follows.

Lemma 3.3. Let D be a weighted oriented graph. Let C be a vertex cover of D. Then there exists a strong vertex cover $C' \subseteq C$ of D, where $x \notin L_1^D(C)$ with $w(x) \neq 1$ implies $x \notin L_1^D(C')$.

Proof. By Lemma 3.2, there exists a strong vertex cover $C' \,\subset C$ of D whose each element of $C \,\smallsetminus\, C'$ does not satisfy the SVC condition on C. We may assume that $C' = C \,\smallsetminus\, \{x_1, \ldots, x_m\}$ of D. Let $x \notin L_1^D(C)$ with $w(x) \neq 1$. Suppose $x \in L_1^D(C')$. That means $x_j \in N_D^+(x) \cap C'^c$ for some $j \in [m]$. Then $x \in N_D^-(x_j) \cap V^+(D) \cap [L_2^D(C) \cup L_3^D(C)]$ and so $x_j \in L_3^D(C)$ satisfies the SVC condition on C, which is a contradiction. Hence $x \notin L_1^D(C')$.

Corollary 3.4. Let D be a weighted oriented graph. Let C be a vertex cover of D with $x_i \notin L_3^D(C)$. Then there exists a strong vertex cover $C' \subseteq C$ of D such that $x_i \in C'$.

Proof. By Lemma 3.2, there exists a strong vertex cover $C' \subseteq C$ of D. By Lemma 2.5, $N_D(x_i) \notin C$ and so $N_D(x_i) \notin C'$. Since C' is a vertex cover of D, we have $x_i \in C'$. \Box

We are now ready for the main result of this section which describes the minimal generators of $I_{\subseteq C}$ for a maximal strong vertex cover C.

Theorem 3.5. Let D be a weighted oriented graph on the vertex set $\{x_1, \ldots, x_n\}$. Let I = I(D) and $w_i = w(x_i)$ for all $x_i \in V(D)$. Let C be a maximal strong vertex cover of D. Then $I_{\subseteq C} = (L_1^D(C)) + (x_i^{w_i} | x_i \in L_2^D(C)) + (x_i x_j^{w_j} | (x_i, x_j) \in E(D), x_i \in L_2^D(C) \cup L_3^D(C)$ and $x_j \in L_3^D(C)$).

Proof. Suppose $x_i \in L_1^D(C)$. Then for any strong vertex cover $C' \subseteq C$, $N_D^+(x_i) \cap C'^c \neq \phi$ and so $x_i \in L_1^D(C')$. Thus $L_1^D(C) \subseteq L_1^D(C')$ and hence $(L_1^D(C)) \subseteq I_{\subseteq C}$. Note that each element of $L_1^D(C)$ is a minimal generator of $I_{\subseteq C}$.

Suppose $x_i \in L_2^D(C)$. Then for any strong vertex cover $C' \subseteq C$, $N_D^-(x_i) \cap {C'}^c \neq \phi$ and so $x_i \in L_1^D(C') \cup L_2^D(C')$. Thus $L_2^D(C) \subseteq L_1^D(C') \cup L_2^D(C')$ and hence $(x_i^{w_i} \mid x_i \in L_2^D(C)) \subseteq I_{\subseteq C}$. Notice that that each element of the set $\{x_i^{w_i} \mid x_i \in L_2^D(C)\}$ is a minimal generator of $I_{\subseteq C}$.

Suppose $(x_i, x_j) \in E(D)$ where $x_j \in L_3^D(C)$. By Lemma 2.5, $N_D[x_j] \subseteq C$ and so $C_1 = C \setminus \{x_j\}$ is a vertex cover of D. By Lemma 2.5, $x_i \notin L_3^D(C_1)$ and hence by Corollary 3.4, there exists a strong vertex cover $C' \subseteq C_1$ such that $x_i \in C'$. Here $x_j \in N_D^+(x_i) \cap C'^c$. Thus $x_i \in L_1^D(C')$ and so $x_i \in \mathcal{G}(I_{C'})$. Note that $x_i \notin I_C$, $x_j^{w_j} \in \mathcal{G}(I_C)$ and $x_j^{w_j} \notin I_{C'}$.

Then $x_i x_j^{w_j} \in \mathcal{G}(I_C \cap I_{C'})$. By Lemma 2.10, for any strong vertex cover $C'' \subseteq C$, $x_i x_j^{w_j} \in I_{C''}$ and hence $x_i x_j^{w_j} \in \mathcal{G}(I_{\subseteq C})$. If $w_j \neq 1$, we have $x_i x_j \notin I_{\subseteq C}$ and so $x_i x_j^{w_j}$ is the only minimal generator of $I_{\subseteq C}$, which involves both x_i and x_j . If $w_j = 1$, $x_i x_j$ is the only minimal generator of $I_{\subseteq C}$, which involves both x_i and x_j .

Suppose $x_i \in L_3^D(C)$. Suppose $(x_i, x_j) \in E(D)$ where $x_j \in L_3^D(C)$. By the previous argument, $C_1 = C \setminus \{x_j\}$ and $C_2 = C \setminus \{x_i\}$ are vertex covers of D. By Lemma 2.5, $x_i \notin L_3^D(C_1)$. Thus by Corollary 3.4, there exists a strong vertex cover $C' \subseteq C_1$ such that $x_i \in C'$. Here $x_j \in N_D^+(x_i) \cap C'^c$ and so $x_i \in L_1^D(C')$. Thus $x_i \in \mathcal{G}(I_{C'})$. By Lemma 2.5, $N_D(x_j) \subset C$. Since $(x_i, x_j) \in E(D)$, we have $N_D^+(x_j) \cap C_2^c = \phi$ and $x_i \in N_D^-(x_j) \cap C_2^c$. Then $x_j \in L_2^D(C_2)$. Here $x_j \notin L_1^D(C_2) \cup L_3^D(C_2)$. By Corollary 3.4, there exists a strong vertex cover $C'' \subseteq C_2$ such that $x_j \in C''$. If $w_j = 1$, then $x_j \in \mathcal{G}(I_{C''})$. If $w_j \neq 1$, by Lemma 3.3, we get $x_j \notin L_1^D(C'')$ and so $x_j^{w_j} \notin \mathcal{G}(I_{C''})$. In both cases $x_j^{w_j} \in \mathcal{G}(I_{C''})$. Note that $x_i \in \mathcal{G}(I_{C'}), x_i \notin I_{C''}$ and $x_j^{w_j} \notin I_{C'}$. Then $x_i x_j^{w_j} \notin \mathcal{G}(I_{C'} \cap I_{C''})$. By Lemma 2.10, for any strong vertex cover $C'' \subseteq C$, which involves both x_i and x_j .

Hence $(L_1^D(C)) + (x_i^{w_i} | x_i \in L_2^D(C)) + (x_i x_j^{w_j} | (x_i, x_j) \in E(D), x_i \in L_2^D(C) \cup L_3^D(C) \text{ and } x_j \in L_3^D(C)) \subseteq I_{\subseteq C}.$

To complete the proof, it is enough to prove the following two statements:

(1) There is no minimal generator of $I_{\subseteq C}$, which involves more than two vertices.

(2) There is no minimal generator of $I_{\subseteq C}$, which involves two non-adjacent vertices.

(1) Suppose there exists a minimal generator f of $I_{\subseteq C}$, which involves more than two vertices. Since f is minimal, we can assume that no element of $supp(f) \in L_1^D(C)$ and no element of supp(f) with trivial weight $\in L_2^D(C)$. Let $f = x_1^{a_1} \cdots x_r^{a_r} y_1^{b_1} \cdots y_s^{b_s} z_1^{c_1} \cdots z_t^{c_t}$ where $\{x_1, \ldots, x_r\} \subseteq L_2^D(C), \{y_1, \ldots, y_s, z_1, \ldots, z_t\} \subseteq L_3^D(C), a_i = 1 \text{ or } w_{x_i} \text{ for } 1 \le i \le r, b_i = 1 \text{ or } w_{x_i}$ w_{y_i} for $1 \le i \le s$, $c_i = 1$ or w_{z_i} for $1 \le i \le t$ and $r+s+t \ge 3$. Without loss of generality we can assume that $b_i = 1$ with $w_{y_i} \neq 1$ for $1 \leq i \leq s$ and $c_i = w_{z_i}$ for $1 \leq i \leq t$. By our assumption $w_{x_i} \neq 1$ and so $x_i^{w_{x_i}} \in \mathcal{G}(I_{\subseteq C})$ for $1 \leq i \leq r$. Since f is minimal, $a_i = 1$ for $1 \leq i \leq r$. Hence $f = x_1 \cdots x_r y_1 \cdots y_s z_1^{w_{z_1}} \cdots z_t^{w_{z_t}}$. If $t = 0, f \notin I_C$, which is a contradiction. Now we assume $t \neq 0$. If two z_i 's (say z_k and z_l) are adjacent, $z_k z_l^{w_{z_l}}$ or $z_l z_k^{w_{z_k}} \in \mathcal{G}(I_{\subseteq C})$ and so f is not minimal. Therefore no two z_i 's are adjacent. By Lemma 2.5, $N_D[z_i] \subset C$ for $1 \leq i \leq t$. Thus $C_1 = C \setminus \{z_1, \ldots, z_t\}$ is a vertex cover of D. If $x_i \in L_1^D(C_1)$ for some $i \in [r]$, that means there exists some $z_i \in N_D^+(x_i) \cap C_1^{c}$. Since $(x_i, z_j) \in E(D), x_i z_i^{w_j} \in \mathcal{G}(I_{\subseteq C})$ and it contradicts the fact that, f is minimal. By the similar argument, if $y_i \in L_1^D(C_1)$ for some $i \in [s]$, we get a contradiction. Therefore each of x_i and $y_i \notin L_1^D(C_1)$. By Lemma 3.3, there exists a strong vertex cover $C' \subseteq C_1$, where each of x_i and $y_i \notin L_1^D(C')$. So $I_{C'} = (x_1^{w_{x_1}}, \dots, x_r^{w_{x_r}}, y_1^{w_{y_1}}, \dots, y_s^{w_{y_s}}, \dots).$ Note that $f \notin I_{C'}$, which is a contradiction. Hence there does not exist any minimal generator of $I_{\subseteq C}$, which involves more than two vertices.

(2) By the similar argument as in (1), we can show that there is no minimal generator of $I_{\subseteq C}$, which involves two non-adjacent vertices of D.

Next we see some applications of the above theorem.

Definition 3.6. A rooted tree is an oriented tree in which all edges are oriented away from the root.

Example 3.7. Consider the weighted rooted tree D with degree of root is 1, as in Figure 1. Let I = I(D). Note that $C = V(D) \setminus \{x_1\}$ is a vertex cover of D. Here $L_1^D(C) = \{x_0\}, L_2^D(C) = \{x_2, x_3\}$ and $L_3^D(C) = V(D) \setminus \{x_0, x_1, x_2, x_3\}$. By the definition of D, we can see that each element of $L_3^D(C)$ satisfies the SVC condition on C and so Cis strong in D. Notice that V(D) is a vertex cover of D and $L_3^D(V(D)) = V(D)$. Since $N_D^-(x_0) = \phi, x_0$ does not satisfy the SVC condition on V(D). Therefore V(D) is not strong in D. Hence C is maximal. Let $D_1 = D \setminus \{x_0, x_1\}$. Then by Theorem 3.5, we have $I_{\subseteq C} = (x_0, x_2^{w_2}, x_3^{w_3}) + I(D_1)$.

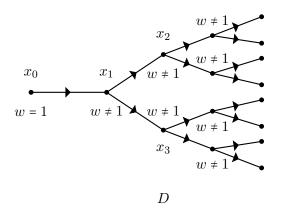


FIGURE 1. A weighted rooted tree D

Remark 3.8. In a weighted oriented graph, if we know all the maximal strong vertex covers, then by the Theorem 3.5 and Lemma 2.13, we can find the symbolic powers of its edge ideal.

4. Symbolic powers of induced weighted oriented graphs

In this section, we see that, by studying the symbolic powers of edge ideal of an induced digraph of a weighted oriented graph D, we can get information about the symbolic powers of edge ideal of D.

By [11, Corollary 2.7], if H be an induced subgraph of a simple graph G, then $I(H)^{(s)} \neq I(H)^s$ for some $s \geq 2$, implies $I(G)^{(s)} \neq I(G)^s$. In general, this property may not hold for weighted oriented graphs. But under certain condition on the strong vertex covers, we extend this result for weighted oriented graphs in Theorem 4.3.

Remark 4.1. Let D' is an induced digraph of D. If one source vertex of D' is not source in D, then $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$, may not imply $I(D)^{(s)} \neq I(D)^s$. For example consider the weighted oriented paths D and D' as in Figure 2. Then $I(D) = (x_1x_2^3, x_2x_3, x_3x_4, x_4^3x_5)$ and $I(D') = (x_1x_2^3, x_2x_3, x_3x_4, x_4^3x_5)$ and $I(D') = (x_1x_2^3, x_2x_3, x_3x_4)$. Note that D' is an induced path of D. Here $w(x_4) = 1$ in D' but $w(x_4) = 3$ in D. Using Macaulay 2, we see that $I(D')^{(2)} \neq I(D')^2$, but $I(D)^{(2)} = I(D)^2$.

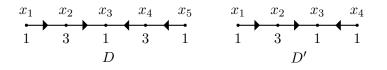


FIGURE 2. A weighted oriented path D containing an induced weighted oriented path D'.

If all source vertices of D' are source in D, we may have $I(D')^{(s)} \neq I(D')^s$, but $I(D)^{(s)} = I(D)^s$ for some s > 1. For example consider the weighted oriented paths D and D' as in Figure 3. Then $I(D) = (x_1x_2^2, x_2x_3, x_3x_4^2, x_5x_4^2, x_6x_5^2)$ and $I(D') = (x_1x_2^2, x_2x_3, x_3x_4^2)$. Note that D' is an induced path of D. Using Macaulay 2, we see that $I(D')^{(2)} \neq I(D')^2$ and $I(D)^{(2)} = I(D)^2$.

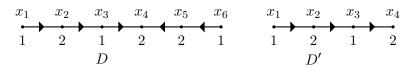


FIGURE 3. A weighted naturally oriented path D containing an induced weighted oriented path D'.

If D' is an induced digraph of D and C is a maximal strong vertex cover of D, we see that $C \cap V(D')$ may not contain any maximal strong vertex cover of D' in the next example.

Example 4.2. Consider the weighted oriented paths D and D' as in Figure 4. Note that D' is an induced path of D. Using Macaulay 2, the strong vertex covers of D are $\{x_1, x_3, x_5\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}, \{x_2, x_4, x_6\}, \{x_1, x_3, x_4, x_6\}, \{x_2, x_3, x_4, x_6\}$ and the strong vertex covers of D' are $\{x_2\}, \{x_1, x_3\}, \{x_2, x_3\}$. Note that $C = \{x_2, x_4, x_5\}$ is a maximal strong vertex cover of D, but $C \cap V(D') = \{x_2\}$ does not contain any maximal strong vertex cover of D'.

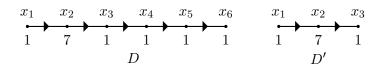


FIGURE 4. A weighted naturally oriented path D containing an induced weighted oriented path D'.

Theorem 4.3. Let D be a weighted oriented graph. Let D' be an induced digraph of D and it satisfies the condition "if C is a maximal strong vertex cover of D, then every strong vertex cover of D' contained in C, is subset of at most one maximal strong vertex cover of D' contained in C". If $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$, then $I(D)^{(s)} \neq I(D)^s$.

Proof. Let I = I(D) and $\tilde{I} = I(D')$. Since $I(D')^s$ is the restriction of $I(D)^s$ to D', it suffices to show that $\tilde{I}^{(s)} \subseteq I^{(s)}$. Equivalently, by Lemma 2.13, it suffices to show that

"if f is a minimal generator of $\tilde{I}^{(s)}$, then $f \in I^s_{\subseteq C}$ for each maximal strong vertex cover C of D."

Let C be a maximal strong vertex cover of D. If C contains two maximal strong vertex covers of D', then those two maximal strong vertex covers can not be subset of one maximal strong vertex cover of D' contained in C, by our assumption. Therefore the proof follows from the following two cases.

Case-I: C contains exactly one maximal strong vertex cover of D'.

Let \tilde{C} be the maximal strong vertex cover of D' contained in C. First we claim that $\tilde{I}_{\subseteq \tilde{C}} \subseteq I_{\subseteq C}$.

Case (1) Let $x_i \in L_1^{D'}(\tilde{C})$.

Case (1.a) Assume that $w_i = 1$ in D'.

Case (1.a.i) Assume $w_i = 1$ in D. We claim $x_i \in L_1^D(C) \cup L_2^D(C)$. Suppose $x_i \in L_3^D(C)$. Since $x_i \in L_1^{D'}(\tilde{C})$, $N_{D'}^+(x_i) \cap \tilde{C}^c \neq \phi$ and so $N_{D'}(x_i) \cap \tilde{C}^c \neq \phi$. Let $N_{D'}(x_i) \cap \tilde{C}^c = \{x_{j_1}, x_{j_2}, \dots, x_{j_q}\}$. Note that $N_{D'}(x_i) \subseteq N_D(x_i)$ and by Lemma 2.5, $N_D(x_i) \subset C$. Then $\{x_{j_1}, x_{j_2}, \dots, x_{j_q}\} \subset C$. Let $C_1 = \tilde{C} \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_q}\}$. Since $N_{D'}[x_i] \subset C_1$, $C_1 \setminus \{x_i\}$ is a vertex cover of D'. Observe that $x_{j_1} \notin L_3^{D'}(C_1 \setminus \{x_i\})$ and hence by Corollary 3.4, there exists a strong vertex cover $\tilde{\tilde{C}} \subseteq [C_1 \setminus \{x_i\}]$ of D' such that $x_{j_1} \in \tilde{\tilde{C}}$. Here $\tilde{\tilde{C}} \subset C$, $x_i \notin \tilde{\tilde{C}}$, $x_{j_1} \in \tilde{\tilde{C}}$ and $x_{j_1} \notin \tilde{C}$. Since \tilde{C} is maximal, \tilde{C} and $\tilde{\tilde{C}}$ can not be subsets of at most one maximal strong vertex cover of D' contained in C. So it contradicts our assumption. Thus the claim follows.

Case (1.a.ii) Assume $w_i \neq 1$ in D. That means $N_D^-(x_i) \neq \phi$. Since $w_i = 1$ in D', $\overline{N_{D'}^-(x_i)} = \phi$. We claim $x_i \in L_1^D(C)$. Suppose $x_i \in L_2^D(C) \cup L_3^D(C)$. Here $N_{D'}^+(x_i) \cap \tilde{C}^c \neq \phi$. Let $N_{D'}^+(x_i) \cap \tilde{C}^c = \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$ and we know $N_{D'}^-(x_i) = \phi$. Note that $N_{D'}^+(x_i) \subseteq N_D^+(x_i)$, and $N_D^+(x_i) \subset C$ because $x_i \notin L_1^D(C)$. Then $\{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} \subset C$. Let $C_1 = \tilde{C} \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$. Since $N_{D'}[x_i] \subset C_1, C_1 \setminus \{x_i\}$ is a vertex cover of D'. By the similar argument as in Case (1.a.i), every strong vertex cover of D' contained in C, which contradicts our assumption. Thus the claim follows.

Case (1.b) Assume that $w_i \neq 1$ in D'. We claim $x_i \in L_1^D(C)$. Suppose $x_i \in L_2^D(C) \cup L_3^D(C)$. Here $N_{D'}^+(x_i) \cap \tilde{C}^c \neq \phi$. Let $N_{D'}^+(x_i) \cap \tilde{C}^c = \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$. Note that $\{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} \subset C$ because $x_i \notin L_1^D(C)$. Let $C_1 = \tilde{C} \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$. Then $x_i \notin L_1^{D'}(C_1)$ and by Lemma 2.5, we have $L_3^{D'}(\tilde{C}) \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} \subseteq L_3^{D'}(C_1)$. Let $x_l \in L_3^{D'}(\tilde{C})$. Then x_l satisfies SVC condition on \tilde{C} because \tilde{C} is strong. By Lemma 3.1, $x_l \in L_3^{D'}(C_1)$ and x_l satisfies SVC condition on C_1 . Since $x_i \in N_{D'}^{-}(x_{j_l}) \cap V^+(D') \cap [L_2^{D'}(C_1) \cup L_3^{D'}(C_1)]$, x_{j_t} satisfies SVC condition on C_1 for $1 \leq t \leq r$. If $L_3^{D'}(\tilde{C}) \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} = L_3^{D'}(C_1)$, then each element of $L_3^{D'}(C_1)$ satisfies SVC condition on C_1 . So C_1 is a strong vertex cover of D'. But it contradicts the fact that \tilde{C} is one maximal

strong vertex cover of D'. Now we assume that $L_3^{D'}(\tilde{C}) \cup \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\} \not\subseteq L_3^{D'}(C_1)$. Let $x_k \in L_3^{D'}(C_1) \setminus [L_3^{D'}(\tilde{C}) \cup \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}]$. That means x_k lies in the neighbourhood of x_{j_t} for some $t \in [r]$. Without loss of generality let $x_k \in N_{D'}(x_{j_1})$. By Lemma 2.5, $C_1 \setminus \{x_k\}$ is a vertex cover of D'. Then $x_{j_1} \notin L_3^{D'}(C_1 \setminus \{x_k\})$ and so by Corollary 3.4, there exists a strong vertex cover $\tilde{\tilde{C}} \subseteq [C_1 \setminus \{x_k\}]$ of D' such that $x_{j_1} \in \tilde{\tilde{C}}$. Here $x_k \in \tilde{C}, x_k \notin \tilde{\tilde{C}}, x_{j_1} \in \tilde{\tilde{C}}$ and $x_{j_1} \notin \tilde{C}$. Then by the same argument as in Case (1.a.i), it contradicts our assumption. Thus the claim follows.

Case (2) Let $x_i \in L_2^{D'}(\tilde{C})$.

We claim $x_i \in L_1^D(C) \cup L_2^D(C)$. Suppose $x_i \in L_3^D(C)$. Since $x_i \in L_2^{D'}(\tilde{C})$, we have $N_{D'}^+(x_i) \cap \tilde{C}^c = \phi$ and $N_{D'}^-(x_i) \cap \tilde{C}^c \neq \phi$. Let $N_{D'}^-(x_i) \cap \tilde{C}^c = \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$. Note that $\{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} \subset C$ because $x_i \in L_3^D(C)$. Let $C_1 = \tilde{C} \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$. Since $N_{D'}[x_i] \subset C_1, C_1 \setminus \{x_i\}$ is a vertex cover of D'. By the similar argument as in Case (1.a.i), we get a contradiction. Thus $x_i \in L_1^D(C) \cup L_2^D(C)$.

Case (3) Let $x_i \in L_3^{D'}(\tilde{C})$. Then $x_i \in L_1^D(C) \cup L_2^D(C) \cup L_3^D(C)$. Here $\tilde{I}_{\subseteq \tilde{C}} = (L_1^{D'}(\tilde{C})) + (x_i^{w_i} | x_i \in L_2^{D'}(\tilde{C})) + (x_i x_j^{w_j} | (x_i, x_j) \in E(D'), x_i \in L_2^{D'}(\tilde{C}) \cup L_3^{D'}(\tilde{C})$ and $x_j \in L_3^{D'}(\tilde{C})$) and $I_{\subseteq C} = (L_1^D(C)) + (x_i^{w_i} | x_i \in L_2^D(C)) + (x_i x_j^{w_j} | (x_i, x_j) \in E(D), x_i \in L_2^D(C) \cup L_3^D(C)$ and $x_j \in L_3^D(C)$). Hence $\tilde{I}_{\subseteq \tilde{C}} \subseteq I_{\subseteq C}$.

If $f \in \mathcal{G}(\tilde{I}^{(s)})$, by Lemma 2.13, we have $f \in \tilde{I}^{s}_{\subset \tilde{C}}$, and so $f \in I^{s}_{\subseteq C}$.

Case-II: C contains no maximal strong vertex cover of D'.

Since D' is an induced digraph of $D, C \cap V(D')$ is a vertex cover of D'. By Lemma 3.2, there exists a strong vertex cover $C' \subseteq [C \cap V(D')]$ of D'. By our assumption, C' is not a maximal strong vertex cover of D'. Thus there exists a maximal strong vertex cover \tilde{C} of D' such that $C' \not\subseteq \tilde{C}$. Again by our assumption, $\tilde{C} \notin C$. We claim $\tilde{I}_{\subset \tilde{C}} \subseteq I_{\subseteq C}$.

Consider one element $x \in \tilde{C} \setminus C$. Since $x \notin C$, we have $x \notin C'$. This implies $N_{D'}(x) \subset C'$ and so $N_{D'}(x) \subset \tilde{C}$ because $C' \subset \tilde{C}$. Thus by Lemma 2.5, $x \in L_3^{D'}(\tilde{C})$. Hence if $x \in L_1^{D'}(\tilde{C}) \cup L_2^{D'}(\tilde{C})$, then $x \in C$.

Case (1) Let $x_i \in L_1^{D'}(\tilde{C})$. Then $x_i \in C$.

Case (1.a) Assume that $w_i = 1$ in D'.

Case (1.a.i) Assume $w_i = 1$ in D. We claim $x_i \in L_1^D(C) \cup L_2^D(C)$. Suppose $x_i \in L_3^D(C)$. Since $C' \subset \tilde{C}$, $x_i \in L_1^{D'}(\tilde{C})$ implies $x_i \in L_1^{D'}(C')$. Note that $N_{D'}(x_i) \cap \tilde{C}^c \neq \phi$ and $N_{D'}(x_i) \cap \tilde{C}^c \subseteq N_{D'}(x_i) \cap C'^c$. Let $N_{D'}(x_i) \cap C'^c = \{x_{j_1}, x_{j_2}, \dots, x_{j_q}\}$. Without loss of generality we can assume that $x_{j_1} \in N_{D'}(x_i) \cap \tilde{C}^c$. Since $x_i \in L_3^D(C)$, we have $\{x_{j_1}, x_{j_2}, \dots, x_{j_q}\} \subset C$. Let $C_1 = C' \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_q}\}$. Since $N_{D'}[x_i] \subset C_1, C_1 \setminus \{x_i\}$ is a vertex cover of D'. Observe that $x_{j_1} \notin L_3^{D'}(C_1 \setminus \{x_i\})$ and hence by Corollary 3.4, there exists a strong vertex cover $\tilde{\tilde{C}} \subseteq [C_1 \setminus \{x_i\}]$ of D' such that $x_{j_1} \in \tilde{\tilde{C}}$. Here $C' \subset \tilde{C}$, $\tilde{\tilde{C}} \subset C, x_i \in C', x_i \notin \tilde{\tilde{C}}, x_{j_1} \in \tilde{\tilde{C}}$ and $x_{j_1} \notin C'$. If C' and $\tilde{\tilde{C}}$ are subset of one maximal

strong vertex cover C'' of D' contained in C, then $C' \notin C''$ and it is a contradiction by Lemma 3.2. Thus the claim follows.

Case (1.a.ii) Assume $w_i \neq 1$ in D. We claim $x_i \in L_1^D(C)$. Suppose $x_i \in L_2^D(C) \cup L_3^D(C)$. Note that $x_i \in L_1^{D'}(C')$. Then by the similar argument as in (1.a.i) of Case-II and Case (1.a.ii) of Case-I, our claim follows.

Case (1.b) Assume that $w_i \neq 1$ in D'. We claim $x_i \in L_1^D(C)$. Suppose $x_i \in L_2^D(C) \cup L_3^D(C)$. Note that $x_i \in L_1^{D'}(C')$. Then by the similar argument as in (1.a.i) of Case-II and Case (1.b) of Case-I, our claim follows.

Case (2) Let $x_i \in L_2^{D'}(\tilde{C})$. Then $x_i \in C$.

We claim $x_i \in L_1^D(C) \cup L_2^D(C)$. Suppose $x_i \in L_3^D(C)$. Since $C' \subset \tilde{C}$, $x_i \in L_1^{D'}(C') \cup L_2^{D'}(C')$. If $x_i \in L_1^{D'}(C')$, by the similar argument as in Case (1.a.i) of Case-II, we get a contradiction. If $x_i \in L_2^{D'}(C')$, by the similar argument as in Case (1.a.i) of Case-II and Case (2) of Case-I, we get a contradiction. Thus the claim follows.

Case (3) Let $x_i \in L_3^{D'}(\tilde{C})$. Then $x_i \in L_1^D(C) \cup L_2^D(C) \cup L_3^D(C)$. By the similar argument as in Case-I, we have $\tilde{I}_{\subset \tilde{C}} \subseteq I_{\subseteq C}$.

If $f \in \mathcal{G}(\tilde{I}^{(s)})$, by Lemma 2.13, we have $f \in \tilde{I}^{s}_{\subset \tilde{C}}$, and so $f \in I^{s}_{\subseteq C}$.

Remark 4.4. The above theorem may not be true if we remove the given condition.

For example consider the weighted oriented paths D and D' as in Figure 4. Here $I(D) = (x_1x_2^3, x_2x_3, x_3x_4, x_4^3x_5)$ and $I(D') = (x_1x_2^3, x_2x_3, x_3x_4)$. Using Macaulay 2, the strong vertex covers of D are $\{x_2, x_4\}$, $\{x_1, x_3, x_4\}$, $\{x_1, x_3, x_5\}$, $\{x_2, x_3, x_4\}$, $\{x_2, x_3, x_5\}$ and the strong vertex covers of D' are $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_2, x_4\}$. Let $C = \{x_2, x_3, x_4\}$. Note that C contains the two maximal strong vertex covers $\{x_2, x_4\}$ and $\{x_2, x_4\}$ of D'. Those two maximal strong vertex covers are not subset of one strong vertex cover of D' contained in C. Using Macaulay 2, we see that $I(D')^{(2)} \neq I(D')^2$ but $I(D)^{(2)} = I(D)^2$.

Next we see some applications of Theorem 4.3 for induced weighted oriented paths.

Theorem 4.5. Let D be a weighted oriented path with $V(D) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$. Let D' be the induced weighted oriented path of D with $V(D') = \{x_1, x_2, \ldots, x_n\}$ and $N_D^-(x_{n-1}) \cap V^+(D) = \phi$. If $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$, then $I(D)^{(s)} \neq I(D)^s$.

Proof. Let C be a maximal strong vertex cover of D. We claim every strong vertex cover of D' contained in C, is subset of at most one maximal strong vertex cover of D' contained in C. Since $N_D^-(x_{n-1}) \cap V^+(D) = \phi$, by Remark 2.7, we have $x_{n-1} \notin L_3^D(C)$. Then by Lemma 2.5, one of x_{n-2} , x_{n-1} and $x_n \notin C$. Therefore we consider the following three cases.

Case (1) Assume that $x_n \notin C$. We claim $C_1 = C \cap V(D')$ is a strong vertex cover of D'. Note that C_1 is a vertex cover of D'. Since $x_n \notin C_1$, $x_{n-1} \in C_1$. Let $x_i \in L_3^{D'}(C_1)$.

Then by Lemma 2.5, $x_i \in \{x_1, x_2, \ldots, x_{n-2}\}$ and $x_i \in L_3^D(C)$. Since C is strong in D, there exists some $x_j \in N_D^-(x_i) \cap V^+(D) \cap [L_2^D(C) \cup L_3^D(C)]$. Note that orientations of edges from x_1 to x_n and weights of vertices from x_1 to x_{n-1} are same in both the paths D and D'. Here $x_j \notin L_1^D(C)$ implies $N_D^+(x_j) \cap C^c = \phi$ and $x_j \in \{x_1, x_2, \ldots, x_{n-1}\}$. Since $C_1 = C \cap V(D')$, we have $N_{D'}^+(x_j) \cap C_1^c = N_D^+(x_j) \cap C^c = \phi$. Thus $x_j \notin L_1^{D'}(C_1)$. So $x_j \in N_{D'}^-(x_i) \cap V^+(D') \cap [L_2^{D'}(C_1) \cup L_3^{D'}(C_1)]$. Hence C_1 is a strong vertex cover of D'and it contains each strong vertex cover of D' contained in C. Therefore every strong vertex cover of D' contained in C, is subset of one strong vertex cover of D' contained in C.

Case (2) Assume that $x_{n-1} \notin C$. We claim $C_1 = C \cap V(D')$ is a strong vertex cover of D'. Note that C_1 is a vertex cover of D'. Here $x_{n-1} \notin C_1$. This implies x_{n-2} and $x_n \in C_1$. Let $x_i \in L_3^{D'}(C_1)$. By Lemma 2.5, $x_i \in \{x_1, x_2, \ldots, x_{n-3}\}$. Then by the same argument as in Case (1), our claim follows and every strong vertex cover of D' contained in C, is subset of one strong vertex cover of D' contained in C.

Case (3) Assume that $x_{n-2} \notin C$. If $x_n \notin C$, then by Case (1), every strong vertex cover of D' contained in C, is subset of one strong vertex cover of D' contained in C. Now we assume $x_n \in C$.

Case (3.a) Suppose $N_{D'}^{-}(x_n) \cap V^{+}(D') \cap [L_2^{D'}(C_1) \cup L_3^{D'}(C_1)] \neq \phi$. We claim $C_1 = C \cap V(D')$ is a strong vertex cover of D'. Note that C_1 is a vertex cover of D'. Here $x_{n-2} \notin C_1$. This implies x_{n-3} and $x_{n-1} \in C_1$. Let $x_i \in L_3^{D'}(C_1)$. By Lemma 2.5, $x_i \in \{x_1, x_2, \ldots, x_{n-4}, x_n\}$. If $x_i \in \{x_1, x_2, \ldots, x_{n-4}\}$, then by the similar argument as in Case (1), x_i satisfies SVC condition on C_1 . Also $x_n \in L_3^{D'}(C_1)$ satisfies SVC condition on C_1 , by our assumption. Hence C_1 is strong in D' and every strong vertex cover of D' contained in C.

Case (3.b) Suppose $N_{D'}^{-}(x_n) \cap V^{+}(D') \cap [L_2^{D'}(C_1) \cup L_3^{D'}(C_1)] = \phi$. Let C' is a strong vertex cover of D' contained in C. Suppose $x_n \in C'$. Since $x_{n-2} \notin C'$, we have $x_{n-1} \in C'$ and so by Lemma 2.5, $x_n \in L_3^{D'}(C')$. Since C' is strong, x_n satisfies SVC condition on C'. By Lemma 3.1, $x_n \in L_3^{D'}(C_1)$ and it satisfies SVC condition on C_1 , i.e., $N_{D'}^{-}(x_n) \cap V^{+}(D') \cap [L_2^{D'}(C_1) \cup L_3^{D'}(C_1)] \neq \phi$. But it contradicts our assumption. Therefore $x_n \notin C'$. Hence we can say that x_n does not belong to any strong vertex cover of D' contained in C. Let $C_2 = [C \cap V(D')] \setminus \{x_n\}$. We claim that C_2 is a strong vertex cover of D'. Since $[C \cap V(D')]$ is a vertex cover of D' and $x_{n-2} \notin [C \cap V(D')]$, we have $x_{n-1} \in [C \cap V(D')]$ and so C_2 is a vertex cover of D'. Here $x_{n-2} \notin C_2$ implies x_{n-3} and $x_{n-1} \in C_2$. Let $x_i \in L_3^{D'}(C_2)$. By Lemma 2.5, $x_i \in \{x_1, x_2, \dots, x_{n-4}\}$. By the similar argument as in Case (1), we can show C_2 is strong and every strong vertex cover of D' contained in C, is subset of one strong vertex cover of D' contained in C.

In all the cases, our claim follows and the proof follows from Theorem 4.3.

Corollary 4.6. Let *D* be a weighted oriented path with $V(D) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$. Let *D'* be the induced weighted oriented path of *D* with $V(D') = \{y_1, y_2, \ldots, y_m\}$ and $N_D^-(y_2) \cap V^+(D) = \phi$. If $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$, then $I(D)^{(s)} \neq I(D)^s$.

Proof. It follows by the similar argument as in Theorem 4.5.

Corollary 4.7. Let *D* be a weighted oriented path with $V(D) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_l\}$. Let *D'* be the induced weighted oriented path of *D* with $V(D') = \{y_1, y_2, \ldots, y_m\}$, where $N_D^-(y_2) \cap V^+(D) = \phi$ and $N_D^-(y_{m-1}) \cap V^+(D) = \phi$. If $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$, then $I(D)^{(s)} \neq I(D)^s$.

Proof. Let D_1 be the induced weighted oriented path of D with $V(D_1) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$. Here D' is an induced weighted oriented path of D_1 . Assume that $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$. Since $N_D^-(y_2) \cap V^+(D) = \phi$, by Corollary 4.6, $I(D_1)^{(s)} \neq I(D_1)^s$. Note that $N_D^-(y_{m-1}) \cap V^+(D) = \phi$. Then by Theorem 4.5, $I(D_1)^{(s)} \neq I(D_1)^s$ implies $I(D)^{(s)} \neq I(D)^s$.

Remark 4.8. When we try to find the necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideals of weighted oriented paths, using the above results, we can show the inequality of ordinary and symbolic powers of edge ideals of a larger class of weighted oriented paths by studying the inequality of ordinary and symbolic powers of edge ideals of a smaller class of weighted oriented paths.

For example consider the weighted oriented path D' as in Figure 5. Then $I(D') = (x_1 x_2^{w_2}, x_2 x_3, x_3 x_4, x_4 x_5)$. By Lemma 2.15, $I(D')^{(3)} \neq I(D')^3$. Now consider the following three class of paths:

Class (1): Set of all weighted oriented paths on the vertex set $\{x_1, x_2, \ldots, x_5, y_1, y_2, \ldots, y_m\}$ containing the induced weighted oriented path D' (as in Figure 5),

Class (2): Set of all weighted oriented paths on the vertex set $\{z_l, \ldots, z_2, z_1, x_1, x_2, \ldots, x_5\}$ containing the induced weighted oriented path D' (as in Figure 5),

Class (3): Set of all weighted oriented paths on the vertex set $\{z_l, \ldots, z_2, z_1, x_1, x_2, \ldots, x_5, y_1, y_2, \ldots, y_m\}$ containing the induced weighted oriented path D' (as in Figure 5),

and in Figure 5, where the directions are not mentioned, it can be any direction.

Let D_1, D_2 and D_3 be any weighted oriented paths of class (1), (2) and (3), respectively. By Theorem 4.5, Corollary 4.6 and Corollary 4.7, $I(D')^{(3)} \neq I(D')^3$ implies $I(D_1)^{(3)} \neq I(D_2)^{(3)} \neq I(D_2)^3$ and $I(D_3)^{(3)} \neq I(D_3)^3$, respectively.

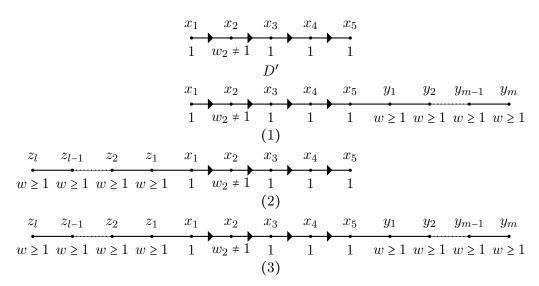


FIGURE 5. Three classes of weighted oriented paths with common induced weighted oriented path D'.

5. Symbolic powers of union of two naturally oriented paths with a common sink vertex

In this section, we give the necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideal of union of two naturally oriented paths with a common sink vertex.

Notation 5.1. Let D be a weighted oriented path with $V(D) = \{y_1, y_2, \dots, y_m, z_1, x_n, \dots, x_2, x_1\}$, $E(D) = \{(y_i, y_{i+1}) \mid 1 \le i \le m-1\} \cup \{(y_m, z_1), (x_n, z_1)\} \cup \{(x_i, x_{i+1}) \mid 1 \le i \le n-1\}$ and $w(z_1) = 1$ (see Figure 6).

 $y_1 \quad y_2 \quad y_l \quad y_{l+1} \quad y_{m-1} \quad y_m \quad z_1 \quad x_n \quad x_{n-1} \quad x_{k+1} \quad x_k \quad x_2 \quad x_1$

FIGURE 6. An oriented path which is union of two naturally oriented paths joined at a sink vertex.

The following lemma is useful for the proof of Theorem 5.3.

Lemma 5.2. Let D be the weighted oriented path same as defined in Notation 5.1. Assume that there exist two indices 1 < l < m and 1 < k < n such that the set of vertices with non-trivial weights are precisely $\{y_l, \ldots, y_m\} \cup \{x_k, \ldots, x_n\}$. Let D_1 be the induced path of D with $V(D_1) = \{x_1, x_2, \ldots, x_n\}$ and $E(D_1) = \{(x_i, x_{i+1}) \mid 1 \le i \le n-1\}$. If C is a maximal strong vertex cover of D, then prove that $C \cap V(D_1)$ is a maximal strong vertex cover of D_1 . *Proof.* Let C be a maximal strong vertex cover of D.

Suppose $z_1 \notin C$. We claim $C' = C \cup \{z_1\}$ is strong. Since $z_1 \notin C$, $N_D(z_1) \subset C$. So by Lemma 2.5, $z_1 \in L_3^D(C')$. Here $y_m \in N_D^-(z_1) \cap V^+(D) \cap [L_2^D(C') \cup L_3^D(C')]$. So z_1 satisfies the SVC condition on C'. If $y_m \in L_3^D(C')$, then $y_{m-1} \in N_D^-(y_m) \cap V^+(D) \cap [L_2^D(C') \cup L_3^D(C')]$. So y_m satisfies the SVC condition on C'. If $x_n \in L_3^D(C')$, by the similar argument, x_n satisfies the SVC condition on C'. If $v \in L_3^D(C')$, where $v \notin \{y_m, z_1, x_n\}$, then by Lemma 2.5, $v \in L_3^D(C)$. Since C is strong, v satisfies the SVC condition on C'and so by Lemma 3.1, v satisfies the SVC condition on C'. Hence C' is strong. It's a contradiction because C is maximal. Therefore $z_1 \in C$.

Let $C_1 = C \cap V(D_1)$. Since $z_1 \in C$, by Lemma 2.5, $v \in L_3^D(C_1)$ implies $v \in L_3^D(C)$. By the similar argument as in Case (1) of Theorem 4.5, we can show that C_1 is strong in D_1 .

Suppose C_1 is not maximal in D_1 . That means there exists a strong vertex cover C_2 of D_1 such that $C_2 = C_1 \sqcup \{x_{i_1}, \ldots, x_{i_m}\}$, where $\{x_{i_1}, \ldots, x_{i_m}\} \in V(D_1)$ for some m > 1. Now we claim $C'' = C \sqcup \{x_{i_1}, \ldots, x_{i_m}\}$ is strong. By the similar argument as in Case (1) of Theorem 4.5, we can show that C'' is strong in D. It's a contradiction because C is maximal. Hence C_1 is a maximal strong vertex cover of D_1 .

Theorem 5.3. Let D be the weighted oriented path same as defined in Notation 5.1. Assume that there exist two indices 1 < l < m and 1 < k < n such that the set of vertices with non-trivial weights are precisely $\{y_l, \ldots, y_m\} \cup \{x_k, \ldots, x_n\}$. Then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$.

Proof. Let D_1 and D_2 be the induced paths of D with $V(D_1) = \{x_1, x_2, \ldots, x_n\}, E(D_1) = \{(x_i, x_{i+1}) \mid 1 \le i \le n-1\}, V(D_2) = \{y_1, y_2, \ldots, y_m\}$ and $E(D_2) = \{(y_i, y_{i+1}) \mid 1 \le i \le m-1\}$. By [1, Theorem 3.6], let $\{C_{\alpha}\}, \{C_{\beta}\}$ and $\{C_{\gamma}\}$ be the collections of all maximal strong vertex covers of D_1 , where

 $C_{\alpha} = \left\{ C_{\alpha}' | C_{\alpha}' \text{ is a minimal vertex cover of the induced path } D(x_1, \dots, x_{k-2}) \right\}$ $\bigcup \{x_{k-1}, x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_n\},$ $C_{\beta} = \left\{ C_{\beta}' | C_{\beta}' \text{ is a minimal vertex cover of the induced path } D(x_1, \dots, x_{k-3}) \right\}$ $\bigcup \{x_{k-2}, x_k, x_{k+1}, x_{k+2}, \dots, x_n\},$

and $C_{\gamma} = \left\{ C_{\gamma}' | C_{\gamma}' \text{ is a minimal vertex cover of the induced path } D(x_1, \dots, x_{k-4}) \right\}$ $\bigcup \{x_{k-3}, x_{k-1}, x_k, x_{k+2}, x_{k+3}, \dots, x_n\}.$

By [1, Theorem 3.6], let $\{C_{\delta}\}, \{C_{\zeta}\}$ and $\{C_{\nu}\}$ be the collections of all maximal strong vertex covers of D_2 , where

$$C_{\delta} = \left\{ C_{\delta}' | C_{\delta}' \text{ is a minimal vertex cover of the induced path } D(y_1, \dots, y_{l-2}) \right\}$$

$$\bigcup \{ y_{l-1}, y_{l+1}, y_{l+2}, y_{l+3}, \dots, y_m \},$$

$$C_{\zeta} = \left\{ C_{\zeta}' | C_{\zeta}' \text{ is a minimal vertex cover of the induced path } D(y_1, \dots, y_{l-3}) \right\}$$

$$\bigcup \{ y_{l-2}, y_l, y_{l+1}, y_{l+2}, \dots, y_m \},$$
and
$$C_{\nu} = \left\{ C_{\nu}' | C_{\nu}' \text{ is a minimal vertex cover of the induced path } D(y_1, \dots, y_{l-4}) \right\}$$

$$\bigcup\{y_{l-3}, y_{l-1}, y_l, y_{l+2}, y_{l+3}, \dots, y_m\}$$

Let C be a maximal strong vertex cover of D. Then by Lemma 5.2, $C \cap V(D_1)$ is a maximal strong vertex cover of D_1 and similarly, $C \cap V(D_2)$ is a maximal strong vertex cover of D_2 . From the proof of Lemma 5.2, we know $z_1 \in C$. Therefore we can say that $\{C_{i,j}\}$ be the collection of all the maximal strong vertex covers of D, where $i \in \{\delta, \zeta, \nu\}$, $j \in \{\alpha, \beta, \gamma\}$ and each $C_{i,j} = C_i \cup \{z_1\} \cup C_j$. By Theorem 2.13, we get

$$I^{(s)} = \bigcap_{\substack{i \in \{\delta, \zeta, \nu\} \\ \text{and } j \in \{\alpha, \beta, \gamma\}}} (I_{\subseteq C_{i,j}})^s.$$

By Theorem 3.5, we know that

$$\begin{split} I_{\subseteq C_{\delta,j}} &= I_{\subseteq C_{\delta}} + (y_m z_1, x_n z_1) + I_{\subseteq C_j} = (C'_{\delta}) + J'_1 + (y_m z_1, x_n z_1) + J_k + (C'_j) \text{ for } j = \alpha, \beta, \gamma \\ \text{when } k = 1, 2, 3, \text{ respectively,} \\ I_{\subseteq C_{\zeta,j}} &= I_{\subseteq C_{\zeta}} + (y_m z_1, x_n z_1) + I_{\subseteq C_j} = (C'_{\zeta}) + J'_2 + (y_m z_1, x_n z_1) + J_k + (C'_j) \text{ for } j = \alpha, \beta, \gamma \\ \text{when } k = 1, 2, 3, \text{ respectively,} \\ I_{\subseteq C_{\nu,j}} &= I_{\subseteq C_{\nu}} + (y_m z_1, x_n z_1) + I_{\subseteq C_j} = (C'_{\nu}) + J'_3 + (y_m z_1, x_n z_1) + J_k + (C'_j) \text{ for } j = \alpha, \beta, \gamma \\ \text{when } k = 1, 2, 3, \text{ respectively,} \\ \text{where } J'_1 = (y_{l-1}, y_{l+1}^{w_{l+1}}, y_{l+1} y_{l+2}^{w_{l+2}}, \dots, y_{m-1} y_m^{w_m}), J_1 = (x_{k-1}, x_{k+1}^{w_{k+1}}, x_{k+1} x_{k+2}^{w_{k+2}}, \dots, x_{n-1} x_n^{w_n}), \\ J'_2 = (y_{l-2}, y_l^{w_l}, y_l y_{l+1}^{w_{l+1}}, y_{l+1} y_{l+2}^{w_{l+2}}, \dots, y_{m-1} y_m^{w_m}), J_2 = (x_{k-2}, x_k^{w}, x_k x_k x_{k+1}^{w_{k+1}}, x_{k+1} x_{k+2}^{w_{k+2}}, \dots, x_{n-1} x_n^{w_n}), \\ J'_3 = (y_{l-3}, y_{l-1}, y_l, y_{l+2}^{w_{l+2}}, y_{l+2} y_{l+3}^{w_{l+3}}, \dots, y_{m-1} y_m^{w_m}), J_3 = (x_{k-3}, x_{k-1}, x_k, x_{k+2}^{w_{k+2}}, x_{k+2}^{w_{k+2}}, x_{k+2} x_{k+3}^{w_{k+3}}, \dots, x_{n-1} x_n^{w_n}). \end{split}$$

The rest of the proof follows with similar argument as in [1, Theorem 3.6].

Lemma 5.4. Let D be the weighted oriented path same as defined in Notation 5.1. Assume that $w(y_m) \ge 2$ and $w(x_n) \ge 2$. If $w(y_l) \ge 2$ for some $1 < l \le m-2$ such that $w(y_{l+1}) = 1$ or $w(x_k) \ge 2$ for some $1 < k \le n-2$ such that $w(x_{k+1}) = 1$, then $I(D)^{(s)} \ne I(D)^s$ for some $s \ge 2$.

Proof. Suppose $w(y_l) \ge 2$ for some $1 < l \le m - 2$ such that $w(y_{l+1}) = 1$. Case (1) Assume that $1 < l \le m - 3$.

Let D' be a weighted naturally oriented path with $V(D') = \{y_1, y_2, \dots, y_l, y_{l+1}, y_{l+2}, y_{l+3}\}$ and $E(D') = \{(y_i, y_{i+1}) \mid 1 \le i \le l+2\}$. Then by Theorem 2.16, $I(D')^{(s)} \neq I(D')^s$ for some $s \ge 2$ and so by Theorem 4.5, we have $I(D)^{(s)} \neq I(D)^s$.

Case (2) Assume that l = m - 2.

Here $w(y_{m-1}) = 1$. Let D' be a weighted naturally oriented path with $V(D') = \{y_1, y_2, \dots, y_{m-2}, y_{m-1}, y_m, z_1\}$ and $E(D') = \{(y_i, y_{i+1}) \mid 1 \le i \le m-1\} \cup \{(y_m, z_1)\}$. Then by Theorem 2.16, $I(D')^{(s)} \ne I(D')^s$ for some $s \ge 2$ and thus by Theorem 4.5, we get $I(D)^{(s)} \ne I(D)^s$. Similarly if $w(x_k) \ge 2$ for some $1 < k \le n-2$ such that $w(x_{k+1}) = 1$, we can show that $I(D)^{(s)} \ne I(D)^s$ for some $s \ge 2$.

Theorem 5.5. Let D be the weighted oriented path same as defined in Notation 5.1. Assume that $w(y_m) \ge 2$ and $w(x_n) \ge 2$. Then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$ if and only if D satisfies the condition "there exist two indices 1 < l < m and 1 < k < n such that the set of vertices with non-trivial weights are precisely $\{y_1, \ldots, y_m\} \cup \{x_k, \ldots, x_n\}$ ".

Proof. It follows from Theorem 5.3 and Lemma 5.4.

Remark 5.6. Let *D* be the weighted oriented path same as defined in Notation 5.1. If $w(x_i) = 1$ for $2 \le i \le n$, then by changing the orientations of edges of the edge set $\{(x_i, x_{i+1}) \mid 1 \le i \le n-1\} \cup \{(x_n, z_1)\}$, we can think *D* as a weighted naturally oriented path. Similarly if $w(y_i) = 1$ for $2 \le i \le m$, we can think *D* as a weighted naturally oriented path.

Notation 5.7. Let D be a weighted oriented path with $V(D) = \{y_1, y_2, \dots, y_m, z_1, z_2, x_n, \dots, x_2, x_1\}$, $E(D) = \{(y_i, y_{i+1}) \mid 1 \le i \le m - 1\} \cup \{(y_m, z_1), (z_1, z_2), (x_n, z_2)\} \cup \{(x_i, x_{i+1}) \mid 1 \le i \le n - 1\}$, $w(z_1) = 1$ and $w(z_2) = 1$ (see Figure 7).

y_1	y_2	y_l	y_{l+1}	y_{m-1}	y_m	z_1	z_2	x_n	x_{n-1}	x_{k+1}	x_k	x_2	x_1
\rightarrow	 •·····		•			$\rightarrow \rightarrow$			← •····	••••••	(•••••	←-•

FIGURE 7. An oriented path which is union of two naturally oriented paths joined at a sink vertex.

Theorem 5.8. Let D be the weighted oriented path same as defined in Notation 5.7. Assume that there exist two indices $1 < l \le m$ and $1 < k \le n$ such that the set of vertices with non-trivial weights are precisely $\{y_l, \ldots, y_m\} \cup \{x_k, \ldots, x_n\}$. Then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$.

Proof. This proof follows by the similar argument as in Theorem 5.3.

Lemma 5.9. Let D be the weighted oriented path same as defined in Notation 5.7. If $w(y_l) \ge 2$ for some $1 < l \le m-1$ such that $w(y_{l+1}) = 1$ or $w(x_k) \ge 2$ for some $1 < k \le n-1$ such that $w(x_{k+1}) = 1$, then $I(D)^{(s)} \ne I(D)^s$ for some $s \ge 2$.

Proof. Suppose $w(y_l) \ge 2$ for some $1 < l \le m - 1$ such that $w(y_{l+1}) = 1$.

Case (1) Assume that $1 < l \le m - 2$.

By the same argument as in Lemma 5.4, we can show that $I(D)^{(s)} \neq I(D)^s$ for some $s \ge 2$.

Case (2) Assume that l = m - 1.

Here $w(y_m) = 1$. Let D' be a weighted naturally oriented path with $V(D') = \{y_1, y_2, \ldots, y_{m-1}, y_m, z_1, z_2\}$ and $E(D') = \{(y_i, y_{i+1}) \mid 1 \leq i \leq m-1\} \cup \{(y_m, z_1), (z_1, z_2)\}$. Then by Theorem 2.16, $I(D')^{(s)} \neq I(D')^s$ for some $s \geq 2$ and thus by Theorem 4.5, we get $I(D)^{(s)} \neq I(D)^s$.

Suppose $w(x_k) \ge 2$ for some $1 < k \le n-1$ such that $w(x_{k+1}) = 1$. Since $w(z_1) = w(z_2) = 1$, we can assume $(z_2, z_1) \in E(D)$. Let D'' be a weighted naturally oriented path with $V(D'') = \{x_1, x_2, \dots, x_{n-1}, x_n, z_2, z_1\}$ and $E(D'') = \{(x_i, x_{i+1}) \mid 1 \le i \le n-1\} \cup \{(x_n, z_2), (z_2, z_1)\}$. By the similar argument as for D', we can show that $I(D)^{(s)} \ne I(D)^s$, for some $s \ge 2$.

Theorem 5.10. Let D be the weighted oriented path same as defined in Notation 5.7. Assume that $w(x_i) \ge 2$ and $w(y_j) \ge 2$ for some i and j. Then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$ if and only if D satisfies the condition "there exist two indices $1 < l \le m$ and $1 < k \le n$ such that the set of vertices with non-trivial weights are precisely $\{y_l, \ldots, y_m\} \cup \{x_k, \ldots, x_n\}$ ".

Proof. It follows from Theorem 5.8 and Lemma 5.9.

Theorem 5.11. Let D be the weighted oriented path same as defined in Notation 5.1. Assume that $w(x_i) \ge 2$ and $w(y_j) \ge 2$ for some i and j. Then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$ if and only if

- When w(x_n) = 1, D satisfies the condition "there exist two indices 1 < l ≤ m and 1 < k < n such that the set of vertices with non-trivial weights are precisely {y_l,...,y_m} ∪ {x_k,...,x_{n-1}}".
- (2) When w(y_m) = 1, D satisfies the condition "there exist two indices 1 < l < m and 1 < k ≤ n such that the set of vertices with non-trivial weights are precisely {y_l,..., y_{m-1}} ∪ {x_k,..., x_n}".
- (3) When w(y_m) ≠ 1 and w(x_n) ≠ 1, D satisfies the condition "there exist two indices 1 < l < m and 1 < k < n such that the set of vertices with non-trivial weights are precisely {y_l,..., y_m} ∪ {x_k,..., x_n}".

Proof. (1) Since $w(x_n) = 1$, we can assume $(z_1, x_n) \in E(D)$. If we rename the vertex x_n by z_2 , then the proof follows from Theorem 5.10.

(2) Note that $w(y_m) = 1$. If we rename the vertices y_m and z_1 by z_1 and z_2 , respectively, then the proof follows from Theorem 5.10.

(3) It follows from Theorem 5.5.

Theorem 5.12. Let D be a weighted oriented path with $V(D) = \{y_1, y_2, \ldots, y_m, z_1, x_n, \ldots, x_2, x_1\}$ and $E(D) = \{(y_i, y_{i+1}) \mid 1 \le i \le m-1\} \cup \{(y_m, z_1), (x_n, z_1)\} \cup \{(x_i, x_{i+1}) \mid 1 \le i \le n-1\}$. Assume that $w(x_i) \ge 2$ and $w(y_j) \ge 2$ for some i and j. Then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$ if and only if

- (1) When $w(x_n) = 1$, D satisfies the condition "there exist two indices $1 < l \le m$ and 1 < k < n such that the set of vertices except z_1 with non-trivial weights are precisely $\{y_1, \ldots, y_m\} \cup \{x_k, \ldots, x_{n-1}\}$ ".
- (2) When w(y_m) = 1, D satisfies the condition "there exist two indices 1 < l < m and 1 < k ≤ n such that the set of vertices except z₁ with non-trivial weights are precisely {y_l,..., y_{m-1}} ∪ {x_k,..., x_n}".
- (3) When w(y_m) ≠ 1 and w(x_n) ≠ 1, D satisfies the condition "there exist two indices 1 < l < m and 1 < k < n such that the set of vertices except z₁ with non-trivial weights are precisely {y_l,..., y_m} ∪ {x_k,..., x_n}".

Proof. Here z_1 is the only sink vertex in D. If $w(z_1) = 1$, the proof follows from Theorem 5.11 and if $w(z_1) \ge 2$, then the proof follows from Lemma 2.17 and Theorem 5.11.

6. Symbolic powers of weighted rooted trees

In the computation of symbolic powers of edge ideals of weighted oriented graphs, we always need to know all the maximal strong vertex covers. In this section, we give a new technique to find all the maximal strong vertex covers of a particular class of weighted oriented graphs. Finally, we show the equality of ordinary and symbolic powers of edge ideals of certain class of weighted rooted trees.

Lemma 6.1. [12, Lemma 47] Let D be a weighted oriented graph such that $I(D) \subseteq (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s})$. Then $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of D.

After fixing the value of each a_i with its corresponding weight, we get strong vertex cover in the following result.

Lemma 6.2. Let D be a weighted oriented graph such that $I(D) \subseteq (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}})$, where s is the least possible value. Then $\{x_{i_1}, \ldots, x_{i_s}\}$ is a maximal strong vertex cover of D.

Proof. Let $C = \{x_{i_1}, \ldots, x_{i_s}\}$. By Lemma 6.1, C is a vertex cover of D. Let $x_{i_j} \in L_3^D(C)$. By Lemma 2.5, $N_D(x_{i_j}) \subset C$. If $N_D^-(x_{i_j}) = \phi$, then $I(D) \subseteq (x_{i_1}^{w_{i_1}}, \ldots, x_{i_{j-1}}^{w_{i_{j-1}}}, x_{i_{j+1}}^{w_{i_{j+1}}}, \ldots, x_{i_s}^{w_{i_s}})$. Since s is the least possible value, its a contradiction. Therefore $N_D^-(x_{i_j}) \neq \phi$. Let $N_D^-(x_{i_j}) = \{x_{k_1}, \ldots, x_{k_t}\}$. If $w_{k_1} = \cdots = w_{k_t} = 1$, then $I(D) \subseteq (x_{i_1}^{w_{i_1}}, \ldots, x_{i_{j-1}}^{w_{i_{j-1}}}, x_{i_{j+1}}^{w_{i_{j+1}}}, \ldots, x_{i_{j-1}}^{w_{i_{j+1}}}, x_{i_{j+1}}^{w_{i_{j+1}}}, \ldots, x_{i_{j+1}}^{w_{j+1}}, \ldots, x_$ $\begin{aligned} x_{i_s}^{w_{i_s}} &) \text{ and its a contradiction by the previous argument. Thus at least one of } x_{k_1}, \ldots, x_{k_t} \\ \text{has non-trivial weight. Without loss of generality let } w_{k_1} \neq 1. \\ \text{Suppose } x_{k_1} \in L_1^D(C). \\ \text{That means there is some } x_{l_1} \in N_D^+(x_{k_1}) \cap C^c. \\ \text{Here } (x_{k_1}, x_{l_1}) \in E(D). \\ \text{Since } x_{k_1} \notin (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}}) \\ \text{and } x_{l_1} \notin C, \\ \text{we have } x_{k_1} x_{l_1}^{w_{l_1}} \notin (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}}). \\ \text{Thus } I(D) \notin (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}}) \\ \text{and } x_{l_1} \notin C, \\ \text{we have } x_{k_1} x_{l_1}^{w_{l_1}} \notin (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}}). \\ \text{Thus } I(D) \notin (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}}). \\ \text{wich is a contradiction. Therefore } x_{k_1} \in L_2^D(C) \cup L_3^D(C). \\ \text{Note that } x_{k_1} \in N_D^-(x_{i_j}) \cap V^+(D) \cap [L_2^D(C) \cup L_3^D(C)], \\ \text{i.e., } x_{i_j} \\ \text{satisfies SVC condition on } C. \\ \text{Hence } C \\ \text{is strong. Suppose } C \\ \text{is not maximal. That means, there exists a strong vertex cover } C' \\ \text{of } D \\ \text{such that } C \\ \notin C'. \\ \text{Let } x_q \\ \in C' \\ \text{Ningle } C \\ \text{Since } x_q \notin C \\ \text{and so by Lemma 2.5, } x_q \\ \in L_3^D(C'). \\ \text{Since } C' \\ \text{is strong, there is some } x_p \\ \in N_D^-(x_q) \\ \cap V^+(D) \\ (L_2^D(C') \\ \cup L_3^D(C')]. \\ \text{Note that } x_p \\ \in C, \\ w(x_p) \\ \geq 2, \\ x_q \notin C \\ \text{and } (x_p, x_q) \\ \in E(D). \\ \text{Then } x_p x_q^{w_q} \\ \notin (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}}) \\ \text{and so } I(D) \\ \notin (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}}), \\ \text{which is a contradiction. Hence } C \\ \text{is maximal. } \\ \square \\ \end{bmatrix}$

Remark 6.3. Converse of the above lemma need not be true in general.

For example consider the weighted oriented path D as in Figure 8. Then $I(D) = (x_1x_2^7, x_2x_3, x_3x_4)$. Using Macaulay 2, the strong vertex covers of D are $\{x_1, x_3\}, \{x_2, x_3\}$ and $\{x_2, x_4\}$. Note that $\{x_2, x_4\}$ is a maximal strong vertex cover, but $I(D) \notin (x_2^7, x_4)$ because $x_2x_3 \notin (x_2^7, x_4)$.

FIGURE 8. A weighted oriented path D of length 3.

We see that the converse of Lemma 6.2 is true under certain condition on weights of vertices and orientation of edges of D. To prove the converse part, the following lemma is important.

Lemma 6.4. Let D be a weighted oriented graph such that at most one edge is oriented into each vertex and $w(x) \ge 2$ if $\deg_D(x) \ge 2$ for all $x \in V(D)$. Let C be a strong vertex cover of D with $x_i \in L_1^D(C)$ and $w(x_i) \ne 1$. Then there exists a strong vertex cover C'of D such that $C \subsetneq C'$.

Proof. Let $L = \{x \mid x \in L_1^D(C) \cap V^+(D)\} = \{x_{i_1}, x_{i_2}, \dots, x_{i_l}\}$. By our assumption $L \neq \phi$. Let $J = [N_D^+(x_{i_1}) \cap C^c] \cup \dots \cup [N_D^+(x_{i_l}) \cap C^c] = \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$. Let $C' = C \cup J$. Then $x_{i_m} \notin L_1^D(C')$ because $N_D^+(x_{i_m}) \cap C'^c = \phi$ for $1 \leq m \leq l$. Since $J \subset C^c$ and C is a vertex cover of D, $N_D(x_{j_n}) \subseteq C \subset C'$ for each $x_{j_n} \in J$. By Lemma 2.5, $J \subset L_3^D(C')$. We claim C' is a strong vertex cover of D. Note that $L_3^D(C) \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} \subseteq L_3^D(C')$. Let

 $x_l \in L_3^D(C)$. Then x_l satisfies SVC condition on C because C is strong. By Lemma 3.1, $x_l \in L_3^D(C')$ and x_l satisfies SVC condition on C'. Without loss of generality let $x_{j_1} \in N_D^+(x_{i_1})$. Since $x_{i_1} \in N_D^-(x_{j_1}) \cap V^+(D) \cap [L_2^D(C') \cup L_3^D(C')]$, x_{j_1} satisfies SVC condition on C'. By the similar argument, we can show x_{j_t} satisfies SVC condition on C' for $2 \leq t \leq r$. If $L_3^D(C) \cup \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\} = L_3^D(C')$, then each element of $L_3^D(C')$ satisfies SVC condition on C' and hence C' is strong. Now we assume that $L_3^D(C) \cup \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\} \notin L_3^D(C')$. Let $x_k \in L_3^D(C') \setminus [L_3^D(C) \cup \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}]$. Then x_k lies in the neighbourhood of x_{j_t} for some $t \in [r]$. Without loss of generality we can assume that $x_{j_t} \in N_D^+(x_{i_1})$. By definition of $D, N_D^-(x_{j_t}) = \{x_{i_1}\}$.

Case (1) Suppose $x_k \in N_D^-(x_{j_t})$. Then $x_k = x_{i_1}$.

Since $w(x_{i_1}) \ge 2$, there is $(x_p, x_{i_1}) \in E(D)$ for some $x_p \in V(D)$. If $\deg_D(x_p) = 1$, then $N_D(x_p) = \{x_{i_1}\}$. Since $x_{i_1} \in L_3^D(C')$, by Lemma 2.5, $N_D(x_{i_1}) \subset C'$ and so $x_p \in C'$. Here $x_p \notin J$ because $N_D^-(x_p) = \phi$. This implies $x_p \in C$. We know $x_{i_1} \in C$. Then by Lemma 2.5, $x_p \in L_3^D(C)$. Since $N_D^-(x_p) = \phi$, x_p does not satisfy the SVC condition on C. Its a contradiction because C is strong. Therefore $\deg_D(x_p) \ge 2$ and by the definition of D, $w(x_p) \ge 2$. Suppose $x_p \in L_1^D(C')$. That means there exists some $x_q \in N_D^+(x_p) \cap C'^c$. Since $C \not\subseteq C'$, $x_q \in N_D^+(x_p) \cap C^c$. Thus $x_p \in L_1^D(C)$ and so $x_p \in L$, which is a contradiction because each element of $L \notin L_1^D(C')$. Therefore $x_p \notin L_1^D(C')$. Then $x_p \in N_D^-(x_{i_1}) \cap V^+(D) \cap [L_2^D(C') \cup L_3^D(C')]$, i.e., $x_{i_1} = x_k$ satisfies SVC condition on C'.

Case (2) Suppose $x_k \in N_D^+(x_{j_t})$. Then $x_k \neq x_{i_1}$.

Note that (x_{i_1}, x_{j_t}) and $(x_{j_t}, x_k) \in E(D)$. Since $\deg_D(x_{j_t}) \ge 2$, by the definition of D, $w(x_{j_t}) \ge 2$. This implies $x_{j_t} \in N_D^-(x_k) \cap V^+(D) \cap L_3^D(C')$, i.e., x_k satisfies SVC condition on C'.

Therefore each element of $L_3^D(C')$ satisfies SVC condition on C'. Hence C' is strong. \Box

Theorem 6.5. Let D be a weighted oriented graph such that at most one edge is oriented into each vertex and $w(x) \ge 2$ if $\deg_D(x) \ge 2$ for all $x \in V(D)$. Then $I(D) \subseteq (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}})$, where s is the least possible value if and only if $\{x_{i_1}, \ldots, x_{i_s}\}$ is a maximal strong vertex cover of D.

Proof. If $I(D) \subseteq (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}})$, where s is the least possible value, then by Lemma 6.2, $\{x_{i_1}, \ldots, x_{i_s}\}$ is a maximal strong vertex cover of D.

Now we assume that $C = \{x_{i_1}, \ldots, x_{i_s}\}$ is a maximal strong vertex cover of D. We claim $I(D) \subseteq (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}})$, where s is the least possible value. Let $x_i x_j^{w_j} \in I(D)$. If $x_j \in C$, then $x_i x_j^{w_j} \in (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}})$. Suppose $x_j \notin C$. Since $(x_i, x_j) \in E(D), x_j \in N_D^+(x_i) \cap C^c$ and it implies $x_i \in L_1^D(C)$. If $w(x_i) \neq 1$, by Lemma 6.4, there exists a strong vertex cover C' of D such that $C \not\subseteq C'$, which is a contradiction because C is maximal. So $w(x_i) = 1$ and $x_i x_j^{w_j} \in (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}})$. Therefore $I(D) \subseteq (x_{i_1}^{w_{i_1}}, \ldots, x_{i_s}^{w_{i_s}})$.

Suppose s is not the least possible value. That means, $I(D) \subseteq (x_{j_1}^{w_{j_1}}, \ldots, x_{j_r}^{w_{j_r}})$ for some $\{x_{j_1}, \ldots, x_{j_r}\} \not\subseteq \{x_{i_1}, \ldots, x_{i_s}\}$, where r < s and r is the least possible value. Then by Lemma 6.2, $\{x_{j_1}, \ldots, x_{j_r}\}$ is a maximal strong vertex cover of D, which is a contradiction because C is maxmal.

Next we prove the equality of ordinary and symbolic powers of edge ideals of some class of weighted rooted trees.

Theorem 6.6. Let D be a weighted rooted tree with root x_0 , $\deg_D(x_0) = 1$ and $w(x) \ge 2$ if $\deg_D(x) \ge 2$ for all $x \in V(D)$. Then $I(D)^{(s)} = I(D)^s$ for all $s \ge 2$.

Proof. Let $N_D(x_0) = \{x_1\}$ and by definition of $D, (x_0, x_1) \in E(D)$. Here $N_D^-(x_1) = \{x_0\}$. Let $N_D^+(x_1) = \{x_2, x_3, \dots, x_r\}$. Note that $I(D) \subseteq (\{x_i^{w_i} \mid x_i \in V(D) \setminus \{x_0\}\})$ and $I(D) \notin$ $(\{x_i^{w_i} \mid x_i \in V(D) \setminus \{x_0, x_p\}\})$ for any $x_p \in V(D) \setminus \{x_0\}$. Thus by Theorem 6.5, $V(D) \setminus \{x_0, x_p\}\}$ $\{x_0\}$ is a maximal strong vertex cover of D. Also $I(D) \subseteq (\{x_i^{w_i} \mid x_i \in V(D) \setminus \{x_1\}\})$ and $I(D) \notin (\{x_i^{w_i} \mid x_i \in V(D) \setminus \{x_1, x_q\}\})$ for any $x_q \in V(D) \setminus \{x_1\}$. Thus by Theorem 6.5, $V(D) \setminus \{x_1\}$ is a maximal strong vertex cover of D. Let $C_1 = V(D) \setminus \{x_0\}$ and $C_2 =$ $V(D) \setminus \{x_1\}$. Suppose there exists a maximal strong vertex cover C of D, which contains both x_0 and x_1 . Then we can write $C = \{x_0, x_1, x_{i_1}, \dots, x_{i_{s-2}}\}$, where $\{x_{i_1}, \dots, x_{i_{s-2}}\} \subseteq \sum_{i=1}^{s-1} \sum_{i=1}^{$ $[V(D) \setminus \{x_0, x_1\}]$ is some vertex set. By Theorem 6.5, $I(D) \subseteq (x_0, x_1^{w_1}, x_{i_1}^{w_{i_1}}, \dots, x_{i_{s-2}}^{w_{i_{s-2}}})$, where |C| = s is the least possible value. Since $x_0 x_1^{w_1}$ is the only minimal generator of I(D), which involves the vertex x_0 , we have $I(D) \subseteq (x_1^{w_1}, x_{i_1}^{w_{i_1}}, \dots, x_{i_{s-2}}^{w_{i_{s-2}}})$. That means s is not the least possible value and by Theorem 6.5, its a contradiction. Hence C is not a maximal strong vertex cover of D. Thus C_1 and C_2 are the only maximal strong vertex covers of D. Let $D' = D \setminus \{x_0\}$ and $D'' = D \setminus \{x_0, x_1\}$ be the induced digraphs of D. Here $L_2^D(C_1) = \{x_1\}, L_3^D(C_1) = V(D) \setminus \{x_0, x_1\} \text{ and } w(x_i) \ge 2 \text{ if } \deg_D(x) \ge 2 \text{ for all } x_i \in C_1.$ Then by Theorem 3.5, $I_{\subseteq C_1} = (x_1^{w_1}) + I(D')$. Here $L_1^D(C_2) = \{x_0\}, L_2^D(C_2) = \{x_2, \dots, x_r\}, I_2^D(C_2) = \{x_1, \dots, x_r\}$ $L_3^D(C_2) = V(D) \setminus \{x_0, x_1, x_2, \dots, x_r\}$ and $w(x_i) \ge 2$ if $\deg_D(x) \ge 2$ for all $x_i \in C_2$. Then by Theorem 3.5, $I_{\subseteq C_2} = (x_0, x_2^{w_2}, \dots, x_r^{w_r}) + I(D'')$. Let $I(D'') = (f_1, \dots, f_t)$. Hence $I_{\subseteq C_1} =$ $(x_1^{w_1}) + I(D') = (x_1^{w_1}, x_1 x_2^{w_2}, \dots, x_1 x_r^{w_r}, f_1, \dots, f_t) \text{ and } I_{\subseteq C_2} = (x_0, x_2^{w_2}, \dots, x_r^{w_r}, f_1, \dots, f_t).$ By Lemma 2.13, we have $I(D)^{(s)} = (I_{\subseteq C_1})^s \cap (I_{\subseteq C_2})^s$. It is enough to prove that $I(D)^{(s)} = (I_{\subseteq C_1})^s \cap (I_{\subseteq C_2})^s \subseteq I(D)^s$. We prove this by induction on s. The case for s = 1 is trivial. Let $m \in \mathcal{G}(I(D)^{(s)})$. Then $m = \operatorname{lcm}(m_1, m_2)$ for some $m_1 \in$ $\mathcal{G}((x_1^{w_1}, x_1 x_2^{w_2}, \dots, x_1 x_r^{w_r}, f_1, \dots, f_t) \text{ and } m_2 \in \mathcal{G}((x_0, x_2^{w_2}, \dots, x_r^{w_r}, f_1, \dots, f_t)). \text{ Thus } m_1 = (x_1^{w_1})^{a_1} (x_1 x_2^{w_2})^{a_2} \dots (x_1 x_r^{w_r})^{a_r} (f_1)^{a_{r+1}} \dots (f_t)^{a_{r+t}} \text{ and } m_2 = (x_0)^{b_1} (x_2^{w_2})^{b_2} \dots (x_r^{w_r})^{b_r} (f_1)^{b_{r+1}}$ $\cdots (f_t)^{b_{r+t}}$ for some $a_i, b_i \ge 0$ with $\sum_{i=1}^{r+t} a_i = s$ and $\sum_{i=1}^{r+t} b_i = s$. **Case (1)** Assume that $a_1 \neq 0$.

If $b_1 \neq 0$, then $x_0 x_1^{w_1}$ is divisible by m and notice that $\frac{m}{x_0 x_1^{w_1}} \in (x_1^{w_1}, x_1 x_2^{w_2}, \dots, x_1 x_r^{w_r}, f_1, \dots, f_t)^{s-1} = I(D)^{(s-1)}$. Hence by induction hypothesis $\frac{m}{x_0 x_1^{w_1}} \in I(D)^{s-1}$ and so $m \in I(D)^s$. If $b_2 \neq 0$, then $x_1 x_2^{w_2}$ is divisible by m and observe that $\frac{m}{x_1 x_2^{w_2}} \in (x_1^{w_1}, x_1 x_2^{w_2}, \dots, x_1 x_r^{w_r}, f_1, \dots, f_t)^{s-1} = I(D)^{(s-1)}$. Hence by induction hypothesis $\frac{m}{x_1 x_2^{w_2}} \in (x_1^{w_1}, x_1 x_2^{w_2}, \dots, x_1 x_r^{w_r}, f_1, \dots, f_t)^{s-1} = I(D)^{(s-1)}$. Hence by induction hypothesis $\frac{m}{x_1 x_2^{w_2}} \in I(D)^{s-1}$ and so $m \in I(D)^s$. Similarly if $b_i \neq 0$ for some $i \in \{3, \dots, r\}$, we can show $m \in I(D)^s$. If $b_1 = \dots = b_r = 0$, then $m_2 \in I(D)^s$ and hence $m \in I(D)^s$.

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