ACYLINDRICITY OF THE ACTION OF RIGHT-ANGLED ARTIN GROUPS ON EXTENSION GRAPHS

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ABSTRACT. The action of a right-angled Artin group on its extension graph is known to be acylindrical because the cardinality of the so-called r-quasi-stabilizer of a pair of distant points is bounded above by a function of r. The known upper bound of the cardinality is an exponential function of r. In this paper we show that the r-quasi-stabilizer is a subset of a cyclic group and its cardinality is bounded above by a linear function of r. This is done by exploring lattice theoretic properties of group elements, studying prefixes of powers and extending the uniqueness of quasi-roots from word length to star length. We also improve the known lower bound for the minimal asymptotic translation length of a right angled Artin group on its extension graph.

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1. INTRODUCTION

Throughout the paper Γ denotes a finite simplicial graph, not necessarily connected, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The *right-angled Artin group* $A(\Gamma)$ with the underlying graph Γ is the group generated by $V(\Gamma)$ such that the defining relations are the commutativity between adjacent vertices, hence $A(\Gamma)$ has the group presentation

$$A(\Gamma) = \langle v \in V(\Gamma) \mid v_i v_j = v_j v_i \text{ for each } \{v_i, v_j\} \in E(\Gamma) \rangle.$$

Right-angled Artin groups are important groups in geometric group theory, which played a key role in Agol's proof of the virtual Haken conjecture [1, 15, 29].

The extension graph Γ^e is the graph such that the vertex set $V(\Gamma^e)$ is the set of all elements of $A(\Gamma)$ that are conjugate to a vertex of Γ , and two vertices $v_1^{g_1}$ and $v_2^{g_2}$ are adjacent in Γ^e if and only if they commute when considered as elements of $A(\Gamma)$. (Here, v^g denotes the conjugate $g^{-1}vg$.) Therefore

$$\begin{split} V(\Gamma^e) &= \{ v^g : v \in V(\Gamma), \ g \in A(\Gamma) \}, \\ E(\Gamma^e) &= \{ \{ v_1^{g_1}, v_2^{g_2} \} : v_1^{g_1} v_2^{g_2} = v_2^{g_2} v_1^{g_1} \text{ in } A(\Gamma) \}. \end{split}$$

Extension graphs are usually infinite and locally infinite. They are very useful in the study of rightangled Artin groups such as the embeddability problem between right-angled Artin groups [17, 19, 23, 24, 16] and the purely loxodromic subgroups which are analogous to convex cocompact subgroups of the mapping class groups of surfaces [21]. It is known that Γ^e is a quasi-tree, hence a δ -hyperbolic graph [17].

Definition 1.1 (acylindrical action). When a group G acts on a path-metric space (X, d) isometrically from the right, the action is called *acylindrical* if for any r > 0, there exist R, N > 0 such that whenever x and y are two points of X with $d(x, y) \ge R$, the cardinality of the set

$$\xi(x,y;r) = \{g \in G : d(xg,x) \leqslant r \text{ and } d(yg,y) \leqslant r\}$$

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is at most N. The set $\xi(x, y; r)$ is called the *r*-quasi-stabilizer of the pair of points (x, y). We sometimes use the notation $\xi_{(X,d)}(x, y; r)$ for the set $\xi(x, y; r)$. Notice that R and N are functions of r. When we need to specify the acylindricity constants R and N, we say that the action is (R, N)-acylindrical.

There have been many works on properties and examples of groups with an acylinrical action on a geodesic hyperbolic metric space. For example, see [5, 27, 8].

Let d denote the graph metric of Γ^e . The right-angled Artin group $A(\Gamma)$ acts on (Γ^e, d) isometrically from the right by conjugation, i.e. the image of the vertex v^h under the action of $g \in A(\Gamma)$ is v^{hg} . The action of $A(\Gamma)$ on Γ^e behaves much like the action of the mapping class group Mod(S) of a hyperbolic surface S on the curve graph $\mathcal{C}(S)$. One of the fundamental properties is that the action of $A(\Gamma)$ on Γ^e is acylindrical, which is shown by Kim and Koberda [18].

Theorem 1.2 ([18, Theorem 30]). The action of $A(\Gamma)$ on Γ^e is acylindrical.

More precisely, it is shown that the action is (R, N)-acylindrical with

$$R = R(r) = D(2r + 4D + 7),$$
$$N = N(r) = \left(V2^{2V}\right)^{r+2D+1},$$

where $D = \operatorname{diam}(\Gamma)$ is the diameter of Γ and $V = |V(\Gamma)|$ is the cardinality of $V(\Gamma)$. Notice that N(r) is an exponential function of r.

For a graph Γ , let $\overline{\Gamma}$ denote the complement graph of Γ , i.e. the graph on the same vertices as Γ such that two distinct vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ .

For the reader's convenience, we give some remarks on the cases where $|V(\Gamma)|$ is small and where Γ or $\overline{\Gamma}$ is disconnected.

The following are known for the extension graph Γ^e [17, Lemma 3.5]: if Γ is disconnected, then Γ^e has countably infinite number of path-components; if $\overline{\Gamma}$ is disconnected, i.e. Γ is a join, then Γ^e is also a join, hence diam(Γ^e) ≤ 2 ; if $|V(\Gamma)| = 1$, then $|V(\Gamma^e)| = 1$. If $|V(\Gamma)| \in \{2,3\}$, then either Γ or $\overline{\Gamma}$ is disconnected. In fact, Γ^e is a connected graph with infinite diameter if and only if $|V(\Gamma)| \geq 4$ and both Γ and $\overline{\Gamma}$ are connected. Therefore, when we consider the action of $A(\Gamma)$ on Γ^e , it is natural to require that $|V(\Gamma)| \geq 4$ and both Γ and $\overline{\Gamma}$ are connected.

In the study of extension graphs, we use the star length metric d_* on $A(\Gamma)$. (See §4 for the definition of star length.) The metric space $(A(\Gamma), d_*)$ is quasi-isometric to the extension graph (Γ^e, d) . If $|V(\Gamma)| = 1$ or if $\overline{\Gamma}$ is disconnected, then $(A(\Gamma), d_*)$ has diameter at most 2, which is not interesting. Therefore, when we consider the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$, it is natural to require that $|V(\Gamma)| \ge 2$ and $\overline{\Gamma}$ is connected (see Remark 6.2).

From the above discussions, the following settings are natural.

- (i) When we consider the action of $A(\Gamma)$ on (Γ^e, d) , we will assume that $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected.
- (ii) When we consider the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$, we will assume that $|V(\Gamma)| \ge 2$ and Γ is connected.

The following is the main result of this paper, which shows that we can take N(r) as a linear function of r and furthermore the quasi-stabilizer $\xi(x, y; r)$ is a subset of a cyclic group.

Theorem A (Theorem 8.2) Let Γ be a finite simplicial graph such that $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected. Then the action of $A(\Gamma)$ on Γ^e is (R, N)-acylindrical with

$$R = R(r) = D(2V + 7)(r + 1) + 10D,$$

$$N = N(r) = 2(V - 2)r - 1,$$

where $D = \operatorname{diam}(\Gamma)$ and $V = |V(\Gamma)|$. Moreover, for any $x, y \in V(\Gamma^e)$ with $d(x, y) \ge R$, if $\xi(x, y; r) \ne \{1\}$, then there exists a loxodromic element $g \in A(\Gamma)$ such that

- (i) $\xi(x, y; r) \subset \{1, g^{\pm 1}, g^{\pm 2}, \dots, g^{\pm k}\}$ for some $1 \leq k \leq (V-2)r 1$;
- (ii) the Hausdorff distance between the $\langle g \rangle$ -orbit of x and that of y is at most D(2r+7).

The following is an easy example to come up with for $g \in \xi(x, y; r)$. Let g be a loxodromic element with a quasi-axis $L = z^{\langle g \rangle} = \{z^{g^m} : m \in \mathbb{Z}\}$ for some $z \in V(\Gamma^e)$ such that $d(z^g, z)$ is sufficiently small. If both x and y are close enough to L, then $d(x^g, x)$ and $d(y^g, y)$ are also small so that $g \in \xi(x, y; r) \setminus \{1\}$. In this case, the Hausdorff distance between the $\langle g \rangle$ -orbits $x^{\langle g \rangle}$ and $y^{\langle g \rangle}$ is small. Theorem 8.2 says that, in some sense, this is the only case where $g \in \xi(x, y; r) \setminus \{1\}$ happens: g is loxodromic and the Hausdorff distance between $x^{\langle g \rangle}$ and $y^{\langle g \rangle}$ is small. Moreover, by Theorem 8.2(i), the set $\xi(x, y; r) \setminus \{1\}$ is purely loxodromic, that is, there is no elliptic element that r-quasi-stabilizes a pair of sufficiently distant points.

In order to prove Theorem A, we develop several tools such as lattice theoretic properties of group elements, decomposition of conjugating elements, properties of prefixes of powers, and then extend the uniqueness of quasi-roots in [25] from word length to star length. Using these tools, we also obtain a new lower bound for the minimal asymptotic translation length of the action of $A(\Gamma)$ on Γ^e .

Definition 1.3. When a group G acts on a connected metric space (X, d) by isometries from right, the *asymptotic translation length* of an element $g \in G$ is defined by

(1)
$$\tau(g) = \tau_{(X,d)}(g) = \lim_{n \to \infty} \frac{d(xg^n, x)}{n}$$

where $x \in X$. This limit always exists, is independent of the choice of $x \in X$, and satisfies $\tau(g^n) = |n|\tau(g)$ and $\tau(h^{-1}gh) = \tau(g)$ for all $g, h \in G$ and $n \in \mathbb{Z}$. If $\tau(g) > 0$, g is called *loxodromic*. If $\{d(xg^n, x)\}_{n=1}^{\infty}$ is bounded, g is called *elliptic*. If $\tau(g) = 0$ and $\{d(xg^n, x)\}_{n=1}^{\infty}$ is unbounded, g is called *parabolic*. For a subgroup H of G, the minimal asymptotic translation length of H for the action on (X, d) is defined by

(2)
$$\mathcal{L}_{(X,d)}(H) = \min\{\tau_{(X,d)}(h) : h \in H, \ \tau_{(X,d)}(h) > 0\}.$$

There have been many works on minimal asymptotic translation lengths of the action of mapping class groups on curve graphs. Let S_g denote a closed orientable surface of genus g. For the action of the mapping class group $Mod(S_g)$ on the curve graph $\mathcal{C}(S_g)$, Gadre and Tsai [12] proved that

$$\mathcal{L}_{\mathcal{C}(S_g)}(\mathrm{Mod}(S_g)) \asymp \frac{1}{g^2}$$

where $f(g) \simeq h(g)$ denotes that there exist positive constants A and B such that $Af(g) \leq h(g) \leq Bf(g)$. The braid group B_n can be regarded as the mapping class group of the *n*-punctured disk D_n fixing boundary pointwise. The pure braid group PB_n is the subgroup of B_n consisting of mapping classes that fix each puncture. Kin and Shin [20] and Baik and Shin [3] showed that

$$\mathcal{L}_{\mathcal{C}(D_n)}(\mathbf{B}_n) \asymp \frac{1}{n^2}, \qquad \mathcal{L}_{\mathcal{C}(D_n)}(\mathbf{PB}_n) \asymp \frac{1}{n}.$$

For the action of $A(\Gamma)$ on Γ^e , it follows from a result of Kim and Koberda [18] that

$$\mathcal{L}_{(\Gamma^e, d)}(A(\Gamma)) \geqslant \frac{1}{2|V(\Gamma)|^2}.$$

Baik, Seo and Shin [2] proved that all loxodromic elements of $A(\Gamma)$ on Γ^e have rational asymptotic translation lengths with a common denominator.

In this paper, we show the following, where the denominator of the lower bound is improved from a quadratic function to a linear function of $|V(\Gamma)|$.

Theorem B (Theorem 6.5) Let Γ be a finite simplicial graph such that $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected. Then

$$\mathcal{L}_{(\Gamma^e, d)}(A(\Gamma)) \ge \frac{1}{|V(\Gamma)| - 2}.$$

In the remaining of this section, we explain briefly our ideas and the structure of this paper.

1.1. Idea for the acylindricity. Let us first explain our idea for the acylindricity. For $g \in A(\Gamma)$, let ||g|| denote the word length of g with respect to the generating set $V(\Gamma)^{\pm 1}$, and let d_{ℓ} denote temporarily the word length metric defined by $d_{\ell}(g,h) = ||gh^{-1}||$ for $g,h \in A(\Gamma)$. The right multiplication induces an isometric action of $A(\Gamma)$ on $(A(\Gamma), d_{\ell})$. Since $\xi(x, y; r) = x^{-1}\xi(1, yx^{-1}; r)x$ for any $x, y \in A(\Gamma)$, it suffices to consider r-quasi-stabilizers of the form $\xi(1, w; r)$ for the acylindricity.

Suppose that we are given R > 0 large, r > 0 small and $w \in A(\Gamma)$ with $||w|| = d_{\ell}(w, 1) \ge R$. Let $g \in \xi(1, w; r) \setminus \{1\}$. Since $||g|| = d_{\ell}(g, 1) \le r$ and $||wgw^{-1}|| = d_{\ell}(wg, w) \le r$, we have

$$\|w\| \ge R, \quad \|g\| \le r, \quad \|wgw^{-1}\| \le r.$$

In other words, ||w|| is large whereas ||g|| and $||wgw^{-1}||$ are small. This happens typically when

$$(**) w = ag^n, n \in \mathbb{Z}, a \in A(\Gamma)$$

with ||a|| small and |n| large. In this case, $d_{\ell}(w, g^n) = d_{\ell}(ag^n, g^n) = ||a||$ is small, hence we can say that w is "close to a power of g".

Even though it is clearly over-optimistic and false, one may hope that the following hold: given a triple (R, r, w) as above (i.e. R > 0 is large, r > 0 is small and $w \in A(\Gamma)$ with $||w|| \ge R$),

- (i) if (*) holds, then (**) holds for some $n \in \mathbb{Z}$ and $a \in A(\Gamma)$ with ||a|| small;
- (ii) only a small number of triples (a, g, n) with ||a|| small and $||g|| \leq r$ satisfy (**).

Of course, the above statements are not true at least as they are written. Moreover, the metric spaces $(A(\Gamma), d_{\ell})$ and (Γ^e, d) are not quasi-isometric, hence the above statements do not imply the acylindricity of (Γ^e, d) . However, we will see that this approach in fact works in the study of the acylindricity of the action of $A(\Gamma)$ on (Γ^e, d) if we replace the word length metric with the star length metric.

1.2. Lattice structure. In §2, we collect basic combinatorial group theoretic properties of rightangled Artin groups. Those properties are stated using lattice theoretic notations.

The motivation comes from Garside groups which are a lattice theoretic generalization of braid groups and finite type Artin groups. For Garside groups, there are elegant tools especially for the word and conjugacy problems and the asymptotic translation length [13, 6, 11, 4, 10, 9, 22, 26]. Right-angled Artin groups are not Garside groups, except free abelian groups, hence we cannot apply Garside theory to right angled-Artin groups. However, some ideas from Garside theory are very useful in our approach.

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For $g \in A(\Gamma)$, the support of g, denoted $\operatorname{supp}(g)$, is the set of generators that appear in a shortest word on $V(\Gamma)^{\pm 1}$ representing g.

For $g_1, g_2 \in A(\Gamma)$, we say that g_1 and g_2 disjointly commute, denoted $g_1 \rightleftharpoons g_2$, if $\operatorname{supp}(g_1) \cap \operatorname{supp}(g_2) = \emptyset$ and each $v_1 \in \operatorname{supp}(g_1)$ commutes with each $v_2 \in \operatorname{supp}(g_2)$.

Let $g = g_1 \cdots g_k$ for some $g, g_1, \ldots, g_k \in A(\Gamma)$. We say that the decomposition is *geodesic* if $||g|| = ||g_1|| + \cdots + ||g_k||$. If $g = g_1g_2$ is geodesic, we say that g_1 is a *prefix* of g, denoted $g_1 \leq_L g$, and that g is a *right multiple* of g_1 .

The relation \leq_L is a partial order on $A(\Gamma)$, hence the notions of gcd $g_1 \wedge_L g_2$ and lcm $g_1 \vee_L g_2$ make sense. Theorem 2.12 shows that for $g_1, g_2 \in A(\Gamma)$, the gcd $g_1 \wedge_L g_2$ always exists and the lcm $g_1 \vee_L g_2$ exists if and only if g_1 and g_2 have a common right multiple. Moreover, in this case, there exist $g'_1, g'_2 \in A(\Gamma)$ such that $g_i = (g_1 \wedge_L g_2)g'_i$ for $i = 1, 2, g'_1 \rightleftharpoons g'_2$ and $g_1 \vee_L g_2 = (g_1 \wedge_L g_2)g'_1g'_2$.

1.3. Cyclic conjugations. In §3, we study conjugations $g^u = u^{-1}gu$. The decomposition $u^{-1}gu$ is not geodesic in general, i.e. $||u^{-1}gu|| \neq ||u^{-1}|| + ||g|| + ||u||$.

Let g be cyclically reduced, i.e. the word length ||g|| is minimal in its conjugacy class. If $u \leq_L g$, then $g = ug_1$ is geodesic for some $g_1 \in A(\Gamma)$ and $g^u = u^{-1}(ug_1)u = g_1u$. In other words, the conjugation of g by u moves the prefix u to the right. An iteration of this type of conjugations is called a *left cyclic conjugation*. The *right cyclic conjugation* is defined similarly. The *cyclic conjugation* is an iteration of left and right cyclic conjugations.

Proposition 3.8 shows that for a cyclically reduced element g, the conjugation g^u is a left cyclic conjugation of g if and only if $u \leq_L g^n$ for some $n \geq 1$.

Theorem 3.9 shows that given $g, u \in A(\Gamma)$ with g cyclically reduced, there exists a unique geodesic decomposition $u = u_1 u_2 u_3$ such that u_1 disjointly commutes with g; g^{u_2} is a cyclic conjugation; $g^u = u_3^{-1} g^{u_2} u_3$ is geodesic, i.e. $||u_3^{-1} g^{u_2} u_3|| = ||u_3^{-1}|| + ||g^{u_2}|| + ||u_3||$. Furthermore, there is a geodesic decomposition $u_2 = u'_2 u''_2$ such that $g^{u'_2}$ (resp. $g^{u''_2}$) is a left (resp. right) cyclic conjugation and $u'_2 \rightleftharpoons u''_2$.

1.4. Star length. An element $g \in A(\Gamma)$ is called a *star-word* if $\operatorname{supp}(g)$ is contained in the star of some vertex. The *star length*, denoted $||g||_*$, of g is the minimum ℓ such that g can be written as a product of ℓ star-words. Let d_* denote the metric on $A(\Gamma)$ induced by the star length: $d_*(g_1, g_2) = ||g_1g_2^{-1}||_*$.

The right multiplication induces an isometric action of $A(\Gamma)$ on $(A(\Gamma), d_*)$. The metric spaces $(A(\Gamma), d_*)$ and (Γ^e, d) are quasi-isometric [18]. It seems that, for some algebraic tools, $(A(\Gamma), d_*)$ is easier to work with than (Γ^e, d) .

In §4, we study basic properties of the star length concerning the prefix order \leq_L and the geodesic decomposition of group elements. For example, Corollary 4.8 shows that if g_1g_2 is geodesic, then $||g_1||_* + ||g_2||_* - 2 \leq ||g_1g_2||_* \leq ||g_1||_* + ||g_2||_*$.

1.5. Prefixes of powers of cyclically reduced elements. Recall that, for a cyclically reduced element g, if g^u is a left cyclic conjugation, then $u \leq_L g^m$ for some $m \geq 1$, i.e. u is a prefix of some power of g. In §5, we study prefixes of powers. In particular, we show that if g is cyclically reduced and non-split and if $u \leq_L g^m$ for some $m \geq 1$, then $u = g^n a$ is geodesic for some $0 \leq n \leq m$ and $a \in A(\Gamma)$ with $||a||_* \leq ||g||_* + 1$ (see Corollary 5.6).

1.6. Asymptotic translation length. In §6, we prove Theorem B by using the results in §5.

1.7. Uniqueness of quasi-roots. An element g is called a *quasi-root* of h if there is a decomposition

 $h = ag^n b$

for some $n \ge 1$ and $a, b \in A(\Gamma)$ such that ||h|| = ||a|| + n||g|| + ||b||. It is called an (A, B, r)-quasi-root if $||a|| \le A$, $||b|| \le B$ and $||g|| \le r$ and an $(A, B, r)^*$ -quasi-root if $||a||_* \le A$, $||b||_* \le B$ and $||g||_* \le r$. The conjugates aga^{-1} and $b^{-1}gb$ are called the *leftward*- and the *rightward-extraction* of the quasi-root g, respectively.

In [25], it is shown that if $||h|| \ge A + B + (2|V(\Gamma)| + 1)r$, then strongly non-split and primitive (A, B, r)-quasi-roots of h are unique up to conjugacy, and their leftward- and rightward-extractions are unique. (See §4 and §7 for the definitions of strongly non-split elements and primitive elements.)

In §7, we extend the above result to $(A, B, r)^*$ -quasi-roots: if $||h||_* \ge 2A+2B+(2|V(\Gamma)|+3)r+2$, then primitive $(A, B, r)^*$ -quasi-roots of h are unique up to conjugacy, and their leftward- and rightward-extractions are unique.

1.8. Proof of the acylindricity. In §8, we first compute the acylindricity constants for the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$ (Theorem 8.1) by combining the results from the previous sections. Then we prove Theorem A using the quasi-isometry between $(A(\Gamma), d_*)$ and (Γ^e, d) .

1.9. Conventions and notations. Throughout the paper, all the group actions are right-actions.

For graphs Γ_1 and Γ_2 , the *disjoint union* $\Gamma_1 \sqcup \Gamma_2$ is the graph such that

$$V(\Gamma_1 \sqcup \Gamma_2) = V(\Gamma_1) \sqcup V(\Gamma_2),$$

$$E(\Gamma_1 \sqcup \Gamma_2) = E(\Gamma_1) \sqcup E(\Gamma_2).$$

The join $\Gamma_1 * \Gamma_2$ is the graph such that $\overline{\Gamma_1 * \Gamma_2} = \overline{\Gamma}_1 \sqcup \overline{\Gamma}_2$, hence

$$V(\Gamma_1 * \Gamma_2) = V(\Gamma_1) \sqcup V(\Gamma_2),$$

$$E(\Gamma_1 * \Gamma_2) = E(\Gamma_1) \sqcup E(\Gamma_2) \sqcup \{ \{v_1, v_2\} : v_1 \in V(\Gamma_1), v_2 \in V(\Gamma_2) \}.$$

A graph is called a *join* if it is the join of two nonempty graphs. A subgraph that is a join is called a *subjoin*.

For $X \subset V(\Gamma)$, $\Gamma[X]$ denotes the subgraph of Γ induced by X, i.e.

$$V(\Gamma[X]) = X, \qquad E(\Gamma[X]) = \{ \{v_1, v_2\} \in E(\Gamma) : v_1, v_2 \in X \}.$$

For $g \in A(\Gamma)$, the subgraphs $\Gamma[\operatorname{supp}(g)]$ and $\overline{\Gamma}[\operatorname{supp}(g)]$ are abbreviated to $\Gamma[g]$ and $\overline{\Gamma}[g]$, respectively. For $v \in V(\Gamma)$ and $X \subset V(\Gamma)$, the sets $\operatorname{Lk}_{\Gamma}(v)$, $\operatorname{St}_{\Gamma}(v)$ and $\operatorname{St}_{\Gamma}(X)$ are defined as follows:

$$Lk_{\Gamma}(v) = \{v_1 \in V(\Gamma) : \{v_1, v\} \in E(\Gamma)\},\$$

$$St_{\Gamma}(v) = \{v\} \cup Lk_{\Gamma}(v),\$$

$$St_{\Gamma}(X) = \bigcup_{v \in X} St_{\Gamma}(v).$$

They are called the *link* of v, the *star* of v and the *star* of X, respectively. They will be written as Lk(v), St(v) and St(X) by omitting Γ whenever the context is clear.

The path graph $P_k = (v_1, v_2, \dots, v_k)$ is the graph with $V(P_k) = \{v_1, \dots, v_k\}$ and $E(P_k) = \{\{v_i, v_{i+1}\}: 1 \leq i \leq k-1\}$, hence P_k looks like $v_1 \quad v_2 \quad \cdots \quad v_{k-1} \quad v_k$.

A path in a graph Γ is a tuple (v_1, v_2, \ldots, v_k) of vertices of Γ such that $\{v_i, v_{i+1}\} \in E(\Gamma)$ for all $1 \leq i \leq k-1$. (We do not assume that the vertices or the edges in the path are mutually distinct. Hence it means the *walk* in the graph theoretical terminology.)

2. LATTICE STRUCTURES

In this section we study lattice structures in right-angled Artin groups.

An element of $V(\Gamma)^{\pm 1} = V(\Gamma) \cup V(\Gamma)^{-1}$ is called a *letter*. A *word* means a finite sequence of letters. For words w_1 and w_2 , the notation $w_1 \equiv w_2$ means that w_1 and w_2 coincide as sequences of letters. A word w' is called a *subword* of a word w if $w \equiv w_1 w' w_2$ for (possibly empty) words w_1 and w_2 .

Suppose that $g \in A(\Gamma)$ is expressed as a word w on $V(\Gamma)^{\pm 1}$. The word w is called *reduced* if w is a shortest word among all the words representing g. In this case, the length of w is called the *word* length of g and denoted by ||g||.

Definition 2.1 (support). For $g \in A(\Gamma)$, the support of g, denoted supp(g), is the set of generators that appear in a reduced word representing g. It is known that supp(g) is well defined (by [14]), i.e. it does not depend on the choice of a reduced word representing g.

Definition 2.2 (disjointly commute). We say that $g_1, g_2 \in A(\Gamma)$ disjointly commute, denoted $g_1 \rightleftharpoons g_2$, if $\operatorname{supp}(g_1) \cap \operatorname{supp}(g_2) = \emptyset$ and each $v_1 \in \operatorname{supp}(g_1)$ commutes with each $v_2 \in \operatorname{supp}(g_2)$. (In particular, the identity element $1 \in A(\Gamma)$ disjointly commutes with any $g \in A(\Gamma)$.)

The notation $\Gamma[g]$ is an abbreviation of $\Gamma[\operatorname{supp}(g)]$, the subgraph of Γ induced by $\operatorname{supp}(g)$. From a graph theoretical viewpoint, $g_1 \rightleftharpoons g_2$ means that $\operatorname{supp}(g_1) \cap \operatorname{supp}(g_2) = \emptyset$ and $\overline{\Gamma}[g_1g_2] = \overline{\Gamma}[g_1] \sqcup \overline{\Gamma}[g_2]$ in the complement graph $\overline{\Gamma}$ (or equivalently $\Gamma[g_1g_2] = \Gamma[g_1] * \Gamma[g_2]$ in the graph Γ). Recall that $\operatorname{St}_{\overline{\Gamma}}(\operatorname{supp}(g))$ denotes the star of $\operatorname{supp}(g)$ in the complement graph $\overline{\Gamma}$. The following lemma is now obvious.

Lemma 2.3. For $g_1, g_2 \in A(\Gamma)$, the following are equivalent:

- (i) $g_1 \rightleftharpoons g_2$ in $A(\Gamma)$;
- (ii) $\operatorname{St}_{\overline{\Gamma}}(\operatorname{supp}(g_1)) \cap \operatorname{supp}(g_2) = \emptyset.$

Let w be a (non-reduced) word on $V(\Gamma)^{\pm 1}$. A subword $v^{\pm 1}w_1v^{\mp 1}$ of w, where $v \in V(\Gamma)$, is called a *cancellation* of v in w if $\operatorname{supp}(w_1) \subset \operatorname{St}_{\Gamma}(v)$, i.e. each $v_1 \in \operatorname{supp}(w_1)$ commutes with v. If, furthermore, no letter in w_1 is equal to v or v^{-1} , it is called an *innermost cancellation* of v in w. It is known that the following are equivalent:

- (i) w is a reduced word;
- (ii) w has no cancellation;
- (iii) w has no innermost cancellation.

Abusing terminology, we do not distinguish between an element $g \in A(\Gamma)$ and a reduced word w representing g if there is no confusion. For example, if there is a cancellation in w_1w_2 , where each w_i is a reduced word representing an element g_i , then we just say that there is a cancellation in g_1g_2 .

Definition 2.4 (geodesic decomposition). For $k \ge 1$ and $g, g_1, \ldots, g_k \in A(\Gamma)$, we say that the decomposition $g = g_1 \cdots g_k$ is geodesic, or $g_1 \cdots g_k$ is geodesic, if $||g|| = ||g_1|| + \cdots + ||g_k||$.

If $g_1 \cdots g_k$ is geodesic, then the following are obvious from the definition:

- (i) $g_k^{-1} g_{k-1}^{-1} \cdots g_1^{-1}$ is geodesic;
- (ii) $g_p g_{p+1} \cdots g_q$ is geodesic for any $1 \leq p < q \leq k$;
- (iii) $\operatorname{supp}(g_1 \cdots g_k) = \operatorname{supp}(g_1) \cup \cdots \cup \operatorname{supp}(g_k).$

Definition 2.5 (prefix order). Let $g = g_1g_2$ be geodesic for $g, g_1, g_2 \in A(\Gamma)$. We say that g_1 is a *prefix* (or a *left divisor*) of g, denoted $g_1 \leq_L g$, and that g is a *right multiple* of g_1 . Similarly, we say that g_2 is a *suffix* (or a *right divisor*) of g, denoted $g_2 \leq_R g$, and that g is a *left multiple* of g_2 .

Clearly both \leq_L and \leq_R are partial orders on $A(\Gamma)$. The following lemma shows their basic properties. The proof is straightforward, hence we omit it.

Lemma 2.6. Let $g, g_1, ..., g_n, h_1, h_2 \in A(\Gamma)$.

- (i) $g_1 \leq_L g_2$ if and only if $g_1^{-1} \leq_R g_2^{-1}$.
- (ii) If gg_1 and gg_2 are geodesic, then $gg_1 \leq_L gg_2$ if and only if $g_1 \leq_L g_2$.
- (iii) $g_1 \cdots g_n$ is geodesic if and only if $g_1 \cdots g_k \leq_L g_1 \cdots g_{k+1}$ for all $1 \leq k \leq n-1$.
- (iv) Suppose $g_1g_2 = h_1h_2$ such that both g_1g_2 and h_1h_2 are geodesic. Then $g_1 \leq_L h_1$ if and only if $h_2 \leq_R g_2$.

Definition 2.7 (gcd and lcm). For $g, h \in A(\Gamma)$, the symbols $g \wedge_L h$ and $g \vee_L h$ denote the greatest common divisor (gcd) and the least common multiple (lcm) with respect to \leq_L . In other words, $g \wedge_L h$ is an element such that (i) $g \wedge_L h \leq_L g$ and $g \wedge_L h \leq_L h$; (ii) if $u \leq_L g$ and $u \leq_L h$ for some $u \in A(\Gamma)$, then $u \leq_L g \wedge_L h$. Similarly, $g \vee_L h$ is an element such that (i) $g \leq_L g \vee_L h$ and $h \leq_L g \vee_L h$; (ii) if $g \leq_L g \vee_L h$ and $h \leq_L g \vee_L h$; (ii) if $g \leq_L u$ and $h \leq_L u$ for some $u \in A(\Gamma)$, then $g \vee_L h \leq_L u$.

The symbols $g \wedge_R h$ and $g \vee_R h$ denote the gcd and lcm respectively with respect to \leq_R .

The elements $g \wedge_L h$ and $g \vee_L h$ are unique if they exist. In Theorem 2.12 we will show that $g \wedge_L h$ always exists and that $g \vee_L h$ exists if and only if g and h admit a common right multiple.

Note that g and h have no nontrivial common prefix if and only if $g \wedge_L h = 1$, i.e. the gcd $g \wedge_L h$ exists and is equal to the identity. Therefore even though we did not prove yet the existence of $g \wedge_L h$ for arbitrary g and h, we can safely use the expression $g \wedge_L h = 1$.

The following lemma is an easy consequence of the fact that a word is reduced if and only if it has no innermost cancellation.

Lemma 2.8. Let $u, g, g_1, \ldots, g_k \in A(\Gamma)$.

(i) Suppose that $g_1 \cdots g_k$ is not geodesic. Then there exist $1 \leq p < q \leq k$ and $x \in V(\Gamma)^{\pm 1}$ such that

$$x^{-1} \leq_R g_p, \quad x \leq_L g_q, \quad x \rightleftharpoons g_j \text{ for all } p < j < q.$$

Furthermore, if both $g_1 \cdots g_{k-1}$ and $g_2 \cdots g_k$ are geodesic, then p = 1 and q = k.

- (ii) Suppose that for each $1 \leq p < q \leq k$, either g_pg_q is geodesic or $g_pg_{j_1} \cdots g_{j_r}g_q$ is geodesic for some $p < j_1 < \cdots < j_r < q$. Then $g_1 \cdots g_k$ is geodesic.
- (iii) Suppose that gg is geodesic. For any n≥ 2 and a, b ∈ A(Γ), the following are equivalent:
 (a) agb is geodesic;
 - (b) $a gg \cdots gb$ is geodesic;

(c)
$$ag^n b^n$$
 is geodesic.

In particular, $g^n = gg \cdots g$ is geodesic for any $n \ge 2$.

(iv) Suppose that g_ig_i is geodesic for all $1 \leq i \leq k$ and that $a_1g_1a_2g_2\cdots a_kg_ka_{k+1}$ is geodesic for some $a_1,\ldots,a_{k+1} \in A(\Gamma)$. Then $a_1g_1^{n_1}a_2g_2^{n_2}\cdots a_kg_k^{n_k}a_{k+1}$ is geodesic for any $n_i \geq 1$.

Proof. (i) Let w_i be a reduced word representing g_i for i = 1, ..., k. Since $g_1 \cdots g_k$ is not geodesic, the word $w \equiv w_1 \cdots w_k$ is not reduced, hence it has an innermost cancellation. Since each w_i is reduced, the cancellation must occur between x^{-1} in w_p and x in w_q for some $1 \leq p < q \leq k$ and $x \in V(\Gamma)^{\pm 1}$. Therefore w_p and w_q are of the form $w_p \equiv w'_p x^{-1} w''_p$ and $w_q \equiv w'_q x w''_q$ such that x disjointly commutes with $w''_p, w_{p+1}, \ldots, w_{q-1}, w'_q$, hence $x^{-1} \leq_R g_p, x \leq_L g_q$ and $x \rightleftharpoons g_j$ for all p < j < q.

If either p > 1 or q < k, then either $g_2 \cdots g_k$ or $g_1 \cdots g_{k-1}$ is not geodesic. Therefore if both $g_1 \cdots g_{k-1}$ and $g_2 \cdots g_k$ are geodesic, then p = 1 and q = k.

(ii) Assume that $g_1 \cdots g_k$ is not geodesic. By (i), there exist $1 \leq p < q \leq k$ and $x \in V(\Gamma)^{\pm 1}$ such that $x^{-1} \leq_R g_p$, $x \leq_L g_q$ and $x \rightleftharpoons g_j$ for all j with p < j < q. Therefore none of $g_p g_q$ and $g_p g_{j_1} \cdots g_{j_r} g_q$ $(p < j_1 < \cdots < j_r < q)$ is geodesic, which contradicts the hypothesis.

(iii) (a) \Rightarrow (b): Let $h_1 = a$, $h_i = g$ for i = 2, ..., n + 1 and $h_{n+2} = b$. Then $h_1, ..., h_{n+2}$ satisfy the hypothesis of (ii), hence $h_1h_2 \cdots h_{n+1}h_{n+2} = a \underbrace{g \cdots g}{g} b$ is geodesic.

(b) \Rightarrow (a): Since $a \underbrace{g \cdots g}_{n} b$ is geodesic, ag and gb are geodesic. If agb is not geodesic, then there

exists $x \in V(\Gamma)^{\pm 1}$ such that $x^{-1} \leq_R a, x \leq_L b$ and $x \rightleftharpoons g$ by (i). Hence $ag \cdots gb$ is not geodesic, which is a contradiction.

(b) \Leftrightarrow (c): From (a) \Rightarrow (b) with a = b = 1, $g^n = gg \cdots g$ is geodesic, i.e. $||g^n|| = n||g||$. Therefore $||ag^nb|| = ||a|| + ||g^n|| + ||b||$ if and only if $||ag^nb|| = ||a|| + n||g|| + ||b||$, i.e. ag^nb is geodesic if and only if $agg \cdots gb$ is geodesic.

(iv) Applying (iii) with $a = a_1$, $g = g_1$, $b = a_2g_2 \cdots a_{k+1}$ and $n = n_1$, we get that $a_1g_1^{n_1}a_2g_2 \cdots a_{k+1}$ is geodesic. Then applying (iii) with $a = a_1g_1^{n_1}a_2$, $g = g_2$, $b = a_3g_3 \cdots a_{k+1}$ and $n = n_2$, we get that $a_1g_1^{n_1}a_2g_2^{n_2}a_3g_3 \cdots a_{k+1}$ is geodesic. Iterating this process, we get that $a_1g_1^{n_1}a_2g_2^{n_2} \cdots a_kg_k^{n_k}a_{k+1}$ is geodesic.

Lemma 2.9. Let $g_1, g_2 \in A(\Gamma)$ and $x \in V(\Gamma)^{\pm 1}$.

- (i) If g_1g_2 is not geodesic, then there exists $y \in V(\Gamma)^{\pm 1}$ such that $y^{-1} \leq_R g_1$ and $y \leq_L g_2$.
- (ii) Let g_1g_2 be geodesic. If $x \leq_L g_1g_2$ and $x \notin_L g_1$, then $x \leq_L g_2$ and $x \rightleftharpoons g_1$.
- (iii) Let g_1g_2 be geodesic. If $x \leq_R g_1g_2$ and $x \notin_R g_2$, then $x \leq_R g_1$ and $x \rightleftharpoons g_2$.

Proof. (i) It follows from Lemma 2.8(i) with k = 2.

(ii) Since $x \not\leq_L g_1$, the decomposition $x^{-1} \cdot g_1$ is geodesic. Since $x \leq_L g_1g_2$, the decomposition $x^{-1} \cdot g_1g_2$ is not geodesic. Since both $x^{-1} \cdot g_1$ and $g_1 \cdot g_2$ are geodesic, there exists $y \in V(\Gamma)^{\pm 1}$ such that $y^{-1} \leq_R x^{-1}$, $y \leq_L g_2$ and $y \rightleftharpoons g_1$ (by Lemma 2.8(i)), hence x = y. Therefore $x \leq_L g_2$ and $x \rightleftharpoons g_1$.

(iii) The proof is analogous to (ii).

Lemma 2.10. Let $g \in A(\Gamma)$ and $x \neq y \in V(\Gamma)^{\pm 1}$ (possibly $y = x^{-1}$).

- (i) If $x \leq_L g$ and $y \leq_R g$, then g = xhy is geodesic for some $h \in A(\Gamma)$.
- (ii) If $x, y \leq_L g$, then $x \rightleftharpoons y$ and g = xyh is geodesic for some $h \in A(\Gamma)$.
- (iii) If $x, y \leq_R g$, then $x \rightleftharpoons y$ and g = hxy is geodesic for some $h \in A(\Gamma)$.

Proof. (i) Since $y \leq_R g$, g = g'y is geodesic for some $g' \in A(\Gamma)$. Since $x \leq_L g = g'y$, if $x \notin_L g'$, then $x \leq_L y$ (by Lemma 2.9(ii)), which contradicts the hypothesis $x \neq y$. Thus $x \leq_L g'$, hence g' = xh is geodesic for some $h \in A(\Gamma)$. Therefore g = g'y = xhy is geodesic.

(ii) Since $x \leq_L g$, g = xg' is geodesic for some $g' \in A(\Gamma)$. Since $y \neq x$ (hence $y \leq_L x$) and $y \leq_L xg'$, we have $y \rightleftharpoons x$ and $y \leq_L g'$ (by Lemma 2.9(ii)), hence g' = yh is geodesic for some $h \in A(\Gamma)$. Therefore g = xg' = xyh is geodesic.

(iii) The proof is analogous to (ii).

Lemma 2.11. Let $g_1, g_2, h_1, h_2, h \in A(\Gamma)$ with both g_1g_2 and h_1h_2 geodesic.

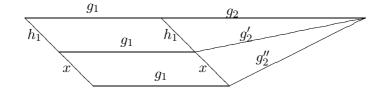


FIGURE 1. van Kampen diagram for Lemma 2.11(ii)

- (i) If $h \wedge_L g_1 = h \wedge_L g_2 = 1$, then $h \wedge_L (g_1g_2) = 1$.
- (ii) If $h \leq_L g_1 g_2$ and $h \wedge_L g_1 = 1$, then $h \rightleftharpoons g_1$ and $h \leq_L g_2$.
- (iii) Let $g_1g_2 = h_1h_2$. If $g_1 \wedge_L h_1 = g_2 \wedge_R h_2 = 1$, then $g_1 \rightleftharpoons h_1$, $g_1 = h_2$ and $g_2 = h_1$.

Proof. (i) If $h \wedge_L (g_1g_2) \neq 1$, then there exists $x \in V(\Gamma)^{\pm 1}$ such that $x \leq_L h$ and $x \leq_L g_1g_2$. Since $x \leq_L h$ and $h \wedge_L g_1 = h \wedge_L g_2 = 1$, we have $x \notin_L g_1$ and $x \notin_L g_2$. Since $x \leq_L g_1g_2$ and $x \notin_L g_1$, we have $x \leq_L g_2$ by Lemma 2.9(ii), which is a contradiction.

(ii) We use induction on ||h||.

If ||h|| = 0, there is nothing to prove. If ||h|| = 1, it holds by Lemma 2.9(ii).

Assume $||h|| \ge 2$. Then $h = h_1 x$ is geodesic for some $h_1 \in A(\Gamma)$ and $x \in V(\Gamma)^{\pm 1}$. See Figure 1. Notice that $h_1 \leq_L g_1 g_2$ and $h_1 \wedge_L g_1 = 1$. By the induction hypothesis, $h_1 \rightleftharpoons g_1$ and $h_1 \leq_L g_2$, hence $g_2 = h_1 g'_2$ is geodesic for some $g'_2 \in A(\Gamma)$.

Since $h_1 \rightleftharpoons g_1$, we have $g_1g_2 = g_1h_1g'_2 = h_1g_1g'_2$. Since g_1g_2 is geodesic, so is $h_1g_1g'_2$. Since $h_1x = h \leq_L g_1g_2 = h_1g_1g'_2$ and both h_1x and $h_1g_1g'_2$ are geodesic, we have $x \leq_L g_1g'_2$.

Observe $x \not\leq_L g_1$. (If $x \leq_L g_1$, then $x \rightleftharpoons h_1$ because $h_1 \rightleftharpoons g_1$. Since $h = h_1 x = xh_1$ is geodesic, we have $x \leq_L h$. Thus x is a common prefix of g_1 and h, which contradicts the hypothesis $h \wedge_L g_1 = 1$.) By Lemma 2.9(ii), we get $x \rightleftharpoons g_1$ and $x \leq_L g'_2$, hence $g'_2 = xg''_2$ is geodesic for some $g''_2 \in A(\Gamma)$.

Since $g_2 = h_1 g'_2 = h_1 x g''_2 = h g''_2$ and since $h g''_2$ is geodesic, we have $h \leq_L g_2$. On the other hand, since $h_1 \rightleftharpoons g_1$ and $x \rightleftharpoons g_1$, we have $h = h_1 x \rightleftharpoons g_1$.

(iii) Since $g_1 \leq_L h_1h_2$, $h_1 \leq_L g_1g_2$ and $g_1 \wedge_L h_1 = 1$, we have $g_1 \rightleftharpoons h_1$, $g_1 \leq_L h_2$ and $h_1 \leq_L g_2$ (by (ii)). Thus $h_2 = g_1h'_2$ and $g_2 = h_1g'_2$ are geodesic for some $g'_2, h'_2 \in A(\Gamma)$.

Observe $g_1h_1g'_2 = g_1g_2 = h_1h_2 = h_1g_1h'_2 = g_1h_1h'_2$, which implies $g'_2 = h'_2$. Since $g_2 \wedge_R h_2 = 1$, we have $g'_2 = h'_2 = 1$, hence $g_1 = h_2$ and $g_2 = h_1$.

The following is the main result of this section.

Theorem 2.12. For $g_1, g_2 \in A(\Gamma)$, the gcd $g_1 \wedge_L g_2$ always exists and the lcm $g_1 \vee_L g_2$ exists if and only if g_1 and g_2 have a common right multiple.

More precisely, if g_0 is a maximal common prefix of g_1 and g_2 , hence $g_1 = g_0g'_1$ and $g_2 = g_0g'_2$ are geodesic for some $g'_1, g'_2 \in A(\Gamma)$ with $g'_1 \wedge_L g'_2 = 1$, then the following hold.

(i) g₁ and g₂ have a common right multiple if and only if g'₁ ⇒ g'₂. In this case, g₁ ∨_L g₂ exists and g₁ ∨_L g₂ = g₁g'₂ = g₂g'₁ = g₀g'₁g'₂. In particular, supp(g₁ ∨_L g₂) = supp(g₁) ∪ supp(g₂).
(ii) g₁ ∧_L g₂ = g₀.

Proof. (i) Assume $g'_1 \rightleftharpoons g'_2$. Then $g'_1g'_2$ is geodesic (otherwise there exists $x \in V(\Gamma)^{\pm 1}$ such that $x^{-1} \leq_R g'_1$ and $x \leq_L g'_2$ by Lemma 2.9(i), hence g'_1 and g'_2 do not disjointly commute). Since $g_0g'_1$, $g_0g'_2$ and $g'_1g'_2$ are all geodesic, $g_0g'_1g'_2$ is geodesic (by Lemma 2.8(ii)). Therefore $g_0g'_1g'_2 = g_1g'_2 = g_2g'_1$ is a common right multiple of g_1 and g_2 .

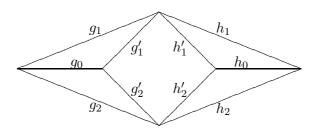


FIGURE 2. van Kampen diagram for Theorem 2.12

Conversely, assume that g_1 and g_2 have a common right multiple h. Then $h = g_1h_1 = g_2h_2$ are geodesic for some $h_1, h_2 \in A(\Gamma)$. We need to show that $g'_1 \rightleftharpoons g'_2$.

Let h_0 be a maximal common suffix of h_1 and h_2 . Then $h_1 = h'_1 h_0$ and $h_2 = h'_2 h_0$ are geodesic for some $h'_1, h'_2 \in A(\Gamma)$ with $h'_1 \wedge_R h'_2 = 1$. See Figure 2. Notice that $g'_1 h'_1 = g'_2 h'_2$ and that g'_1, g'_2, h'_1, h'_2 satisfy the hypotheses of Lemma 2.11(iii). Therefore $g'_1 \rightleftharpoons g'_2$.

Lemma 2.11(iii) also claims $g'_1 = h'_2$ and $g'_2 = h'_1$, hence $h = g_1h_1 = g_0g'_1h'_1h_0 = g_0g'_1g'_2h_0$. Therefore $g_0g'_1g'_2$ is a prefix of any common right multiple h of g_1 and g_2 , namely, $g_1 \vee_L g_2 = g_0g'_1g'_2$. It follows immediately that $\operatorname{supp}(g_1 \vee_L g_2) = \operatorname{supp}(g_1) \cup \operatorname{supp}(g_2)$. Since $g_0g'_1g'_2 = g_0g'_2g'_1$, we have $g_1 \vee_L g_2 = g_0g'_1g'_2 = g_1g'_2 = g_2g'_1$.

(ii) Let u_0 be a common prefix of g_1 and g_2 . Since g_1 and g_2 are common right multiples of g_0 and u_0 , the lcm $g_0 \lor_L u_0$ exists (by (i)) and is a prefix of both g_1 and g_2 , hence $g_0 \lor_L u_0$ is a common prefix of g_1 and g_2 . Since $g_0 \lor_L u_0$ is a right multiple of g_0 and since g_0 is a maximal common prefix of g_1 and g_2 , we have $g_0 = g_0 \lor_L u_0$, hence $u_0 \leqslant_L g_0$. Therefore $g_0 = g_1 \land_L g_2$.

Obviously we can replace (\wedge_L, \vee_L) in Theorem 2.12 with (\wedge_R, \vee_R) as follows.

Theorem 2.13. For $g_1, g_2 \in A(\Gamma)$, the gcd $g_1 \wedge_R g_2$ always exists and the lcm $g_1 \vee_R g_2$ exists if and only if g_1 and g_2 have a common left multiple.

More precisely, if g_0 is a maximal common suffix of g_1 and g_2 , hence $g_1 = g'_1g_0$ and $g_2 = g'_2g_0$ are geodesic for some $g'_1, g'_2 \in A(\Gamma)$ with $g'_1 \wedge_R g'_2 = 1$, then the following hold.

(i) g₁ and g₂ have a common left multiple if and only if g'₁ ≓ g'₂. In this case, g₁ ∨_R g₂ exists and g₁ ∨_R g₂ = g'₂g₁ = g'₁g₂ = g'₁g'₂g₀. In particular, supp(g₁ ∨_R g₂) = supp(g₁) ∪ supp(g₂).
(ii) g₁ ∧_R g₂ = g₀.

Observe that the gcds $g_1 \wedge_L g_2$ and $g_1 \wedge_R g_2$ exist for any $g_1, g_2 \in A(\Gamma)$ by the above theorems. The following lemma is obvious, hence we omit the proof.

Lemma 2.14. Let $g_1, g_2 \in A(\Gamma)$.

- (i) $(g_1 \wedge_L g_2)^{-1} = g_1^{-1} \wedge_R g_2^{-1}$.
- (ii) If $g_1 = g_0 g'_1$ and $g_2 = g_0 g'_2$ are geodesic, then $g_1 \wedge_L g_2 = g_0 (g'_1 \wedge_L g'_2)$. In particular, if $g'_1 \wedge_L g'_2 = 1$, then $g_1 \wedge_L g_2 = g_0$.
- (iii) If $g_1 \leq_L g_2$, then $(h \wedge_L g_1) \leq_L (h \wedge_L g_2)$ for any $h \in A(\Gamma)$.
- (iv) The statements analogous to (ii) and (iii) also hold for (\leq_R, \wedge_R) .

Lemma 2.15. Let $g_1, g_2, h \in A(\Gamma)$ with g_1g_2 geodesic.

- (i) If $h \rightleftharpoons g_1$, then $h \wedge_L (g_1g_2) = h \wedge_L g_2$.
- (ii) If $\operatorname{supp}(h) \cap \operatorname{supp}(g_2) = \emptyset$, then $h \wedge_L (g_1g_2) = h \wedge_L g_1$.
- (iii) If $h \leq_L g_1 g_2$ and $h \rightleftharpoons g_1$, then $h \leq_L g_2$.

(iv) If $h \leq_L g_1 g_2$ and $\operatorname{supp}(h) \cap \operatorname{supp}(g_2) = \emptyset$, then $h \leq_L g_1$.

(v) The statements analogous to (i)–(iv) also hold for (\leq_R, \wedge_R) .

Proof. (i) Let $h_0 = h \wedge_L g_2$. Then $h = h_0 h'$ and $g_2 = h_0 g'_2$ are geodesic for some $h', g'_2 \in A(\Gamma)$ with $h' \wedge_L g'_2 = 1$. Since $h \rightleftharpoons g_1$ and $h = h_0 h'$ is geodesic, we have $h_0 \rightleftharpoons g_1$ and $h' \rightleftharpoons g_1$, hence $h' \wedge_L g_1 = 1$.

Notice that $g_1g'_2$ is geodesic because $g_1g_2(=g_1h_0g'_2) = h_0g_1g'_2$ is geodesic. Since $h' \wedge_L g_1 = h' \wedge_L g'_2 = 1$, we have $h' \wedge_L (g_1g'_2) = 1$ (by Lemma 2.11(i)). Therefore by Lemma 2.14.(ii)

$$h \wedge_L (g_1 g_2) = (h_0 h') \wedge_L (g_1 h_0 g'_2) = (h_0 h') \wedge_L (h_0 g_1 g'_2)$$
$$= h_0 (h' \wedge_L (g_1 g'_2)) = h_0 = h \wedge_L g_2.$$

(ii) Let $h_0 = h \wedge_L g_1$. Then $h = h_0 h'$ and $g_1 = h_0 g'_1$ are geodesic for some $h', g'_1 \in A(\Gamma)$ with $h' \wedge_L g'_1 = 1$. Since $\operatorname{supp}(h) \cap \operatorname{supp}(g_2) = \emptyset$ and since $h = h_0 h'$ is geodesic, we have $\operatorname{supp}(h') \cap \operatorname{supp}(g_2) = \emptyset$, hence $h' \wedge_L g_2 = 1$.

Notice that g'_1g_2 is geodesic because $g_1g_2 = h_0g'_1g_2$ is geodesic. Since $h' \wedge_L g'_1 = h' \wedge_L g_2 = 1$, we have $h' \wedge_L (g'_1g_2) = 1$ (by Lemma 2.11(i)). Therefore by Lemma 2.14.(ii)

$$h \wedge_L (g_1g_2) = (h_0h') \wedge_L (h_0g'_1g_2) = h_0(h' \wedge_L (g'_1g_2)) = h_0 = h \wedge_L g_1.$$

- (iii) and (iv) are direct consequences of (i) and (ii), respectively.
- (v) The proof is analogous to (i)–(iv).

Corollary 2.16. Suppose that a set $C \subset A(\Gamma)$ satisfies the following conditions.

- (P1) C is prefix-closed, i.e. if $g \in C$ and $h \leq_L g$, then $h \in C$.
- (P2) For $g \in A(\Gamma)$ and $x, y \in V(\Gamma)^{\pm 1}$ such that both gx and gy are geodesic, if $gx, gy \in C$ and $x \rightleftharpoons y$, then $gxy \in C$.

Then C is lcm-closed, i.e. if $g_1, g_2 \in C$ and $g_1 \vee_L g_2$ exists, then $g_1 \vee_L g_2 \in C$.

Proof. Let $g_1, g_2 \in C$ such that $g_1 \vee_L g_2$ exists. Let $g_0 = g_1 \wedge_L g_2$. Then

$$g_1 = g_0 g'_1$$
 and $g_2 = g_0 g'_2$

are geodesic for some $g'_1, g'_2 \in A(\Gamma)$. By Theorem 2.12, $g'_1 \rightleftharpoons g'_2$ and $g_1 \lor_L g_2 = g_0 g'_1 g'_2$.

We use induction on $||g'_1|| + ||g'_2||$. If $||g'_1|| = 0$ or $||g'_2|| = 0$, then $g_1 \vee_L g_2$ is either g_2 or g_1 , respectively, hence there is nothing to prove. If $||g'_1|| = ||g'_2|| = 1$, then $g_1 \vee_L g_2 = g_0 g'_1 g'_2 \in C$ by (P2). Therefore we may assume $||g'_1|| + ||g'_2|| \ge 3$ and $||g'_1||, ||g'_2|| \ge 1$.

Then $g'_1 = g''_1 x_1$ and $g'_2 = g''_2 x_2$ are geodesic for some $g''_1, g''_2 \in A(\Gamma)$ and $x_1, x_2 \in V(\Gamma)^{\pm 1}$. Thus

$$g_1 = g_0 g_1'' x_1$$
 and $g_2 = g_0 g_2'' x_2$

are geodesic, where $g_1''x_1 \rightleftharpoons g_2''x_2$.

Since $g_1, g_2 \in C$, we have $g_0 g_1'', g_0 g_2'' \in C$ by (P1), hence by the induction hypothesis we have

$$g_1 \vee_L (g_0 g_2'') = g_0 g_1'' g_2'' x_1 \in C$$
 and $g_0 g_1'' \vee_L g_2 = g_0 g_1'' g_2'' x_2 \in C.$

Therefore $g_1 \vee_L g_2 = g_0 g_1'' g_2'' x_1 x_2 \in C$ by (P2).

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3. Cyclic conjugations

Definition 3.1 (cyclically reduced). An element $g \in A(\Gamma)$ is called *cyclically reduced* if it has the minimal word length in its conjugacy class.

Servatius [28, Proposition on p. 38] showed that every $g \in A(\Gamma)$ has a unique geodesic decomposition

$$g = u^{-1}hu$$

with h cyclically reduced. The following lemma shows that u is determined from g by $u = g \wedge_R g^{-1}$.

Lemma 3.2. Let $g, h, u \in A(\Gamma)$.

- (i) If $g = u^{-1}hu$ is geodesic with h cyclically reduced, then $u = g \wedge_R g^{-1}$.
- (ii) g is cyclically reduced if and only if $g \wedge_R g^{-1} = 1$.

Proof. (i) We have two geodesic decompositions $g = u^{-1}hu$ and $g^{-1} = u^{-1}h^{-1}u$. By Lemma 2.14, it suffices to show $(u^{-1}h) \wedge_R (u^{-1}h^{-1}) = 1$ or equivalently $(hu) \wedge_L (h^{-1}u) = 1$.

Assume $(hu) \wedge_L (h^{-1}u) \neq 1$. Then there exists $x \in V(\Gamma)^{\pm 1}$ with $x \leq_L hu$ and $x \leq_L h^{-1}u$.

If $x \leq L h$, then $x \rightleftharpoons h$ (hence $x \rightleftharpoons h^{-1}$) and $x \leq L u$ (by Lemma 2.9(ii)). Let $u = xu_1$ be geodesic for some $u_1 \in A(\Gamma)$. Then $g = u^{-1}hu = u_1^{-1}x^{-1}hxu_1 = u_1^{-1}hu_1$. This contradicts that $g = u^{-1}hu$ is geodesic. Therefore $x \leq L h$. By the same reason, $x \leq L h^{-1}$, hence $x^{-1} \leq R h$.

Since $x \leq_L h$ and $x^{-1} \leq_R h$, $h = xh_1x^{-1}$ is geodesic for some h_1 (by Lemma 2.10(i)), which contradicts that h is cyclically reduced. Therefore $(hu) \wedge_L (h^{-1}u) = 1$.

(ii) It follows from (i).

Definition 3.3 (starting set). For $g \in A(\Gamma)$, the starting set S(g) of g is defined as

$$S(g) = \{ x \in V(\Gamma)^{\pm 1} : x \leq_L g \}.$$

Lemma 3.4. The following hold.

- (i) For any $g \in A(\Gamma)$, the following are equivalent.
 - (a) g is cyclically reduced.
 - (b) There is no geodesic decomposition such as $g = u^{-1}hu$, where $u, h \in A(\Gamma)$ with $u \neq 1$.
 - (c) For any geodesic decomposition $g = g_1g_2$, g_2g_1 is geodesic.
 - (d) For any $g_1 \in A(\Gamma)$ with $g_1 \leq _L g$, gg_1 is geodesic.
 - (e) $g^n = gg \cdots g$ is geodesic (i.e. $||g^n|| = n||g||$) for some $n \ge 2$.
 - (f) $g^n = gg \cdots g$ is geodesic (i.e. $||g^n|| = n||g||$) for all $n \ge 2$.
 - (g) q^n is cyclically reduced for some $n \ge 2$.
 - (h) g^n is cyclically reduced for all $n \ge 2$.
- (ii) Let $g_1 \cdots g_k$ be geodesic (i.e. $||g_1 \cdots g_k|| = ||g_1|| + \cdots + ||g_k||$). Then $g_1^{n_1} \cdots g_k^{n_k}$ is geodesic (i.e. $||g_1^{n_1} \cdots g_k^{n_k}|| = ||g_1^{n_1}|| + \cdots + ||g_k^{n_k}||$) for any positive integers n_i .
- (iii) For any $g \in A(\Gamma)$ and $n \ge 2$, $\operatorname{supp}(g^n) = \operatorname{supp}(g)$ and $S(g^n) = S(g)$.

Proof. (i) The equivalences between (a), (b), (c), (e), (f) are easy to prove. For example, see [25, Lemma 2.1]. We show the remaining equivalences assuming the known equivalences.

(a) \Rightarrow (d): Assume that $g_1 \leq_L g$ but gg_1 is not geodesic. Then there exists a letter $x \in V(\Gamma)^{\pm 1}$ such that $x \leq_R g$ and $x^{-1} \leq_L g_1$ (by Lemma 2.9(i)). Since $g_1 \leq_L g$, we have $x^{-1} \leq_L g$, hence $x \leq_R g^{-1}$. Now $x \leq_R g \wedge_R g^{-1}$, hence $g \wedge_R g^{-1} \neq 1$. By Lemma 3.2(ii), g is not cyclically reduced.

(d) \Rightarrow (e): Since $g \leq_L g$, gg is geodesic.

(f) \Rightarrow (h): Let $n \ge 2$. Since $||g^{2n}|| = 2n||g||$ and $||g^n|| = n||g||$, we have $||g^{2n}|| = 2||g^n||$, hence $g^n \cdot g^n$ is geodesic. Because (a) and (e) are equivalent, g^n is cyclically reduced.

(h) \Rightarrow (g): It is obvious.

(g) \Rightarrow (a): Assume that g is not cyclically reduced. Then $g = u^{-1}hu$ is geodesic for some $u, h \in A(\Gamma)$ such that $u \neq 1$ and h is cyclically reduced [28]. Observe that hh is geodesic (by (a) \Leftrightarrow (f)). Hence g^n has a geodesic decomposition $u^{-1}h^n u$ for any $n \geq 2$ (by Lemma 2.8). Therefore g^n is not cyclically reduced for any $n \geq 2$ (by (a) \Leftrightarrow (b)).

(ii) Let $g_i = u_i^{-1} h_i u_i$ be a geodesic decomposition of g_i with h_i cyclically reduced for i = 1, ..., k. Then

$$g_1 \cdots g_k = u_1^{-1} h_1 u_1 \cdots u_k^{-1} h_k u_k,$$

$$g_1^{n_1} \cdots g_k^{n_k} = u_1^{-1} h_1^{n_1} u_1 \cdots u_k^{-1} h_k^{n_k} u_k.$$

In particular, $u_1^{-1}h_1u_1\cdots u_k^{-1}h_ku_k$ is geodesic because $g_1\cdots g_k$ and each $g_i = u_i^{-1}h_iu_i$ are geodesic. Notice that each h_ih_i is geodesic by (i). Applying Lemma 2.8(iv), we get that $u_1^{-1}h_1^{n_1}u_1\cdots u_k^{-1}h_k^{n_k}u_k$ is geodesic. Therefore $g_1^{n_1}\cdots g_k^{n_k}$ is geodesic.

(iii) Let $g = u^{-1}hu$ be a geodesic decomposition of g with h cyclically reduced. Then

(3)
$$g^n = u^{-1}h^n u = u^{-1}h \cdots hu$$

are each geodesic decompositions of g^n (by Lemma 2.8(iii)).

Notice that $\operatorname{supp}(u) = \operatorname{supp}(u^{-1})$ and that if $g_1 \cdots g_k$ is a geodesic decomposition, then $\operatorname{supp}(g_1 \cdots g_k) = \operatorname{supp}(g_1) \cup \cdots \cup \operatorname{supp}(g_k)$. Therefore $\operatorname{supp}(g) = \operatorname{supp}(u) \cup \operatorname{supp}(h) = \operatorname{supp}(g^n)$ from (3).

Observe that $x \leq_L h^n$ if and only if $x \leq_L h$: if $x \leq_L h$, then it is obvious that $x \leq_L h^n$; if $x \notin_L h$, then $x \notin_L h^n$ (otherwise, $x \leq_L h^n = h \cdot h^{n-1}$ implies $x \rightleftharpoons h$ and $x \leq_L h^{n-1}$, which contradicts that $\operatorname{supp}(h) = \operatorname{supp}(h^{n-1})$).

Since $g = u^{-1}hu$ is geodesic, $x \leq_L g$ if and only if one of the following holds (by Lemma 2.9(ii)): (i) $x \leq_L u^{-1}$; (ii) $x \leq_L h$ and $x \rightleftharpoons u^{-1}$; (iii) $x \leq_L u$ and $x \rightleftharpoons u^{-1}h$. Notice that (iii) cannot happen. Since $x \leq_L h$ if and only if $x \leq_L h^n$, we can conclude that $x \leq_L g$ if and only if $x \leq_L g^n$. Therefore $S(g) = S(g^n)$.

Definition 3.5 (cycling, cyclic conjugation). Let $g \in A(\Gamma)$ be cyclically reduced.

- (i) For a letter $x \in V(\Gamma)^{\pm 1}$, the conjugation $g^x = x^{-1}gx$ is called a *left* (resp. *right*) cycling if $x \leq_L g$ (resp. $x^{-1} \leq_R g$). Left and right cyclings are collectively called cyclings.
- (ii) For an element $u \in A(\Gamma)$, the conjugation $g^u = u^{-1}gu$ is called a *cyclic conjugation* of g by u if $||g^u|| = ||g||$ and $\operatorname{supp}(u) \subset \operatorname{supp}(g)$. A cyclic conjugation g^u is called a *left* (resp. *right*) *cyclic conjugation* if gu (resp. $u^{-1}g$) is geodesic.

For $g \in A(\Gamma)$ and $x \in V(\Gamma)^{\pm 1}$, if g^x is a left cycling, i.e. $x \leq_L g$, then g = xh is geodesic for some $h \in A(\Gamma)$ and $g^x = x^{-1}gx = hx$ is geodesic. Therefore the left cycling g^x is obtained from g = xh by moving the first letter x to the last. Similarly, if g^x is a right cycling, then g^x is obtained from $g = hx^{-1}$ by moving the last letter x^{-1} to the first.

If g^x is a cycling, then it is easy to see that $||g^x|| = ||g||$ and $\operatorname{supp}(x) \subset \operatorname{supp}(g)$, hence g^x is a cyclic conjugation. Conversely, we will show in Lemma 3.7 that a cyclic conjugation g^u is obtained by iterated application of cyclings.

If $g \in A(\Gamma)$ is cyclically reduced and g^u is a cyclic conjugation, then $||g^u|| = ||g||$, hence g^u is also cyclically reduced.

Lemma 3.6. Let $g \in A(\Gamma)$ and $x, y \in V(\Gamma)^{\pm 1}$ with g cyclically reduced.

- (i) The conjugation g^x cannot be both a left cycling and a right cycling.
- (ii) Let $y \neq x^{-1}$. If g^x and $(g^x)^y$ are cyclings of different type, then $x \rightleftharpoons y$.
- (iii) Let $x \rightleftharpoons y$. If both g^x and $(g^x)^y$ are cyclings, then so are g^y and $(g^y)^x$.
- (iv) Let $x \rightleftharpoons y$. If both g^x and g^y are cyclings, then so are $(g^x)^y$ and $(g^y)^x$.

In (iii) and (iv), the types of cyclings depend only on the conjugating letters. For example, if g^x is a left cycling, then $(g^y)^x$ is also a left cycling, and so on.

Proof. (i) If g^x is both a left cycling and a right cycling, then $x \leq_L g$ and $x^{-1} \leq_R g$, hence $g = xhx^{-1}$ is geodesic for some $h \in A(\Gamma)$ (by Lemmas 2.10(i)). Thus g is not cyclically reduced (by Lemma 3.4(i)).

(ii) Assume that g^x is a left cycling and $(g^x)^y$ is a right cycling. (An analogous argument applies to the case where g^x is a right cycling and $(g^x)^y$ is a left cycling.)

Since g^x is a left cycling, we have $x \leq_L g$, hence g = xh is geodesic for some $h \in A(\Gamma)$. Notice that $g^x = hx$ is geodesic. Since $(g^x)^y$ is a right cycling, $y^{-1} \leq_R g^x = hx$.

Since $y^{-1} \neq x$, we have $x \rightleftharpoons y^{-1}$ (by Lemma 2.9(iii)) and hence $x \rightleftharpoons y$.

(iii) and (iv) Assume that g^x and $(g^x)^y$ are left cyclings, hence $x \leq_L g$ and $y \leq_L g^x$. Since $x \leq_L g$, $g = xh_1$ is geodesic for some $h_1 \in A(\Gamma)$, hence $g^x = h_1 x$ is also geodesic. Since $y \leq_L g^x = h_1 x$ and $y \rightleftharpoons x$ (hence $y \leq_L x$), we have $y \leq_L h_1$, hence $h_1 = yh_2$ is geodesic for some $h_2 \in A(\Gamma)$. Now we know that

$$g = xh_1 = xyh_2 = yxh_2$$

and $g^y = xh_2y$ are all geodesic, hence $y \leq_L g$ and $x \leq_L g^y$. This means that g^y and $(g^y)^x$ are left cyclings.

For the other cases, it is easy to see that g has a geodesic decomposition as one of xyh, xhy^{-1} , yhx^{-1} and $hx^{-1}y^{-1}$ depending on the types of cyclings, from which the conclusions follow.

Lemma 3.7. Let $g, u, u_1, u_2 \in A(\Gamma)$ with g cyclically reduced.

- (i) The following are equivalent:
 - (a) g^u is a cyclic (resp. left cyclic, right cyclic) conjugation;
 - (b) there exists a reduced word $w_0 \equiv y_1 \cdots y_k$ representing u such that $(g^{y_1 \cdots y_{i-1}})^{y_i}$ is a cycling (resp. left cycling, right cycling) for all $1 \leq i \leq k$;
 - (c) for any reduced word $w \equiv x_1 \cdots x_k$ representing u, $(g^{x_1 \cdots x_{i-1}})^{x_i}$ is a cycling (resp. left cycling, right cycling) for all $1 \leq i \leq k$.

In particular, if g^u is a cyclic conjugation, then $\operatorname{supp}(g^u) = \operatorname{supp}(g)$.

- (ii) Let $u = u_1 u_2$ be geodesic. Then g^u is a cyclic (resp. left cyclic, right cyclic) conjugation if and only if both g^{u_1} and $(g^{u_1})^{u_2}$ are cyclic (resp. left cyclic, right cyclic) conjugations.
- (iii) If g^{u_1} and g^{u_2} are cyclic (resp. left cyclic, right cyclic) conjugations and $u_1 \vee_L u_2$ exists, then $g^{u_1 \vee_L u_2}$ is also a cyclic (resp. left cyclic, right cyclic) conjugation.
- (iv) Let $u_1 \rightleftharpoons u_2$. Suppose that g^{u_1} and g^{u_2} are a left cyclic conjugation and a right cyclic conjugation, respectively. Then $(g^{u_2})^{u_1}$ and $(g^{u_1})^{u_2}$ are a left cyclic conjugation and a right cyclic conjugation, respectively. Moreover, $u_2^{-1}gu_1$ is geodesic.
- (v) Suppose that g^u is a cyclic conjugation. Then there is a geodesic decomposition $u = u_1 u_2$ such that $u_1 \rightleftharpoons u_2$ and g^{u_1} (resp. g^{u_2}) is a left (resp. right) cyclic conjugation. Moreover, $u_2^{-1}gu_1$ is geodesic.

Proof. The statements (i)–(iii) concern three types of cyclic conjugations. We prove only the case of cyclic conjugations. The other cases (i.e. left and right cyclic conjugations) can be proved analogously.

We use the following claim.

Claim 1. If g^u is a cyclic conjugation for some $u \in A(\Gamma) \setminus \{1\}$, then there exists $y_1 \in V(\Gamma)^{\pm 1}$ and $u_1 \in A(\Gamma)$ such that $u = y_1 u_1$ is geodesic, g^{y_1} is a cycling and $(g^{y_1})^{u_1}$ is a cyclic conjugation.

Proof of Claim 1. Since $||g^u|| = ||g||$, the decomposition $u^{-1}gu$ is not geodesic. If both $u^{-1}g$ and gu are geodesic, then there exists $x \in V(\Gamma)^{\pm 1}$ such that $x^{-1} \leq_R u^{-1}$, $x \leq_L u$ and $x \rightleftharpoons g$ (by Lemma 2.8(i)). However, the relation $x \rightleftharpoons g$ is impossible because $x \leq_L u$ and $\supp(u) \subset \supp(g)$. Hence either $u^{-1}g$ or gu is not geodesic, i.e. there exists $y_1 \in V(\Gamma)^{\pm 1}$ such that either $y_1^{-1} \leq_R u^{-1}$ and $y_1 \leq_L g$ or $y_1 \leq_L u$ and $y_1^{-1} \leq_R g$. This means that g^{y_1} is a cycling and that $u = y_1u_1$ is geodesic for some $u_1 \in A(\Gamma)$. Therefore $(g^{y_1})^{u_1}$ is a cyclic conjugation because $||(g^{y_1})^{u_1}|| = ||g^u|| = ||g|| = ||g^{y_1}||$ and $\supp(u_1) \subset \supp(u) \subset \supp(g) = \supp(g^{y_1})$.

(i) We may assume $u \neq 1$ because otherwise there is nothing to prove.

(a) \Rightarrow (b): Suppose that g^u is a cyclic conjugation. By Claim 1, there is a geodesic decomposition $u = y_1 u_1$ such that g^{y_1} is a cycling and $(g^{y_1})^{u_1}$ is a cyclic conjugation. Applying Claim 1 again to $(g^{y_1})^{u_1}$, we have a geodesic decomposition $u_1 = y_2 u_2$ such that $(g^{y_1})^{y_2}$ is a cycling and $(g^{y_1y_2})^{u_2}$ is a cyclic conjugation. Iterating this process, we get a desired reduced word $w_0 \equiv y_1 \dots y_k$.

(b) \Rightarrow (c): Let $w \equiv x_1 \cdots x_k$ be a reduced word representing u. Notice that the word $w_0 \equiv y_1 \cdots y_k$ can be transformed into the word $w \equiv x_1 \cdots x_k$ by using only commutation relations. Therefore each $(g^{x_1 \cdots x_{i-1}})^{x_i}$ is a cycling (by Lemma 3.6(iii)).

(c) \Rightarrow (a): Let $w \equiv x_1 \cdots x_k$ be a reduced word representing u, where $x_i = v_i^{\epsilon_i}$, $v_i \in V(\Gamma)$ and $\epsilon_i = \pm 1$ for all $1 \leq i \leq k$. Then, for each $1 \leq i \leq k$, $(g^{x_1 \cdots x_{i-1}})^{x_i}$ is a cycling, hence

$$||g^{x_1\cdots x_{i-1}}|| = ||g^{x_1\cdots x_i}||$$
 and $v_i \in \operatorname{supp}(g^{x_1\cdots x_{i-1}}) = \operatorname{supp}(g^{x_1\cdots x_i})$

Thus $||g^{x_1\cdots x_k}|| = ||g||$ and $\{v_1, \ldots, v_k\} \subset \operatorname{supp}(g^{x_1\cdots x_k}) = \operatorname{supp}(g)$. Therefore $||g^u|| = ||g||$ and $\operatorname{supp}(u) \subset \operatorname{supp}(g)$, hence g^u is a cyclic conjugation.

(ii) Let $u_1 = x_1 \cdots x_j$ and $u_2 = x_{j+1} \cdots x_k$ be geodesic decompositions, where $x_1 \cdots x_k \in V(\Gamma)^{\pm 1}$. Then $u = x_1 \cdots x_k$ is also geodesic because $u = u_1 u_2$ is geodesic. By (i), g^u is a cyclic conjugation if and only if $(g^{x_1 \cdots x_{i-1}})^{x_i}$ is a cycling for each $1 \leq i \leq k$, and this happens if and only if both g^{u_1} and $(g^{u_1})^{u_2}$ are cyclic conjugations.

(iii) Let C(g) be the set of all $u \in A(\Gamma)$ such that g^u is a cyclic conjugation. Then C(g) satisfies (P1) in Corollary 2.16 by (ii) in this lemma. Therefore it suffices to show that C(g) satisfies (P2) in Corollary 2.16.

Let $ux, uy \in C(g)$ (i.e. g^{ux} and g^{uy} are cyclic conjugations) such that ux and uy are geodesic and $x \rightleftharpoons y$, where $u \in A(\Gamma)$ and $x, y \in V(\Gamma)^{\pm 1}$. Then both $(g^u)^x$ and $(g^u)^y$ are cyclings of g^u (by (ii)). By Lemma 3.6(iv), $(g^{ux})^y$ is a cycling, hence g^{uxy} is a cyclic conjugation (by (ii)). Therefore $uxy \in C(g)$, hence C(g) satisfies (P2) in Corollary 2.16.

(iv) Notice that $u_1 \vee_L u_2 = u_1 u_2 = u_2 u_1$ and that both $u_1 u_2$ and $u_2 u_1$ are geodesic, because $u_1 \rightleftharpoons u_2$. Since both g^{u_1} and g^{u_2} are cyclic conjugations, so are $g^{u_1 u_2}$, $(g^{u_1})^{u_2}$ and $(g^{u_2})^{u_1}$ (by (ii) and (iii)).

Let us show that the cyclic conjugation $(g^{u_2})^{u_1}$ is a left cyclic conjugation, i.e. the decomposition $g^{u_2}u_1$ is geodesic. (The proof for $(g^{u_1})^{u_2}$ is analogous.) Observe

$$u_2^{-1}gu_1 = g^{u_2}u_1u_2^{-1}.$$

Since both $u_2^{-1}g$ and gu_1 are geodesic and since $u_1 \rightleftharpoons u_2$, $u_2^{-1}gu_1$ is geodesic (by Lemma 2.8(ii)). Since $||g^{u_2}|| = ||g||$, the decomposition $g^{u_2}u_1u_2^{-1}$ is also geodesic. Therefore $g^{u_2}u_1$ is geodesic.

(v) We use induction on ||u||. If ||u|| = 1, there is nothing to prove.

Suppose that u = u'x is geodesic for some $u' \in A(\Gamma) \setminus \{1\}$ and $x \in V(\Gamma)^{\pm 1}$. Then $g^{u'}$ is a cyclic conjugation and $(g^{u'})^x$ is a cycling (by (ii)). Suppose that $(g^{u'})^x$ is a left cycling. (The proof is analogous for the case where $(g^{u'})^x$ is a right cycling.) By the induction hypothesis, we have a geodesic decomposition $u' = u'_1 u'_2$ such that $u'_1 \rightleftharpoons u'_2$ and $g^{u'_1}$ (resp. $g^{u'_2}$) is a left (resp. right) cyclic conjugation.

Claim 2. $x \rightleftharpoons u'_2$, and $u = u'_1 x u'_2$ is geodesic.

Proof of Claim 2. Let $u'_2 = y_1 \cdots y_k$ be geodesic, where $y_1, \ldots, y_k \in V(\Gamma)^{\pm 1}$. Then u has the following three geodesic decompositions:

$$u = u'x = u'_1u'_2x = u'_1y_1\cdots y_kx.$$

Let $h_0 = g^{u'_1}$ and $h_i = g^{u'_1 y_1 \cdots y_i}$ for $1 \leq i \leq k$. Then each h_i is cyclically reduced (by (ii)), and $h_i = h_{i-1}^{y_i}$. Since $(g^{u'_1})^{u'_2}$ is a right cyclic conjugation (by (iv)), each $h_{i-1}^{y_i} = (g^{u'_1 y_1 \cdots y_{i-1}})^{y_i}$ is a right cycling (by (i)).

Since $u = u'_1 y_1 \cdots y_k x$ is geodesic, we have $y_k \neq x^{-1}$. We know that $h_{k-1}^{y_k}$ is a right cycling and that $(h_{k-1}^{y_k})^x = (g^{u'})^x$ is a left cycling, hence $x \rightleftharpoons y_k$ (by Lemma 3.6(ii)). Therefore $u = u'_1 y_1 \cdots y_{k-1} x y_k$ and h_{k-1}^x is a left cycling (by Lemma 3.6(iii)).

Applying the above argument to the right cyclings $h_{k-2}^{y_{k-1}}, \ldots, h_0^{y_1}$ in this order iteratively, we obtain $x \rightleftharpoons y_i$ for all $1 \le i \le k$. Therefore $x \rightleftharpoons u'_2$ and hence $u = u'_1 u'_2 x = u'_1 x u'_2$. Since $u'_1 u'_2 x$ is geodesic, so is $u'_1 x u'_2$.

Let $u_1 = u'_1 x$ and $u_2 = u'_2$. Then $u = u_1 u_2$ is geodesic, $u_1 \rightleftharpoons u_2$ and g^{u_1} (resp. g^{u_2}) is a left (resp. right) cyclic conjugation. Moreover, $u_2^{-1}gu_1$ is geodesic (by (iv))

For a cyclically reduced $g \in A(\Gamma)$, if $u \leq_L g$, then g^u is obviously a left cyclic conjugation. The following proposition is concerned with the opposite direction.

Proposition 3.8. Let $g, u \in A(\Gamma)$ with g cyclically reduced. Then the following are equivalent.

- (i) g^u is a left (resp. right) cyclic conjugation.
- (ii) $u \leq_L g^n$ (resp. $u^{-1} \leq_R g^n$) for some $n \geq 1$.

Proof. We prove the equivalence only for the left cyclic conjugation. The proof for the right cyclic conjugation is analogous. We may assume $||g|| \ge 2$ and $||u|| \ge 1$ (otherwise it is obvious).

(ii) \Rightarrow (i): We may assume $n \ge 2$ (otherwise it is obvious). We proceed by induction on ||u||. If ||u|| = 1, then u is a letter. In this case, $u \le_L g^n$ implies $u \le_L g$ (by Lemma 3.4(iii)), hence g^u is a left cycling.

Suppose $||u|| \ge 2$. Then $u = xu_1$ is geodesic for some $x \in V(\Gamma)^{\pm 1}$ and $u_1 \in A(\Gamma) \setminus \{1\}$. Since $xu_1 = u \leq_L g^n$, we get $x \leq_L g^n$ and hence $x \leq_L g$ (by Lemma 3.4(iii)). Therefore $g = xg_1$ is geodesic for some $g_1 \in A(\Gamma)$, and $g^x = g_1 x$ is also geodesic. Since both $g \cdots g$ and $g = xg_1$ are geodesic, $xg_1xg_1 \cdots xg_1x$ is geodesic, hence the following three decompositions are all geodesic.

$$g^n x = xg_1 xg_1 \cdots xg_1 x = x(g^x)^n$$

Since $xu_1 = u \leq_L g^n \leq_L g^n x = x(g^x)^n$, we have $u_1 \leq_L (g^x)^n$ (by Lemma 2.6(ii)). By the induction hypothesis, $(g^x)^{u_1}$ is a left cyclic conjugation. And g^x is also a left cyclic conjugation because $x \leq_L g$. Therefore g^u is a left cyclic conjugation (by Lemma 3.7(ii)).

(i) \Rightarrow (ii): As before, we use induction on ||u||. If ||u|| = 1, g^u is a left cycling, hence $u \leq_L g$.

Suppose $||u|| \ge 2$. Then $u = xu_1$ is geodesic for some $x \in V(\Gamma)^{\pm}$ and $u_1 \in A(\Gamma) \setminus \{1\}$. Since g^u is a left cyclic conjugation, both g^x and $(g^x)^{u_1}$ are left cyclic conjugations (by Lemma 3.7(ii)). Since g^x is a left cyclic conjugation and x is a letter, we have $x \leq_L g$. Since $(g^x)^{u_1}$ is a left cyclic conjugation, $u_1 \leq_L (g^x)^n$ for some n by the induction hypothesis. Using a similar argument as above, we get that $x(g^x)^n$ and $g^n x$ are geodesic, hence $u = xu_1 \leq_L x(g^x)^n = g^n x \leq_L g^{n+1}$.

From the above proposition, g^u is a right cyclic conjugation if and only if $(g^{-1})^u$ is a left cyclic conjugation.

Theorem 3.9. Let $g, u \in A(\Gamma)$ with g cyclically reduced. Then there exists a unique geodesic decomposition $u = u_1 u_2 u_3$ such that

- (i) u_1 disjointly commutes with g;
- (ii) g^{u_2} is a cyclic conjugation;
- (iii) $g^u = u_3^{-1} g^{u_2} u_3$ is geodesic, i.e. $\|g^u\| = \|u_3^{-1}\| + \|g^{u_2}\| + \|u_3\| = \|g\| + 2\|u_3\|.$

Moreover, the following hold: u_1 is the maximal prefix of u that disjointly commutes with g; u_2 is the maximal prefix of u such that g^{u_2} is a cyclic conjugation; $u_3 = g^u \wedge_R (g^u)^{-1}$. In particular, $u_1 \rightleftharpoons u_2$.

Proof. We first prove the existence of the decomposition $u = u_1 u_2 u_3$.

If u'_1 and u''_1 are prefixes of u such that $u'_1 \rightleftharpoons g$ and $u''_1 \rightleftharpoons g$, then $u'_1 \lor_L u''_1$ exists (because u'_1 and u''_1 have a common right multiple u). Observe that $u'_1 \lor_L u''_1$ is also a prefix of u and also disjointly commutes with g (by Theorem 2.12). Therefore there exists a unique maximal prefix u_1 of u that disjointly commutes with g.

If u'_2 and u''_2 are prefixes of u such that $g^{u'_2}$ and $g^{u''_2}$ are cyclic conjugations, then $u'_2 \vee_L u''_2$ exists (because u'_2 and u''_2 have a common right multiple u) and is also a prefix of u, and $g^{u'_2 \vee_L u''_2}$ is also a cyclic conjugation (by Lemma 3.7(iii)). Therefore there exists a unique maximal prefix u_2 of u such that g^{u_2} is a cyclic conjugation.

Notice that $u_1 \rightleftharpoons u_2$ because $\operatorname{supp}(u_2) \subset \operatorname{supp}(g)$ and $u_1 \rightleftharpoons g$. Thus $u_1 \lor_L u_2 = u_1 u_2$ is a prefix of u and $u_1 u_2$ is geodesic, hence $u = u_1 u_2 u_3$ is geodesic for some $u_3 \in A(\Gamma)$. Observe

$$g^{u} = u_{3}^{-1}u_{2}^{-1}u_{1}^{-1}gu_{1}u_{2}u_{3} = u_{3}^{-1}u_{2}^{-1}gu_{2}u_{3} = u_{3}^{-1}g^{u_{2}}u_{3}.$$

Let us show that $u_3^{-1}g^{u_2}u_3$ is geodesic.

If $g^{u_2}u_3$ is not geodesic, then there exists $x \in V(\Gamma)^{\pm 1}$ such that $x \leq_L u_3$ and $x^{-1} \leq_R g^{u_2}$ (by Lemma 2.9(i)), hence $(g^{u_2})^x$ is a cyclic conjugation. Notice that u_2x is geodesic. By Lemma 3.7(ii), g^{u_2x} is also a cyclic conjugation, hence $x \in \text{supp}(g)$, which implies $x \rightleftharpoons u_1$. Consequently, $u_2x \rightleftharpoons u_1$ and hence $u_2x \leq_L u$. This contradicts the maximality of u_2 . Therefore $g^{u_2}u_3$ is geodesic. Similarly $u_3^{-1}g^{u_2}$ is geodesic.

Since both $u_3^{-1}g^{u_2}$ and $g^{u_2}u_3$ are geodesic, if $u_3^{-1}g^{u_2}u_3$ is not geodesic, then there exists $x \in V(\Gamma)^{\pm 1}$ such that $x \leq_L u_3$, $x^{-1} \leq_R u_3^{-1}$ and $x \rightleftharpoons g^{u_2}$ (by Lemma 2.8(i)). Since $\operatorname{supp}(g) = \operatorname{supp}(g^{u_2})$, we have $x \rightleftharpoons g$, hence $x \rightleftharpoons u_2$ and $u_1 x \rightleftharpoons g$. Notice that $u_1 x$ is geodesic. Since $u_1 x$ is a prefix of u, this contradicts the maximality of u_1 . Therefore $u_3^{-1}g^{u_2}u_3$ is geodesic.

Since $g^u = u_3^{-1} g^{u_2} u_3$ is geodesic such that g^{u_2} is cyclically reduced, u_3 satisfies the formula $u_3 = g^u \wedge_R (g^u)^{-1}$ (by Lemma 3.2(i)).

So far we have shown that $u = u_1 u_2 u_3$ is a desired decomposition. We will now show the uniqueness of the decomposition. Let $u = u'_1 u'_2 u'_3$ be another geodesic decomposition satisfying the conditions (i), (ii) and (iii) of the theorem. Since u'_2 and u'_3 satisfy the conditions (ii) and (iii), we have $u'_3 = g^u \wedge_R (g^u)^{-1}$ (by Lemma 3.2(i)), hence

$$u_3 = u'_3$$
, $g^{u_2} = g^{u'_2}$ and $u_1 u_2 = u'_1 u'_2$.

Since both u_1 and u'_1 are prefixes of u that disjointly commute with g, so is $u_1 \vee_L u'_1$ (by Theorem 2.12). By the maximality of u_1 , we have $u_1 \vee_L u'_1 \leq_L u_1$, hence $u'_1 \leq_L u_1$.

Similarly, since g^{u_2} and $g^{u'_2}$ are cyclic conjugations, so is $g^{u_2 \vee_L u'_2}$ (by Lemma 3.7). By the maximality of u_2 , we have $u_2 \vee_L u'_2 \leq_L u_2$, hence $u'_2 \leq_L u_2$.

Since $u'_1 \leq u_1, u'_2 \leq u_2$ and $u_1 u_2 = u'_1 u'_2$, we have $u_1 = u'_1$ and $u_2 = u'_2$.

The following seems to be well known to experts.

Corollary 3.10. Let $g_1, g_2 \in A(\Gamma)$ be cyclically reduced. If g_1 and g_2 are conjugate, then they are cyclically conjugate.

Proof. Since g_1 and g_2 are conjugate, $g_2 = g_1^u$ for some $u \in A(\Gamma)$. Let $u = u_1 u_2 u_3$ be the geodesic decomposition for g_1^u as in Theorem 3.9. Since $u_1 \rightleftharpoons g_1$, we may assume $u_1 = 1$. Since $u_3^{-1} g_1^{u_2} u_3$ is a geodesic decomposition of g_2 and since $||g_2|| = ||g_1|| = ||g_1^{u_2}||$, we have $u_3 = 1$. Therefore $u = u_2$, hence g_1 is cyclically conjugate to g_2 .

4. Star length

Star lengths of elements of $A(\Gamma)$, introduced in [18], induce a metric d_* on $A(\Gamma)$ such that the metric space $(A(\Gamma), d_*)$ is quasi-isometric to the extension graph (Γ^e, d) , preserving the right action of $A(\Gamma)$. In this section, we study basic properties of star lengths.

It is known that the centralizer Z(v) of $v \in V(\Gamma)$ in $A(\Gamma)$ is generated by the vertices in $St_{\Gamma}(v)$.

Definition 4.1 (star-word, star length). An element in the centralizer Z(v) of some vertex v is called a *star-word*. The *star length* of $g \in A(\Gamma)$, denoted $||g||_*$, is the minimum ℓ such that g is written as a product of ℓ star-words. Let d_* denote the right-invariant metric on $A(\Gamma)$ induced by the star length: $d_*(g_1, g_2) = ||g_1g_2^{-1}||_*$.

The following example illustrates that the decompositions into star-words are not unique.

Example 4.2. Let $\Gamma = \overline{P}_5$, where $P_5 = (v_1, \ldots, v_5)$ is a path graph, and let the underlying rightangled Artin group here be $A(\Gamma)$, hence $v_i v_j = v_j v_i$ whenever $|i - j| \ge 2$. Let $g = v_1 v_3 v_5 v_2 v_4$. The following shows various decompositions of g into two star-words.

$$g = (v_1 v_3 v_5)(v_2 v_4) = (v_1 v_3 v_5 v_2)(v_4) = (v_1 v_3 v_5 v_4)(v_2)$$

= $(v_1 v_3 v_2)(v_5 v_4) = (v_3 v_5 v_4)(v_1 v_2) = (v_3 v_5)(v_4 v_1 v_2).$

Notice that all the parenthesized words are star-words. For example, $v_1v_3v_5 \in Z(v_i)$ for i = 1, 3, 5, $v_4 \in Z(v_i)$ for i = 1, 2, 4, $v_1v_3v_5v_2 \in Z(v_5)$ and so on. Since $\operatorname{supp}(g) = \{v_1, \ldots, v_5\}$ is not contained in $\operatorname{St}(v_i)$ for any $1 \leq i \leq 5$, we have $||g||_* = 2$.

The group $A(\Gamma)$ acts on $(A(\Gamma), d_*)$ by right multiplication $w \mapsto wg$. Recall that $A(\Gamma)$ acts on (Γ^e, d) by conjugation $v^w \mapsto v^{wg}$. For any $v \in V(\Gamma)$, the following map is equivariant.

$$\phi_v : A(\Gamma) \to \Gamma^e, \qquad \phi_v(w) = v^w$$

Lemma 4.3. [18, Lemma 19] Let Γ be connected and let $D = \operatorname{diam}(\Gamma)$. The following holds between the metric d on Γ^e and the star length $\|\cdot\|_*$ on $A(\Gamma)$: for any $g \in A(\Gamma)$ and $v \in V(\Gamma)$,

$$||g||_* - 1 \le d(v^g, v) \le D(||g||_* + 1)$$

Notice that $d(\phi_v(g), \phi_v(h)) = d(v^g, v^h) = d(v^{gh^{-1}}, v)$ and $d_*(g, h) = ||gh^{-1}||_*$. Therefore the above lemma implies that $d_*(g, h) - 1 \leq d(v^g, v^h) \leq D(d_*(g, h) + 1)$, and hence that ϕ_v is a quasi-isometry. The above lemma also yields the following corollary for the asymptotic translation length.

Corollary 4.4. Let Γ be connected and let $D = \operatorname{diam}(\Gamma)$. For every $g \in A(\Gamma)$,

$$\tau_{(A(\Gamma),d_*)}(g) \leqslant \tau_{(\Gamma^e,d)}(g) \leqslant D\tau_{(A(\Gamma),d_*)}(g)$$

Proof. Notice that

$$\tau_{(A(\Gamma),d_*)}(g) = \lim_{n \to \infty} \frac{d_*(g^n, 1)}{n} = \lim_{n \to \infty} \frac{\|g^n\|_*}{n}$$
$$\tau_{(\Gamma^e,d)}(g) = \lim_{n \to \infty} \frac{d(v^{g^n}, v)}{n},$$

where v is any vertex of Γ . By Lemma 4.3,

$$\frac{\|g^n\|_* - 1}{n} \leqslant \frac{d_*(v^{g^n}, v)}{n} \leqslant \frac{D(\|g^n\|_* + 1)}{n}.$$

By taking n to infinity, we get the desired inequalities.

The following lemma shows basic properties of star length.

Lemma 4.5. Let $g_1, g_2, g_3, g, h \in A(\Gamma)$.

- (i) If $g_1g_2g_3$ is geodesic, then $||g_1g_3||_* \leq ||g_1g_2g_3||_*$. In particular, if $g \leq_L h$ or $g \leq_R h$, then $||g||_* \leq ||h||_*$.
- (ii) $||g^m||_* \leq ||g^n||_*$ for all $1 \leq m \leq n$.
- (iii) If $g \rightleftharpoons h$ and $h \neq 1$, then $||g||_* \leq 1$.

Proof. Let us denote $g \preccurlyeq_0 h$ if a reduced word representing g can be obtained by deleting some letters from a reduced word representing h. For example, if v_i 's are distinct vertices, then $v_1v_3 \preccurlyeq_0 v_1v_2v_3v_4$. It is proved in [18, Lemma 20(i)] that if $g \preccurlyeq_0 h$, then $\|g\|_* \leq \|h\|_*$.

(i) Since $g_1g_2g_3$ is geodesic, we have $g_1g_3 \preccurlyeq_0 g_1g_2g_3$, hence $||g_1g_3||_* \leqslant ||g_1g_2g_3||_*$.

(ii) Let $g = u^{-1}hu$ be geodesic such that h is cyclically reduced. Then $g^k = u^{-1}\underbrace{h\cdots h}_{h}u$ is also

geodesic for all $k \ge 1$ (by Lemma 2.8(iii)). Therefore $g^m \preccurlyeq_0 g^n$, hence $\|g^m\|_* \le \|g^n\|_*$.

(iii) Since $h \neq 1$, there is a vertex $v \in \text{supp}(h)$. Then $g \in Z(v)$, namely $||g||_* \leq 1$.

Lemma 4.6. Suppose that $g_1, g_2 \in A(\Gamma)$ have a common right multiple and that none of them is a prefix of the other, i.e. $g_1 \not\leq_L g_2$ and $g_2 \not\leq_L g_1$. Then $\|g_1^{-1}g_2\|_* \leq 2$ and $\|g_1\|_* - \|g_2\|_* \in \{0, \pm 1\}$.

Proof. Let $g_i = (g_1 \wedge_L g_2)g'_i$ for i = 1, 2. Since g_1 and g_2 have a common right multiple, $g'_1 \rightleftharpoons g'_2$ (by Theorem 2.12). Since $g_1 \not\leq_L g_2$ and $g_2 \not\leq_L g_1$, both g'_1 and g'_2 are nontrivial, hence $||g'_1||_* = ||g'_2||_* = 1$ (by Lemma 4.5(iii)). Therefore

$$||g_1^{-1}g_2||_* = ||g_1'^{-1}g_2'||_* \leq ||g_1'||_* + ||g_2'||_* = 1 + 1 = 2.$$

Furthermore, for each i = 1, 2,

 $||g_1 \wedge_L g_2||_* \leq ||g_i||_* \leq ||g_1 \wedge_L g_2||_* + ||g_i'||_* = ||g_1 \wedge_L g_2||_* + 1,$

hence $||g_i||_* = ||g_1 \wedge_L g_2||_* + \epsilon_i$, where $\epsilon_i \in \{0, 1\}$. Therefore $||g_1||_* - ||g_2||_* = \epsilon_1 - \epsilon_2 \in \{0, \pm 1\}$. \Box

Corollary 4.7. Let $g_1, g_2, h \in A(\Gamma)$ with g_1g_2 geodesic. If $h \leq_L g_1g_2$ and $||g_1||_* \geq ||h||_* + 2$, then $h \leq_L g_1$.

Proof. Observe that $g_1 \not\leq_L h$ (otherwise $||g_1||_* \leq ||h||_*$). Assume $h \not\leq_L g_1$. Since g_1 and h have a common right multiple, say g_1g_2 , we have $||g_1||_* - ||h||_* \in \{0, \pm 1\}$ (by Lemma 4.6). This contradicts that $||g_1||_* \geq ||h||_* + 2$.

Corollary 4.8. Let $g_1, g_2 \in A(\Gamma)$. If g_1g_2 is geodesic, then

$$||g_1||_* + ||g_2||_* - 2 \leq ||g_1g_2||_* \leq ||g_1||_* + ||g_2||_*.$$

Proof. Let $r = ||g_1||_*$, $s = ||g_2||_*$ and $t = ||g_1g_2||_*$. Then it is obvious that $t \leq r + s$, hence it suffices to show $t \geq r + s - 2$. Since $g_2 \leq_R g_1g_2$, we have $t = ||g_1g_2||_* \geq ||g_2||_* = s$ (by Lemma 4.5). We may assume $r \geq 3$ because otherwise $t \geq s \geq r + s - 2$. Let

$$g_1g_2 = w_1w_2\cdots w_t$$

be a geodesic decomposition of g_1g_2 into star-words. Then $w_1 \cdots w_{r-2} \leq_L g_1$ (by Corollary 4.7), hence $g_2 \leq_R w_{r-1} \cdots w_t$. Therefore $s = ||g_2||_* \leq ||w_{r-1} \cdots w_t||_* = t - r + 2$, namely $t \geq r + s - 2$.

The following example shows that the upper and lower bounds in the above corollary are sharp.

Example 4.9. Let $\Gamma = \overline{P}_5$, where $P_5 = (v_1, \ldots, v_5)$, and let the underlying right-angled Artin group here be $A(\Gamma)$.

(i) Let $g_1 = v_1 v_2$ and $g_2 = v_3 v_4$. Then $g_1 \rightleftharpoons v_5$ and $g_2 \rightleftharpoons v_1$ and hence $||g_1||_* = ||g_2||_* = 1$. Since $g_1 g_2 = v_1 v_2 v_3 v_4 \notin Z(v_i)$ for any $1 \le i \le 5$, we have $||g_1 g_2||_* \ge 2$. Since $||g_1 g_2||_* \le ||g_1||_* + ||g_2||_* = 2$, we have $||g_1 g_2||_* = ||g_1||_* + ||g_2||_* = 1$.

(ii) Let $g_1 = g_2 = v_2 v_3 v_4$. Then $g_1 g_2 = v_2 v_3 v_4 \cdot v_2 v_3 v_4 = v_2 v_3 v_2 \cdot v_4 v_3 v_4$. Since $v_2 v_3 v_2 \in Z(v_5)$ and $v_4 v_3 v_4 \in Z(v_1)$, we have $||v_2 v_3 v_2||_* = ||v_4 v_3 v_4||_* = 1$. It is easy to see that $||g_1||_* = ||g_2||_* = ||g_1g_2||_* = 2$. Therefore $||g_1g_2||_* = ||g_1||_* + ||g_2||_* - 2$ in this case.

The following is an immediate consequence of Lemma 4.5(ii) and Corollary 4.8.

Corollary 4.10. Let $g \in A(\Gamma)$ be cyclically reduced. Then $\{\|g^n\|_*\}_{n=0}^{\infty}$ is an increasing sequence such that the following hold.

- (i) If $||g||_* = 1$, then $||g^n||_* = 1$ for all $n \ge 1$.
- (ii) If $||g||_* = 2$, then $||g^{n-1}||_* \leq ||g^n||_* \leq ||g^{n-1}||_* + 2$ for all $n \ge 1$.
- (iii) If $||g||_* \ge 3$, then $||g^n||_* \ge ||g^{n-1}||_* + 1$ and hence $||g^n||_* \ge n+2$ for all $n \ge 1$.

Corollary 4.11. Let $g, u \in A(\Gamma)$ with g cyclically reduced. If $||g||_* \ge 3$ and $g \not\leq_L u \leq_L g^n$ for some $n \ge 1$, then $u \leq_L g^2$.

Proof. We may assume $n \ge 3$ and $u \le L g$ (otherwise it is obvious). Since g and u have a common right multiple, say g^n , there exist g' and u' such that $gg' = uu' = g \lor_L u \le L g^n$ and $u' \rightleftharpoons g'$ (by Theorem 2.12), where gg' and uu' are geodesic. Since $gg' \le L g^n = gg^{n-1}$, (by Lemma 2.6)

$$g' \leqslant_L g^{n-1} = g \cdot g^{n-2}.$$

Since $u \not\leq_L g$ and $g \not\leq_L u$, both g' and u' are nontrivial, hence $||g'||_* = ||u'||_* = 1$. Since $||g'||_* = 1$ and $||g||_* \geq 3$, we get $g' \leq_L g$ (by Corollary 4.7). Therefore $u \leq_L uu' = gg' \leq_L g^2$.

Lemma 4.12. Let $g_1, g_2, g_3 \in A(\Gamma)$ be such that both g_1g_2 and g_2g_3 are geodesic. If $||g_2||_* \ge 2$, then $g_1g_2g_3$ is geodesic.

$g\in A(\Gamma)$	Г	$\bar{\Gamma}$
g is split	$\Gamma[g]$ is a join	$\bar{\Gamma}[g]$ is disconnected
g is non-split	$\Gamma[g]$ is not a join	$\overline{\Gamma}[g]$ is connected
g is strongly non-split	$\Gamma[g]$ is not contained in	$\overline{\Gamma}[g]$ is connected and
	a subjoin of Γ	$\operatorname{St}_{\overline{\Gamma}}(\operatorname{supp}(g)) = V(\Gamma)$

TABLE 1. Equivalent conditions for $g \in A(\Gamma)$ to be split, non-split and strongly non-split

Proof. Assume that $g_1g_2g_3$ is not geodesic. Since g_1g_2 and g_2g_3 are geodesic, there exists $x \in V(\Gamma)^{\pm 1}$ such that $x^{-1} \leq_R g_1$, $x \leq_L g_3$ and $x \rightleftharpoons g_2$ (by Lemma 2.8(i)). Observe that $x \rightleftharpoons g_2$ implies $||g_2||_* \leq 1$, which contradicts the hypothesis $||g_2||_* \geq 2$.

We introduce the notion of strongly non-split elements. We will see (in Lemma 6.3 and Remark 6.4) that if $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected, then a cyclically reduced element $g \in A(\Gamma)$ is strongly non-split if and only if g is loxodromic on the extension graph Γ^e .

Definition 4.13 (non-split, strongly non-split). Let $g \in A(\Gamma) \setminus \{1\}$.

- (i) g is called *split* if g has a nontrivial geodesic decomposition $g = g_1 g_2$ with $g_1 \rightleftharpoons g_2$.
- (ii) g is called *non-split* if it is not split.
- (iii) g is called strongly non-split if g is non-split and $g \neq v$ for any $v \in V(\Gamma)$.

It is easy to see that $g \in A(\Gamma)$ is split if and only if $\Gamma[g]$ is a join (equivalently, $\Gamma[g]$ is disconnected). Similarly, one can characterize the property of being non-split and strongly non-split using the graphs $\Gamma[g]$ and $\overline{\Gamma}[g]$ as shown in Table 1.

From definition, the existence of a strongly non-split element implies that Γ is connected.

Remark 4.14. Let $n \ge 2$ and $g, h \in A(\Gamma) \setminus \{1\}$. Observe that strongly non-splitness of an element depends only on its support. Note that $\operatorname{supp}(g^{-1}) = \operatorname{supp}(g) = \operatorname{supp}(g^n)$ (by Lemma 3.4), and that if either $g \leq_L h$ or $g \leq_R h$, then $\operatorname{supp}(g) \subset \operatorname{supp}(h)$. Therefore

- (i) g is strongly non-split if and only if g^{-1} is strongly non-split;
- (ii) g is strongly non-split if and only if g^n is strongly non-split;
- (iii) if g is strongly non-split and either $g \leq_L h$ or $g \leq_R h$, then h is also strongly non-split.

Strongly non-splitness is related to the star length as follows.

Lemma 4.15. Let $g \in A(\Gamma) \setminus \{1\}$.

- (i) If $||g||_* \ge 3$, then g is strongly non-split.
- (ii) g is strongly non-split with $|\operatorname{supp}(g)| \ge 2$ if and only if g is non-split with $||g||_* \ge 2$.

Proof. (i) Assume that g is not strongly non-split. If g is split, then clearly $||g||_* \leq 2$. If g is non-split but not strongly non-split, then there is $v \in V(\Gamma) \setminus \operatorname{supp}(g)$ with $v \rightleftharpoons g$, hence $||g||_* = 1$. In either case, $||g||_* \leq 2$.

(ii) Suppose that g is strongly non-split with $|\operatorname{supp}(g)| \ge 2$. Then g is non-split by definition. Assume $||g||_* = 1$. Then there exists $v \in V(\Gamma)$ with $\operatorname{supp}(g) \subset Z(v)$. Since g is strongly non-split, $v \in \operatorname{supp}(g)$. Since $|\operatorname{supp}(g)| \ge 2$ and $\operatorname{supp}(g) \subset Z(v)$, $g = v^n g_1$ is geodesic for some $n \ne 0$ and $g_1 \in A(\Gamma) \setminus \{1\}$ with $g_1 \rightleftharpoons v$. Namely, g is split, which is a contradiction. Therefore $||g||_* \ge 2$.

Conversely, suppose that g is non-split with $||g||_* \ge 2$. Then $|\operatorname{supp}(g)| \ge 2$ and there does not exit $v \in V(\Gamma) \setminus \operatorname{supp}(g)$ with $v \rightleftharpoons g$. Therefore g is strongly non-split.

5. Prefixes of powers of cyclically reduced elements

In this section, we study prefixes of powers of cyclically reduced elements. The main result is Theorem 5.3, which plays important roles in the study of the asymptotic translation length and the acylindricity of the action of $A(\Gamma)$ on Γ^e .

Lemma 5.1. Let $u, g_1, g_2, \ldots, g_m \in A(\Gamma)$. If $g_1g_2 \cdots g_m$ is geodesic, then for each $1 \leq k \leq m$ there exists a geodesic decomposition $g_k = a_k b_k$ such that

- (i) $u \wedge_L (g_1 \cdots g_k) = a_1 \cdots a_k;$
- (ii) $a_k \rightleftharpoons b_j$ for all $1 \leq j \leq k-1$;
- (iii) $a_1 \cdots a_k b_1 \dots b_k$ is a geodesic decomposition of $g_1 \cdots g_k$.

Proof. The relation $u \wedge_L (g_1 \cdots g_k) = a_1 \cdots a_k$ determines the elements a_k inductively for $k = 1, \ldots, m$. Then the relation $g_k = a_k b_k$ determines the elements b_k for all $1 \leq k \leq m$. Therefore we get elements $a_1, \ldots, a_m, b_1, \ldots, b_m$ such that $u \wedge_L (g_1 \cdots g_k) = a_1 \cdots a_k$ and $g_k = a_k b_k$ for all $1 \leq k \leq m$.

Since $g_1 \cdots g_m$ is geodesic, $g_1 \cdots g_k \leq_L g_1 \cdots g_{k+1}$ for each $1 \leq k \leq m-1$ (by Lemma 2.6), hence

$$a_1 \cdots a_k = u \wedge_L (g_1 \cdots g_k) \leqslant_L u \wedge_L (g_1 \cdots g_{k+1}) = a_1 \cdots a_{k+1}.$$

Therefore $a_1 \cdots a_m$ and hence each $a_1 \cdots a_k$ are geodesic (by Lemma 2.6 again).

For each $1 \leq k \leq m$, let $u_k \in A(\Gamma)$ be the element such that $u = a_1 \cdots a_k u_k$. Then each $a_1 \cdots a_k u_k$ is geodesic because $a_1 \cdots a_k \leq_L u$.

Claim. For each $1 \leq k \leq m$,

- (a) $a_k \leq _L g_k$, hence $g_k = a_k b_k$ is geodesic;
- (b) $a_k \rightleftharpoons b_j$ for all $1 \leq j \leq k-1$;
- (c) $a_1 \cdots a_k b_1 \cdots b_k$ is a geodesic decomposition of $g_1 \cdots g_k$.

Proof of Claim. We use induction on k.

For k = 1, (a) and (c) hold because $a_1 = u \wedge_L g_1 \leq_L g_1$ and $g_1 = a_1 b_1$, and (b) is vacuously true.

Assume that the claim holds for some $1 \leq k < m$. We now have the following geodesic decompositions at hand:

$$u = (a_1 \cdots a_k)u_k,$$

$$g_1 \cdots g_k = (a_1 \cdots a_k)(b_1 \cdots b_k),$$

$$g_1 \cdots g_{k+1} = (a_1 \cdots a_k)(b_1 \cdots b_k)g_{k+1}.$$

Since $u \wedge_L (g_1 \cdots g_k) = a_1 \cdots a_k$, we have $u_k \wedge_L (b_1 \cdots b_k) = 1$ (by Lemma 2.14). Since $u \wedge_L (g_1 \cdots g_{k+1}) = a_1 \cdots a_{k+1}$, we have $a_{k+1} = u_k \wedge_L (b_1 \cdots b_k g_{k+1})$, hence

 $a_{k+1} \leq_L u_k$ and $a_{k+1} \leq_L (b_1 \cdots b_k) g_{k+1}$.

Since $a_{k+1} \leq_L u_k$, we have $a_{k+1} \wedge_L (b_1 \cdots b_k) \leq_L u_k \wedge_L (b_1 \cdots b_k) = 1$ (by Lemma 2.14). Since $a_{k+1} \leq_L (b_1 \cdots b_k) g_{k+1}$ and $a_{k+1} \wedge_L (b_1 \cdots b_k) = 1$, we have

$$a_{k+1} \rightleftharpoons b_1 \cdots b_k$$
 and $a_{k+1} \leqslant_L g_{k+1}$

(by Lemma 2.11(ii)). In particular, $a_{k+1} \rightleftharpoons b_j$ for all $1 \le j \le k$. Therefore (a) and (b) hold for k+1. Since $a_{k+1} \rightleftharpoons b_j$ for all $1 \le j \le k$, we have

$$g_1 \cdots g_k g_{k+1} = (a_1 \cdots a_k)(b_1 \cdots b_k)(a_{k+1}b_{k+1})$$
$$= (a_1 \cdots a_{k+1})(b_1 \cdots b_{k+1}).$$

The above three decompositions are all geodesic because $g_1 \cdots g_k g_{k+1}$, $g_{k+1} = a_{k+1}b_{k+1}$ and $g_1 \cdots g_k = a_1 \cdots a_k b_1 \cdots b_k$ are all geodesic. Therefore (c) holds for k+1.

The above claim completes the proof.

In the following, we frequently use the notation $\operatorname{St}_{\overline{\Gamma}[g]}(X)$, for $X \subset \operatorname{supp}(g)$, which denotes the star of X in $\overline{\Gamma}[g] = \overline{\Gamma}[\operatorname{supp}(g)]$. Hence, $v \in \operatorname{St}_{\overline{\Gamma}[g]}(X)$ if and only if either $v \in X$ or $v \in \operatorname{supp}(g)$ and $\{v, v_1\}$ is an edge in $\overline{\Gamma}$ for some $v_1 \in X$. Therefore

$$\operatorname{St}_{\overline{\Gamma}[g]}(X) = \operatorname{St}_{\overline{\Gamma}}(X) \cap \operatorname{supp}(g)$$

When $g_1 = \cdots = g_m$ in Lemma 5.1, we have the following.

Corollary 5.2. Let $m \ge 1$ and $g, u \in A(\Gamma)$ with g cyclically reduced. Then for each $1 \le k \le m$ there exists a geodesic decomposition $g = a_k b_k$ such that

- (i) $u \wedge_L g^k = a_1 a_2 \cdots a_k, a_k \rightleftharpoons b_j$ for all $1 \leq j < k$ and $a_1 \cdots a_k b_1 \ldots b_k$ is a geodesic decomposition of g^k ;
- (ii) $\{a_k\}_{k=1}^m$ is descending with respect to \leq_L such that

$$1 \leqslant_L a_m \leqslant_L \dots \leqslant_L a_2 \leqslant_L a_1 \leqslant_L g,$$

St_{\bar{\bar{\nu}}[a]}(supp(a_{k+1})) \supp(a_k);

(iii) $\{b_k\}_{k=1}^m$ is ascending with respect to \leq_R such that

$$1 \leqslant_R b_1 \leqslant_R b_2 \leqslant_R \dots \leqslant_R b_m \leqslant_R g,$$

St_{\bar{\Gamma}[q]}(supp(b_k)) \supp(b_{k+1}).

In (ii) and (iii), we let $a_{m+1} = 1$ and $b_{m+1} = g$ for notational convenience.

Proof. Since g is cyclically reduced, $gg \cdots g$ is geodesic. By Lemma 5.1, there exists a geodesic decomposition $g = a_k b_k$ for $1 \le k \le m$ satisfying (i).

Since $g = a_k b_k = a_{k+1} b_{k+1}$, we have $a_{k+1} \leq_L a_k b_k$. Since $a_{k+1} \rightleftharpoons b_k$, we have $a_{k+1} \leq_L a_k$ (by Lemma 2.15(iv)), hence $\{a_k\}_{k=1}^m$ is descending with respect to \leq_L . Since $a_1 \leq_L g$ and $1 \leq_L a_m$,

 $1 \leqslant_L a_m \leqslant_L \cdots \leqslant_L a_2 \leqslant_L a_1 \leqslant_L g.$

Since $g = a_k b_k$, it follows immediately from the above inequalities that the sequence $\{b_k\}_{k=1}^m$ is ascending with respect to \leq_R such that

$$1 \leqslant_R b_1 \leqslant_R b_2 \leqslant_R \cdots \leqslant_R b_m \leqslant_R g.$$

Since $\operatorname{supp}(g) = \operatorname{supp}(a_j) \cup \operatorname{supp}(b_j)$ for j = k, k + 1,

$$\operatorname{supp}(g) - \operatorname{supp}(a_{k+1}) \subset \operatorname{supp}(b_{k+1}),$$

 $\operatorname{supp}(g) - \operatorname{supp}(b_k) \subset \operatorname{supp}(a_k).$

Since $a_{k+1} \rightleftharpoons b_k$, by Lemma 2.3

$$\sup(b_k) \cap \operatorname{St}_{\bar{\Gamma}}(\operatorname{supp}(a_{k+1})) = \emptyset,$$

$$\sup(a_{k+1}) \cap \operatorname{St}_{\bar{\Gamma}}(\operatorname{supp}(b_k)) = \emptyset.$$

Hence

$$\begin{aligned} \operatorname{St}_{\bar{\Gamma}[g]}(\operatorname{supp}(a_{k+1})) &= \operatorname{supp}(g) \cap \operatorname{St}_{\bar{\Gamma}}(\operatorname{supp}(a_{k+1})) \\ &\subset \operatorname{supp}(g) - \operatorname{supp}(b_k) \subset \operatorname{supp}(a_k), \\ \operatorname{St}_{\bar{\Gamma}[g]}(\operatorname{supp}(b_k)) &= \operatorname{supp}(g) \cap \operatorname{St}_{\bar{\Gamma}}(\operatorname{supp}(b_k)) \\ &\subset \operatorname{supp}(g) - \operatorname{supp}(a_{k+1}) \subset \operatorname{supp}(b_{k+1}). \end{aligned}$$

Therefore (ii) and (iii) are proved.

Theorem 5.3. Let $m \ge 2$ and $g, u \in A(\Gamma)$ with g cyclically reduced and non-split. If

 $g \not\leq_L u \not\leq_L g^{m-1}$ and $u \leq_L g^m$,

then the following hold.

(i) $m \leq \operatorname{diam}(\overline{\Gamma}[g])$. In particular, $m \leq |\operatorname{supp}(g)| - 1 \leq |V(\Gamma)| - 1$.

(ii) There is a geodesic decomposition g = g_mg_{m-1} ··· g₁g₀ such that
(a) g_k ≠ 1 for all 0 ≤ k ≤ m;

- (b) $g_i \rightleftharpoons g_j$ whenever $|i j| \ge 2$;
- (c) $u \wedge_L g^k = (g_m \cdots g_1)(g_m \cdots g_2) \cdots (g_m \cdots g_k)$ for all $1 \leq k \leq m$.
- In particular, $u = u \wedge_L g^m = (g_m \cdots g_1)(g_m \cdots g_2) \cdots (g_m g_{m-1})(g_m).$
- (iii) $||u||_* \leq ||g||_* + 1.$
- (iv) If $||g||_* \ge 3$, then m = 2. (Equivalently, if $m \ge 3$, then $||g||_* \le 2$.)

Proof. For each $k \ge 1$, let $g = a_k b_k$ be the geodesic decomposition given by Corollary 5.2. Then

- $u \wedge_L g^k = a_1 \cdots a_k$ is geodesic and $a_k \rightleftharpoons b_j$ for all $1 \leq j < k$;
- $\{a_k\}_{k=1}^{\infty}$ is descending with respect to \leq_L such that $1 \leq_L \cdots \leq_L a_2 \leq_L a_1 \leq_L g$;
- $\{b_k\}_{k=1}^{\infty}$ is ascending with respect to \leq_R such that $1 \leq_R b_1 \leq_R b_2 \leq_R \cdots \leq_R g$.

The following claim is a result of the hypothesis that $g \not\leq_L u \not\leq_L g^{m-1}$ and $u \leq_L g^m$.

Claim 1. For each $1 \leq k \leq m$, $a_k \notin \{1, g, a_{k+1}\}$ and hence $b_k \notin \{1, g, b_{k+1}\}$. For each k > m, $a_k = 1$ and hence $b_k = g$.

Proof of Claim 1. For each $k \ge 1$, $a_1 \cdots a_k \le_L u$ and $a_1 \cdots a_k \le_L g^k$ (because $a_1 \cdots a_k = u \wedge_L g^k$). Furthermore, $u = u \wedge_L g^m = a_1 \cdots a_m$ (from the hypothesis $u \le_L g^m$). Therefore

 $a_1 \leq_L u, \quad a_1 \cdots a_{m-1} \leq_L g^{m-1}, \quad a_1 \cdots a_m = u.$

If $a_1 = g$, then $g \leq_L u$, which contradicts the hypothesis $g \leq_L u$. If $a_m = 1$, then $u = a_1 \cdots a_{m-1} a_m = a_1 \cdots a_{m-1} \leq_L g^{m-1}$, which contradicts the hypothesis $u \leq_L g^{m-1}$. Thus $a_1 \neq g$ and $a_m \neq 1$. Therefore, for each $1 \leq k \leq m$, we get $a_k \notin \{1, g\}$ (because $1 \leq_L a_m \leq_L a_k \leq_L a_1 \leq_L g$) and hence $b_k \notin \{1, g\}$ (because $g = a_k b_k$).

Assume that $a_k = a_{k+1}$ for some $1 \leq k \leq m$. Then $a_k \rightleftharpoons b_k$ because $a_{k+1} \rightleftharpoons b_k$. Since $g = a_k b_k$ and both a_k and b_k are nontrivial, this contradicts that g is non-split. Therefore $a_k \neq a_{k+1}$ and hence $b_k \neq b_{k+1}$ for all $1 \leq k \leq m$.

Let $j \ge 1$. Since $u \le_L g^m$ and g is cyclically reduced, we have $u \le_L g^{m+j}$, hence $u \wedge_L g^m = u = u \wedge_L g^{m+j}$. Therefore $a_1 \cdots a_m = a_1 \cdots a_m a_{m+1} \cdots a_{m+j}$, hence $a_{m+1} \cdots a_{m+j} = 1$. Since the decomposition $a_{m+1} \cdots a_{m+j}$ is geodesic, we have $a_{m+j} = 1$. Namely, for all k > m, $a_k = 1$ and hence $b_k = g$.

Define $\{g_k\}_{k=0}^m$ by $g_0 = b_1$ and $g_k = a_{k+1}^{-1}a_k$ (hence $a_k = a_{k+1}g_k$) for $1 \leq k \leq m$. Then $a_k = a_{k+1}g_k$ is geodesic for all $1 \leq k \leq m$ because $a_{k+1} \leq a_k$.

Claim 2. The decomposition $g = g_m g_{m-1} \cdots g_1 g_0$ is geodesic such that

- (a) $g_k \neq 1$ for all $0 \leq k \leq m$;
- (b) $g_i \rightleftharpoons g_j$ whenever $|i j| \ge 2$;
- (c) $u \wedge_L g^k = (g_m \cdots g_1)(g_m \cdots g_2) \cdots (g_m \cdots g_k)$ for all $1 \leq k \leq m$;
- (d) $a_k = g_m g_{m-1} \cdots g_k$ and $b_k = g_{k-1} g_{k-2} \cdots g_0$ for all $1 \le k \le m$.

Proof of Claim 2. Since $a_k = a_{k+1}g_k$ is geodesic, $||g_k|| = ||a_k|| - ||a_{k+1}||$ for all $1 \le k \le m$. Since $g_0 = b_1$, $a_{m+1} = 1$ and $g = a_1b_1$ is geodesic,

$$||g_0|| + ||g_1|| + \dots + ||g_m||$$

= $||b_1|| + (||a_1|| - ||a_2||) + \dots + (||a_m|| - ||a_{m+1}||)$
= $||b_1|| + ||a_1|| - ||a_{m+1}|| = ||g||.$

Consequently, $||g|| = ||g_0|| + ||g_1|| + \dots + ||g_m||$.

For $1 \leq k \leq m$, $a_k = a_{k+1}g_k = a_{k+2}g_{k+1}g_k = \cdots = a_{m+1}g_m \cdots g_k = g_m \cdots g_k$ because $a_{m+1} = 1$. Therefore we have the following decompositions:

$$a_{k} = g_{m} \cdots g_{k} \quad \text{for all } 1 \leq k \leq m,$$

$$g = a_{1}b_{1} = (g_{m} \cdots g_{1})g_{0} = g_{m} \cdots g_{0},$$

$$b_{k} = a_{k}^{-1}g = g_{k-1} \cdots g_{0} \quad \text{for all } 1 \leq k \leq m$$

Observe that $g = g_m \cdots g_0$ is geodesic because $||g|| = ||g_0|| + ||g_1|| + \cdots + ||g_m||$.

The decompositions for a_k and b_k in the above prove (d).

For each $1 \leq k \leq m$, $u \wedge_L g^k = a_1 a_2 \cdots a_k = (g_m \cdots g_1)(g_m \cdots g_2) \cdots (g_m \cdots g_k)$. This proves (c). By Claim 1, $g_0 = b_1 \neq 1$ and $g_k = a_{k+1}^{-1} a_k \neq 1$ for all $1 \leq k \leq m$. This proves (a).

For each (i, j) with $0 \leq j < j + 2 \leq i \leq m$, we know that $a_i \rightleftharpoons b_{j+1}$. Since $a_i = g_m \cdots g_i$ and $b_{j+1} = g_j \cdots g_0$ are geodesic, we have $g_i \leq_R a_i$ and $g_j \leq_L b_{j+1}$, hence $g_i \rightleftharpoons g_j$. This proves (b). \Box

Recall from Claim 2 that both $g_0 = b_1$ and $g_m = a_m$ are nontrivial.

Claim 3. For any path $(v_0, v_1, \ldots, v_{r-1}, v_r)$ in $\overline{\Gamma}[g]$ such that $v_0 \in \operatorname{supp}(g_0) = \operatorname{supp}(b_1)$ and $v_r \in \operatorname{supp}(g_m) = \operatorname{supp}(a_m)$, we have $m \leq r$. In particular, $m \leq \operatorname{diam}(\overline{\Gamma}[g])$.

Proof of Claim 3. Using induction on k, we first show that

$$v_k \in \operatorname{supp}(b_{k+1})$$

for all $0 \leq k \leq \min\{m-1, r-1\}$. By the hypothesis of the claim, $v_0 \in \operatorname{supp}(b_1)$. Assume that $v_k \in \operatorname{supp}(b_{k+1})$ for some $0 \leq k \leq \min\{m-2, r-2\}$. Since $\{v_k, v_{k+1}\}$ is an edge in $\overline{\Gamma}[g]$, we have $v_{k+1} \in \operatorname{St}_{\overline{\Gamma}[g]}(v_k)$. Since $v_k \in \operatorname{supp}(b_{k+1})$ by induction hypothesis, $\operatorname{St}_{\overline{\Gamma}[g]}(v_k) \subset \operatorname{St}_{\overline{\Gamma}[g]}(\operatorname{supp}(b_{k+1}))$, hence $v_{k+1} \in \operatorname{St}_{\overline{\Gamma}[g]}(\operatorname{supp}(b_{k+1}))$. By Corollary 5.2, $\operatorname{St}_{\overline{\Gamma}[g]}(\operatorname{supp}(b_{k+1})) \subset \operatorname{supp}(b_{k+2})$, hence $v_{k+1} \in \operatorname{supp}(b_{k+2})$.

If m > r, then $a_m \rightleftharpoons b_r$ (by Corollary 5.2). Since $v_r \in \operatorname{supp}(a_m)$ and $v_{r-1} \in \operatorname{supp}(b_r)$, we have $v_r \rightleftharpoons v_{r-1}$, which contradicts that $\{v_{r-1}, v_r\}$ is an edge in $\overline{\Gamma}$. Therefore $m \leq r$.

Since $\Gamma[g]$ is connected and both g_0 and g_m are nontrivial (by Claim 2), we may assume that (v_0, \ldots, v_r) is a shortest path from $v_0 \in \operatorname{supp}(g_0)$ to $v_r \in \operatorname{supp}(g_m)$ in $\overline{\Gamma}[g]$, hence $r \leq \operatorname{diam}(\overline{\Gamma}[g])$. Therefore $m \leq r \leq \operatorname{diam}(\overline{\Gamma}[g])$.

Claim 3 proves (i) and Claim 2 proves (ii).

Since $g_0 \neq 1$ and $g_j \rightleftharpoons g_0$ for all $j \ge 2$, we have $||(g_m \cdots g_2) \cdots (g_m g_{m-1})g_m||_* \le 1$. Since $g_m \cdots g_1 \le L$ g, we have $||g_m \cdots g_1||_* \le ||g||_*$. Therefore

$$||u||_* \leq ||g_m \cdots g_1||_* + ||(g_m \cdots g_2) \cdots (g_m g_{m-1})g_m||_* \leq ||g||_* + 1.$$

This proves (iii).

Assume $m \ge 3$. Since $g_0 \ne 1$, $g_m \ne 1$, $g_m \cdots g_2 \rightleftharpoons g_0$ and $g_1g_0 \rightleftharpoons g_m$, we have $||g_m \cdots g_2||_* \le 1$ and $||g_1g_0||_* \le 1$. Therefore $||g||_* \le ||g_m \cdots g_2||_* + ||g_1g_0||_* \le 2$. This proves (iv).

Remark 5.4. From the disjoint commutativity $g_i \rightleftharpoons g_j$ for $|i - j| \ge 2$, the following decompositions are geodesic for all $1 \le k \le m$.

$$g^{k} = (g_{m} \cdots g_{0})(g_{m} \cdots g_{0}) \cdots (g_{m} \cdots g_{0})$$

= $((g_{m} \cdots g_{1}) \cdot (g_{0}))((g_{m} \cdots g_{2}) \cdot (g_{1}g_{0})) \cdots ((g_{m} \cdots g_{k}) \cdot (g_{k-1} \cdots g_{0}))$
= $(g_{m} \cdots g_{1})(g_{m} \cdots g_{2}) \cdots (g_{m} \cdots g_{k}) \cdot (g_{0})(g_{1}g_{0}) \cdots (g_{k-1} \cdots g_{0})$
= $(u \wedge_{L} g^{k})(g_{0})(g_{1}g_{0}) \cdots (g_{k-1} \cdots g_{0})$

In particular, $g^m = uu'$ is geodesic, where

$$u = (g_m \cdots g_1)(g_m \cdots g_2) \cdots (g_m), u' = (g_0)(g_1g_0) \cdots (g_{m-1} \cdots g_0).$$

The following example shows that the upper bounds $m \leq \operatorname{diam}(\overline{\Gamma}[g])$ and $m \leq |V(\Gamma)| - 1$ in Theorem 5.3(i) are sharp.

Example 5.5. Let $\Gamma = \overline{P}_4$, where $P_4 = (v_1, \ldots, v_4)$ is a path graph, and let the underlying rightangled Artin group here be $A(\Gamma)$. Let $g = v_1^2 v_2 v_3 v_4$ and $u = v_1^2 v_2 v_3 v_1^2 v_2 v_1$. Then g is clearly cyclically reduced and non-split. It is easy to see that $g \not\leq_L u$ and $u \not\leq_L g^2$. (By Lemma 2.6, if $g \leq_L u$ then $v_4 \leq_L v_1^2 v_2 v_1$, and if $u \leq_L g^2$ then $v_1 \leq_L v_4 v_3 v_4$.) On the other hand, $u \leq_L g^3$ because

$$g^{3} = (v_{1}^{2}v_{2}v_{3}v_{4})(v_{1}^{2}v_{2}v_{3}v_{4})(v_{1}^{2}v_{2}v_{3}v_{4})$$

= $(v_{1}^{2}v_{2}v_{3} \cdot v_{4})(v_{1}^{2}v_{2} \cdot v_{3}v_{4})(v_{1} \cdot v_{1}v_{2}v_{3}v_{4})$
= $(v_{1}^{2}v_{2}v_{3} \cdot v_{1}^{2}v_{2} \cdot v_{1})(v_{4} \cdot v_{3}v_{4} \cdot v_{1}v_{2}v_{3}v_{4})$
= $u(v_{4} \cdot v_{3}v_{4} \cdot v_{1}v_{2}v_{3}v_{4}).$

In the notation of Theorem 5.3,

$$m = 3 = \operatorname{diam}(\Gamma) = \operatorname{diam}(\Gamma[g]) = |\operatorname{supp}(g)| - 1 = |V(\Gamma)| - 1.$$

Thus the bounds of m in Theorem 5.3(i) are sharp.

Corollary 5.6. Let $g, u \in A(\Gamma)$ with g cyclically reduced and non-split. If $u \leq_L g^m$ for some $m \geq 1$, then $u = g^k a$ is geodesic for some $0 \leq k \leq m$ and $a \in A(\Gamma)$ with $||a||_* \leq ||g||_* + 1$.

Proof. We may assume that $u \not\leq_L g^{m-1}$. Let $k = \max\{l \geq 0 : g^l \leq_L u\}$. Then $0 \leq k \leq m$ and $u = g^k a$ is geodesic for some $a \in A(\Gamma)$ with $g \not\leq_L a \not\leq_L g^{m-k-1}$ and $a \leq_L g^{m-k}$.

If $m - k \leq 1$, then it is obvious that $||a||_* \leq ||g||_* \leq ||g||_* + 1$.

If $m - k \ge 2$, then $||a||_* \le ||g||_* + 1$ by Theorem 5.3.

For a cyclically reduced element $g \in A(\Gamma)$, we have seen in Corollary 4.10 that the sequence $\{\|g^n\|_*\}_{n=0}^{\infty}$ is increasing. In particular, if $\|g\|_* \ge 3$, then $\|g^{n+1}\|_* \ge \|g^n\|_* + 1$ for all $n \ge 0$. However, if $\|g\|_* = 2$, then it may happen that $\|g\|_* = \|g^2\|_* = \cdots = \|g^n\|_* = 2$ for some $n \ge 2$. The following proposition finds m with $\|g^m\|_* \ge 3$ when $\|g\|_* = 2$.

Proposition 5.7. Let $g \in A(\Gamma)$ be cyclically reduced and non-split with $||g||_* = 2$. Then the following hold.

- (i) Let $m \ge 2$. If either $m \ge |V(\Gamma)| 2$ or $m \ge \operatorname{diam}(\overline{\Gamma}[g]) + 1$, then $||g^m||_* \ge 3$.
- (ii) Let $m \ge 2$. If $||g^m||_* = 2$, then $m \le |V(\Gamma)| 3$ and $m \le \operatorname{diam}(\overline{\Gamma}[g])$.
- (iii) If $|V(\Gamma)| \leq 4$, then $||g^2||_* \geq 3$.

Proof. The statements (i) and (ii) are equivalent, and (iii) follows from (i). Therefore we prove only (ii).

Since g is non-split, $\Gamma[g]$ is connected. Suppose that $||g^m||_* = 2$ for some $m \ge 2$. Then there is a geodesic decomposition

$$q^m = uu$$

for some $u, u' \in A(\Gamma)$ with $||u||_* = ||u'||_* = 1$.

Claim 1. $g \not\leq_L u \not\leq_L g^{m-1}$ and $u \leq_L g^m$.

Proof of Claim 1. Since $g^m = uu'$ is geodesic, $u \leq_L g^m$. Since $||u||_* = 1$ and $||g||_* = 2$, $g \leq_L u$. If $u \leq_L g^{m-1}$, then $g \leq_R u'$ (by Lemma 2.6(iv)), which is impossible because $||u'||_* = 1$ and $||g||_* = 2$. Therefore $u \leq_L g^{m-1}$.

By Claim 1, we can apply Theorem 5.3, hence $m \leq \operatorname{diam}(\overline{\Gamma}[g])$, which is the second inequality of (ii).

By Theorem 5.3 and Remark 5.4, there is a geodesic decomposition $g = g_m g_{m-1} \cdots g_0$ such that $g_i \neq 1$ for all $0 \leq i \leq m, g_i \rightleftharpoons g_j$ whenever $|i - j| \geq 2$ and

$$u = (g_m \dots g_1)(g_m \dots g_2) \dots (g_m),$$

 $u' = (g_0)(g_1g_0) \dots (g_{m-1} \dots g_0).$

Since $||u||_* = ||u'||_* = 1$, there exist vertices $x, y \in V(\Gamma)$ such that $u \in Z(x)$ and $u' \in Z(y)$, where $Z(\cdot)$ denotes the centralizer. Since $||uu'||_* = 2, x \neq y$. Notice that

$$\operatorname{supp}(g_1), \dots, \operatorname{supp}(g_m) \subset Z(x),$$

 $\operatorname{supp}(g_0), \dots, \operatorname{supp}(g_{m-1}) \subset Z(y).$

Claim 2. There is a path $(x, v_0, v_1, v_2, \ldots, v_{r-1}, v_r, y)$ in $\overline{\Gamma}$ such that

(a) $v_0 \in \operatorname{supp}(g_0)$ and $v_r \in \operatorname{supp}(g_m)$;

(b) the subpath (v_0, \ldots, v_r) is a shortest path from v_0 to v_r in $\Gamma[g]$;

(c) all the vertices on the path are mutually distinct.

Proof of Claim 2. If either supp $(g) \subset Z(x)$ or supp $(g) \subset Z(y)$, then $||g||_* = 1$, hence supp $(g) \setminus Z(x) \neq \emptyset$ and supp $(g) \setminus Z(y) \neq \emptyset$.

Choose any vertices $v_x \in \operatorname{supp}(g) \setminus Z(x)$ and $v_y \in \operatorname{supp}(g) \setminus Z(y)$, equivalently, $v_x, v_y \in \operatorname{supp}(g)$ such that $\{v_x, x\}, \{v_y, y\} \in E(\bar{\Gamma})$. Since $v_x \in \operatorname{supp}(g) \setminus Z(x) = (\bigcup_{k=0}^m \operatorname{supp}(g_k)) \setminus Z(x)$ and $\bigcup_{k=1}^m \operatorname{supp}(g_k) \subset Z(x)$, we have $v_x \in \operatorname{supp}(g_0)$. Similarly, $v_y \in \operatorname{supp}(g_m)$. Furthermore, $v_x \neq v_y$ and $\{v_x, v_y\} \notin E(\bar{\Gamma})$ because $g_0 \rightleftharpoons g_m$.

Since $\{v_x, x\}, \{v_y, y\} \in E(\overline{\Gamma})$ and $\overline{\Gamma}[g]$ is connected, there is a path in $\overline{\Gamma}$ from x to y

$$(x, v_0 = v_x, v_1, v_2, \dots, v_{r-1}, v_r = v_y, y)$$

such that $v_k \in \operatorname{supp}(g)$ for all $0 \leq k \leq r$. Observe that $v_0 = v_x \in \operatorname{supp}(g_0)$ and $v_r = v_y \in \operatorname{supp}(g_m)$.

We may assume that it is a shortest path among all the paths from x to y such that $v_k \in \text{supp}(g)$ for all $0 \leq k \leq r$. Then the subpath (v_0, \dots, v_r) must be a shortest path from v_0 to v_r in $\overline{\Gamma}[g]$, hence v_0, \dots, v_r are mutually distinct.

If $x = v_j$ for some $0 \leq j \leq r-1$, then the path $(x, v_{j+1}, \ldots, v_r, y)$ is shorter than the original one, all of whose middle vertices belong to $\operatorname{supp}(g)$. This contradicts that (x, v_0, \cdots, v_r, y) is a shortest path among such paths. If $x = v_r$ then $\{x, v_x\} \notin E(\overline{\Gamma})$ (because $x = v_r \in \operatorname{supp}(g_m), v_x = v_0 \in \operatorname{supp}(g_0)$ and $g_0 \rightleftharpoons g_m$). This is a contradiction. Therefore $x \neq v_j$ for any $0 \leq j \leq r$. Similarly, $y \neq v_j$ for any $0 \leq j \leq r$. Since $x \neq y$, all the vertices on the path $(x, v_0, v_1, v_2, \ldots, v_{r-1}, v_r, y)$ are mutually distinct.

Since the (r+3) points on the path in Claim 2 are mutually distinct, $|V(\Gamma)| \ge r+3$. By Claim 3 in the proof of Theorem 5.3, we have $m \le r$. Therefore $|V(\Gamma)| \ge r+3 \ge m+3$. This proves the first inequality of (ii), hence (ii) is proved.

The following example illustrates that the upper bounds $m \leq |V(\Gamma)| - 3$ and $m \leq \text{diam}(\overline{\Gamma}[g])$ in Proposition 5.7(ii) are sharp.

Example 5.8. Let $\Gamma = \overline{P}_6$, where $P_6 = (v_0, v_1, \dots, v_5)$ is a path graph, and let the underlying right-angled Artin group here be $A(\Gamma)$. Let $g = v_1 v_2 v_3 v_4$. It is easy to see that $||g||_* = 2$. Since $\operatorname{supp}(g) = \{v_1, v_2, v_3, v_4\}$, $\operatorname{diam}(\overline{\Gamma}[g]) = 3$. Observe

$$g^{3} = (v_{1}v_{2}v_{3}v_{4}) \cdot (v_{1}v_{2}v_{3}v_{4}) \cdot (v_{1}v_{2}v_{3}v_{4})$$

= $(v_{1}v_{2}v_{3} \cdot v_{4}) \cdot (v_{1}v_{2} \cdot v_{3}v_{4}) \cdot (v_{1} \cdot v_{2}v_{3}v_{4})$
= $(v_{1}v_{2}v_{3} \cdot v_{1}v_{2} \cdot v_{1})(v_{4} \cdot v_{3}v_{4} \cdot v_{2}v_{3}v_{4}).$

Let $u = v_1 v_2 v_3 v_1 v_2 v_1$ and $u' = v_4 v_3 v_4 v_2 v_3 v_4$. Then $g^3 = uu'$ is geodesic. Since $u \in Z(v_5)$ and $u' \in Z(v_0)$, we have $||u||_* = ||u'||_* = 1$, hence $||g^3||_* = 2$. Notice that $3 = |V(\Gamma)| - 3 = \text{diam}(\bar{\Gamma}[g])$.

6. Asymptotic translation length

In this section, we study asymptotic translation lengths of elements of $A(\Gamma)$ on $(A(\Gamma), d_*)$ and on (Γ^e, d) , and then find a lower bound of the minimal asymptotic translation length of $A(\Gamma)$ on Γ^e .

Proposition 6.1. If $g \in A(\Gamma)$ is cyclically reduced and non-split with $||g||_* \ge 2$, then

$$\tau_{(A(\Gamma),d_*)}(g) \ge \frac{1}{\max\{2, |V(\Gamma)|-2\}}.$$

Proof. Let τ_* denote $\tau_{(A(\Gamma),d_*)}$, and let $V = |V(\Gamma)|$.

Notice that if $||g||_* \ge 3$, then $||g^n||_* \ge n+2$ for all $n \ge 1$ (by Corollary 4.10), hence

$$\tau_*(g) = \lim_{n \to \infty} \frac{\|g^n\|_*}{n} \ge \lim_{n \to \infty} \frac{n+2}{n} \ge 1.$$

Suppose $||g||_* = 2$. By Proposition 5.7, if $V \leq 4$ then $||g^2||_* \geq 3$, and if $V \geq 5$ then $||g^{V-2}||_* \geq 3$. Therefore $||g^{\max\{2,V-2\}}||_* \geq 3$. From the above discussion, $\tau_*(g^{\max\{2,V-2\}}) \geq 1$ and hence $\tau_*(g) \geq \frac{1}{\max\{2,V-2\}}$. **Remark 6.2.** When we study the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$, we will assume that " $|V(\Gamma)| \ge 2$ and $\overline{\Gamma}$ is connected" because otherwise $\|g\|_* \leq 2$ for all $g \in A(\Gamma)$ and hence $(A(\Gamma), d_*)$ has diameter at most 2: if $|V(\Gamma)| = 1$, then $||g||_* \leq 1$ for all $g \in A(\Gamma)$; if $\overline{\Gamma}$ is disconnected (i.e. Γ is a join), then $||g||_* \leq 2$ for all $g \in A(\Gamma)$.

Lemma 6.3. Suppose that $|V(\Gamma)| \ge 2$ and $\overline{\Gamma}$ is connected. The following are equivalent for a cyclically reduced element $g \in A(\Gamma)$.

- (i) g is strongly non-split and $|\operatorname{supp}(g)| \ge 2$.
- (ii) g is non-split and $||g||_* \ge 2$.
- (iii) $||g^n||_* \ge 3$ for some $n \ge 1$.
- $\begin{array}{ll} (\mathrm{iv}) \ g \ is \ loxodromic \ on \ (A(\Gamma), d_*), \ i.e. \ \tau_{(A(\Gamma), d_*)}(g) > 0. \\ (\mathrm{v}) \ \tau_{(A(\Gamma), d_*)}(g) \geqslant \frac{1}{\max\{2, |V(\Gamma)| 2\}}. \end{array} \end{array}$

Proof of Lemma 6.3. (i) \Leftrightarrow (ii) follows from Lemma 4.15.

(ii) \Rightarrow (v) follows from Proposition 6.1.

 $(v) \Rightarrow (iv)$ and $(iv) \Rightarrow (iii)$ are obvious.

(iii) \Rightarrow (i): Since $||g^n||_* \ge 3$, g^n is strongly non-split (by Lemma 4.15), hence g is also strongly non-split (see Remark 4.14). It is obvious that $|\operatorname{supp}(g)| \ge 2$.

Remark 6.4. Suppose that $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected. Then the condition $|\operatorname{supp}(g)| \ge 2$ of Lemma 6.3(i) is not necessary because all strongly non-split elements g must have $|\operatorname{supp}(g)| \ge 2$. Moreover, g is loxodromic on $(A(\Gamma), d_*)$ if and only if it is loxodromic on (Γ^e, d) by Corollary 4.4. Therefore, if $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected, then (i) and (iv) in the above lemma are equivalent to the following (i') and (iv') respectively.

- (i') g is strongly non-split.
- (iv') g is loxodromic on (Γ^e, d) , i.e. $\tau_{(\Gamma^e, d)}(g) > 0$.

Kim and Koberda [18, Lemma 33] showed that if $g \in A(\Gamma)$ is cyclically reduced and strongly non-split, then $\|g^{2n|V(\Gamma)|^2}\|_* \ge n$ for all $n \ge 1$. Therefore (by Corollary 4.4)

$$\tau_{(\Gamma^e,d)}(g) \ge \tau_{(A(\Gamma),d_*)}(g) \ge \frac{1}{2|V(\Gamma)|^2}$$

From this, a lower bound of the minimal asymptotic translation length of $A(\Gamma)$ on Γ^e follows:

$$\mathcal{L}_{(\Gamma^e, d)}(A(\Gamma)) \geqslant \frac{1}{2|V(\Gamma)|^2}$$

We improve the denominator of the lower bound from a quadratic function to a linear function of $|V(\Gamma)|$ as follows.

Theorem 6.5. Let Γ be a finite simplicial graph such that $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected. Then

$$\mathcal{L}_{(\Gamma^e, d)}(A(\Gamma)) \ge \frac{1}{|V(\Gamma)| - 2}.$$

Proof. Since $|V(\Gamma)| \ge 4$, $|V(\Gamma)| - 2 = \max\{2, |V(\Gamma)| - 2\}$. Let $g \in A(\Gamma)$ be loxodromic on (Γ^e, d) and hence on $(A(\Gamma), d_*)$ (by Remark 6.4). We may assume that g is cyclically reduced because asymptotic translation lengths are invariant under conjugation. By Corollary 4.4 and Lemma 6.3,

$$\tau_{(\Gamma^e,d)}(g) \ge \tau_{(A(\Gamma),d_*)}(g) \ge \frac{1}{|V(\Gamma)| - 2}$$

Therefore $\mathcal{L}_{(\Gamma^e, d)}(A(\Gamma)) \ge \frac{1}{|V(\Gamma)|-2}$.

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7. UNIQUENESS OF QUASI-ROOTS

The notion of quasi-roots in $A(\Gamma)$ was introduced in [25], where the quasi-roots are defined using word length. The uniqueness up to conjugacy was established by using the normal form of elements introduced by Crisp, Godelle and Wiest [7]. In this section, we extend the uniqueness of quasi-roots from word length to star length.

Definition 7.1. (quasi-root) An element $g \in A(\Gamma) \setminus \{1\}$ is called a *quasi-root* of $h \in A(\Gamma)$ if there is a decomposition

$$h = ag^n b$$

for some $n \ge 1$ and $a, b \in A(\Gamma)$ such that ||h|| = ||a|| + ||b|| + n||g||. The decomposition is called a *quasi-root decomposition* of h. The conjugates aga^{-1} and $b^{-1}gb$ are called the *leftward-extraction* and the *rightward-extraction* of the quasi-root g, respectively. We consider the following two cases.

- (i) g is called an (A, B, r)-quasi-root of h if $||a|| \leq A$, $||b|| \leq B$ and $||g|| \leq r$.
- (ii) g is called an $(A, B, r)^*$ -quasi-root of h if $||a||_* \leq A$, $||b||_* \leq B$ and $||g||_* \leq r$.

In the above definition, the condition ||h|| = ||a|| + ||b|| + n||g|| implies $||g^n|| = n||g||$, hence g is cyclically reduced when $n \ge 2$.

Notice that if $g_1 = aga^{-1}$ and $g_2 = b^{-1}gb$ are respectively the leftward- and the rightward-extractions of g, then we have decompositions $h = g_1^n ab = abg_2^n$, which are not necessarily geodesic.

Definition 7.2 (primitive). An element $g \in A(\Gamma) \setminus \{1\}$ is called *primitive* if g is not a nontrivial power of another element, i.e. $g = h^n$ never holds for any $n \ge 2$ and $h \in A(\Gamma)$.

The following proposition is Proposition 3.5 in [25] written in the setting of this paper. It shows a kind of uniqueness property of quasi-roots in right-angled Artin groups.

Proposition 7.3 ([25, Proposition 3.5]). Let $h \in A(\Gamma)$, $A, B \ge 0$ and $r \ge 1$. If

$$\|h\| \ge A + B + (2V+1)r$$

where $V = |V(\Gamma)|$, then strongly non-split and primitive (A, B, r)-quasi-roots of h are conjugate to each other, and moreover, their leftward- and rightward-extractions are unique.

In other words, Proposition 7.3 shows that if

$$h = a_1 g_1^{n_1} b_1 = a_2 g_2^{n_2} b_2$$

are two quasi-root decompositions of h such that for each $i = 1, 2, g_i$ is strongly non-split and primitive,

$$\begin{aligned} \|a_i\| \leqslant A, \quad \|b_i\| \leqslant B, \quad \|g_i\| \leqslant r, \\ \|h\| \geqslant A + B + (2V+1)r, \end{aligned}$$

then g_1 and g_2 are conjugate, and moreover, $a_1g_1a_1^{-1} = a_2g_2a_2^{-1}$ and $b_1^{-1}g_1b_1 = b_2^{-1}g_2b_2$.

The following theorem is the main result of this section. It is a star length version of Proposition 7.3, which plays an important role in the proof of Theorem 8.2. We remark that the word length and the star length are quite independent, hence the word length version does not naively extend to a star length version. We exploit lattice structures developed in §2.

We also remark that in the following theorem since $||h||_* \ge 2A + 2B + (2V+3)r + 2 \ge 3r + 2 \ge 5$, the existence of such an element h implies that $|V(\Gamma)| \ge 2$ and $\overline{\Gamma}$ is connected (as explained in Remark 6.2).

Theorem 7.4. Let $h \in A(\Gamma)$, $A, B \ge 0$ and $r \ge 1$. If

$$||h||_* \ge 2A + 2B + (2V + 3)r + 2,$$

where $V = |V(\Gamma)|$, then primitive $(A, B, r)^*$ -quasi-roots of h are conjugate to each other, and moreover, their leftward- and rightward-extractions are unique. In other words, if

$$h = a_1 g_1^{n_1} b_1 = a_2 g_2^{n_2} b_2$$

are two quasi-root decompositions of h such that for each $i = 1, 2, g_i$ is primitive and

$$\begin{aligned} \|a_i\|_* \leqslant A, \quad \|b_i\|_* \leqslant B, \quad \|g_i\|_* \leqslant r, \\ \|h\|_* \ge 2A + 2B + (2V+3)r + 2, \end{aligned}$$

then g_1 and g_2 are conjugate to each other such that

$$a_1g_1a_1^{-1} = a_2g_2a_2^{-1}$$
 and $b_1^{-1}g_1b_1 = b_2^{-1}g_2b_2$.

Proof. Let i = 1, 2.

Claim 1. $n_i \ge 4$ and g_i is cyclically reduced and strongly non-split with $||g_i||_* \ge 2$.

Proof of Claim 1. If $n_i \leq 3$, then

$$||h||_* = ||a_i g_i^{n_i} b_i||_* \leq ||a_i||_* + n_i ||g_i||_* + ||b_i||_* \leq A + 3r + B.$$

This contradicts the hypothesis $||h||_* \ge 2A + 2B + (2V + 3)r + 2$. Therefore $n_i \ge 4$ and hence g_i is cyclically reduced (see the paragraph following Definition 7.1).

Observe that $||g_i^{n_i}||_* \ge 5$ because

$$||g_i^{n_i}||_* \ge ||h||_* - ||a_i||_* - ||b_i||_* \ge A + B + (2V + 3)r + 2 \ge 3r + 2 \ge 5.$$

Therefore g_i is strongly non-split and $||g_i||_* \ge 2$ (by Lemma 6.3).

Let α_i and β_i be integers defined by

$$\alpha_i = \min\{k \ge 1 : \|g_i^k\|_* \ge A+2\},$$

$$\beta_i = \min\{k \ge 1 : \|g_i^k\|_* \ge B+2\}.$$

The numbers α_i and β_i are well-defined because the sequence $\{\|g_i^k\|_*\}_{k=1}^{\infty}$ is increasing such that $\lim_{k\to\infty} \|g_i^k\|_* = \infty$ (by Claim 1, Lemmas 4.5 and 6.3). Since $\|g_i^k\|_* - \|g_i^{k-1}\|_* \leq \|g_i\|_* \leq r$ for all $k \geq 1$, we get

$$\begin{aligned} A+2 &\leqslant \|g_i^{\alpha_i}\|_* \leqslant A+1+r, \\ B+2 &\leqslant \|g_i^{\beta_i}\|_* \leqslant B+1+r. \end{aligned}$$

Claim 2. $n_i - \alpha_i - \beta_i \ge 2V + 1$.

Proof of Claim 2. Observe that

$$\begin{split} \|g_i^{n_i}\|_* &\ge \|h\|_* - \|a_i\|_* - \|b_i\|_* \ge \|h\|_* - A - B, \\ \|g_i^{n_i}\|_* - \|g_i^{\alpha_i + \beta_i}\|_* &\ge \|g_i^{n_i}\|_* - \|g_i^{\alpha_i}\|_* - \|g_i^{\beta_i}\|_* \\ &\ge (\|h\|_* - A - B) - (A + 1 + r) - (B + 1 + r) \\ &= \|h\|_* - 2A - 2B - 2r - 2 \ge (2V + 1)r. \end{split}$$

Since $\{\|g_i^k\|_*\}_{k=1}^{\infty}$ is increasing and $\|g_i^{n_i}\|_* - \|g_i^{\alpha_i+\beta_i}\|_* \ge (2V+1)r > 0$, we have $n_i > \alpha_i + \beta_i$. Since $(n_i - \alpha_i - \beta_i)\|g_i\|_* \ge \|g_i^{n_i - \alpha_i - \beta_i}\|_* \ge \|g_i^{n_i}\|_* - \|g_i^{\alpha_i + \beta_i}\|_* \ge (2V+1)r \ge (2V+1)\|g_i\|_*$,

we get $n_i - \alpha_i - \beta_i \ge 2V + 1$ as desired.

Let $a_0 = a_1 \wedge_L a_2$ and $b_0 = b_1 \wedge_R b_2$. Then we have geodesic decompositions

$$\begin{cases} a_1 = a_0 a'_1, \\ a_2 = a_0 a'_2, \end{cases} \begin{cases} b_1 = b'_1 b_0, \\ b_2 = b'_2 b_0 \end{cases}$$

for some $a'_1, a'_2, b'_1, b'_2 \in A(\Gamma)$ with $a'_1 \wedge_L a'_2 = 1$ and $b'_1 \wedge_R b'_2 = 1$. Observe that

$$||a_i'||_* \leq ||a_i||_* \leq A, ||b_i'||_* \leq ||b_i||_* \leq B.$$

Since $h = a_1 g_1^{n_1} b_1 = a_2 g_2^{n_2} b_2$, we have $h = a_0 (a'_1 g_1^{n_1} b'_1) b_0 = a_0 (a'_2 g_2^{n_2} b'_2) b_0$.

Let $h_0 = a_0^{-1} h b_0^{-1}$. Then h_0 has the following two geodesic decompositions.

(4)
$$h_0 = a_1' g_1^{n_1} b_1' = a_2' g_2^{n_2} b_2'$$

On the other hand, since a_1 and a_2 have a common right multiple, say h, we have $a'_1 \rightleftharpoons a'_2$ (by Theorem 2.12). Since $a'_1 \leq a'_2 g_2^{n_2} b'_2$ and $a'_2 \leq a'_1 g_1^{n_1} b'_1$, we have (by Lemma 2.15)

$$a'_1 \leqslant_L g_2^{n_2} b'_2$$
 and $a'_2 \leqslant_L g_1^{n_1} b'_1$.
Let $A' = ||a'_1|| + ||a'_2||$ and $B' = ||b'_1|| + ||b'_2||$.

Claim 3. $||h_0|| \ge A' + B' + (2V+1)||g_i||$.

Proof of Claim 3. We know that $n_1 - \alpha_1 > 0$ (by Claim 2) and $a'_2 \leq_L g_1^{n_1} b'_1 = g_1^{\alpha_1} \cdot g_1^{n_1 - \alpha_1} b'_1$. Since $g_1^{\alpha_1} \cdot g_1^{n_1 - \alpha_1} b'_1$ is geodesic and $\|g_1^{\alpha_1}\|_* \geq A + 2 \geq \|a'_2\|_* + 2$, we have $a'_2 \leq_L g_1^{\alpha_1}$ (by Corollary 4.7). Similarly, $b'_2 \leq_R g_1^{\beta_1}$. (In other words, $g_1^{n_1}$ has a geodesic decomposition $g_1^{n_1} = g_1^{\alpha_1} \cdot g_1^{n_1 - \alpha_1 - \beta_1} \cdot g_1^{\beta_1}$ such that $a'_2 \leq_L g_1^{\alpha_1}$ and $b'_2 \leq_R g_1^{\beta_1}$.) Therefore

$$||a_2'|| \le ||g_1^{\alpha_1}|| = \alpha_1 ||g_1||,$$

$$||b_2'|| \le ||g_1^{\beta_1}|| = \beta_1 ||g_1||.$$

Since $h_0 = a'_1 g_1^{n_1} b'_1 = a'_2 g_2^{n_2} b'_2$, we get

$$||h_0|| - (A' + B') = (||a_1'|| + n_1||g_1|| + ||b_1'||) - (||a_1'|| + ||a_2'||) - (||b_1'|| + ||b_2'||)$$

= $n_1||g_1|| - ||a_2'|| - ||b_2'|| \ge n_1||g_1|| - \alpha_1||g_1|| - \beta_1||g_1||$
= $(n_1 - \alpha_1 - \beta_1)||g_1|| \ge (2V + 1)||g_1||.$

In the same way, we get $||h_0|| - (A' + B') \ge (2V + 1)||g_2||$.

Notice that $||a'_i|| \leq A'$ and $||b'_i|| \leq B'$. Let $r' = \max\{||g_1||, ||g_2||\}$. Then each $a'_i g_i^{n_i} b'_i$ in (4) is a (A', B', r')-quasi-root decomposition of h_0 such that $||h_0|| \geq A' + B' + (2V+1)r'$.

Applying Proposition 7.3 to (4) yields $a'_1g_1a'_1^{-1} = a'_2g_2a'_2^{-1}$ and $b'_1{}^{-1}g_1b'_1 = b'_2{}^{-1}g_2b'_2$. Consequently

$$a_1g_1a_1^{-1} = a_0(a_1'g_1a_1'^{-1})a_0^{-1} = a_0(a_2'g_2a_2'^{-1})a_0^{-1} = a_2g_2a_2^{-1},$$

$$b_1^{-1}g_1b_1 = b_0^{-1}(b_1'^{-1}g_1b_1')b_0 = b_0^{-1}(b_2'^{-1}g_2b_2')b_0 = b_2^{-1}g_2b_2.$$

8. Acylindricity of the action of $A(\Gamma)$ on Γ^e

In this section, we prove the following two theorems.

Theorem 8.1. Let Γ be a finite simplicial graph such that $|V(\Gamma)| \ge 2$ and $\overline{\Gamma}$ is connected. Then the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$ is (R, N)-acylindrical with

$$R = R(r) = (2V + 7)r + 8,$$

$$N = N(r) = 2(V - 2)(r - 1) - 1,$$

where $V = \max\{4, |V(\Gamma)|\}$. Moreover, for any $x, y \in A(\Gamma)$ with $d_*(x, y) \ge R$, if $\xi(x, y; r) \ne \{1\}$, then there exists a loxodromic element $g \in A(\Gamma)$ such that

- (i) $\xi(x,y;r) = \{1, g^{\pm 1}, g^{\pm 2}, \dots, g^{\pm k}\}$ for some $1 \le k \le (V-2)(r-1)-1$;
- (ii) the Hausdorff distance between the $\langle g \rangle$ -orbit of x and that of y is at most 2r + 3.

Theorem 8.2. Let Γ be a finite simplicial graph such that $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected. Then the action of $A(\Gamma)$ on Γ^e is (R, N)-acylindrical with

$$R = R(r) = D(2V + 7)(r + 1) + 10D,$$

$$N = N(r) = 2(V - 2)r - 1,$$

where $D = \operatorname{diam}(\Gamma)$ and $V = |V(\Gamma)|$. Moreover, for any $x, y \in V(\Gamma^e)$ with $d(x, y) \ge R$, if $\xi(x, y; r) \ne \{1\}$, then there exists a loxodromic element $g \in A(\Gamma)$ such that

- (i) $\xi(x, y; r) \subset \{1, g^{\pm 1}, g^{\pm 2}, \dots, g^{\pm k}\}$ for some $1 \leq k \leq (V-2)r 1$;
- (ii) the Hausdorff distance between the $\langle g \rangle$ -orbit of x and that of y is at most D(2r+7).

The following lemma connects the acylindricities of the actions of $A(\Gamma)$ on $(A(\Gamma), d_*)$ and on (Γ^e, d) . It is an improvement of the argument of Kim and Koberda in the proof of Theorem 30 in [18].

Lemma 8.3. Suppose that $|V(\Gamma)| \ge 4$ and both Γ and $\overline{\Gamma}$ are connected. Let $D = \operatorname{diam}(\Gamma)$. If the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$ is $(R_1(r), N_1(r))$ -acylindrical, then the action of $A(\Gamma)$ on (Γ^e, d) is $(R_2(r), N_2(r))$ -acylindrical with

$$R_2(r) = D \cdot R_1(r+1) + 2D,$$

 $N_2(r) = N_1(r+1).$

More precisely, for any $v_1^{w_1}, v_2^{w_2} \in V(\Gamma^e)$, where $v_1, v_2 \in V(\Gamma)$ and $w_1, w_2 \in A(\Gamma)$, if $d(v_1^{w_1}, v_2^{w_2}) \ge R_2(r)$, then

- (i) $d_*(w_1, w_2) \ge R_1(r+1);$
- (ii) $\xi_{(\Gamma^e,d)}(v_1^{w_1}, v_2^{w_2}; r)$ is contained in $\xi_{(A(\Gamma),d_*)}(w_1, w_2; r+1)$.

Proof. Note that $D = \operatorname{diam}(\Gamma) \neq 0$. Let $d(v_1^{w_1}, v_2^{w_2}) \geq R_2(r)$ for $v_1^{w_1}, v_2^{w_2} \in V(\Gamma^e)$. Since Γ is connected, we can apply Lemma 4.3 and obtain

$$d_*(w_1, w_2) = \|w_2 w_1^{-1}\|_*$$

$$\geqslant \frac{d(v_2, v_2^{w_2 w_1^{-1}})}{D} - 1 \geqslant \frac{d(v_1, v_2^{w_2 w_1^{-1}}) - d(v_1, v_2)}{D} - 1$$

$$\geqslant \frac{d(v_1^{w_1}, v_2^{w_2}) - D}{D} - 1 \geqslant \frac{R_2(r) - 2D}{D} = R_1(r+1),$$

which proves (i).

Let $g \in \xi_{(\Gamma^e,d)}(v_1^{w_1}, v_2^{w_2}; r)$. Then $d(v_i^{w_ig}, v_i^{w_i}) \leqslant r$ for i = 1, 2. By Lemma 4.3 again,

$$d_*(w_ig, w_i) = \|w_igw_i^{-1}\|_* \leq d(v_i^{w_igw_i^{-1}}, v_i) + 1$$
$$= d(v_i^{w_ig}, v_i^{w_i}) + 1 \leq r + 1$$

for i = 1, 2, hence $g \in \xi_{(A(\Gamma), d_*)}(w_1, w_2; r+1)$. This shows that the set $\xi_{(\Gamma^e, d)}(v_1^{w_1}, v_2^{w_2}; r)$ is contained in $\xi_{(A(\Gamma), d_*)}(w_1, w_2; r+1)$, hence (ii) is proved.

Since $\xi_{(\Gamma^e,d)}(v_1^{w_1}, v_2^{w_2}; r) \subset \xi_{(A(\Gamma),d_*)}(w_1, w_2; r+1)$ and $d_*(w_1, w_2) \ge R_1(r+1)$, the (R_1, N_1) -acylindricity of the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$ implies that

$$|\xi_{(\Gamma^e,d)}(v_1^{w_1}, v_2^{w_2}; r)| \leq |\xi_{(A(\Gamma),d_*)}(w_1, w_2; r+1)| \leq N_1(r+1) = N_2(r).$$

Therefore the action of $A(\Gamma)$ on (Γ^e, d) is $(R_2(r), N_2(r))$ -acylindrical.

Proposition 8.4. Let $g, w \in A(\Gamma) \setminus \{1\}$ and $r, R \ge 1$ such that

$$||g||_* \leq r, ||w^{-1}gw||_* \leq r, ||w||_* \ge R, R \ge 3r + 7.$$

Then there exists a quasi-root decomposition

$$w = a(g_1^{\epsilon})^n b,$$

where $a, b, g_1 \in A(\Gamma)$, $\epsilon \in \{\pm 1\}$ and $n \ge 2$ such that

- (i) $||a||_* \leq \frac{1}{2}r + 1$ and $||b||_* \leq \frac{3}{2}r + 2;$
- (ii) g_1 is cyclically reduced and $g = ag_1a^{-1}$ is geodesic.

Notice that $||w||_* \ge R \ge 3r+7 \ge 7$, hence the existence of such an element w implies that $|V(\Gamma)| \ge 2$ and $\overline{\Gamma}$ is connected (as explained in Remark 6.2).

Proof. Let $g = ag_1a^{-1}$ be the geodesic decomposition such that g_1 is cyclically reduced. Let $h = w^{-1}gw$. Then

$$h = w^{-1}ag_1a^{-1}w = (a^{-1}w)^{-1}g_1(a^{-1}w)$$

By Theorem 3.9, there exists a geodesic decomposition of $a^{-1}w$

(5)
$$a^{-1}w = w_1w_2w_3$$

such that (i) $w_1 \rightleftharpoons g_1$; (ii) $g_1^{w_2}$ is a cyclic conjugation; (iii) $h = w_3^{-1} \cdot g_1^{w_2} \cdot w_3$ is geodesic.

Claim 1. The following hold.

- (i) $||w_2||_* \ge 3$, hence w_2 is strongly non-split.
- (ii) $g_1^{w_2}$ is either a left cyclic conjugation or a right cyclic conjugation.

Proof of Claim 1. Since both $g = ag_1a^{-1}$ and $h = w_3^{-1}g_1^{w_2}w_3$ are geodesic decompositions,

$$||g_1||_* + 2||a||_* - 4 \leq ||g||_* \leq ||g_1||_* + 2||a||_*,$$

$$||g_1^{w_2}||_* + 2||w_3||_* - 4 \leq ||h||_* \leq ||g_1^{w_2}||_* + 2||w_3||_*$$

(by Corollary 4.8), whence

$$\frac{\|g\|_{*} - \|g_{1}\|_{*}}{2} \leq \|a\|_{*} \leq \frac{\|g\|_{*} - \|g_{1}\|_{*}}{2} + 2,$$
$$\frac{\|h\|_{*} - \|g_{1}^{w_{2}}\|_{*}}{2} \leq \|w_{3}\|_{*} \leq \frac{\|h\|_{*} - \|g_{1}^{w_{2}}\|_{*}}{2} + 2.$$

Since $w_1 \rightleftharpoons g_1 \neq 1$, we have $||w_1||_* \leq 1$. Since $g_1 \neq 1$ and both $g = ag_1a^{-1}$ and $h = w_3^{-1}g_1^{w_2}w_3$ are geodesic, we have $1 \leq ||g_1||_* \leq ||g||_* \leq r$ and $1 \leq ||g_1^{w_2}||_* \leq ||h||_* \leq r$. Since $a^{-1}w = w_1w_2w_3$,

$$\begin{split} w \|_{*} &\leq \|a\|_{*} + \|w_{1}\|_{*} + \|w_{2}\|_{*} + \|w_{3}\|_{*} \\ &\leq \left(\frac{\|g\|_{*} - \|g_{1}\|_{*}}{2} + 2\right) + 1 + \|w_{2}\|_{*} + \left(\frac{\|h\|_{*} - \|g_{1}^{w_{2}}\|_{*}}{2} + 2\right) \\ &\leq \left(\frac{r-1}{2} + 2\right) + 1 + \|w_{2}\|_{*} + \left(\frac{r-1}{2} + 2\right) \\ &= \|w_{2}\|_{*} + r + 4. \end{split}$$

Therefore $||w_2||_* \ge ||w||_* - r - 4 \ge R - r - 4 \ge 2r + 3 \ge 3$, hence w_2 is strongly non-split (by Lemma 4.15). This proves (i).

Assume that the cyclic conjugation $g_1^{w_2}$ is neither a left cyclic conjugation nor a right cyclic conjugation. Then, by Proposition 3.7(v), $w_2 = w'_2 w''_2$ is geodesic for some $w'_2, w''_2 \in A(\Gamma) \setminus \{1\}$ such that $g_1^{w'_2}$ (resp. $g_1^{w''_2}$) is a left (resp. right) cyclic conjugation and $w'_2 \rightleftharpoons w''_2$. Since both w'_2 and w''_2 are nontrivial, we have $||w'_2||_* = ||w''_2||_* = 1$, hence $||w_2||_* = ||w'_2w''_2||_* \leq ||w'_2||_* + ||w''_2||_* = 2$, which contradicts $||w_2||_* \geq 3$. Therefore $g_1^{w_2}$ is either a left cyclic conjugation or a right cyclic conjugation. This proves (ii).

Claim 2. The following hold.

- (i) g_1 is strongly non-split with $|\operatorname{supp}(g_1)| \ge 2$ and $2 \le ||g_1||_* \le r$.
- (ii) $||a||_* \leq \frac{1}{2}r+1$, $w_1 = 1$, $||w_2||_* \geq R-r-2$ and $||w_3||_* \leq \frac{1}{2}r+1$.

Proof of Claim 2. (i) Since $g_1^{w_2}$ is either a left or a right cyclic conjugation (by Claim 1),

$$w_2 \leqslant_L g_1^n$$
 or $w_2^{-1} \leqslant_R g_1^n$

for some $n \ge 1$ (by Proposition 3.8). Since $||w_2||_* \ge 3$, we have $||g_1^n||_* \ge 3$. By Lemma 6.3, g_1 is strongly non-split with $|\operatorname{supp}(g_1)| \ge 2$ and $||g_1||_* \ge 2$. On the other hand, $||g_1||_* \le ||g||_* \le r$ because $g = ag_1a^{-1}$ is geodesic.

(ii) If $w_1 \neq 1$, then $||g_1||_* \leq 1$ because $g_1 \rightleftharpoons w_1$, which contradicts $||g_1||_* \geq 2$. Therefore $w_1 = 1$.

Since g_1 is strongly non-split and $g_1^{w_2}$ is a cyclic conjugation, $g_1^{w_2}$ is also strongly non-split and $|\operatorname{supp}(g_1^{w_2})| = |\operatorname{supp}(g_1)| \ge 2$. Therefore $||g_1^{w_2}||_* \ge 2$ (by Lemma 4.15).

Since $w_1 = 1$, $||g||_* \leq r$, $||h||_* \leq r$, $||g_1||_* \geq 2$ and $||g_1^{w_2}||_* \geq 2$, using the inequalities in the proof of Claim 1, we have

$$\begin{aligned} \|a\|_{*} &\leq \frac{\|g\|_{*} - \|g_{1}\|_{*}}{2} + 2 \leq \frac{r-2}{2} + 2 = \frac{r}{2} + 1, \\ \|w_{3}\|_{*} &\leq \frac{\|h\|_{*} - \|g_{1}^{w_{2}}\|_{*}}{2} + 2 \leq \frac{r-2}{2} + 2 = \frac{r}{2} + 1, \\ \|w\|_{*} &\leq \|a\|_{*} + \|w_{1}\|_{*} + \|w_{2}\|_{*} + \|w_{3}\|_{*} \\ &\leq \left(\frac{r}{2} + 1\right) + 0 + \|w_{2}\|_{*} + \left(\frac{r}{2} + 1\right) = \|w_{2}\|_{*} + r + 2. \\ &+ 1, \|w_{3}\|_{*} \leq \frac{1}{2}r + 1 \text{ and } \|w_{2}\|_{*} \geq \|w\|_{*} - r - 2 \geq R - r - 2. \end{aligned}$$

Therefore $||a||_* \leq \frac{1}{2}r+1$, $||w_3||_* \leq \frac{1}{2}r+1$ and $||w_2||_* \geq ||w||_* - r - 2 \geq R - r - 2$.

Claim 3. Let $\epsilon = 1$ (resp. $\epsilon = -1$) if $g_1^{w_2}$ is a left (resp. right) cyclic conjugation. Then there exists a quasi-root decomposition

$$w = a(g_1^{\epsilon})^n b$$

such that $n \ge 2$ and $||b||_* \le \frac{3}{2}r + 2$.

Proof of Claim 3. Suppose that $g_1^{w_2}$ is a left cyclic conjugation. Then $w_2 \leq_L g_1^k$ for some $k \geq 1$ (by Proposition 3.8). Hence

$$w_2 = g_1^n d$$

is geodesic for some $0 \leq n \leq k$ and $d \in A(\Gamma)$ with $||d||_* \leq ||g_1||_* + 1$ (by Corollary 5.6). Notice that $n \geq 2$ because $||g_1||_* \leq r$ whereas

$$||g_1^n||_* \ge ||w_2||_* - ||d||_* \ge (R - r - 2) - (r + 1) = R - 2r - 3 \ge r + 4.$$

Since $a^{-1}w = w_1w_2w_3$ and $w_1 = 1$, we have

$$w = aw_2w_3 = ag_1^n dw_3.$$

We will now prove that the decomposition $w = ag_1^n dw_3$ is geodesic. Since $g = ag_1a^{-1}$ is geodesic and g_1 is cyclically reduced, ag_1^n is geodesic (by Lemmas 2.8 and 3.4). Since both w_2w_3 and $w_2 = g_1^n d$ are geodesic, $g_1^n dw_3$ is also geodesic. Recall $||g_1^n||_* \ge r + 4 \ge 2$. Therefore $w = ag_1^n dw_3$ is geodesic (by Lemma 4.12).

Let $b = dw_3$. Then $w = ag_1^n b$ is geodesic and

$$||b||_* \leq ||d||_* + ||w_3||_* \leq \left(||g_1||_* + 1\right) + \left(\frac{r}{2} + 1\right) \leq r + 1 + \frac{r}{2} + 1 = \frac{3}{2}r + 2.$$

Therefore $w = ag_1^n b$ is a quasi-root decomposition with the desired properties.

Now suppose that $g_1^{w_2}$ is a right cyclic conjugation. Then $(g_1^{-1})^{w_2}$ is a left cyclic conjugation (by Proposition 3.8). From the above argument, $w = a(g_1^{-1})^n b$ is a quasi-root decomposition with the desired properties.

The proof is now completed.

Remark 8.5. In Proposition 8.4, notice that

$$w = a(g_1^{\epsilon})^n b = (a(g_1^{\epsilon})^n a^{-1})ab = (g^{\epsilon})^n ab,$$

$$\|ab\|_* \leq \|a\|_* + \|b\|_* \leq (\frac{1}{2}r+1) + (\frac{3}{2}r+2) = 2r+3.$$

Thus one could understand Proposition 8.4 as follows: if $||w||_*$ is large but both $||g||_*$ and $||w^{-1}gw||_*$ are small, then $w = g^n c$ for some integer n and $c \in A(\Gamma)$ with $||c||_*$ small.

If $||w^{-1}gw||_* \leq r$ and $g = ag_1a^{-1}$ in the statement of Proposition 8.4 are respectively replaced with $||wgw^{-1}||_* \leq r$ and $g = a^{-1}g_1a$, then we have the following corollary.

Corollary 8.6. Let $g, w \in A(\Gamma) \setminus \{1\}$ and $r, R \ge 1$ such that

$$||g||_* \leq r, ||wgw^{-1}||_* \leq r, ||w||_* \ge R, R \ge 3r+7.$$

Then there exists a quasi-root decomposition

$$w = b(g_1^{\epsilon})^n a_{\epsilon}$$

where $a, b, g_1 \in A(\Gamma)$, $\epsilon \in \{\pm 1\}$ and $n \ge 2$ such that

- (i) $||a||_* \leq \frac{1}{2}r + 1$ and $||b||_* \leq \frac{3}{2}r + 2;$
- (ii) g_1 is cyclically reduced and $g = a^{-1}g_1a$ is geodesic.

We will now prove Theorem 8.1.

Proof of Theorem 8.1. Choose $x, y \in A(\Gamma)$ with $d_*(x, y) \ge R$. Let $w = yx^{-1}$, hence $||w||_* = d_*(x, y) \ge R$. R.

We may assume $\xi(1, w; r) \neq \{1\}$ because otherwise $\xi(x, y; r) = x^{-1}\xi(1, w; r)x = \{1\}$ and there is nothing to prove.

By Lemma 4.5(ii), the set $\xi(1, w; r)$ is closed under taking a root, i.e. if $h^k \in \xi(1, w; r)$ for some $h \in A(\Gamma)$ and $k \ge 1$, then $h \in \xi(1, w; r)$. Therefore there exists a primitive element g_0 in $\xi(1, w; r) \setminus \{1\}$, hence $||g_0||_* = d_*(g_0, 1) \leq r$ and $||wg_0w^{-1}||_* = d_*(wg_0, w) \leq r$.

We will now show that $g_0^{\pm 1}$ is uniquely determined from $w = yx^{-1}$. Let

$$g_0 = a^{-1}g_1a$$

be the geodesic decomposition such that g_1 is cyclically reduced. Then g_1 is also primitive and $||g_1||_* \leq ||g_0||_* \leq r$. Since $R = (2V+7)r + 8 \geq 3r + 7$, (w, g_0, g_1, a, R, r) satisfies the conditions on (w, g, g_1, a, R, r) in Corollary 8.6, hence there exists a quasi-root decomposition

$$w = b(g_1^{\epsilon})^n a$$

where $b \in A(\Gamma)$, $\epsilon \in \{\pm 1\}$, $n \ge 2$, $||a||_* \le \frac{1}{2}r + 1$ and $||b||_* \le \frac{3}{2}r + 2$.

Let $A = \frac{1}{2}r + 1$ and $B = \frac{3}{2}r + 2$. Then g_1^{ϵ} is a primitive $(B, A, r)^*$ -quasi-root of w. Observe that 2A + 2B + (2V + 3)r + 2 = (r + 2) + (3r + 4) + (2V + 3)r + 2 = (2V + 7)r + 8 = R, hence

$$||w||_* \ge R = 2A + 2B + (2V + 3)r + 2.$$

The tuple $(w, g_1^{\epsilon}, b, a, B, A, r)$ now satisfies the conditions on $(h, g_1, a_1, b_1, A, B, r)$ in Theorem 7.4. Therefore the primitive element g_0^{ϵ} is uniquely determined from w because $g_0^{\epsilon} = a^{-1}g_1^{\epsilon}a$ is the rightward-extraction of the $(B, A, r)^*$ -quasi-root g_1^{ϵ} . This means that each element of $\xi(1, w; r)$ is a power of g_0 , hence $\xi(1, w; r) \subset \langle g_0 \rangle$. Since

$$||g_1^n||_* = ||(g_1^{\epsilon})^n||_* \ge ||w||_* - ||b||_* - ||a||_* \ge R - B - A$$
$$\ge A + B + (2V + 3)r + 2 \ge 3,$$

the cyclically reduced element g_1 is loxodromic (by Lemma 6.3), hence $\|g_1^{(V-2)j}\|_* \ge j+2$ for all $\begin{aligned} j &\geq 1 \text{ (by Lemma 6.3, Proposition 5.7 and Corollary 4.10). Since } g_0 \text{ is conjugate to } g_1, g_0 \text{ is} \\ also loxodromic. Since } g_0^{(V-2)j} &= a^{-1}g_1^{(V-2)j}a \text{ is a geodesic decomposition (by Lemma 2.8(iii)),} \\ \|g_0^{(V-2)j}\|_* &\geq \|g_1^{(V-2)j}\|_* \geq j+2 \text{ for all } j \geq 1. \\ \text{If } k \geq (V-2)(r-1), \text{ then } \|g_0^k\|_* \geq \|g_0^{(V-2)(r-1)}\|_* \geq (r-1)+2 = r+1, \text{ hence } g_0^k \notin \xi(1,w;r). \text{ From} \end{aligned}$

this fact and Lemma 4.5, it follows that

$$\xi(1, w; r) = \{1, g_0^{\pm 1}, \dots, g_0^{\pm k}\}$$

for some $1 \le k \le (V - 2)(r - 1) - 1$.

Let $g = x^{-1}g_0x$. Then g is also loxodromic. Since $\xi(x,y;r) = x^{-1} \cdot \xi(1,w;r) \cdot x$,

$$\xi(x,y;r) = \{1, g^{\pm 1}, \dots, g^{\pm k}\}$$

hence (i) is proved.

Let N(r) = 2(V-2)(r-1) - 1. Since $|\xi(x,y;r)| = 2k + 1 \le 2(V-2)(r-1) - 1 = N(r)$, the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$ is (R(r), N(r))-acylindrical.

Since $g_0 = a^{-1}g_1a$, $g = x^{-1}g_0x$ and $yx^{-1} = w = b(g_1^{\epsilon})^n a$, we get

$$y = wx = b(g_1^{\epsilon})^n ax = bax \cdot x^{-1}a^{-1}(g_1^{\epsilon})^n ax = bax(g^{\epsilon})^n.$$

Hence $d_*(y, x(g^{\epsilon})^n) = d_*(bax(g^{\epsilon})^n, x(g^{\epsilon})^n) = ||ba||_* \leq ||b||_* + ||a||_* \leq 2r + 3$. Therefore the Hausdorff distance between the $\langle g \rangle$ -orbits $x \langle g \rangle$ and $y \langle g \rangle$ is at most 2r + 3, hence (ii) is proved.

Remark 8.7. The above proof shows that g_1^{ϵ} is a primitive $\left(\frac{3}{2}r+2, \frac{1}{2}r+1, r\right)^*$ -quasi-root of $w = b(g_1^{\epsilon})^n a$. Notice that the rightward-extraction of g_1^{ϵ} is $a^{-1}g_1^{\epsilon}a$, and that $xgx^{-1} = a^{-1}g_1a$. Therefore either xgx^{-1} or $xg^{-1}x^{-1}$ is the rightward-extraction of a primitive $\left(\frac{3}{2}r+2, \frac{1}{2}r+1, r\right)^*$ -quasi-root of yx^{-1} .

We are now ready to prove Theorem 8.2.

Proof of Theorem 8.2. By Theorem 8.1, the action of $A(\Gamma)$ on $(A(\Gamma), d_*)$ is $(R_1(r), N_1(r))$ -acylindrical with

$$R_1(r) = (2V+7)r + 8,$$

$$N_1(r) = 2(V-2)(r-1) - 1$$

Applying Lemma 8.3 to the above, the action of $A(\Gamma)$ on (Γ^e, d) is (R(r), N(r))-acylindrical with

$$R(r) = D \cdot R_1(r+1) + 2D = D(2V+7)(r+1) + 10D$$
$$N(r) = N_1(r+1) = 2(V-2)r - 1.$$

Choose $x, y \in V(\Gamma^e)$ with $d(x, y) \ge R(r)$ and $\xi_{(\Gamma^e, d)}(x, y; r) \ne \{1\}$. Then there exist $v_1, v_2 \in V(\Gamma)$ and $w_1, w_2 \in A(\Gamma)$ such that $x = v_1^{w_1}$ and $y = v_2^{w_2}$. By Lemma 8.3,

$$\begin{split} & d_*(w_1,w_2) \geqslant R_1(r+1), \\ & \xi_{(\Gamma^e,d)}(v_1^{w_1},v_2^{w_2};r) \subset \xi_{(A(\Gamma),d_*)}(w_1,w_2;r+1). \end{split}$$

Since $\xi_{(\Gamma^e,d)}(v_1^{w_1}, v_2^{w_2}; r) \neq \{1\}$, we have $\xi_{(A(\Gamma),d_*)}(w_1, w_2; r+1) \neq \{1\}$. Hence (w_1, w_2) satisfies all the conditions on (x, y) in Theorem 8.1. Therefore, by Theorem 8.1(i),

$$\xi_{(\Gamma^e,d)}(x,y;r) \subset \xi_{(A(\Gamma),d_*)}(w_1,w_2;r+1) = \{1,g^{\pm 1},g^{\pm 2},\ldots,g^{\pm k}\}$$

for some loxodromic element $g \in A(\Gamma)$ and $1 \leq k \leq (V-2)r - 1$, hence (i) is proved.

Since the Hausdorff distance between the $\langle g \rangle$ -orbits of w_1 and w_2 is at most 2(r+1) + 3 = 2r + 5(by Theorem 8.1(ii)), $w_2 = cw_1g^n$ for some $n \in \mathbb{Z}$ and $c \in A(\Gamma)$ with $||c||_* \leq 2r + 5$. Hence we get (by Lemma 4.3)

$$d(x^{g^n}, y) = d(v_1^{w_1g^n}, v_2^{w_2}) = d(v_1, v_2^{w_2g^{-n}w_1^{-1}}) = d(v_1, v_2^c)$$

$$\leqslant d(v_1, v_2) + d(v_2, v_2^c) \leqslant D + D(||c||_* + 1)$$

$$= D(||c||_* + 2) \leqslant D(2r + 7).$$

Therefore the Hausdorff distance between $x^{\langle g \rangle}$ and $y^{\langle g \rangle}$ is at most D(2r+7), hence (ii) is proved. \Box

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