# ACYLINDRICITY OF THE ACTION OF RIGHT-ANGLED ARTIN GROUPS ON EXTENSION GRAPHS 

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#### Abstract

The action of a right-angled Artin group on its extension graph is known to be acylindrical because the cardinality of the so-called $r$-quasi-stabilizer of a pair of distant points is bounded above by a function of $r$. The known upper bound of the cardinality is an exponential function of $r$. In this paper we show that the $r$-quasi-stabilizer is a subset of a cyclic group and its cardinality is bounded above by a linear function of $r$. This is done by exploring lattice theoretic properties of group elements, studying prefixes of powers and extending the uniqueness of quasi-roots from word length to star length. We also improve the known lower bound for the minimal asymptotic translation length of a right angled Artin group on its extension graph.


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## 1. Introduction

Throughout the paper $\Gamma$ denotes a finite simplicial graph, not necessarily connected, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The right-angled Artin group $A(\Gamma)$ with the underlying graph $\Gamma$ is the group generated by $V(\Gamma)$ such that the defining relations are the commutativity between adjacent vertices, hence $A(\Gamma)$ has the group presentation

$$
\left.A(\Gamma)=\langle v \in V(\Gamma)| v_{i} v_{j}=v_{j} v_{i} \text { for each }\left\{v_{i}, v_{j}\right\} \in E(\Gamma)\right\rangle
$$

Right-angled Artin groups are important groups in geometric group theory, which played a key role in Agol's proof of the virtual Haken conjecture [1, 15, 29,

The extension graph $\Gamma^{e}$ is the graph such that the vertex set $V\left(\Gamma^{e}\right)$ is the set of all elements of $A(\Gamma)$ that are conjugate to a vertex of $\Gamma$, and two vertices $v_{1}^{g_{1}}$ and $v_{2}^{g_{2}}$ are adjacent in $\Gamma^{e}$ if and only if they commute when considered as elements of $A(\Gamma)$. (Here, $v^{g}$ denotes the conjugate $g^{-1} v g$.) Therefore

$$
\begin{aligned}
& V\left(\Gamma^{e}\right)=\left\{v^{g}: v \in V(\Gamma), g \in A(\Gamma)\right\}, \\
& E\left(\Gamma^{e}\right)=\left\{\left\{v_{1}^{g_{1}}, v_{2}^{g_{2}}\right\}: v_{1}^{g_{1}} v_{2}^{g_{2}}=v_{2}^{g_{2}} v_{1}^{g_{1}} \text { in } A(\Gamma)\right\} .
\end{aligned}
$$

Extension graphs are usually infinite and locally infinite. They are very useful in the study of rightangled Artin groups such as the embeddability problem between right-angled Artin groups [17, 19, [23, 24, 16] and the purely loxodromic subgroups which are analogous to convex cocompact subgroups of the mapping class groups of surfaces [21]. It is known that $\Gamma^{e}$ is a quasi-tree, hence a $\delta$-hyperbolic graph [17.

Definition 1.1 (acylindrical action). When a group $G$ acts on a path-metric space ( $X, d$ ) isometrically from the right, the action is called acylindrical if for any $r>0$, there exist $R, N>0$ such that whenever $x$ and $y$ are two points of $X$ with $d(x, y) \geqslant R$, the cardinality of the set

$$
\xi(x, y ; r)=\{g \in G: d(x g, x) \leqslant r \text { and } d(y g, y) \leqslant r\}
$$

is at most $N$. The set $\xi(x, y ; r)$ is called the $r$-quasi-stabilizer of the pair of points $(x, y)$. We sometimes use the notation $\xi_{(X, d)}(x, y ; r)$ for the set $\xi(x, y ; r)$. Notice that $R$ and $N$ are functions of $r$. When we need to specify the acylindricity constants $R$ and $N$, we say that the action is $(R, N)$-acylindrical.

There have been many works on properties and examples of groups with an acylinrical action on a geodesic hyperbolic metric space. For example, see [5, 27, 8].

Let $d$ denote the graph metric of $\Gamma^{e}$. The right-angled Artin group $A(\Gamma)$ acts on $\left(\Gamma^{e}, d\right)$ isometrically from the right by conjugation, i.e. the image of the vertex $v^{h}$ under the action of $g \in A(\Gamma)$ is $v^{h g}$. The action of $A(\Gamma)$ on $\Gamma^{e}$ behaves much like the action of the mapping class group $\operatorname{Mod}(S)$ of a hyperbolic surface $S$ on the curve graph $\mathcal{C}(S)$. One of the fundamental properties is that the action of $A(\Gamma)$ on $\Gamma^{e}$ is acylindrical, which is shown by Kim and Koberda [18].

Theorem 1.2 ([18, Theorem 30]). The action of $A(\Gamma)$ on $\Gamma^{e}$ is acylindrical.
More precisely, it is shown that the action is $(R, N)$-acylindrical with

$$
\begin{aligned}
& R=R(r)=D(2 r+4 D+7), \\
& N=N(r)=\left(V 2^{2 V}\right)^{r+2 D+1},
\end{aligned}
$$

where $D=\operatorname{diam}(\Gamma)$ is the diameter of $\Gamma$ and $V=|V(\Gamma)|$ is the cardinality of $V(\Gamma)$. Notice that $N(r)$ is an exponential function of $r$.

For a graph $\Gamma$, let $\bar{\Gamma}$ denote the complement graph of $\Gamma$, i.e. the graph on the same vertices as $\Gamma$ such that two distinct vertices are adjacent in $\bar{\Gamma}$ if and only if they are not adjacent in $\Gamma$.

For the reader's convenience, we give some remarks on the cases where $|V(\Gamma)|$ is small and where $\Gamma$ or $\bar{\Gamma}$ is disconnected.

The following are known for the extension graph $\Gamma^{e}$ [17, Lemma 3.5]: if $\Gamma$ is disconnected, then $\Gamma^{e}$ has countably infinite number of path-components; if $\bar{\Gamma}$ is disconnected, i.e. $\Gamma$ is a join, then $\Gamma^{e}$ is also a join, hence $\operatorname{diam}\left(\Gamma^{e}\right) \leqslant 2$; if $|V(\Gamma)|=1$, then $\left|V\left(\Gamma^{e}\right)\right|=1$. If $|V(\Gamma)| \in\{2,3\}$, then either $\Gamma$ or $\bar{\Gamma}$ is disconnected. In fact, $\Gamma^{e}$ is a connected graph with infinite diameter if and only if $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected. Therefore, when we consider the action of $A(\Gamma)$ on $\Gamma^{e}$, it is natural to require that $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected.

In the study of extension graphs, we use the star length metric $d_{*}$ on $A(\Gamma)$. (See $\S 44$ for the definition of star length.) The metric space $\left(A(\Gamma), d_{*}\right)$ is quasi-isometric to the extension graph $\left(\Gamma^{e}, d\right)$. If $|V(\Gamma)|=1$ or if $\bar{\Gamma}$ is disconnected, then $\left(A(\Gamma), d_{*}\right)$ has diameter at most 2, which is not interesting. Therefore, when we consider the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$, it is natural to require that $|V(\Gamma)| \geqslant 2$ and $\bar{\Gamma}$ is connected (see Remark (6.2).

From the above discussions, the following settings are natural.
(i) When we consider the action of $A(\Gamma)$ on $\left(\Gamma^{e}, d\right)$, we will assume that $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected.
(ii) When we consider the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$, we will assume that $|V(\Gamma)| \geqslant 2$ and $\bar{\Gamma}$ is connected.

The following is the main result of this paper, which shows that we can take $N(r)$ as a linear function of $r$ and furthermore the quasi-stabilizer $\xi(x, y ; r)$ is a subset of a cyclic group.

Theorem A (Theorem 8.2) Let $\Gamma$ be a finite simplicial graph such that $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected. Then the action of $A(\Gamma)$ on $\Gamma^{e}$ is $(R, N)$-acylindrical with

$$
\begin{aligned}
& R=R(r)=D(2 V+7)(r+1)+10 D, \\
& N=N(r)=2(V-2) r-1,
\end{aligned}
$$

where $D=\operatorname{diam}(\Gamma)$ and $V=|V(\Gamma)|$. Moreover, for any $x, y \in V\left(\Gamma^{e}\right)$ with $d(x, y) \geqslant R$, if $\xi(x, y ; r) \neq$ $\{1\}$, then there exists a loxodromic element $g \in A(\Gamma)$ such that
(i) $\xi(x, y ; r) \subset\left\{1, g^{ \pm 1}, g^{ \pm 2}, \ldots, g^{ \pm k}\right\}$ for some $1 \leqslant k \leqslant(V-2) r-1$;
(ii) the Hausdorff distance between the $\langle g\rangle$-orbit of $x$ and that of $y$ is at most $D(2 r+7)$.

The following is an easy example to come up with for $g \in \xi(x, y ; r)$. Let $g$ be a loxodromic element with a quasi-axis $L=z^{\langle g\rangle}=\left\{z^{g^{m}}: m \in \mathbb{Z}\right\}$ for some $z \in V\left(\Gamma^{e}\right)$ such that $d\left(z^{g}, z\right)$ is sufficiently small. If both $x$ and $y$ are close enough to $L$, then $d\left(x^{g}, x\right)$ and $d\left(y^{g}, y\right)$ are also small so that $g \in \xi(x, y ; r) \backslash\{1\}$. In this case, the Hausdorff distance between the $\langle g\rangle$-orbits $x^{\langle g\rangle}$ and $y^{\langle g\rangle}$ is small. Theorem 8.2 says that, in some sense, this is the only case where $g \in \xi(x, y ; r) \backslash\{1\}$ happens: $g$ is loxodromic and the Hausdorff distance between $x^{\langle g\rangle}$ and $y^{\langle g\rangle}$ is small. Moreover, by Theorem 8.2(i), the set $\xi(x, y ; r) \backslash\{1\}$ is purely loxodromic, that is, there is no elliptic element that $r$-quasi-stabilizes a pair of sufficiently distant points.

In order to prove Theorem A, we develop several tools such as lattice theoretic properties of group elements, decomposition of conjugating elements, properties of prefixes of powers, and then extend the uniqueness of quasi-roots in [25] from word length to star length. Using these tools, we also obtain a new lower bound for the minimal asymptotic translation length of the action of $A(\Gamma)$ on $\Gamma^{e}$.

Definition 1.3. When a group $G$ acts on a connected metric space $(X, d)$ by isometries from right, the asymptotic translation length of an element $g \in G$ is defined by

$$
\begin{equation*}
\tau(g)=\tau_{(X, d)}(g)=\lim _{n \rightarrow \infty} \frac{d\left(x g^{n}, x\right)}{n} \tag{1}
\end{equation*}
$$

where $x \in X$. This limit always exists, is independent of the choice of $x \in X$, and satisfies $\tau\left(g^{n}\right)=$ $|n| \tau(g)$ and $\tau\left(h^{-1} g h\right)=\tau(g)$ for all $g, h \in G$ and $n \in \mathbb{Z}$. If $\tau(g)>0, g$ is called loxodromic. If $\left\{d\left(x g^{n}, x\right)\right\}_{n=1}^{\infty}$ is bounded, $g$ is called elliptic. If $\tau(g)=0$ and $\left\{d\left(x g^{n}, x\right)\right\}_{n=1}^{\infty}$ is unbounded, $g$ is called parabolic. For a subgroup $H$ of $G$, the minimal asymptotic translation length of $H$ for the action on $(X, d)$ is defined by

$$
\begin{equation*}
\mathcal{L}_{(X, d)}(H)=\min \left\{\tau_{(X, d)}(h): h \in H, \tau_{(X, d)}(h)>0\right\} . \tag{2}
\end{equation*}
$$

There have been many works on minimal asymptotic translation lengths of the action of mapping class groups on curve graphs. Let $S_{g}$ denote a closed orientable surface of genus $g$. For the action of the mapping class group $\operatorname{Mod}\left(S_{g}\right)$ on the curve graph $\mathcal{C}\left(S_{g}\right)$, Gadre and Tsai [12 proved that

$$
\mathcal{L}_{\mathcal{C}\left(S_{g}\right)}\left(\operatorname{Mod}\left(S_{g}\right)\right) \asymp \frac{1}{g^{2}},
$$

where $f(g) \asymp h(g)$ denotes that there exist positive constants $A$ and $B$ such that $A f(g) \leqslant h(g) \leqslant$ $B f(g)$. The braid group $\mathrm{B}_{n}$ can be regarded as the mapping class group of the $n$-punctured disk $D_{n}$ fixing boundary pointwise. The pure braid group $\mathrm{PB}_{n}$ is the subgroup of $\mathrm{B}_{n}$ consisting of mapping classes that fix each puncture. Kin and Shin [20] and Baik and Shin [3] showed that

$$
\mathcal{L}_{\mathcal{C}\left(D_{n}\right)}\left(\mathrm{B}_{n}\right) \asymp \frac{1}{n^{2}}, \quad \mathcal{L}_{\mathcal{C}\left(D_{n}\right)}\left(\mathrm{PB}_{n}\right) \asymp \frac{1}{n} .
$$

For the action of $A(\Gamma)$ on $\Gamma^{e}$, it follows from a result of Kim and Koberda [18] that

$$
\mathcal{L}_{\left(\Gamma^{e}, d\right)}(A(\Gamma)) \geqslant \frac{1}{2|V(\Gamma)|^{2}}
$$

Baik, Seo and Shin [2] proved that all loxodromic elements of $A(\Gamma)$ on $\Gamma^{e}$ have rational asymptotic translation lengths with a common denominator.

In this paper, we show the following, where the denominator of the lower bound is improved from a quadratic function to a linear function of $|V(\Gamma)|$.

Theorem B (Theorem 6.5) Let $\Gamma$ be a finite simplicial graph such that $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected. Then

$$
\mathcal{L}_{\left(\Gamma^{e}, d\right)}(A(\Gamma)) \geqslant \frac{1}{|V(\Gamma)|-2} .
$$

In the remaining of this section, we explain briefly our ideas and the structure of this paper.
1.1. Idea for the acylindricity. Let us first explain our idea for the acylindricity. For $g \in A(\Gamma)$, let $\|g\|$ denote the word length of $g$ with respect to the generating set $V(\Gamma)^{ \pm 1}$, and let $d_{\ell}$ denote temporarily the word length metric defined by $d_{\ell}(g, h)=\left\|g h^{-1}\right\|$ for $g, h \in A(\Gamma)$. The right multiplication induces an isometric action of $A(\Gamma)$ on $\left(A(\Gamma), d_{\ell}\right)$. Since $\xi(x, y ; r)=x^{-1} \xi\left(1, y x^{-1} ; r\right) x$ for any $x, y \in A(\Gamma)$, it suffices to consider $r$-quasi-stabilizers of the form $\xi(1, w ; r)$ for the acylindricity.

Suppose that we are given $R>0$ large, $r>0$ small and $w \in A(\Gamma)$ with $\|w\|=d_{\ell}(w, 1) \geqslant R$. Let $g \in \xi(1, w ; r) \backslash\{1\}$. Since $\|g\|=d_{\ell}(g, 1) \leqslant r$ and $\left\|w g w^{-1}\right\|=d_{\ell}(w g, w) \leqslant r$, we have

$$
\begin{equation*}
\|w\| \geqslant R, \quad\|g\| \leqslant r, \quad\left\|w g w^{-1}\right\| \leqslant r . \tag{*}
\end{equation*}
$$

In other words, $\|w\|$ is large whereas $\|g\|$ and $\left\|w g w^{-1}\right\|$ are small. This happens typically when

$$
\begin{equation*}
w=a g^{n}, \quad n \in \mathbb{Z}, \quad a \in A(\Gamma) \tag{**}
\end{equation*}
$$

with $\|a\|$ small and $|n|$ large. In this case, $d_{\ell}\left(w, g^{n}\right)=d_{\ell}\left(a g^{n}, g^{n}\right)=\|a\|$ is small, hence we can say that $w$ is "close to a power of $g$ ".

Even though it is clearly over-optimistic and false, one may hope that the following hold: given a triple $(R, r, w)$ as above (i.e. $R>0$ is large, $r>0$ is small and $w \in A(\Gamma)$ with $\|w\| \geqslant R$ ),
(i) if (*) holds, then (**) holds for some $n \in \mathbb{Z}$ and $a \in A(\Gamma)$ with $\|a\|$ small;
(ii) only a small number of triples ( $a, g, n$ ) with $\|a\|$ small and $\|g\| \leqslant r$ satisfy ( $* *$ ).

Of course, the above statements are not true at least as they are written. Moreover, the metric spaces $\left(A(\Gamma), d_{\ell}\right)$ and $\left(\Gamma^{e}, d\right)$ are not quasi-isometric, hence the above statements do not imply the acylindricity of $\left(\Gamma^{e}, d\right)$. However, we will see that this approach in fact works in the study of the acylindricity of the action of $A(\Gamma)$ on $\left(\Gamma^{e}, d\right)$ if we replace the word length metric with the star length metric.
1.2. Lattice structure. In $\$ 2$ we collect basic combinatorial group theoretic properties of rightangled Artin groups. Those properties are stated using lattice theoretic notations.

The motivation comes from Garside groups which are a lattice theoretic generalization of braid groups and finite type Artin groups. For Garside groups, there are elegant tools especially for the word and conjugacy problems and the asymptotic translation length [13, 6, 11, 4, 10, 9, 22, 26]. Right-angled Artin groups are not Garside groups, except free abelian groups, hence we cannot apply Garside theory to right angled-Artin groups. However, some ideas from Garside theory are very useful in our approach.

For $g \in A(\Gamma)$, the support of $g$, denoted $\operatorname{supp}(g)$, is the set of generators that appear in a shortest word on $V(\Gamma)^{ \pm 1}$ representing $g$.

For $g_{1}, g_{2} \in A(\Gamma)$, we say that $g_{1}$ and $g_{2}$ disjointly commute, denoted $g_{1} \rightleftharpoons g_{2}$, if $\operatorname{supp}\left(g_{1}\right) \cap$ $\operatorname{supp}\left(g_{2}\right)=\emptyset$ and each $v_{1} \in \operatorname{supp}\left(g_{1}\right)$ commutes with each $v_{2} \in \operatorname{supp}\left(g_{2}\right)$.

Let $g=g_{1} \cdots g_{k}$ for some $g, g_{1}, \ldots, g_{k} \in A(\Gamma)$. We say that the decomposition is geodesic if $\|g\|=\left\|g_{1}\right\|+\cdots+\left\|g_{k}\right\|$. If $g=g_{1} g_{2}$ is geodesic, we say that $g_{1}$ is a prefix of $g$, denoted $g_{1} \leqslant L g$, and that $g$ is a right multiple of $g_{1}$.

The relation $\leqslant_{L}$ is a partial order on $A(\Gamma)$, hence the notions of gcd $g_{1} \wedge_{L} g_{2}$ and lcm $g_{1} \vee_{L} g_{2}$ make sense. Theorem 2.12 shows that for $g_{1}, g_{2} \in A(\Gamma)$, the gcd $g_{1} \wedge_{L} g_{2}$ always exists and the lcm $g_{1} \vee_{L} g_{2}$ exists if and only if $g_{1}$ and $g_{2}$ have a common right multiple. Moreover, in this case, there exist $g_{1}^{\prime}, g_{2}^{\prime} \in A(\Gamma)$ such that $g_{i}=\left(g_{1} \wedge_{L} g_{2}\right) g_{i}^{\prime}$ for $i=1,2, g_{1}^{\prime} \rightleftharpoons g_{2}^{\prime}$ and $g_{1} \vee_{L} g_{2}=\left(g_{1} \wedge_{L} g_{2}\right) g_{1}^{\prime} g_{2}^{\prime}$.
1.3. Cyclic conjugations. In 93 , we study conjugations $g^{u}=u^{-1} g u$. The decomposition $u^{-1} g u$ is not geodesic in general, i.e. $\left\|u^{-1} g u\right\| \neq\left\|u^{-1}\right\|+\|g\|+\|u\|$.

Let $g$ be cyclically reduced, i.e. the word length $\|g\|$ is minimal in its conjugacy class. If $u \leqslant_{L} g$, then $g=u g_{1}$ is geodesic for some $g_{1} \in A(\Gamma)$ and $g^{u}=u^{-1}\left(u g_{1}\right) u=g_{1} u$. In other words, the conjugation of $g$ by $u$ moves the prefix $u$ to the right. An iteration of this type of conjugations is called a left cyclic conjugation. The right cyclic conjugation is defined similarly. The cyclic conjugation is an iteration of left and right cyclic conjugations.

Proposition 3.8 shows that for a cyclically reduced element $g$, the conjugation $g^{u}$ is a left cyclic conjugation of $g$ if and only if $u \leqslant_{L} g^{n}$ for some $n \geqslant 1$.

Theorem 3.9 shows that given $g, u \in A(\Gamma)$ with $g$ cyclically reduced, there exists a unique geodesic decomposition $u=u_{1} u_{2} u_{3}$ such that $u_{1}$ disjointly commutes with $g ; g^{u_{2}}$ is a cyclic conjugation; $g^{u}=u_{3}^{-1} g^{u_{2}} u_{3}$ is geodesic, i.e. $\left\|u_{3}^{-1} g^{u_{2}} u_{3}\right\|=\left\|u_{3}^{-1}\right\|+\left\|g^{u_{2}}\right\|+\left\|u_{3}\right\|$. Furthermore, there is a geodesic decomposition $u_{2}=u_{2}^{\prime} u_{2}^{\prime \prime}$ such that $g^{u_{2}^{\prime}}$ (resp. $g^{u_{2}^{\prime \prime}}$ ) is a left (resp. right) cyclic conjugation and $u_{2}^{\prime} \rightleftharpoons u_{2}^{\prime \prime}$.
1.4. Star length. An element $g \in A(\Gamma)$ is called a star-word if $\operatorname{supp}(g)$ is contained in the star of some vertex. The star length, denoted $\|g\|_{*}$, of $g$ is the minimum $\ell$ such that $g$ can be written as a product of $\ell$ star-words. Let $d_{*}$ denote the metric on $A(\Gamma)$ induced by the star length: $d_{*}\left(g_{1}, g_{2}\right)=\left\|g_{1} g_{2}^{-1}\right\|_{*}$.

The right multiplication induces an isometric action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$. The metric spaces $\left(A(\Gamma), d_{*}\right)$ and $\left(\Gamma^{e}, d\right)$ are quasi-isometric [18]. It seems that, for some algebraic tools, $\left(A(\Gamma), d_{*}\right)$ is easier to work with than $\left(\Gamma^{e}, d\right)$.

In 84 , we study basic properties of the star length concerning the prefix order $\leqslant_{L}$ and the geodesic decomposition of group elements. For example, Corollary 4.8 shows that if $g_{1} g_{2}$ is geodesic, then $\left\|g_{1}\right\|_{*}+\left\|g_{2}\right\|_{*}-2 \leqslant\left\|g_{1} g_{2}\right\|_{*} \leqslant\left\|g_{1}\right\|_{*}+\left\|g_{2}\right\|_{*}$.
1.5. Prefixes of powers of cyclically reduced elements. Recall that, for a cyclically reduced element $g$, if $g^{u}$ is a left cyclic conjugation, then $u \leqslant_{L} g^{m}$ for some $m \geqslant 1$, i.e. $u$ is a prefix of some power of $g$. In 55 , we study prefixes of powers. In particular, we show that if $g$ is cyclically reduced and non-split and if $u \leqslant L g^{m}$ for some $m \geqslant 1$, then $u=g^{n} a$ is geodesic for some $0 \leqslant n \leqslant m$ and $a \in A(\Gamma)$ with $\|a\|_{*} \leqslant\|g\|_{*}+1$ (see Corollary (5.6).
1.6. Asymptotic translation length. In $\sqrt[4]{6}$, we prove Theorem B by using the results in $\sqrt{5} 5$,
1.7. Uniqueness of quasi-roots. An element $g$ is called a quasi-root of $h$ if there is a decomposition

$$
h=a g^{n} b
$$

for some $n \geqslant 1$ and $a, b \in A(\Gamma)$ such that $\|h\|=\|a\|+n\|g\|+\|b\|$. It is called an $(A, B, r)$-quasi-root if $\|a\| \leqslant A,\|b\| \leqslant B$ and $\|g\| \leqslant r$ and an $(A, B, r)^{*}$-quasi-root if $\|a\|_{*} \leqslant A,\|b\|_{*} \leqslant B$ and $\|g\|_{*} \leqslant r$. The conjugates $a g a^{-1}$ and $b^{-1} g b$ are called the leftward- and the rightward-extraction of the quasi-root $g$, respectively.

In [25], it is shown that if $\|h\| \geqslant A+B+(2|V(\Gamma)|+1) r$, then strongly non-split and primitive ( $A, B, r$ )-quasi-roots of $h$ are unique up to conjugacy, and their leftward- and rightward-extractions are unique. (See $\$ \mathbb{4}$ and $\$ 7$ for the definitions of strongly non-split elements and primitive elements.)

In 87, we extend the above result to $(A, B, r)^{*}$-quasi-roots: if $\|h\|_{*} \geqslant 2 A+2 B+(2|V(\Gamma)|+3) r+2$, then primitive $(A, B, r)^{*}$-quasi-roots of $h$ are unique up to conjugacy, and their leftward- and rightwardextractions are unique.
1.8. Proof of the acylindricity. In $\mathbb{8} 8$, we first compute the acylindricity constants for the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$ (Theorem [8.1) by combining the results from the previous sections. Then we prove Theorem A using the quasi-isometry between $\left(A(\Gamma), d_{*}\right)$ and $\left(\Gamma^{e}, d\right)$.
1.9. Conventions and notations. Throughout the paper, all the group actions are right-actions.

For graphs $\Gamma_{1}$ and $\Gamma_{2}$, the disjoint union $\Gamma_{1} \sqcup \Gamma_{2}$ is the graph such that

$$
\begin{aligned}
& V\left(\Gamma_{1} \sqcup \Gamma_{2}\right)=V\left(\Gamma_{1}\right) \sqcup V\left(\Gamma_{2}\right), \\
& E\left(\Gamma_{1} \sqcup \Gamma_{2}\right)=E\left(\Gamma_{1}\right) \sqcup E\left(\Gamma_{2}\right) .
\end{aligned}
$$

The join $\Gamma_{1} * \Gamma_{2}$ is the graph such that $\overline{\Gamma_{1} * \Gamma_{2}}=\bar{\Gamma}_{1} \sqcup \bar{\Gamma}_{2}$, hence

$$
\begin{aligned}
& V\left(\Gamma_{1} * \Gamma_{2}\right)=V\left(\Gamma_{1}\right) \sqcup V\left(\Gamma_{2}\right), \\
& E\left(\Gamma_{1} * \Gamma_{2}\right)=E\left(\Gamma_{1}\right) \sqcup E\left(\Gamma_{2}\right) \sqcup\left\{\left\{v_{1}, v_{2}\right\}: v_{1} \in V\left(\Gamma_{1}\right), v_{2} \in V\left(\Gamma_{2}\right)\right\} .
\end{aligned}
$$

A graph is called a join if it is the join of two nonempty graphs. A subgraph that is a join is called a subjoin.

For $X \subset V(\Gamma), \Gamma[X]$ denotes the subgraph of $\Gamma$ induced by $X$, i.e.

$$
V(\Gamma[X])=X, \quad E(\Gamma[X])=\left\{\left\{v_{1}, v_{2}\right\} \in E(\Gamma): v_{1}, v_{2} \in X\right\} .
$$

For $g \in A(\Gamma)$, the subgraphs $\Gamma[\operatorname{supp}(g)]$ and $\bar{\Gamma}[\operatorname{supp}(g)]$ are abbreviated to $\Gamma[g]$ and $\bar{\Gamma}[g]$, respectively. For $v \in V(\Gamma)$ and $X \subset V(\Gamma)$, the sets $\mathrm{Lk}_{\Gamma}(v), \mathrm{St}_{\Gamma}(v)$ and $\mathrm{St}_{\Gamma}(X)$ are defined as follows:

$$
\begin{aligned}
\operatorname{Lk}_{\Gamma}(v) & =\left\{v_{1} \in V(\Gamma):\left\{v_{1}, v\right\} \in E(\Gamma)\right\} \\
\operatorname{St}_{\Gamma}(v) & =\{v\} \cup \operatorname{Lk}_{\Gamma}(v) \\
\operatorname{St}_{\Gamma}(X) & =\bigcup_{v \in X} \operatorname{St}_{\Gamma}(v)
\end{aligned}
$$

They are called the link of $v$, the star of $v$ and the star of $X$, respectively. They will be written as $\mathrm{Lk}(v), \operatorname{St}(v)$ and $\operatorname{St}(X)$ by omitting $\Gamma$ whenever the context is clear.

The path graph $P_{k}=\left(v_{1}, v_{2}, \ldots, v_{k}\right)$ is the graph with $V\left(P_{k}\right)=\left\{v_{1}, \ldots, v_{k}\right\}$ and $E\left(P_{k}\right)=\left\{\left\{v_{i}, v_{i+1}\right\}\right.$ : $1 \leqslant i \leqslant k-1\}$, hence $P_{k}$ looks like $\stackrel{\bullet-}{v_{1}} \stackrel{\bullet}{v_{2}} \cdots \xrightarrow[v_{k-1}]{\bullet} \stackrel{\bullet}{v}_{k}$.

A path in a graph $\Gamma$ is a tuple $\left(v_{1}, v_{2}, \ldots, v_{k}\right)$ of vertices of $\Gamma$ such that $\left\{v_{i}, v_{i+1}\right\} \in E(\Gamma)$ for all $1 \leqslant i \leqslant k-1$. (We do not assume that the vertices or the edges in the path are mutually distinct. Hence it means the walk in the graph theoretical terminology.)

## 2. Lattice structures

In this section we study lattice structures in right-angled Artin groups.
An element of $V(\Gamma)^{ \pm 1}=V(\Gamma) \cup V(\Gamma)^{-1}$ is called a letter. A word means a finite sequence of letters. For words $w_{1}$ and $w_{2}$, the notation $w_{1} \equiv w_{2}$ means that $w_{1}$ and $w_{2}$ coincide as sequences of letters. A word $w^{\prime}$ is called a subword of a word $w$ if $w \equiv w_{1} w^{\prime} w_{2}$ for (possibly empty) words $w_{1}$ and $w_{2}$.

Suppose that $g \in A(\Gamma)$ is expressed as a word $w$ on $V(\Gamma)^{ \pm 1}$. The word $w$ is called reduced if $w$ is a shortest word among all the words representing $g$. In this case, the length of $w$ is called the word length of $g$ and denoted by $\|g\|$.

Definition 2.1 (support). For $g \in A(\Gamma)$, the support of $g$, denoted $\operatorname{supp}(g)$, is the set of generators that appear in a reduced word representing $g$. It is known that $\operatorname{supp}(g)$ is well defined (by [14]), i.e. it does not depend on the choice of a reduced word representing $g$.

Definition 2.2 (disjointly commute). We say that $g_{1}, g_{2} \in A(\Gamma)$ disjointly commute, denoted $g_{1} \rightleftharpoons g_{2}$, if $\operatorname{supp}\left(g_{1}\right) \cap \operatorname{supp}\left(g_{2}\right)=\emptyset$ and each $v_{1} \in \operatorname{supp}\left(g_{1}\right)$ commutes with each $v_{2} \in \operatorname{supp}\left(g_{2}\right)$. (In particular, the identity element $1 \in A(\Gamma)$ disjointly commutes with any $g \in A(\Gamma)$.)

The notation $\Gamma[g]$ is an abbreviation of $\Gamma[\operatorname{supp}(g)]$, the subgraph of $\Gamma$ induced by $\operatorname{supp}(g)$. From a graph theoretical viewpoint, $g_{1} \rightleftharpoons g_{2}$ means that $\operatorname{supp}\left(g_{1}\right) \cap \operatorname{supp}\left(g_{2}\right)=\emptyset$ and $\bar{\Gamma}\left[g_{1} g_{2}\right]=\bar{\Gamma}\left[g_{1}\right] \sqcup \bar{\Gamma}\left[g_{2}\right]$ in the complement graph $\bar{\Gamma}$ (or equivalently $\Gamma\left[g_{1} g_{2}\right]=\Gamma\left[g_{1}\right] * \Gamma\left[g_{2}\right]$ in the graph $\Gamma$ ). Recall that $\mathrm{St}_{\bar{\Gamma}}(\operatorname{supp}(g))$ denotes the star of $\operatorname{supp}(g)$ in the complement graph $\bar{\Gamma}$. The following lemma is now obvious.

Lemma 2.3. For $g_{1}, g_{2} \in A(\Gamma)$, the following are equivalent:
(i) $g_{1} \rightleftharpoons g_{2}$ in $A(\Gamma)$;
(ii) $\mathrm{St}_{\bar{\Gamma}}\left(\operatorname{supp}\left(g_{1}\right)\right) \cap \operatorname{supp}\left(g_{2}\right)=\emptyset$.

Let $w$ be a (non-reduced) word on $V(\Gamma)^{ \pm 1}$. A subword $v^{ \pm 1} w_{1} v^{\mp 1}$ of $w$, where $v \in V(\Gamma)$, is called a cancellation of $v$ in $w$ if $\operatorname{supp}\left(w_{1}\right) \subset \operatorname{St}_{\Gamma}(v)$, i.e. each $v_{1} \in \operatorname{supp}\left(w_{1}\right)$ commutes with $v$. If, furthermore, no letter in $w_{1}$ is equal to $v$ or $v^{-1}$, it is called an innermost cancellation of $v$ in $w$. It is known that the following are equivalent:
(i) $w$ is a reduced word;
(ii) $w$ has no cancellation;
(iii) $w$ has no innermost cancellation.

Abusing terminology, we do not distinguish between an element $g \in A(\Gamma)$ and a reduced word $w$ representing $g$ if there is no confusion. For example, if there is a cancellation in $w_{1} w_{2}$, where each $w_{i}$ is a reduced word representing an element $g_{i}$, then we just say that there is a cancellation in $g_{1} g_{2}$.

Definition 2.4 (geodesic decomposition). For $k \geqslant 1$ and $g, g_{1}, \ldots, g_{k} \in A(\Gamma)$, we say that the decomposition $g=g_{1} \cdots g_{k}$ is geodesic, or $g_{1} \cdots g_{k}$ is geodesic, if $\|g\|=\left\|g_{1}\right\|+\cdots+\left\|g_{k}\right\|$.

If $g_{1} \cdots g_{k}$ is geodesic, then the following are obvious from the definition:
(i) $g_{k}^{-1} g_{k-1}^{-1} \cdots g_{1}^{-1}$ is geodesic;
(ii) $g_{p} g_{p+1} \cdots g_{q}$ is geodesic for any $1 \leqslant p<q \leqslant k$;
(iii) $\operatorname{supp}\left(g_{1} \cdots g_{k}\right)=\operatorname{supp}\left(g_{1}\right) \cup \cdots \cup \operatorname{supp}\left(g_{k}\right)$.

Definition 2.5 (prefix order). Let $g=g_{1} g_{2}$ be geodesic for $g, g_{1}, g_{2} \in A(\Gamma)$. We say that $g_{1}$ is a prefix (or a left divisor) of $g$, denoted $g_{1} \leqslant_{L} g$, and that $g$ is a right multiple of $g_{1}$. Similarly, we say that $g_{2}$ is a suffix (or a right divisor) of $g$, denoted $g_{2} \leqslant_{R} g$, and that $g$ is a left multiple of $g_{2}$.

Clearly both $\leqslant_{L}$ and $\leqslant_{R}$ are partial orders on $A(\Gamma)$. The following lemma shows their basic properties. The proof is straightforward, hence we omit it.

Lemma 2.6. Let $g, g_{1}, \ldots, g_{n}, h_{1}, h_{2} \in A(\Gamma)$.
(i) $g_{1} \leqslant L g_{2}$ if and only if $g_{1}^{-1} \leqslant R g_{2}^{-1}$.
(ii) If $g g_{1}$ and $g g_{2}$ are geodesic, then $g g_{1} \leqslant L g g_{2}$ if and only if $g_{1} \leqslant L g_{2}$.
(iii) $g_{1} \cdots g_{n}$ is geodesic if and only if $g_{1} \cdots g_{k} \leqslant L g_{1} \cdots g_{k+1}$ for all $1 \leqslant k \leqslant n-1$.
(iv) Suppose $g_{1} g_{2}=h_{1} h_{2}$ such that both $g_{1} g_{2}$ and $h_{1} h_{2}$ are geodesic. Then $g_{1} \leqslant_{L} h_{1}$ if and only if $h_{2} \leqslant R g_{2}$.

Definition 2.7 (gcd and lcm). For $g, h \in A(\Gamma)$, the symbols $g \wedge_{L} h$ and $g \vee_{L} h$ denote the greatest common divisor (gcd) and the least common multiple (lcm) with respect to $\leqslant_{L}$. In other words, $g \wedge_{L} h$ is an element such that (i) $g \wedge_{L} h \leqslant_{L} g$ and $g \wedge_{L} h \leqslant_{L} h$; (ii) if $u \leqslant_{L} g$ and $u \leqslant_{L} h$ for some $u \in A(\Gamma)$, then $u \leqslant_{L} g \wedge_{L} h$. Similarly, $g \vee_{L} h$ is an element such that (i) $g \leqslant_{L} g \vee_{L} h$ and $h \leqslant_{L} g \vee_{L} h$; (ii) if $g \leqslant_{L} u$ and $h \leqslant_{L} u$ for some $u \in A(\Gamma)$, then $g \vee_{L} h \leqslant_{L} u$.

The symbols $g \wedge_{R} h$ and $g \vee_{R} h$ denote the gcd and lcm respectively with respect to $\leqslant_{R}$.
The elements $g \wedge_{L} h$ and $g \vee_{L} h$ are unique if they exist. In Theorem 2.12 we will show that $g \wedge_{L} h$ always exists and that $g \vee_{L} h$ exists if and only if $g$ and $h$ admit a common right multiple.

Note that $g$ and $h$ have no nontrivial common prefix if and only if $g \wedge_{L} h=1$, i.e. the gcd $g \wedge_{L} h$ exists and is equal to the identity. Therefore even though we did not prove yet the existence of $g \wedge_{L} h$ for arbitrary $g$ and $h$, we can safely use the expression $g \wedge_{L} h=1$.

The following lemma is an easy consequence of the fact that a word is reduced if and only if it has no innermost cancellation.

Lemma 2.8. Let $u, g, g_{1}, \ldots, g_{k} \in A(\Gamma)$.
(i) Suppose that $g_{1} \cdots g_{k}$ is not geodesic. Then there exist $1 \leqslant p<q \leqslant k$ and $x \in V(\Gamma)^{ \pm 1}$ such that

$$
x^{-1} \leqslant_{R} g_{p}, \quad x \leqslant_{L} g_{q}, \quad x \rightleftharpoons g_{j} \text { for all } p<j<q .
$$

Furthermore, if both $g_{1} \cdots g_{k-1}$ and $g_{2} \cdots g_{k}$ are geodesic, then $p=1$ and $q=k$.
(ii) Suppose that for each $1 \leqslant p<q \leqslant k$, either $g_{p} g_{q}$ is geodesic or $g_{p} g_{j_{1}} \cdots g_{j_{r}} g_{q}$ is geodesic for some $p<j_{1}<\cdots<j_{r}<q$. Then $g_{1} \cdots g_{k}$ is geodesic.
(iii) Suppose that $g g$ is geodesic. For any $n \geqslant 2$ and $a, b \in A(\Gamma)$, the following are equivalent:
(a) agb is geodesic;
(b) $a \underbrace{g g \cdots g}_{n} b$ is geodesic;
(c) $a g^{n} b$ is geodesic.

In particular, $g^{n}=g g \cdots g$ is geodesic for any $n \geqslant 2$.
(iv) Suppose that $g_{i} g_{i}$ is geodesic for all $1 \leqslant i \leqslant k$ and that $a_{1} g_{1} a_{2} g_{2} \cdots a_{k} g_{k} a_{k+1}$ is geodesic for some $a_{1}, \ldots, a_{k+1} \in A(\Gamma)$. Then $a_{1} g_{1}^{n_{1}} a_{2} g_{2}^{n_{2}} \cdots a_{k} g_{k}^{n_{k}} a_{k+1}$ is geodesic for any $n_{i} \geqslant 1$.

Proof. (i) Let $w_{i}$ be a reduced word representing $g_{i}$ for $i=1, \ldots, k$. Since $g_{1} \cdots g_{k}$ is not geodesic, the word $w \equiv w_{1} \cdots w_{k}$ is not reduced, hence it has an innermost cancellation. Since each $w_{i}$ is reduced, the cancellation must occur between $x^{-1}$ in $w_{p}$ and $x$ in $w_{q}$ for some $1 \leqslant p<q \leqslant k$ and $x \in V(\Gamma)^{ \pm 1}$. Therefore $w_{p}$ and $w_{q}$ are of the form $w_{p} \equiv w_{p}^{\prime} x^{-1} w_{p}^{\prime \prime}$ and $w_{q} \equiv w_{q}^{\prime} x w_{q}^{\prime \prime}$ such that $x$ disjointly commutes with $w_{p}^{\prime \prime}, w_{p+1}, \ldots, w_{q-1}, w_{q}^{\prime}$, hence $x^{-1} \leqslant_{R} g_{p}, x \leqslant_{L} g_{q}$ and $x \rightleftharpoons g_{j}$ for all $p<j<q$.

If either $p>1$ or $q<k$, then either $g_{2} \cdots g_{k}$ or $g_{1} \cdots g_{k-1}$ is not geodesic. Therefore if both $g_{1} \cdots g_{k-1}$ and $g_{2} \cdots g_{k}$ are geodesic, then $p=1$ and $q=k$.
(ii) Assume that $g_{1} \cdots g_{k}$ is not geodesic. By (i), there exist $1 \leqslant p<q \leqslant k$ and $x \in V(\Gamma)^{ \pm 1}$ such that $x^{-1} \leqslant_{R} g_{p}, x \leqslant_{L} g_{q}$ and $x \rightleftharpoons g_{j}$ for all $j$ with $p<j<q$. Therefore none of $g_{p} g_{q}$ and $g_{p} g_{j_{1}} \cdots g_{j_{r}} g_{q}$ ( $p<j_{1}<\cdots<j_{r}<q$ ) is geodesic, which contradicts the hypothesis.
(iii) (a) $\Rightarrow$ (b): Let $h_{1}=a, h_{i}=g$ for $i=2, \ldots, n+1$ and $h_{n+2}=b$. Then $h_{1}, \ldots, h_{n+2}$ satisfy the hypothesis of (ii), hence $h_{1} h_{2} \cdots h_{n+1} h_{n+2}=a \underbrace{g \cdots g}_{n} b$ is geodesic.
(b) $\Rightarrow(\mathrm{a})$ : Since $a \underbrace{g \cdots g}_{n} b$ is geodesic, $a g$ and $g b$ are geodesic. If $a g b$ is not geodesic, then there exists $x \in V(\Gamma)^{ \pm 1}$ such that $x^{-1} \leqslant_{R} a, x \leqslant_{L} b$ and $x \rightleftharpoons g$ by (i). Hence $a g \cdots g b$ is not geodesic, which is a contradiction.
(b) $\Leftrightarrow(\mathrm{c})$ : $\operatorname{From}(\mathrm{a}) \Rightarrow(\mathrm{b})$ with $a=b=1, g^{n}=g g \cdots g$ is geodesic, i.e. $\left\|g^{n}\right\|=n\|g\|$. Therefore $\left\|a g^{n} b\right\|=\|a\|+\left\|g^{n}\right\|+\|b\|$ if and only if $\left\|a g^{n} b\right\|=\|a\|+n\|g\|+\|b\|$, i.e. $a g^{n} b$ is geodesic if and only if $a \underbrace{g g \cdots g}_{n} b$ is geodesic.
(iv) Applying (iii) with $a=a_{1}, g=g_{1}, b=a_{2} g_{2} \cdots a_{k+1}$ and $n=n_{1}$, we get that $a_{1} g_{1}^{n_{1}} a_{2} g_{2} \cdots a_{k+1}$ is geodesic. Then applying (iii) with $a=a_{1} g_{1}^{n_{1}} a_{2}, g=g_{2}, b=a_{3} g_{3} \cdots a_{k+1}$ and $n=n_{2}$, we get that $a_{1} g_{1}^{n_{1}} a_{2} g_{2}^{n_{2}} a_{3} g_{3} \cdots a_{k+1}$ is geodesic. Iterating this process, we get that $a_{1} g_{1}^{n_{1}} a_{2} g_{2}^{n_{2}} \cdots a_{k} g_{k}^{n_{k}} a_{k+1}$ is geodesic.

Lemma 2.9. Let $g_{1}, g_{2} \in A(\Gamma)$ and $x \in V(\Gamma)^{ \pm 1}$.
(i) If $g_{1} g_{2}$ is not geodesic, then there exists $y \in V(\Gamma)^{ \pm 1}$ such that $y^{-1} \leqslant_{R} g_{1}$ and $y \leqslant L g_{2}$.
(ii) Let $g_{1} g_{2}$ be geodesic. If $x \leqslant_{L} g_{1} g_{2}$ and $x \not{ }_{L} g_{1}$, then $x \leqslant_{L} g_{2}$ and $x \rightleftharpoons g_{1}$.
(iii) Let $g_{1} g_{2}$ be geodesic. If $x \leqslant_{R} g_{1} g_{2}$ and $x \not \bigotimes_{R} g_{2}$, then $x \leqslant_{R} g_{1}$ and $x \rightleftharpoons g_{2}$.

Proof. (i) It follows from Lemma 2.8(i) with $k=2$.
(ii) Since $x \not{ }_{L} g_{1}$, the decomposition $x^{-1} \cdot g_{1}$ is geodesic. Since $x \leqslant L g_{1} g_{2}$, the decomposition $x^{-1} \cdot g_{1} g_{2}$ is not geodesic. Since both $x^{-1} \cdot g_{1}$ and $g_{1} \cdot g_{2}$ are geodesic, there exists $y \in V(\Gamma)^{ \pm 1}$ such that $y^{-1} \leqslant_{R} x^{-1}, y \leqslant_{L} g_{2}$ and $y \rightleftharpoons g_{1}$ (by Lemma [2.8(i)), hence $x=y$. Therefore $x \leqslant_{L} g_{2}$ and $x \rightleftharpoons g_{1}$.
(iii) The proof is analogous to (ii).

Lemma 2.10. Let $g \in A(\Gamma)$ and $x \neq y \in V(\Gamma)^{ \pm 1}$ (possibly $y=x^{-1}$ ).
(i) If $x \leqslant_{L} g$ and $y \leqslant_{R} g$, then $g=x h y$ is geodesic for some $h \in A(\Gamma)$.
(ii) If $x, y \leqslant_{L} g$, then $x \rightleftharpoons y$ and $g=x y h$ is geodesic for some $h \in A(\Gamma)$.
(iii) If $x, y \leqslant_{R} g$, then $x \rightleftharpoons y$ and $g=h x y$ is geodesic for some $h \in A(\Gamma)$.

Proof. (i) Since $y \leqslant_{R} g, g=g^{\prime} y$ is geodesic for some $g^{\prime} \in A(\Gamma)$. Since $x \leqslant_{L} g=g^{\prime} y$, if $x \not \mathbb{k}_{L} g^{\prime}$, then $x \leqslant_{L} y$ (by Lemma 2.9(ii)), which contradicts the hypothesis $x \neq y$. Thus $x \leqslant_{L} g^{\prime}$, hence $g^{\prime}=x h$ is geodesic for some $h \in A(\Gamma)$. Therefore $g=g^{\prime} y=x h y$ is geodesic.
(ii) Since $x \leqslant_{L} g, g=x g^{\prime}$ is geodesic for some $g^{\prime} \in A(\Gamma)$. Since $y \neq x$ (hence $y \not{ }_{L} x$ ) and $y \leqslant L x g^{\prime}$, we have $y \rightleftharpoons x$ and $y \leqslant_{L} g^{\prime}$ (by Lemma 2.9(ii)), hence $g^{\prime}=y h$ is geodesic for some $h \in A(\Gamma)$. Therefore $g=x g^{\prime}=x y h$ is geodesic.
(iii) The proof is analogous to (ii).

Lemma 2.11. Let $g_{1}, g_{2}, h_{1}, h_{2}, h \in A(\Gamma)$ with both $g_{1} g_{2}$ and $h_{1} h_{2}$ geodesic.


Figure 1. van Kampen diagram for Lemma 2.11(ii)
(i) If $h \wedge_{L} g_{1}=h \wedge_{L} g_{2}=1$, then $h \wedge_{L}\left(g_{1} g_{2}\right)=1$.
(ii) If $h \leqslant_{L} g_{1} g_{2}$ and $h \wedge_{L} g_{1}=1$, then $h \rightleftharpoons g_{1}$ and $h \leqslant_{L} g_{2}$.
(iii) Let $g_{1} g_{2}=h_{1} h_{2}$. If $g_{1} \wedge_{L} h_{1}=g_{2} \wedge_{R} h_{2}=1$, then $g_{1} \rightleftharpoons h_{1}, g_{1}=h_{2}$ and $g_{2}=h_{1}$.

Proof. (i) If $h \wedge_{L}\left(g_{1} g_{2}\right) \neq 1$, then there exists $x \in V(\Gamma)^{ \pm 1}$ such that $x \leqslant_{L} h$ and $x \leqslant_{L} g_{1} g_{2}$. Since $x \leqslant_{L} h$ and $h \wedge_{L} g_{1}=h \wedge_{L} g_{2}=1$, we have $x \not{ }_{L} g_{1}$ and $x \not{ }_{L} g_{2}$. Since $x \leqslant_{L} g_{1} g_{2}$ and $x \not \forall_{L} g_{1}$, we have $x \leqslant L g_{2}$ by Lemma 2.9(ii), which is a contradiction.
(ii) We use induction on $\|h\|$.

If $\|h\|=0$, there is nothing to prove. If $\|h\|=1$, it holds by Lemma 2.9(ii).
Assume $\|h\| \geqslant 2$. Then $h=h_{1} x$ is geodesic for some $h_{1} \in A(\Gamma)$ and $x \in V(\Gamma)^{ \pm 1}$. See Figure 1 . Notice that $h_{1} \leqslant L g_{1} g_{2}$ and $h_{1} \wedge_{L} g_{1}=1$. By the induction hypothesis, $h_{1} \rightleftharpoons g_{1}$ and $h_{1} \leqslant L g_{2}$, hence $g_{2}=h_{1} g_{2}^{\prime}$ is geodesic for some $g_{2}^{\prime} \in A(\Gamma)$.

Since $h_{1} \rightleftharpoons g_{1}$, we have $g_{1} g_{2}=g_{1} h_{1} g_{2}^{\prime}=h_{1} g_{1} g_{2}^{\prime}$. Since $g_{1} g_{2}$ is geodesic, so is $h_{1} g_{1} g_{2}^{\prime}$. Since $h_{1} x=h \leqslant_{L} g_{1} g_{2}=h_{1} g_{1} g_{2}^{\prime}$ and both $h_{1} x$ and $h_{1} g_{1} g_{2}^{\prime}$ are geodesic, we have $x \leqslant_{L} g_{1} g_{2}^{\prime}$.

Observe $x \not{ }_{L} g_{1}$. (If $x \leqslant_{L} g_{1}$, then $x \rightleftharpoons h_{1}$ because $h_{1} \rightleftharpoons g_{1}$. Since $h=h_{1} x=x h_{1}$ is geodesic, we have $x \leqslant_{L} h$. Thus $x$ is a common prefix of $g_{1}$ and $h$, which contradicts the hypothesis $h \wedge_{L} g_{1}=1$.) By Lemma 2.9(ii), we get $x \rightleftharpoons g_{1}$ and $x \leqslant_{L} g_{2}^{\prime}$, hence $g_{2}^{\prime}=x g_{2}^{\prime \prime}$ is geodesic for some $g_{2}^{\prime \prime} \in A(\Gamma)$.

Since $g_{2}=h_{1} g_{2}^{\prime}=h_{1} x g_{2}^{\prime \prime}=h g_{2}^{\prime \prime}$ and since $h g_{2}^{\prime \prime}$ is geodesic, we have $h \leqslant L g_{2}$. On the other hand, since $h_{1} \rightleftharpoons g_{1}$ and $x \rightleftharpoons g_{1}$, we have $h=h_{1} x \rightleftharpoons g_{1}$.
(iii) Since $g_{1} \leqslant_{L} h_{1} h_{2}, h_{1} \leqslant_{L} g_{1} g_{2}$ and $g_{1} \wedge_{L} h_{1}=1$, we have $g_{1} \rightleftharpoons h_{1}, g_{1} \leqslant_{L} h_{2}$ and $h_{1} \leqslant_{L} g_{2}$ (by (ii)). Thus $h_{2}=g_{1} h_{2}^{\prime}$ and $g_{2}=h_{1} g_{2}^{\prime}$ are geodesic for some $g_{2}^{\prime}, h_{2}^{\prime} \in A(\Gamma)$.

Observe $g_{1} h_{1} g_{2}^{\prime}=g_{1} g_{2}=h_{1} h_{2}=h_{1} g_{1} h_{2}^{\prime}=g_{1} h_{1} h_{2}^{\prime}$, which implies $g_{2}^{\prime}=h_{2}^{\prime}$. Since $g_{2} \wedge_{R} h_{2}=1$, we have $g_{2}^{\prime}=h_{2}^{\prime}=1$, hence $g_{1}=h_{2}$ and $g_{2}=h_{1}$.

The following is the main result of this section.
Theorem 2.12. For $g_{1}, g_{2} \in A(\Gamma)$, the $g c d g_{1} \wedge_{L} g_{2}$ always exists and the lcm $g_{1} \vee_{L} g_{2}$ exists if and only if $g_{1}$ and $g_{2}$ have a common right multiple.

More precisely, if $g_{0}$ is a maximal common prefix of $g_{1}$ and $g_{2}$, hence $g_{1}=g_{0} g_{1}^{\prime}$ and $g_{2}=g_{0} g_{2}^{\prime}$ are geodesic for some $g_{1}^{\prime}, g_{2}^{\prime} \in A(\Gamma)$ with $g_{1}^{\prime} \wedge_{L} g_{2}^{\prime}=1$, then the following hold.
(i) $g_{1}$ and $g_{2}$ have a common right multiple if and only if $g_{1}^{\prime} \rightleftharpoons g_{2}^{\prime}$. In this case, $g_{1} \vee_{L} g_{2}$ exists and $g_{1} \vee_{L} g_{2}=g_{1} g_{2}^{\prime}=g_{2} g_{1}^{\prime}=g_{0} g_{1}^{\prime} g_{2}^{\prime}$. In particular, $\operatorname{supp}\left(g_{1} \vee_{L} g_{2}\right)=\operatorname{supp}\left(g_{1}\right) \cup \operatorname{supp}\left(g_{2}\right)$.
(ii) $g_{1} \wedge_{L} g_{2}=g_{0}$.

Proof. (i) Assume $g_{1}^{\prime} \rightleftharpoons g_{2}^{\prime}$. Then $g_{1}^{\prime} g_{2}^{\prime}$ is geodesic (otherwise there exists $x \in V(\Gamma)^{ \pm 1}$ such that $x^{-1} \leqslant_{R} g_{1}^{\prime}$ and $x \leqslant_{L} g_{2}^{\prime}$ by Lemma 2.9(i), hence $g_{1}^{\prime}$ and $g_{2}^{\prime}$ do not disjointly commute). Since $g_{0} g_{1}^{\prime}$, $g_{0} g_{2}^{\prime}$ and $g_{1}^{\prime} g_{2}^{\prime}$ are all geodesic, $g_{0} g_{1}^{\prime} g_{2}^{\prime}$ is geodesic (by Lemma 2.8(ii)). Therefore $g_{0} g_{1}^{\prime} g_{2}^{\prime}=g_{1} g_{2}^{\prime}=g_{2} g_{1}^{\prime}$ is a common right multiple of $g_{1}$ and $g_{2}$.


Figure 2. van Kampen diagram for Theorem 2.12
Conversely, assume that $g_{1}$ and $g_{2}$ have a common right multiple $h$. Then $h=g_{1} h_{1}=g_{2} h_{2}$ are geodesic for some $h_{1}, h_{2} \in A(\Gamma)$. We need to show that $g_{1}^{\prime} \rightleftharpoons g_{2}^{\prime}$.

Let $h_{0}$ be a maximal common suffix of $h_{1}$ and $h_{2}$. Then $h_{1}=h_{1}^{\prime} h_{0}$ and $h_{2}=h_{2}^{\prime} h_{0}$ are geodesic for some $h_{1}^{\prime}, h_{2}^{\prime} \in A(\Gamma)$ with $h_{1}^{\prime} \wedge_{R} h_{2}^{\prime}=1$. See Figure 2. Notice that $g_{1}^{\prime} h_{1}^{\prime}=g_{2}^{\prime} h_{2}^{\prime}$ and that $g_{1}^{\prime}, g_{2}^{\prime}, h_{1}^{\prime}, h_{2}^{\prime}$ satisfy the hypotheses of Lemma [2.11(iii). Therefore $g_{1}^{\prime} \rightleftharpoons g_{2}^{\prime}$.

Lemma 2.11(iii) also claims $g_{1}^{\prime}=h_{2}^{\prime}$ and $g_{2}^{\prime}=h_{1}^{\prime}$, hence $h=g_{1} h_{1}=g_{0} g_{1}^{\prime} h_{1}^{\prime} h_{0}=g_{0} g_{1}^{\prime} g_{2}^{\prime} h_{0}$. Therefore $g_{0} g_{1}^{\prime} g_{2}^{\prime}$ is a prefix of any common right multiple $h$ of $g_{1}$ and $g_{2}$, namely, $g_{1} \vee_{L} g_{2}=g_{0} g_{1}^{\prime} g_{2}^{\prime}$. It follows immediately that $\operatorname{supp}\left(g_{1} \vee_{L} g_{2}\right)=\operatorname{supp}\left(g_{1}\right) \cup \operatorname{supp}\left(g_{2}\right)$. Since $g_{0} g_{1}^{\prime} g_{2}^{\prime}=g_{0} g_{2}^{\prime} g_{1}^{\prime}$, we have $g_{1} \vee_{L} g_{2}=g_{0} g_{1}^{\prime} g_{2}^{\prime}=g_{1} g_{2}^{\prime}=g_{2} g_{1}^{\prime}$.
(ii) Let $u_{0}$ be a common prefix of $g_{1}$ and $g_{2}$. Since $g_{1}$ and $g_{2}$ are common right multiples of $g_{0}$ and $u_{0}$, the lcm $g_{0} \vee_{L} u_{0}$ exists (by (i)) and is a prefix of both $g_{1}$ and $g_{2}$, hence $g_{0} \vee_{L} u_{0}$ is a common prefix of $g_{1}$ and $g_{2}$. Since $g_{0} \vee_{L} u_{0}$ is a right multiple of $g_{0}$ and since $g_{0}$ is a maximal common prefix of $g_{1}$ and $g_{2}$, we have $g_{0}=g_{0} \vee_{L} u_{0}$, hence $u_{0} \leqslant L g_{0}$. Therefore $g_{0}=g_{1} \wedge_{L} g_{2}$.

Obviously we can replace $\left(\wedge_{L}, \vee_{L}\right)$ in Theorem 2.12 with $\left(\wedge_{R}, \vee_{R}\right)$ as follows.
Theorem 2.13. For $g_{1}, g_{2} \in A(\Gamma)$, the gcd $g_{1} \wedge_{R} g_{2}$ always exists and the lcm $g_{1} \vee_{R} g_{2}$ exists if and only if $g_{1}$ and $g_{2}$ have a common left multiple.

More precisely, if $g_{0}$ is a maximal common suffix of $g_{1}$ and $g_{2}$, hence $g_{1}=g_{1}^{\prime} g_{0}$ and $g_{2}=g_{2}^{\prime} g_{0}$ are geodesic for some $g_{1}^{\prime}, g_{2}^{\prime} \in A(\Gamma)$ with $g_{1}^{\prime} \wedge_{R} g_{2}^{\prime}=1$, then the following hold.
(i) $g_{1}$ and $g_{2}$ have a common left multiple if and only if $g_{1}^{\prime} \rightleftharpoons g_{2}^{\prime}$. In this case, $g_{1} \vee_{R} g_{2}$ exists and $g_{1} \vee_{R} g_{2}=g_{2}^{\prime} g_{1}=g_{1}^{\prime} g_{2}=g_{1}^{\prime} g_{2}^{\prime} g_{0}$. In particular, $\operatorname{supp}\left(g_{1} \vee_{R} g_{2}\right)=\operatorname{supp}\left(g_{1}\right) \cup \operatorname{supp}\left(g_{2}\right)$.
(ii) $g_{1} \wedge_{R} g_{2}=g_{0}$.

Observe that the gcds $g_{1} \wedge_{L} g_{2}$ and $g_{1} \wedge_{R} g_{2}$ exist for any $g_{1}, g_{2} \in A(\Gamma)$ by the above theorems.
The following lemma is obvious, hence we omit the proof.
Lemma 2.14. Let $g_{1}, g_{2} \in A(\Gamma)$.
(i) $\left(g_{1} \wedge_{L} g_{2}\right)^{-1}=g_{1}^{-1} \wedge_{R} g_{2}^{-1}$.
(ii) If $g_{1}=g_{0} g_{1}^{\prime}$ and $g_{2}=g_{0} g_{2}^{\prime}$ are geodesic, then $g_{1} \wedge_{L} g_{2}=g_{0}\left(g_{1}^{\prime} \wedge_{L} g_{2}^{\prime}\right)$. In particular, if $g_{1}^{\prime} \wedge_{L} g_{2}^{\prime}=1$, then $g_{1} \wedge_{L} g_{2}=g_{0}$.
(iii) If $g_{1} \leqslant L g_{2}$, then $\left(h \wedge_{L} g_{1}\right) \leqslant_{L}\left(h \wedge_{L} g_{2}\right)$ for any $h \in A(\Gamma)$.
(iv) The statements analogous to (ii) and (iii) also hold for $\left(\leqslant_{R}, \wedge_{R}\right)$.

Lemma 2.15. Let $g_{1}, g_{2}, h \in A(\Gamma)$ with $g_{1} g_{2}$ geodesic.
(i) If $h \rightleftharpoons g_{1}$, then $h \wedge_{L}\left(g_{1} g_{2}\right)=h \wedge_{L} g_{2}$.
(ii) If $\operatorname{supp}(h) \cap \operatorname{supp}\left(g_{2}\right)=\emptyset$, then $h \wedge_{L}\left(g_{1} g_{2}\right)=h \wedge_{L} g_{1}$.
(iii) If $h \leqslant_{L} g_{1} g_{2}$ and $h \rightleftharpoons g_{1}$, then $h \leqslant_{L} g_{2}$.
(iv) If $h \leqslant_{L} g_{1} g_{2}$ and $\operatorname{supp}(h) \cap \operatorname{supp}\left(g_{2}\right)=\emptyset$, then $h \leqslant_{L} g_{1}$.
(v) The statements analogous to (i)-(iv) also hold for $\left(\leqslant_{R}, \wedge_{R}\right)$.

Proof. (i) Let $h_{0}=h \wedge_{L} g_{2}$. Then $h=h_{0} h^{\prime}$ and $g_{2}=h_{0} g_{2}^{\prime}$ are geodesic for some $h^{\prime}, g_{2}^{\prime} \in A(\Gamma)$ with $h^{\prime} \wedge_{L} g_{2}^{\prime}=1$. Since $h \rightleftharpoons g_{1}$ and $h=h_{0} h^{\prime}$ is geodesic, we have $h_{0} \rightleftharpoons g_{1}$ and $h^{\prime} \rightleftharpoons g_{1}$, hence $h^{\prime} \wedge_{L} g_{1}=1$.

Notice that $g_{1} g_{2}^{\prime}$ is geodesic because $g_{1} g_{2}\left(=g_{1} h_{0} g_{2}^{\prime}\right)=h_{0} g_{1} g_{2}^{\prime}$ is geodesic. Since $h^{\prime} \wedge_{L} g_{1}=h^{\prime} \wedge_{L} g_{2}^{\prime}=$ 1, we have $h^{\prime} \wedge_{L}\left(g_{1} g_{2}^{\prime}\right)=1$ (by Lemma 2.11(i)). Therefore by Lemma 2.14)(ii)

$$
\begin{aligned}
h \wedge_{L}\left(g_{1} g_{2}\right) & =\left(h_{0} h^{\prime}\right) \wedge_{L}\left(g_{1} h_{0} g_{2}^{\prime}\right)=\left(h_{0} h^{\prime}\right) \wedge_{L}\left(h_{0} g_{1} g_{2}^{\prime}\right) \\
& =h_{0}\left(h^{\prime} \wedge_{L}\left(g_{1} g_{2}^{\prime}\right)\right)=h_{0}=h \wedge_{L} g_{2} .
\end{aligned}
$$

(ii) Let $h_{0}=h \wedge_{L} g_{1}$. Then $h=h_{0} h^{\prime}$ and $g_{1}=h_{0} g_{1}^{\prime}$ are geodesic for some $h^{\prime}, g_{1}^{\prime} \in A(\Gamma)$ with $h^{\prime} \wedge_{L} g_{1}^{\prime}=1$. Since $\operatorname{supp}(h) \cap \operatorname{supp}\left(g_{2}\right)=\emptyset$ and since $h=h_{0} h^{\prime}$ is geodesic, we have $\operatorname{supp}\left(h^{\prime}\right) \cap \operatorname{supp}\left(g_{2}\right)=$ $\emptyset$, hence $h^{\prime} \wedge_{L} g_{2}=1$.

Notice that $g_{1}^{\prime} g_{2}$ is geodesic because $g_{1} g_{2}=h_{0} g_{1}^{\prime} g_{2}$ is geodesic. Since $h^{\prime} \wedge_{L} g_{1}^{\prime}=h^{\prime} \wedge_{L} g_{2}=1$, we have $h^{\prime} \wedge_{L}\left(g_{1}^{\prime} g_{2}\right)=1$ (by Lemma 2.11(i)). Therefore by Lemma 2.14.(ii)

$$
h \wedge_{L}\left(g_{1} g_{2}\right)=\left(h_{0} h^{\prime}\right) \wedge_{L}\left(h_{0} g_{1}^{\prime} g_{2}\right)=h_{0}\left(h^{\prime} \wedge_{L}\left(g_{1}^{\prime} g_{2}\right)\right)=h_{0}=h \wedge_{L} g_{1} .
$$

(iii) and (iv) are direct consequences of (i) and (ii), respectively.
(v) The proof is analogous to (i)-(iv).

Corollary 2.16. Suppose that a set $C \subset A(\Gamma)$ satisfies the following conditions.
(P1) $C$ is prefix-closed, i.e. if $g \in C$ and $h \leqslant_{L} g$, then $h \in C$.
(P2) For $g \in A(\Gamma)$ and $x, y \in V(\Gamma)^{ \pm 1}$ such that both $g x$ and $g y$ are geodesic, if $g x, g y \in C$ and $x \rightleftharpoons y$, then $g x y \in C$.

Then $C$ is lcm-closed, i.e. if $g_{1}, g_{2} \in C$ and $g_{1} \vee_{L} g_{2}$ exists, then $g_{1} \vee_{L} g_{2} \in C$.

Proof. Let $g_{1}, g_{2} \in C$ such that $g_{1} \vee_{L} g_{2}$ exists. Let $g_{0}=g_{1} \wedge_{L} g_{2}$. Then

$$
g_{1}=g_{0} g_{1}^{\prime} \quad \text { and } \quad g_{2}=g_{0} g_{2}^{\prime}
$$

are geodesic for some $g_{1}^{\prime}, g_{2}^{\prime} \in A(\Gamma)$. By Theorem (2.12, $g_{1}^{\prime} \rightleftharpoons g_{2}^{\prime}$ and $g_{1} \vee_{L} g_{2}=g_{0} g_{1}^{\prime} g_{2}^{\prime}$.
We use induction on $\left\|g_{1}^{\prime}\right\|+\left\|g_{2}^{\prime}\right\|$. If $\left\|g_{1}^{\prime}\right\|=0$ or $\left\|g_{2}^{\prime}\right\|=0$, then $g_{1} \vee_{L} g_{2}$ is either $g_{2}$ or $g_{1}$, respectively, hence there is nothing to prove. If $\left\|g_{1}^{\prime}\right\|=\left\|g_{2}^{\prime}\right\|=1$, then $g_{1} \vee_{L} g_{2}=g_{0} g_{1}^{\prime} g_{2}^{\prime} \in C$ by (P2). Therefore we may assume $\left\|g_{1}^{\prime}\right\|+\left\|g_{2}^{\prime}\right\| \geqslant 3$ and $\left\|g_{1}^{\prime}\right\|,\left\|g_{2}^{\prime}\right\| \geqslant 1$.

Then $g_{1}^{\prime}=g_{1}^{\prime \prime} x_{1}$ and $g_{2}^{\prime}=g_{2}^{\prime \prime} x_{2}$ are geodesic for some $g_{1}^{\prime \prime}, g_{2}^{\prime \prime} \in A(\Gamma)$ and $x_{1}, x_{2} \in V(\Gamma)^{ \pm 1}$. Thus

$$
g_{1}=g_{0} g_{1}^{\prime \prime} x_{1} \quad \text { and } \quad g_{2}=g_{0} g_{2}^{\prime \prime} x_{2}
$$

are geodesic, where $g_{1}^{\prime \prime} x_{1} \rightleftharpoons g_{2}^{\prime \prime} x_{2}$.
Since $g_{1}, g_{2} \in C$, we have $g_{0} g_{1}^{\prime \prime}, g_{0} g_{2}^{\prime \prime} \in C$ by (P1), hence by the induction hypothesis we have

$$
g_{1} \vee_{L}\left(g_{0} g_{2}^{\prime \prime}\right)=g_{0} g_{1}^{\prime \prime} g_{2}^{\prime \prime} x_{1} \in C \quad \text { and } \quad g_{0} g_{1}^{\prime \prime} \vee_{L} g_{2}=g_{0} g_{1}^{\prime \prime} g_{2}^{\prime \prime} x_{2} \in C
$$

Therefore $g_{1} \vee_{L} g_{2}=g_{0} g_{1}^{\prime \prime} g_{2}^{\prime \prime} x_{1} x_{2} \in C$ by (P2).

## 3. Cyclic conjugations

Definition 3.1 (cyclically reduced). An element $g \in A(\Gamma)$ is called cyclically reduced if it has the minimal word length in its conjugacy class.

Servatius [28, Proposition on p. 38] showed that every $g \in A(\Gamma)$ has a unique geodesic decomposition

$$
g=u^{-1} h u
$$

with $h$ cyclically reduced. The following lemma shows that $u$ is determined from $g$ by $u=g \wedge_{R} g^{-1}$.
Lemma 3.2. Let $g, h, u \in A(\Gamma)$.
(i) If $g=u^{-1} h u$ is geodesic with $h$ cyclically reduced, then $u=g \wedge_{R} g^{-1}$.
(ii) $g$ is cyclically reduced if and only if $g \wedge_{R} g^{-1}=1$.

Proof. (i) We have two geodesic decompositions $g=u^{-1} h u$ and $g^{-1}=u^{-1} h^{-1} u$. By Lemma 2.14, it suffices to show $\left(u^{-1} h\right) \wedge_{R}\left(u^{-1} h^{-1}\right)=1$ or equivalently $(h u) \wedge_{L}\left(h^{-1} u\right)=1$.

Assume $(h u) \wedge_{L}\left(h^{-1} u\right) \neq 1$. Then there exists $x \in V(\Gamma)^{ \pm 1}$ with $x \leqslant_{L} h u$ and $x \leqslant_{L} h^{-1} u$.
If $x \not \mathbb{K}_{L} h$, then $x \rightleftharpoons h$ (hence $x \rightleftharpoons h^{-1}$ ) and $x \leqslant_{L} u$ (by Lemma 2.9(ii)). Let $u=x u_{1}$ be geodesic for some $u_{1} \in A(\Gamma)$. Then $g=u^{-1} h u=u_{1}^{-1} x^{-1} h x u_{1}=u_{1}^{-1} h u_{1}$. This contradicts that $g=u^{-1} h u$ is geodesic. Therefore $x \leqslant_{L} h$. By the same reason, $x \leqslant_{L} h^{-1}$, hence $x^{-1} \leqslant_{R} h$.

Since $x \leqslant_{L} h$ and $x^{-1} \leqslant_{R} h, h=x h_{1} x^{-1}$ is geodesic for some $h_{1}$ (by Lemma 2.10(i)), which contradicts that $h$ is cyclically reduced. Therefore $(h u) \wedge_{L}\left(h^{-1} u\right)=1$.
(ii) It follows from (i).

Definition 3.3 (starting set). For $g \in A(\Gamma)$, the starting set $S(g)$ of $g$ is defined as

$$
S(g)=\left\{x \in V(\Gamma)^{ \pm 1}: x \leqslant_{L} g\right\}
$$

Lemma 3.4. The following hold.
(i) For any $g \in A(\Gamma)$, the following are equivalent.
(a) $g$ is cyclically reduced.
(b) There is no geodesic decomposition such as $g=u^{-1} h u$, where $u, h \in A(\Gamma)$ with $u \neq 1$.
(c) For any geodesic decomposition $g=g_{1} g_{2}, g_{2} g_{1}$ is geodesic.
(d) For any $g_{1} \in A(\Gamma)$ with $g_{1} \leqslant_{L} g$, $g g_{1}$ is geodesic.
(e) $g^{n}=g g \cdots g$ is geodesic (i.e. $\left\|g^{n}\right\|=n\|g\|$ ) for some $n \geqslant 2$.
(f) $g^{n}=g g \cdots g$ is geodesic (i.e. $\left\|g^{n}\right\|=n\|g\|$ ) for all $n \geqslant 2$.
(g) $g^{n}$ is cyclically reduced for some $n \geqslant 2$.
(h) $g^{n}$ is cyclically reduced for all $n \geqslant 2$.
(ii) Let $g_{1} \cdots g_{k}$ be geodesic (i.e. $\left\|g_{1} \cdots g_{k}\right\|=\left\|g_{1}\right\|+\cdots+\left\|g_{k}\right\|$ ). Then $g_{1}^{n_{1}} \cdots g_{k}^{n_{k}}$ is geodesic (i.e. $\left.\left\|g_{1}^{n_{1}} \cdots g_{k}^{n_{k}}\right\|=\left\|g_{1}^{n_{1}}\right\|+\cdots+\left\|g_{k}^{n_{k}}\right\|\right)$ for any positive integers $n_{i}$.
(iii) For any $g \in A(\Gamma)$ and $n \geqslant 2, \operatorname{supp}\left(g^{n}\right)=\operatorname{supp}(g)$ and $S\left(g^{n}\right)=S(g)$.

Proof. (i) The equivalences between (a), (b), (c), (e), (f) are easy to prove. For example, see [25, Lemma 2.1]. We show the remaining equivalences assuming the known equivalences.
$(\mathrm{a}) \Rightarrow(\mathrm{d})$ : Assume that $g_{1} \leqslant L g$ but $g g_{1}$ is not geodesic. Then there exists a letter $x \in V(\Gamma)^{ \pm 1}$ such that $x \leqslant_{R} g$ and $x^{-1} \leqslant_{L} g_{1}$ (by Lemma 2.9(i)). Since $g_{1} \leqslant_{L} g$, we have $x^{-1} \leqslant_{L} g$, hence $x \leqslant_{R} g^{-1}$. Now $x \leqslant_{R} g \wedge_{R} g^{-1}$, hence $g \wedge_{R} g^{-1} \neq 1$. By Lemma 3.2(ii), $g$ is not cyclically reduced.
$(\mathrm{d}) \Rightarrow(\mathrm{e})$ : Since $g \leqslant_{L} g, g g$ is geodesic.
$(\mathrm{f}) \Rightarrow(\mathrm{h})$ : Let $n \geqslant 2$. Since $\left\|g^{2 n}\right\|=2 n\|g\|$ and $\left\|g^{n}\right\|=n\|g\|$, we have $\left\|g^{2 n}\right\|=2\left\|g^{n}\right\|$, hence $g^{n} \cdot g^{n}$ is geodesic. Because (a) and (e) are equivalent, $g^{n}$ is cyclically reduced.
$(\mathrm{h}) \Rightarrow(\mathrm{g})$ : It is obvious.
$(\mathrm{g}) \Rightarrow(\mathrm{a})$ : Assume that $g$ is not cyclically reduced. Then $g=u^{-1} h u$ is geodesic for some $u, h \in A(\Gamma)$ such that $u \neq 1$ and $h$ is cyclically reduced [28]. Observe that $h h$ is geodesic (by (a) $\Leftrightarrow(\mathrm{f})$ ). Hence $g^{n}$ has a geodesic decomposition $u^{-1} h^{n} u$ for any $n \geqslant 2$ (by Lemma 2.8). Therefore $g^{n}$ is not cyclically reduced for any $n \geqslant 2$ (by (a) $\Leftrightarrow(\mathrm{b})$ ).
(ii) Let $g_{i}=u_{i}^{-1} h_{i} u_{i}$ be a geodesic decomposition of $g_{i}$ with $h_{i}$ cyclically reduced for $i=1, \ldots, k$. Then

$$
\begin{aligned}
g_{1} \cdots g_{k} & =u_{1}^{-1} h_{1} u_{1} \cdots u_{k}^{-1} h_{k} u_{k} \\
g_{1}^{n_{1}} \cdots g_{k}^{n_{k}} & =u_{1}^{-1} h_{1}^{n_{1}} u_{1} \cdots u_{k}^{-1} h_{k}^{n_{k}} u_{k}
\end{aligned}
$$

In particular, $u_{1}^{-1} h_{1} u_{1} \cdots u_{k}^{-1} h_{k} u_{k}$ is geodesic because $g_{1} \cdots g_{k}$ and each $g_{i}=u_{i}^{-1} h_{i} u_{i}$ are geodesic. Notice that each $h_{i} h_{i}$ is geodesic by (i). Applying Lemma 2.8(iv), we get that $u_{1}^{-1} h_{1}^{n_{1}} u_{1} \cdots u_{k}^{-1} h_{k}^{n_{k}} u_{k}$ is geodesic. Therefore $g_{1}^{n_{1}} \cdots g_{k}^{n_{k}}$ is geodesic.
(iii) Let $g=u^{-1} h u$ be a geodesic decomposition of $g$ with $h$ cyclically reduced. Then

$$
\begin{equation*}
g^{n}=u^{-1} h^{n} u=u^{-1} h \cdots h u \tag{3}
\end{equation*}
$$

are each geodesic decompositions of $g^{n}$ (by Lemma 2.8(iii)).
Notice that $\operatorname{supp}(u)=\operatorname{supp}\left(u^{-1}\right)$ and that if $g_{1} \cdots g_{k}$ is a geodesic decomposition, then $\operatorname{supp}\left(g_{1} \cdots g_{k}\right)=$ $\operatorname{supp}\left(g_{1}\right) \cup \cdots \cup \operatorname{supp}\left(g_{k}\right)$. Therefore $\operatorname{supp}(g)=\operatorname{supp}(u) \cup \operatorname{supp}(h)=\operatorname{supp}\left(g^{n}\right)$ from (3).

Observe that $x \leqslant_{L} h^{n}$ if and only if $x \leqslant_{L} h$ : if $x \leqslant_{L} h$, then it is obvious that $x \leqslant_{L} h^{n}$; if $x \nless_{L} h$, then $x \not \underbrace{}_{L} h^{n}$ (otherwise, $x \leqslant_{L} h^{n}=h \cdot h^{n-1}$ implies $x \rightleftharpoons h$ and $x \leqslant_{L} h^{n-1}$, which contradicts that $\left.\operatorname{supp}(h)=\operatorname{supp}\left(h^{n-1}\right)\right)$.

Since $g=u^{-1} h u$ is geodesic, $x \leqslant_{L} g$ if and only if one of the following holds (by Lemma 2.9(ii)): (i) $x \leqslant_{L} u^{-1}$; (ii) $x \leqslant_{L} h$ and $x \rightleftharpoons u^{-1}$; (iii) $x \leqslant_{L} u$ and $x \rightleftharpoons u^{-1} h$. Notice that (iii) cannot happen. Since $x \leqslant_{L} h$ if and only if $x \leqslant_{L} h^{n}$, we can conclude that $x \leqslant_{L} g$ if and only if $x \leqslant_{L} g^{n}$. Therefore $S(g)=S\left(g^{n}\right)$.

Definition 3.5 (cycling, cyclic conjugation). Let $g \in A(\Gamma)$ be cyclically reduced.
(i) For a letter $x \in V(\Gamma)^{ \pm 1}$, the conjugation $g^{x}=x^{-1} g x$ is called a left (resp. right) cycling if $x \leqslant_{L} g$ (resp. $x^{-1} \leqslant_{R} g$ ). Left and right cyclings are collectively called cyclings.
(ii) For an element $u \in A(\Gamma)$, the conjugation $g^{u}=u^{-1} g u$ is called a cyclic conjugation of $g$ by $u$ if $\left\|g^{u}\right\|=\|g\|$ and $\operatorname{supp}(u) \subset \operatorname{supp}(g)$. A cyclic conjugation $g^{u}$ is called a left (resp. right) cyclic conjugation if $g u$ (resp. $u^{-1} g$ ) is geodesic.

For $g \in A(\Gamma)$ and $x \in V(\Gamma)^{ \pm 1}$, if $g^{x}$ is a left cycling, i.e. $x \leqslant_{L} g$, then $g=x h$ is geodesic for some $h \in A(\Gamma)$ and $g^{x}=x^{-1} g x=h x$ is geodesic. Therefore the left cycling $g^{x}$ is obtained from $g=x h$ by moving the first letter $x$ to the last. Similarly, if $g^{x}$ is a right cycling, then $g^{x}$ is obtained from $g=h x^{-1}$ by moving the last letter $x^{-1}$ to the first.

If $g^{x}$ is a cycling, then it is easy to see that $\left\|g^{x}\right\|=\|g\|$ and $\operatorname{supp}(x) \subset \operatorname{supp}(g)$, hence $g^{x}$ is a cyclic conjugation. Conversely, we will show in Lemma 3.7 that a cyclic conjugation $g^{u}$ is obtained by iterated application of cyclings.

If $g \in A(\Gamma)$ is cyclically reduced and $g^{u}$ is a cyclic conjugation, then $\left\|g^{u}\right\|=\|g\|$, hence $g^{u}$ is also cyclically reduced.

Lemma 3.6. Let $g \in A(\Gamma)$ and $x, y \in V(\Gamma)^{ \pm 1}$ with $g$ cyclically reduced.
(i) The conjugation $g^{x}$ cannot be both a left cycling and a right cycling.
(ii) Let $y \neq x^{-1}$. If $g^{x}$ and $\left(g^{x}\right)^{y}$ are cyclings of different type, then $x \rightleftharpoons y$.
(iii) Let $x \rightleftharpoons y$. If both $g^{x}$ and $\left(g^{x}\right)^{y}$ are cyclings, then so are $g^{y}$ and $\left(g^{y}\right)^{x}$.
(iv) Let $x \rightleftharpoons y$. If both $g^{x}$ and $g^{y}$ are cyclings, then so are $\left(g^{x}\right)^{y}$ and $\left(g^{y}\right)^{x}$.

In (iii) and (iv), the types of cyclings depend only on the conjugating letters. For example, if $g^{x}$ is a left cycling, then $\left(g^{y}\right)^{x}$ is also a left cycling, and so on.

Proof. (i) If $g^{x}$ is both a left cycling and a right cycling, then $x \leqslant_{L} g$ and $x^{-1} \leqslant_{R} g$, hence $g=x h x^{-1}$ is geodesic for some $h \in A(\Gamma)$ (by Lemmas 2.10(i)). Thus $g$ is not cyclically reduced (by Lemma 3.4(i)).
(ii) Assume that $g^{x}$ is a left cycling and $\left(g^{x}\right)^{y}$ is a right cycling. (An analogous argument applies to the case where $g^{x}$ is a right cycling and $\left(g^{x}\right)^{y}$ is a left cycling.)

Since $g^{x}$ is a left cycling, we have $x \leqslant_{L} g$, hence $g=x h$ is geodesic for some $h \in A(\Gamma)$. Notice that $g^{x}=h x$ is geodesic. Since $\left(g^{x}\right)^{y}$ is a right cycling, $y^{-1} \leqslant_{R} g^{x}=h x$.

Since $y^{-1} \neq x$, we have $x \rightleftharpoons y^{-1}$ (by Lemma 2.9(iii)) and hence $x \rightleftharpoons y$.
(iii) and (iv) Assume that $g^{x}$ and $\left(g^{x}\right)^{y}$ are left cyclings, hence $x \leqslant_{L} g$ and $y \leqslant_{L} g^{x}$. Since $x \leqslant_{L} g$, $g=x h_{1}$ is geodesic for some $h_{1} \in A(\Gamma)$, hence $g^{x}=h_{1} x$ is also geodesic. Since $y \leqslant L g^{x}=h_{1} x$ and $y \rightleftharpoons x$ (hence $y \not \star_{L} x$ ), we have $y \leqslant_{L} h_{1}$, hence $h_{1}=y h_{2}$ is geodesic for some $h_{2} \in A(\Gamma)$. Now we know that

$$
g=x h_{1}=x y h_{2}=y x h_{2}
$$

and $g^{y}=x h_{2} y$ are all geodesic, hence $y \leqslant_{L} g$ and $x \leqslant_{L} g^{y}$. This means that $g^{y}$ and $\left(g^{y}\right)^{x}$ are left cyclings.

For the other cases, it is easy to see that $g$ has a geodesic decomposition as one of $x y h, x h y^{-1}$, $y h x^{-1}$ and $h x^{-1} y^{-1}$ depending on the types of cyclings, from which the conclusions follow.

Lemma 3.7. Let $g, u, u_{1}, u_{2} \in A(\Gamma)$ with $g$ cyclically reduced.
(i) The following are equivalent:
(a) $g^{u}$ is a cyclic (resp. left cyclic, right cyclic) conjugation;
(b) there exists a reduced word $w_{0} \equiv y_{1} \cdots y_{k}$ representing $u$ such that $\left(g^{y_{1} \cdots y_{i-1}}\right)^{y_{i}}$ is a cycling (resp. left cycling, right cycling) for all $1 \leqslant i \leqslant k$;
(c) for any reduced word $w \equiv x_{1} \cdots x_{k}$ representing $u$, $\left(g^{x_{1} \cdots x_{i-1}}\right)^{x_{i}}$ is a cycling (resp. left cycling, right cycling) for all $1 \leqslant i \leqslant k$.
In particular, if $g^{u}$ is a cyclic conjugation, then $\operatorname{supp}\left(g^{u}\right)=\operatorname{supp}(g)$.
(ii) Let $u=u_{1} u_{2}$ be geodesic. Then $g^{u}$ is a cyclic (resp. left cyclic, right cyclic) conjugation if and only if both $g^{u_{1}}$ and $\left(g^{u_{1}}\right)^{u_{2}}$ are cyclic (resp. left cyclic, right cyclic) conjugations.
(iii) If $g^{u_{1}}$ and $g^{u_{2}}$ are cyclic (resp. left cyclic, right cyclic) conjugations and $u_{1} \vee_{L} u_{2}$ exists, then $g^{u_{1} \vee_{L} u_{2}}$ is also a cyclic (resp. left cyclic, right cyclic) conjugation.
(iv) Let $u_{1} \rightleftharpoons u_{2}$. Suppose that $g^{u_{1}}$ and $g^{u_{2}}$ are a left cyclic conjugation and a right cyclic conjugation, respectively. Then $\left(g^{u_{2}}\right)^{u_{1}}$ and $\left(g^{u_{1}}\right)^{u_{2}}$ are a left cyclic conjugation and a right cyclic conjugation, respectively. Moreover, $u_{2}^{-1} g u_{1}$ is geodesic.
(v) Suppose that $g^{u}$ is a cyclic conjugation. Then there is a geodesic decomposition $u=u_{1} u_{2}$ such that $u_{1} \rightleftharpoons u_{2}$ and $g^{u_{1}}$ (resp. $g^{u_{2}}$ ) is a left (resp. right) cyclic conjugation. Moreover, $u_{2}^{-1} g u_{1}$ is geodesic.

Proof. The statements (i)-(iii) concern three types of cyclic conjugations. We prove only the case of cyclic conjugations. The other cases (i.e. left and right cyclic conjugations) can be proved analogously.

We use the following claim.
Claim 1. If $g^{u}$ is a cyclic conjugation for some $u \in A(\Gamma) \backslash\{1\}$, then there exists $y_{1} \in V(\Gamma)^{ \pm 1}$ and $u_{1} \in A(\Gamma)$ such that $u=y_{1} u_{1}$ is geodesic, $g^{y_{1}}$ is a cycling and $\left(g^{y_{1}}\right)^{u_{1}}$ is a cyclic conjugation.

Proof of Claim 1. Since $\left\|g^{u}\right\|=\|g\|$, the decomposition $u^{-1} g u$ is not geodesic. If both $u^{-1} g$ and $g u$ are geodesic, then there exists $x \in V(\Gamma)^{ \pm 1}$ such that $x^{-1} \leqslant_{R} u^{-1}, x \leqslant_{L} u$ and $x \rightleftharpoons g$ (by Lemma 2.8(i)). However, the relation $x \rightleftharpoons g$ is impossible because $x \leqslant_{L} u$ and $\operatorname{supp}(u) \subset \operatorname{supp}(g)$. Hence either $u^{-1} g$ or $g u$ is not geodesic, i.e. there exists $y_{1} \in V(\Gamma)^{ \pm 1}$ such that either $y_{1}^{-1} \leqslant R u^{-1}$ and $y_{1} \leqslant L g$ or $y_{1} \leqslant_{L} u$ and $y_{1}^{-1} \leqslant_{R} g$. This means that $g^{y_{1}}$ is a cycling and that $u=y_{1} u_{1}$ is geodesic for some $u_{1} \in A(\Gamma)$. Therefore $\left(g^{y_{1}}\right)^{u_{1}}$ is a cyclic conjugation because $\left\|\left(g^{y_{1}}\right)^{u_{1}}\right\|=\left\|g^{u}\right\|=\|g\|=\left\|g^{y_{1}}\right\|$ and $\operatorname{supp}\left(u_{1}\right) \subset \operatorname{supp}(u) \subset \operatorname{supp}(g)=\operatorname{supp}\left(g^{y_{1}}\right)$.
(i) We may assume $u \neq 1$ because otherwise there is nothing to prove.
$(\mathrm{a}) \Rightarrow(\mathrm{b})$ : Suppose that $g^{u}$ is a cyclic conjugation. By Claim 1, there is a geodesic decomposition $u=y_{1} u_{1}$ such that $g^{y_{1}}$ is a cycling and $\left(g^{y_{1}}\right)^{u_{1}}$ is a cyclic conjugation. Applying Claim 1 again to $\left(g^{y_{1}}\right)^{u_{1}}$, we have a geodesic decomposition $u_{1}=y_{2} u_{2}$ such that $\left(g^{y_{1}}\right)^{y_{2}}$ is a cycling and $\left(g^{y_{1} y_{2}}\right)^{u_{2}}$ is a cyclic conjugation. Iterating this process, we get a desired reduced word $w_{0} \equiv y_{1} \ldots y_{k}$.
$(\mathrm{b}) \Rightarrow(\mathrm{c})$ : Let $w \equiv x_{1} \cdots x_{k}$ be a reduced word representing $u$. Notice that the word $w_{0} \equiv y_{1} \cdots y_{k}$ can be transformed into the word $w \equiv x_{1} \cdots x_{k}$ by using only commutation relations. Therefore each $\left(g^{x_{1} \cdots x_{i-1}}\right)^{x_{i}}$ is a cycling (by Lemma 3.6(iii)).
$(\mathrm{c}) \Rightarrow(\mathrm{a})$ : Let $w \equiv x_{1} \cdots x_{k}$ be a reduced word representing $u$, where $x_{i}=v_{i}^{\epsilon_{i}}, v_{i} \in V(\Gamma)$ and $\epsilon_{i}= \pm 1$ for all $1 \leqslant i \leqslant k$. Then, for each $1 \leqslant i \leqslant k,\left(g^{x_{1} \cdots x_{i-1}}\right)^{x_{i}}$ is a cycling, hence

$$
\left\|g^{x_{1} \cdots x_{i-1}}\right\|=\left\|g^{x_{1} \cdots x_{i}}\right\| \quad \text { and } \quad v_{i} \in \operatorname{supp}\left(g^{x_{1} \cdots x_{i-1}}\right)=\operatorname{supp}\left(g^{x_{1} \cdots x_{i}}\right)
$$

Thus $\left\|g^{x_{1} \cdots x_{k}}\right\|=\|g\|$ and $\left\{v_{1}, \ldots, v_{k}\right\} \subset \operatorname{supp}\left(g^{x_{1} \cdots x_{k}}\right)=\operatorname{supp}(g)$. Therefore $\left\|g^{u}\right\|=\|g\|$ and $\operatorname{supp}(u) \subset \operatorname{supp}(g)$, hence $g^{u}$ is a cyclic conjugation.
(ii) Let $u_{1}=x_{1} \cdots x_{j}$ and $u_{2}=x_{j+1} \cdots x_{k}$ be geodesic decompositions, where $x_{1} \cdots x_{k} \in V(\Gamma)^{ \pm 1}$. Then $u=x_{1} \cdots x_{k}$ is also geodesic because $u=u_{1} u_{2}$ is geodesic. By (i), $g^{u}$ is a cyclic conjugation if and only if $\left(g^{x_{1} \cdots x_{i-1}}\right)^{x_{i}}$ is a cycling for each $1 \leqslant i \leqslant k$, and this happens if and only if both $g^{u_{1}}$ and $\left(g^{u_{1}}\right)^{u_{2}}$ are cyclic conjugations.
(iii) Let $C(g)$ be the set of all $u \in A(\Gamma)$ such that $g^{u}$ is a cyclic conjugation. Then $C(g)$ satisfies (P1) in Corollary 2.16 by (ii) in this lemma. Therefore it suffices to show that $C(g)$ satisfies (P2) in Corollary 2.16,

Let $u x, u y \in C(g)$ (i.e. $g^{u x}$ and $g^{u y}$ are cyclic conjugations) such that $u x$ and $u y$ are geodesic and $x \rightleftharpoons y$, where $u \in A(\Gamma)$ and $x, y \in V(\Gamma)^{ \pm 1}$. Then both $\left(g^{u}\right)^{x}$ and $\left(g^{u}\right)^{y}$ are cyclings of $g^{u}$ (by (ii)). By Lemma 3.6(iv), $\left(g^{u x}\right)^{y}$ is a cycling, hence $g^{u x y}$ is a cyclic conjugation (by (ii)). Therefore $u x y \in C(g)$, hence $C(g)$ satisfies (P2) in Corollary 2.16.
(iv) Notice that $u_{1} \vee_{L} u_{2}=u_{1} u_{2}=u_{2} u_{1}$ and that both $u_{1} u_{2}$ and $u_{2} u_{1}$ are geodesic, because $u_{1} \rightleftharpoons u_{2}$. Since both $g^{u_{1}}$ and $g^{u_{2}}$ are cyclic conjugations, so are $g^{u_{1} u_{2}},\left(g^{u_{1}}\right)^{u_{2}}$ and $\left(g^{u_{2}}\right)^{u_{1}}$ (by (ii) and (iii)).

Let us show that the cyclic conjugation $\left(g^{u_{2}}\right)^{u_{1}}$ is a left cyclic conjugation, i.e. the decomposition $g^{u_{2}} u_{1}$ is geodesic. (The proof for $\left(g^{u_{1}}\right)^{u_{2}}$ is analogous.) Observe

$$
u_{2}^{-1} g u_{1}=g^{u_{2}} u_{1} u_{2}^{-1}
$$

Since both $u_{2}^{-1} g$ and $g u_{1}$ are geodesic and since $u_{1} \rightleftharpoons u_{2}, u_{2}^{-1} g u_{1}$ is geodesic (by Lemma 2.8(ii)). Since $\left\|g^{u_{2}}\right\|=\|g\|$, the decomposition $g^{u_{2}} u_{1} u_{2}^{-1}$ is also geodesic. Therefore $g^{u_{2}} u_{1}$ is geodesic.
(v) We use induction on $\|u\|$. If $\|u\|=1$, there is nothing to prove.

Suppose that $u=u^{\prime} x$ is geodesic for some $u^{\prime} \in A(\Gamma) \backslash\{1\}$ and $x \in V(\Gamma)^{ \pm 1}$. Then $g^{u^{\prime}}$ is a cyclic conjugation and $\left(g^{u^{\prime}}\right)^{x}$ is a cycling (by (ii)). Suppose that $\left(g^{u^{\prime}}\right)^{x}$ is a left cycling. (The proof is analogous for the case where $\left(g^{u^{\prime}}\right)^{x}$ is a right cycling.) By the induction hypothesis, we have a geodesic decomposition $u^{\prime}=u_{1}^{\prime} u_{2}^{\prime}$ such that $u_{1}^{\prime} \rightleftharpoons u_{2}^{\prime}$ and $g^{u_{1}^{\prime}}$ (resp. $g^{u_{2}^{\prime}}$ ) is a left (resp. right) cyclic conjugation.
Claim 2. $x \rightleftharpoons u_{2}^{\prime}$, and $u=u_{1}^{\prime} x u_{2}^{\prime}$ is geodesic.
Proof of Claim 2. Let $u_{2}^{\prime}=y_{1} \cdots y_{k}$ be geodesic, where $y_{1}, \ldots, y_{k} \in V(\Gamma)^{ \pm 1}$. Then $u$ has the following three geodesic decompositions:

$$
u=u^{\prime} x=u_{1}^{\prime} u_{2}^{\prime} x=u_{1}^{\prime} y_{1} \cdots y_{k} x .
$$

Let $h_{0}=g^{u_{1}^{\prime}}$ and $h_{i}=g^{u_{1}^{\prime} y_{1} \cdots y_{i}}$ for $1 \leqslant i \leqslant k$. Then each $h_{i}$ is cyclically reduced (by (ii)), and $h_{i}=h_{i-1}^{y_{i}}$. Since $\left(g^{u_{1}^{\prime}}\right)^{u_{2}^{\prime}}$ is a right cyclic conjugation (by (iv)), each $h_{i-1}^{y_{i}}=\left(g^{u_{1}^{\prime} y_{1} \cdots y_{i-1}}\right)^{y_{i}}$ is a right cycling (by (i)).

Since $u=u_{1}^{\prime} y_{1} \cdots y_{k} x$ is geodesic, we have $y_{k} \neq x^{-1}$. We know that $h_{k-1}^{y_{k}}$ is a right cycling and that $\left(h_{k-1}^{y_{k}}\right)^{x}=\left(g^{u^{\prime}}\right)^{x}$ is a left cycling, hence $x \rightleftharpoons y_{k}$ (by Lemma 3.6(ii)). Therefore $u=u_{1}^{\prime} y_{1} \cdots y_{k-1} x y_{k}$ and $h_{k-1}^{x}$ is a left cycling (by Lemma 3.6(iii)).

Applying the above argument to the right cyclings $h_{k-2}^{y_{k-1}}, \ldots, h_{0}^{y_{1}}$ in this order iteratively, we obtain $x \rightleftharpoons y_{i}$ for all $1 \leqslant i \leqslant k$. Therefore $x \rightleftharpoons u_{2}^{\prime}$ and hence $u=u_{1}^{\prime} u_{2}^{\prime} x=u_{1}^{\prime} x u_{2}^{\prime}$. Since $u_{1}^{\prime} u_{2}^{\prime} x$ is geodesic, so is $u_{1}^{\prime} x u_{2}^{\prime}$.

Let $u_{1}=u_{1}^{\prime} x$ and $u_{2}=u_{2}^{\prime}$. Then $u=u_{1} u_{2}$ is geodesic, $u_{1} \rightleftharpoons u_{2}$ and $g^{u_{1}}$ (resp. $g^{u_{2}}$ ) is a left (resp. right) cyclic conjugation. Moreover, $u_{2}^{-1} g u_{1}$ is geodesic (by (iv))

For a cyclically reduced $g \in A(\Gamma)$, if $u \leqslant_{L} g$, then $g^{u}$ is obviously a left cyclic conjugation. The following proposition is concerned with the opposite direction.
Proposition 3.8. Let $g, u \in A(\Gamma)$ with $g$ cyclically reduced. Then the following are equivalent.
(i) $g^{u}$ is a left (resp. right) cyclic conjugation.
(ii) $u \leqslant L g^{n}$ (resp. $u^{-1} \leqslant R g^{n}$ ) for some $n \geqslant 1$.

Proof. We prove the equivalence only for the left cyclic conjugation. The proof for the right cyclic conjugation is analogous. We may assume $\|g\| \geqslant 2$ and $\|u\| \geqslant 1$ (otherwise it is obvious).
(ii) $\Rightarrow$ (i): We may assume $n \geqslant 2$ (otherwise it is obvious). We proceed by induction on $\|u\|$. If $\|u\|=1$, then $u$ is a letter. In this case, $u \leqslant_{L} g^{n}$ implies $u \leqslant_{L} g$ (by Lemma 3.4(iii)), hence $g^{u}$ is a left cycling.

Suppose $\|u\| \geqslant 2$. Then $u=x u_{1}$ is geodesic for some $x \in V(\Gamma)^{ \pm 1}$ and $u_{1} \in A(\Gamma) \backslash\{1\}$. Since $x u_{1}=u \leqslant_{L} g^{n}$, we get $x \leqslant_{L} g^{n}$ and hence $x \leqslant_{L} g$ (by Lemma 3.4(iii)). Therefore $g=x g_{1}$ is geodesic for some $g_{1} \in A(\Gamma)$, and $g^{x}=g_{1} x$ is also geodesic. Since both $g \cdots g$ and $g=x g_{1}$ are geodesic, $x g_{1} x g_{1} \cdots x g_{1} x$ is geodesic, hence the following three decompositions are all geodesic.

$$
g^{n} x=x g_{1} x g_{1} \cdots x g_{1} x=x\left(g^{x}\right)^{n}
$$

Since $x u_{1}=u \leqslant_{L} g^{n} \leqslant_{L} g^{n} x=x\left(g^{x}\right)^{n}$, we have $u_{1} \leqslant_{L}\left(g^{x}\right)^{n}$ (by Lemma [2.6(ii)). By the induction hypothesis, $\left(g^{x}\right)^{u_{1}}$ is a left cyclic conjugation. And $g^{x}$ is also a left cyclic conjugation because $x \leqslant_{L} g$. Therefore $g^{u}$ is a left cyclic conjugation (by Lemma 3.7(ii)).
(i) $\Rightarrow$ (ii): As before, we use induction on $\|u\|$. If $\|u\|=1, g^{u}$ is a left cycling, hence $u \leqslant_{L} g$.

Suppose $\|u\| \geqslant 2$. Then $u=x u_{1}$ is geodesic for some $x \in V(\Gamma)^{ \pm}$and $u_{1} \in A(\Gamma) \backslash\{1\}$. Since $g^{u}$ is a left cyclic conjugation, both $g^{x}$ and $\left(g^{x}\right)^{u_{1}}$ are left cyclic conjugations (by Lemma 3.7(ii)). Since $g^{x}$ is a left cyclic conjugation and $x$ is a letter, we have $x \leqslant_{L} g$. Since $\left(g^{x}\right)^{u_{1}}$ is a left cyclic conjugation, $u_{1} \leqslant_{L}\left(g^{x}\right)^{n}$ for some $n$ by the induction hypothesis. Using a similar argument as above, we get that $x\left(g^{x}\right)^{n}$ and $g^{n} x$ are geodesic, hence $u=x u_{1} \leqslant_{L} x\left(g^{x}\right)^{n}=g^{n} x \leqslant_{L} g^{n+1}$.

From the above proposition, $g^{u}$ is a right cyclic conjugation if and only if $\left(g^{-1}\right)^{u}$ is a left cyclic conjugation.

Theorem 3.9. Let $g, u \in A(\Gamma)$ with $g$ cyclically reduced. Then there exists a unique geodesic decomposition $u=u_{1} u_{2} u_{3}$ such that
(i) $u_{1}$ disjointly commutes with $g$;
(ii) $g^{u_{2}}$ is a cyclic conjugation;
(iii) $g^{u}=u_{3}^{-1} g^{u_{2}} u_{3}$ is geodesic, i.e. $\left\|g^{u}\right\|=\left\|u_{3}^{-1}\right\|+\left\|g^{u_{2}}\right\|+\left\|u_{3}\right\|=\|g\|+2\left\|u_{3}\right\|$.

Moreover, the following hold: $u_{1}$ is the maximal prefix of $u$ that disjointly commutes with $g ; u_{2}$ is the maximal prefix of $u$ such that $g^{u_{2}}$ is a cyclic conjugation; $u_{3}=g^{u} \wedge_{R}\left(g^{u}\right)^{-1}$. In particular, $u_{1} \rightleftharpoons u_{2}$.

Proof. We first prove the existence of the decomposition $u=u_{1} u_{2} u_{3}$.
If $u_{1}^{\prime}$ and $u_{1}^{\prime \prime}$ are prefixes of $u$ such that $u_{1}^{\prime} \rightleftharpoons g$ and $u_{1}^{\prime \prime} \rightleftharpoons g$, then $u_{1}^{\prime} \vee_{L} u_{1}^{\prime \prime}$ exists (because $u_{1}^{\prime}$ and $u_{1}^{\prime \prime}$ have a common right multiple $u$ ). Observe that $u_{1}^{\prime} \vee_{L} u_{1}^{\prime \prime}$ is also a prefix of $u$ and also disjointly commutes with $g$ (by Theorem (2.12). Therefore there exists a unique maximal prefix $u_{1}$ of $u$ that disjointly commutes with $g$.

If $u_{2}^{\prime}$ and $u_{2}^{\prime \prime}$ are prefixes of $u$ such that $g^{u_{2}^{\prime}}$ and $g^{u_{2}^{\prime \prime}}$ are cyclic conjugations, then $u_{2}^{\prime} \vee_{L} u_{2}^{\prime \prime}$ exists (because $u_{2}^{\prime}$ and $u_{2}^{\prime \prime}$ have a common right multiple $u$ ) and is also a prefix of $u$, and $g^{u_{2}^{\prime} V_{L} u_{2}^{\prime \prime}}$ is also a cyclic conjugation (by Lemma 3.7(iii)). Therefore there exists a unique maximal prefix $u_{2}$ of $u$ such that $g^{u_{2}}$ is a cyclic conjugation.

Notice that $u_{1} \rightleftharpoons u_{2}$ because $\operatorname{supp}\left(u_{2}\right) \subset \operatorname{supp}(g)$ and $u_{1} \rightleftharpoons g$. Thus $u_{1} \vee_{L} u_{2}=u_{1} u_{2}$ is a prefix of $u$ and $u_{1} u_{2}$ is geodesic, hence $u=u_{1} u_{2} u_{3}$ is geodesic for some $u_{3} \in A(\Gamma)$. Observe

$$
g^{u}=u_{3}^{-1} u_{2}^{-1} u_{1}^{-1} g u_{1} u_{2} u_{3}=u_{3}^{-1} u_{2}^{-1} g u_{2} u_{3}=u_{3}^{-1} g^{u_{2}} u_{3} .
$$

Let us show that $u_{3}^{-1} g^{u_{2}} u_{3}$ is geodesic.
If $g^{u_{2}} u_{3}$ is not geodesic, then there exists $x \in V(\Gamma)^{ \pm 1}$ such that $x \leqslant_{L} u_{3}$ and $x^{-1} \leqslant_{R} g^{u_{2}}$ (by Lemma 2.9(i)), hence $\left(g^{u_{2}}\right)^{x}$ is a cyclic conjugation. Notice that $u_{2} x$ is geodesic. By Lemma 3.7(ii), $g^{u_{2} x}$ is also a cyclic conjugation, hence $x \in \operatorname{supp}(g)$, which implies $x \rightleftharpoons u_{1}$. Consequently, $u_{2} x \rightleftharpoons u_{1}$ and hence $u_{2} x \leqslant_{L} u$. This contradicts the maximality of $u_{2}$. Therefore $g^{u_{2}} u_{3}$ is geodesic. Similarly $u_{3}^{-1} g^{u_{2}}$ is geodesic.

Since both $u_{3}^{-1} g^{u_{2}}$ and $g^{u_{2}} u_{3}$ are geodesic, if $u_{3}^{-1} g^{u_{2}} u_{3}$ is not geodesic, then there exists $x \in V(\Gamma)^{ \pm 1}$ such that $x \leqslant_{L} u_{3}, x^{-1} \leqslant_{R} u_{3}^{-1}$ and $x \rightleftharpoons g^{u_{2}}$ (by Lemma 2.8(i)). Since $\operatorname{supp}(g)=\operatorname{supp}\left(g^{u_{2}}\right)$, we have $x \rightleftharpoons g$, hence $x \rightleftharpoons u_{2}$ and $u_{1} x \rightleftharpoons g$. Notice that $u_{1} x$ is geodesic. Since $u_{1} x$ is a prefix of $u$, this contradicts the maximality of $u_{1}$. Therefore $u_{3}^{-1} g^{u_{2}} u_{3}$ is geodesic.

Since $g^{u}=u_{3}^{-1} g^{u_{2}} u_{3}$ is geodesic such that $g^{u_{2}}$ is cyclically reduced, $u_{3}$ satisfies the formula $u_{3}=$ $g^{u} \wedge_{R}\left(g^{u}\right)^{-1}$ (by Lemma 3.2(i)).

So far we have shown that $u=u_{1} u_{2} u_{3}$ is a desired decomposition. We will now show the uniqueness of the decomposition. Let $u=u_{1}^{\prime} u_{2}^{\prime} u_{3}^{\prime}$ be another geodesic decomposition satisfying the conditions (i), (ii) and (iii) of the theorem.

Since $u_{2}^{\prime}$ and $u_{3}^{\prime}$ satisfy the conditions (ii) and (iii), we have $u_{3}^{\prime}=g^{u} \wedge_{R}\left(g^{u}\right)^{-1}$ (by Lemma 3.2(i)), hence

$$
u_{3}=u_{3}^{\prime}, \quad g^{u_{2}}=g^{u_{2}^{\prime}} \quad \text { and } \quad u_{1} u_{2}=u_{1}^{\prime} u_{2}^{\prime} .
$$

Since both $u_{1}$ and $u_{1}^{\prime}$ are prefixes of $u$ that disjointly commute with $g$, so is $u_{1} \vee_{L} u_{1}^{\prime}$ (by Theorem (2.12). By the maximality of $u_{1}$, we have $u_{1} \vee_{L} u_{1}^{\prime} \leqslant_{L} u_{1}$, hence $u_{1}^{\prime} \leqslant_{L} u_{1}$.

Similarly, since $g^{u_{2}}$ and $g^{u_{2}^{\prime}}$ are cyclic conjugations, so is $g^{u_{2} \vee_{L} u_{2}^{\prime}}$ (by Lemma 3.7). By the maximality of $u_{2}$, we have $u_{2} \vee_{L} u_{2}^{\prime} \leqslant L u_{2}$, hence $u_{2}^{\prime} \leqslant L u_{2}$.

Since $u_{1}^{\prime} \leqslant_{L} u_{1}, u_{2}^{\prime} \leqslant_{L} u_{2}$ and $u_{1} u_{2}=u_{1}^{\prime} u_{2}^{\prime}$, we have $u_{1}=u_{1}^{\prime}$ and $u_{2}=u_{2}^{\prime}$.
The following seems to be well known to experts.
Corollary 3.10. Let $g_{1}, g_{2} \in A(\Gamma)$ be cyclically reduced. If $g_{1}$ and $g_{2}$ are conjugate, then they are cyclically conjugate.

Proof. Since $g_{1}$ and $g_{2}$ are conjugate, $g_{2}=g_{1}^{u}$ for some $u \in A(\Gamma)$. Let $u=u_{1} u_{2} u_{3}$ be the geodesic decomposition for $g_{1}^{u}$ as in Theorem [3.9. Since $u_{1} \rightleftharpoons g_{1}$, we may assume $u_{1}=1$. Since $u_{3}^{-1} g_{1}^{u_{2}} u_{3}$ is a geodesic decomposition of $g_{2}$ and since $\left\|g_{2}\right\|=\left\|g_{1}\right\|=\left\|g_{1}^{u_{2}}\right\|$, we have $u_{3}=1$. Therefore $u=u_{2}$, hence $g_{1}$ is cyclically conjugate to $g_{2}$.

## 4. Star length

Star lengths of elements of $A(\Gamma)$, introduced in [18], induce a metric $d_{*}$ on $A(\Gamma)$ such that the metric space $\left(A(\Gamma), d_{*}\right)$ is quasi-isometric to the extension graph $\left(\Gamma^{e}, d\right)$, preserving the right action of $A(\Gamma)$. In this section, we study basic properties of star lengths.

It is known that the centralizer $Z(v)$ of $v \in V(\Gamma)$ in $A(\Gamma)$ is generated by the vertices in $\mathrm{St}_{\Gamma}(v)$.
Definition 4.1 (star-word, star length). An element in the centralizer $Z(v)$ of some vertex $v$ is called a star-word. The star length of $g \in A(\Gamma)$, denoted $\|g\|_{*}$, is the minimum $\ell$ such that $g$ is written as a product of $\ell$ star-words. Let $d_{*}$ denote the right-invariant metric on $A(\Gamma)$ induced by the star length: $d_{*}\left(g_{1}, g_{2}\right)=\left\|g_{1} g_{2}^{-1}\right\|_{*}$.

The following example illustrates that the decompositions into star-words are not unique.
Example 4.2. Let $\Gamma=\bar{P}_{5}$, where $P_{5}=\left(v_{1}, \ldots, v_{5}\right)$ is a path graph, and let the underlying rightangled Artin group here be $A(\Gamma)$, hence $v_{i} v_{j}=v_{j} v_{i}$ whenever $|i-j| \geqslant 2$. Let $g=v_{1} v_{3} v_{5} v_{2} v_{4}$. The following shows various decompositions of $g$ into two star-words.

$$
\begin{aligned}
g & =\left(v_{1} v_{3} v_{5}\right)\left(v_{2} v_{4}\right)=\left(v_{1} v_{3} v_{5} v_{2}\right)\left(v_{4}\right)=\left(v_{1} v_{3} v_{5} v_{4}\right)\left(v_{2}\right) \\
& =\left(v_{1} v_{3} v_{2}\right)\left(v_{5} v_{4}\right)=\left(v_{3} v_{5} v_{4}\right)\left(v_{1} v_{2}\right)=\left(v_{3} v_{5}\right)\left(v_{4} v_{1} v_{2}\right) .
\end{aligned}
$$

Notice that all the parenthesized words are star-words. For example, $v_{1} v_{3} v_{5} \in Z\left(v_{i}\right)$ for $i=1,3,5$, $v_{4} \in Z\left(v_{i}\right)$ for $i=1,2,4, v_{1} v_{3} v_{5} v_{2} \in Z\left(v_{5}\right)$ and so on. Since $\operatorname{supp}(g)=\left\{v_{1}, \ldots, v_{5}\right\}$ is not contained in $\operatorname{St}\left(v_{i}\right)$ for any $1 \leqslant i \leqslant 5$, we have $\|g\|_{*}=2$.

The group $A(\Gamma)$ acts on $\left(A(\Gamma), d_{*}\right)$ by right multiplication $w \mapsto w g$. Recall that $A(\Gamma)$ acts on $\left(\Gamma^{e}, d\right)$ by conjugation $v^{w} \mapsto v^{w g}$. For any $v \in V(\Gamma)$, the following map is equivariant.

$$
\phi_{v}: A(\Gamma) \rightarrow \Gamma^{e}, \quad \phi_{v}(w)=v^{w}
$$

Lemma 4.3. [18, Lemma 19] Let $\Gamma$ be connected and let $D=\operatorname{diam}(\Gamma)$. The following holds between the metric $d$ on $\Gamma^{e}$ and the star length $\|\cdot\|_{*}$ on $A(\Gamma)$ : for any $g \in A(\Gamma)$ and $v \in V(\Gamma)$,

$$
\|g\|_{*}-1 \leqslant d\left(v^{g}, v\right) \leqslant D\left(\|g\|_{*}+1\right) .
$$

Notice that $d\left(\phi_{v}(g), \phi_{v}(h)\right)=d\left(v^{g}, v^{h}\right)=d\left(v^{g h^{-1}}, v\right)$ and $d_{*}(g, h)=\left\|g h^{-1}\right\|_{*}$. Therefore the above lemma implies that $d_{*}(g, h)-1 \leqslant d\left(v^{g}, v^{h}\right) \leqslant D\left(d_{*}(g, h)+1\right)$, and hence that $\phi_{v}$ is a quasi-isometry. The above lemma also yields the following corollary for the asymptotic translation length.

Corollary 4.4. Let $\Gamma$ be connected and let $D=\operatorname{diam}(\Gamma)$. For every $g \in A(\Gamma)$,

$$
\tau_{\left(A(\Gamma), d_{*}\right)}(g) \leqslant \tau_{\left(\Gamma^{e}, d\right)}(g) \leqslant D \tau_{\left(A(\Gamma), d_{*}\right)}(g)
$$

Proof. Notice that

$$
\begin{aligned}
\tau_{\left(A(\Gamma), d_{*}\right)}(g) & =\lim _{n \rightarrow \infty} \frac{d_{*}\left(g^{n}, 1\right)}{n}=\lim _{n \rightarrow \infty} \frac{\left\|g^{n}\right\|_{*}}{n} \\
\tau_{\left(\Gamma^{e}, d\right)}(g) & =\lim _{n \rightarrow \infty} \frac{d\left(v^{g^{n}}, v\right)}{n}
\end{aligned}
$$

where $v$ is any vertex of $\Gamma$. By Lemma 4.3,

$$
\frac{\left\|g^{n}\right\|_{*}-1}{n} \leqslant \frac{d_{*}\left(v^{g^{n}}, v\right)}{n} \leqslant \frac{D\left(\left\|g^{n}\right\|_{*}+1\right)}{n}
$$

By taking $n$ to infinity, we get the desired inequalities.
The following lemma shows basic properties of star length.
Lemma 4.5. Let $g_{1}, g_{2}, g_{3}, g, h \in A(\Gamma)$.
(i) If $g_{1} g_{2} g_{3}$ is geodesic, then $\left\|g_{1} g_{3}\right\|_{*} \leqslant\left\|g_{1} g_{2} g_{3}\right\|_{*}$. In particular, if $g \leqslant_{L} h$ or $g \leqslant_{R} h$, then $\|g\|_{*} \leqslant\|h\|_{*}$.
(ii) $\left\|g^{m}\right\|_{*} \leqslant\left\|g^{n}\right\|_{*}$ for all $1 \leqslant m \leqslant n$.
(iii) If $g \rightleftharpoons h$ and $h \neq 1$, then $\|g\|_{*} \leqslant 1$.

Proof. Let us denote $g \preccurlyeq{ }_{0} h$ if a reduced word representing $g$ can be obtained by deleting some letters from a reduced word representing $h$. For example, if $v_{i}$ 's are distinct vertices, then $v_{1} v_{3} \preccurlyeq_{0} v_{1} v_{2} v_{3} v_{4}$. It is proved in [18, Lemma 20(i)] that if $g \preccurlyeq_{0} h$, then $\|g\|_{*} \leqslant\|h\|_{*}$.
(i) Since $g_{1} g_{2} g_{3}$ is geodesic, we have $g_{1} g_{3} \preccurlyeq 0 g_{1} g_{2} g_{3}$, hence $\left\|g_{1} g_{3}\right\|_{*} \leqslant\left\|g_{1} g_{2} g_{3}\right\|_{*}$.
(ii) Let $g=u^{-1} h u$ be geodesic such that $h$ is cyclically reduced. Then $g^{k}=u^{-1} \underbrace{h \cdots h}_{k} u$ is also geodesic for all $k \geqslant 1$ (by Lemma [2.8(iii)). Therefore $g^{m} \preccurlyeq 0 g^{n}$, hence $\left\|g^{m}\right\|_{*} \leqslant\left\|g^{n}\right\|_{*}$.
(iii) Since $h \neq 1$, there is a vertex $v \in \operatorname{supp}(h)$. Then $g \in Z(v)$, namely $\|g\|_{*} \leqslant 1$.

Lemma 4.6. Suppose that $g_{1}, g_{2} \in A(\Gamma)$ have a common right multiple and that none of them is a prefix of the other, i.e. $g_{1} \not{ }_{L} g_{2}$ and $g_{2} \not{ }_{L} g_{1}$. Then $\left\|g_{1}^{-1} g_{2}\right\|_{*} \leqslant 2$ and $\left\|g_{1}\right\|_{*}-\left\|g_{2}\right\|_{*} \in\{0, \pm 1\}$.

Proof. Let $g_{i}=\left(g_{1} \wedge_{L} g_{2}\right) g_{i}^{\prime}$ for $i=1,2$. Since $g_{1}$ and $g_{2}$ have a common right multiple, $g_{1}^{\prime} \rightleftharpoons g_{2}^{\prime}$ (by Theorem 2.12). Since $g_{1} \not \star_{L} g_{2}$ and $g_{2} \not \star_{L} g_{1}$, both $g_{1}^{\prime}$ and $g_{2}^{\prime}$ are nontrivial, hence $\left\|g_{1}^{\prime}\right\|_{*}=\left\|g_{2}^{\prime}\right\|_{*}=1$ (by Lemma 4.5(iii)). Therefore

$$
\left\|g_{1}^{-1} g_{2}\right\|_{*}=\left\|g_{1}^{\prime-1} g_{2}^{\prime}\right\|_{*} \leqslant\left\|g_{1}^{\prime}\right\|_{*}+\left\|g_{2}^{\prime}\right\|_{*}=1+1=2
$$

Furthermore, for each $i=1,2$,

$$
\left\|g_{1} \wedge_{L} g_{2}\right\|_{*} \leqslant\left\|g_{i}\right\|_{*} \leqslant\left\|g_{1} \wedge_{L} g_{2}\right\|_{*}+\left\|g_{i}^{\prime}\right\|_{*}=\left\|g_{1} \wedge_{L} g_{2}\right\|_{*}+1
$$

hence $\left\|g_{i}\right\|_{*}=\left\|g_{1} \wedge_{L} g_{2}\right\|_{*}+\epsilon_{i}$, where $\epsilon_{i} \in\{0,1\}$. Therefore $\left\|g_{1}\right\|_{*}-\left\|g_{2}\right\|_{*}=\epsilon_{1}-\epsilon_{2} \in\{0, \pm 1\}$.
Corollary 4.7. Let $g_{1}, g_{2}, h \in A(\Gamma)$ with $g_{1} g_{2}$ geodesic. If $h \leqslant{ }_{L} g_{1} g_{2}$ and $\left\|g_{1}\right\|_{*} \geqslant\|h\|_{*}+2$, then $h \leqslant_{L} g_{1}$.

Proof. Observe that $g_{1} \not{ }_{L} h$ (otherwise $\left\|g_{1}\right\|_{*} \leqslant\|h\|_{*}$ ). Assume $h \not \mathbb{k}_{L} g_{1}$. Since $g_{1}$ and $h$ have a common right multiple, say $g_{1} g_{2}$, we have $\left\|g_{1}\right\|_{*}-\|h\|_{*} \in\{0, \pm 1\}$ (by Lemma 4.6). This contradicts that $\left\|g_{1}\right\|_{*} \geqslant\|h\|_{*}+2$.

Corollary 4.8. Let $g_{1}, g_{2} \in A(\Gamma)$. If $g_{1} g_{2}$ is geodesic, then

$$
\left\|g_{1}\right\|_{*}+\left\|g_{2}\right\|_{*}-2 \leqslant\left\|g_{1} g_{2}\right\|_{*} \leqslant\left\|g_{1}\right\|_{*}+\left\|g_{2}\right\|_{*}
$$

Proof. Let $r=\left\|g_{1}\right\|_{*}, s=\left\|g_{2}\right\|_{*}$ and $t=\left\|g_{1} g_{2}\right\|_{*}$. Then it is obvious that $t \leqslant r+s$, hence it suffices to show $t \geqslant r+s-2$. Since $g_{2} \leqslant R g_{1} g_{2}$, we have $t=\left\|g_{1} g_{2}\right\|_{*} \geqslant\left\|g_{2}\right\|_{*}=s$ (by Lemma 4.5). We may assume $r \geqslant 3$ because otherwise $t \geqslant s \geqslant r+s-2$. Let

$$
g_{1} g_{2}=w_{1} w_{2} \cdots w_{t}
$$

be a geodesic decomposition of $g_{1} g_{2}$ into star-words. Then $w_{1} \cdots w_{r-2} \leqslant L g_{1}$ (by Corollary 4.7), hence $g_{2} \leqslant R w_{r-1} \cdots w_{t}$. Therefore $s=\left\|g_{2}\right\|_{*} \leqslant\left\|w_{r-1} \cdots w_{t}\right\|_{*}=t-r+2$, namely $t \geqslant r+s-2$.

The following example shows that the upper and lower bounds in the above corollary are sharp.
Example 4.9. Let $\Gamma=\bar{P}_{5}$, where $P_{5}=\left(v_{1}, \ldots, v_{5}\right)$, and let the underlying right-angled Artin group here be $A(\Gamma)$.
(i) Let $g_{1}=v_{1} v_{2}$ and $g_{2}=v_{3} v_{4}$. Then $g_{1} \rightleftharpoons v_{5}$ and $g_{2} \rightleftharpoons v_{1}$ and hence $\left\|g_{1}\right\|_{*}=\left\|g_{2}\right\|_{*}=1$. Since $g_{1} g_{2}=v_{1} v_{2} v_{3} v_{4} \notin Z\left(v_{i}\right)$ for any $1 \leqslant i \leqslant 5$, we have $\left\|g_{1} g_{2}\right\|_{*} \geqslant 2$. Since $\left\|g_{1} g_{2}\right\|_{*} \leqslant\left\|g_{1}\right\|_{*}+\left\|g_{2}\right\|_{*}=2$, we have $\left\|g_{1} g_{2}\right\|_{*}=\left\|g_{1}\right\|_{*}+\left\|g_{2}\right\|_{*}$ in this case.
(ii) Let $g_{1}=g_{2}=v_{2} v_{3} v_{4}$. Then $g_{1} g_{2}=v_{2} v_{3} v_{4} \cdot v_{2} v_{3} v_{4}=v_{2} v_{3} v_{2} \cdot v_{4} v_{3} v_{4}$. Since $v_{2} v_{3} v_{2} \in Z\left(v_{5}\right)$ and $v_{4} v_{3} v_{4} \in Z\left(v_{1}\right)$, we have $\left\|v_{2} v_{3} v_{2}\right\|_{*}=\left\|v_{4} v_{3} v_{4}\right\|_{*}=1$. It is easy to see that $\left\|g_{1}\right\|_{*}=\left\|g_{2}\right\|_{*}=\left\|g_{1} g_{2}\right\|_{*}=2$. Therefore $\left\|g_{1} g_{2}\right\|_{*}=\left\|g_{1}\right\|_{*}+\left\|g_{2}\right\|_{*}-2$ in this case.

The following is an immediate consequence of Lemma 4.5(ii) and Corollary 4.8,
Corollary 4.10. Let $g \in A(\Gamma)$ be cyclically reduced. Then $\left\{\left\|g^{n}\right\|_{*}\right\}_{n=0}^{\infty}$ is an increasing sequence such that the following hold.
(i) If $\|g\|_{*}=1$, then $\left\|g^{n}\right\|_{*}=1$ for all $n \geqslant 1$.
(ii) If $\|g\|_{*}=2$, then $\left\|g^{n-1}\right\|_{*} \leqslant\left\|g^{n}\right\|_{*} \leqslant\left\|g^{n-1}\right\|_{*}+2$ for all $n \geqslant 1$.
(iii) If $\|g\|_{*} \geqslant 3$, then $\left\|g^{n}\right\|_{*} \geqslant\left\|g^{n-1}\right\|_{*}+1$ and hence $\left\|g^{n}\right\|_{*} \geqslant n+2$ for all $n \geqslant 1$.

Corollary 4.11. Let $g, u \in A(\Gamma)$ with $g$ cyclically reduced. If $\|g\|_{*} \geqslant 3$ and $g \not \mathbb{k}_{L} u \leqslant_{L} g^{n}$ for some $n \geqslant 1$, then $u \leqslant_{L} g^{2}$.

Proof. We may assume $n \geqslant 3$ and $u \star_{L} g$ (otherwise it is obvious). Since $g$ and $u$ have a common right multiple, say $g^{n}$, there exist $g^{\prime}$ and $u^{\prime}$ such that $g g^{\prime}=u u^{\prime}=g \vee_{L} u \leqslant_{L} g^{n}$ and $u^{\prime} \rightleftharpoons g^{\prime}$ (by Theorem (2.12), where $g g^{\prime}$ and $u u^{\prime}$ are geodesic. Since $g g^{\prime} \leqslant L g^{n}=g g^{n-1}$, (by Lemma 2.6)

$$
g^{\prime} \leqslant L g^{n-1}=g \cdot g^{n-2} .
$$

Since $u \not \mathbb{k}_{L} g$ and $g \not \mathbb{k}_{L} u$, both $g^{\prime}$ and $u^{\prime}$ are nontrivial, hence $\left\|g^{\prime}\right\|_{*}=\left\|u^{\prime}\right\|_{*}=1$. Since $\left\|g^{\prime}\right\|_{*}=1$ and $\|g\|_{*} \geqslant 3$, we get $g^{\prime} \leqslant_{L} g$ (by Corollary 4.7). Therefore $u \leqslant_{L} u u^{\prime}=g g^{\prime} \leqslant_{L} g^{2}$.

Lemma 4.12. Let $g_{1}, g_{2}, g_{3} \in A(\Gamma)$ be such that both $g_{1} g_{2}$ and $g_{2} g_{3}$ are geodesic. If $\left\|g_{2}\right\|_{*} \geqslant 2$, then $g_{1} g_{2} g_{3}$ is geodesic.

| $g \in A(\Gamma)$ | $\Gamma$ | $\bar{\Gamma}$ |
| :--- | :--- | :--- |
| $g$ is split | $\Gamma[g]$ is a join | $\bar{\Gamma}[g]$ is disconnected |
| $g$ is non-split | $\Gamma[g]$ is not a join | $\bar{\Gamma}[g]$ is connected |
| $g$ is strongly non-split | $\Gamma[g]$ is not contained in | $\bar{\Gamma}[g]$ is connected and |
|  | a subjoin of $\Gamma$ | $\mathrm{St}_{\bar{\Gamma}}(\operatorname{supp}(g))=V(\Gamma)$ |

Table 1. Equivalent conditions for $g \in A(\Gamma)$ to be split, non-split and strongly non-split

Proof. Assume that $g_{1} g_{2} g_{3}$ is not geodesic. Since $g_{1} g_{2}$ and $g_{2} g_{3}$ are geodesic, there exists $x \in V(\Gamma)^{ \pm 1}$ such that $x^{-1} \leqslant R g_{1}, x \leqslant_{L} g_{3}$ and $x \rightleftharpoons g_{2}$ (by Lemma 2.8(i)). Observe that $x \rightleftharpoons g_{2}$ implies $\left\|g_{2}\right\|_{*} \leqslant 1$, which contradicts the hypothesis $\left\|g_{2}\right\|_{*} \geqslant 2$.

We introduce the notion of strongly non-split elements. We will see (in Lemma 6.3 and Remark 6.4) that if $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected, then a cyclically reduced element $g \in A(\Gamma)$ is strongly non-split if and only if $g$ is loxodromic on the extension graph $\Gamma^{e}$.

Definition 4.13 (non-split, strongly non-split). Let $g \in A(\Gamma) \backslash\{1\}$.
(i) $g$ is called split if $g$ has a nontrivial geodesic decomposition $g=g_{1} g_{2}$ with $g_{1} \rightleftharpoons g_{2}$.
(ii) $g$ is called non-split if it is not split.
(iii) $g$ is called strongly non-split if $g$ is non-split and $g \neq v$ for any $v \in V(\Gamma)$.

It is easy to see that $g \in A(\Gamma)$ is split if and only if $\Gamma[g]$ is a join (equivalently, $\bar{\Gamma}[g]$ is disconnected). Similarly, one can characterize the property of being non-split and strongly non-split using the graphs $\Gamma[g]$ and $\bar{\Gamma}[g]$ as shown in Table 1.

From definition, the existence of a strongly non-split element implies that $\bar{\Gamma}$ is connected.
Remark 4.14. Let $n \geqslant 2$ and $g, h \in A(\Gamma) \backslash\{1\}$. Observe that strongly non-splitness of an element depends only on its support. Note that $\operatorname{supp}\left(g^{-1}\right)=\operatorname{supp}(g)=\operatorname{supp}\left(g^{n}\right)$ (by Lemma 3.4), and that if either $g \leqslant_{L} h$ or $g \leqslant_{R} h$, then $\operatorname{supp}(g) \subset \operatorname{supp}(h)$. Therefore
(i) $g$ is strongly non-split if and only if $g^{-1}$ is strongly non-split;
(ii) $g$ is strongly non-split if and only if $g^{n}$ is strongly non-split;
(iii) if $g$ is strongly non-split and either $g \leqslant_{L} h$ or $g \leqslant_{R} h$, then $h$ is also strongly non-split.

Strongly non-splitness is related to the star length as follows.
Lemma 4.15. Let $g \in A(\Gamma) \backslash\{1\}$.
(i) If $\|g\|_{*} \geqslant 3$, then $g$ is strongly non-split.
(ii) $g$ is strongly non-split with $|\operatorname{supp}(g)| \geqslant 2$ if and only if $g$ is non-split with $\|g\|_{*} \geqslant 2$.

Proof. (i) Assume that $g$ is not strongly non-split. If $g$ is split, then clearly $\|g\|_{*} \leqslant 2$. If $g$ is non-split but not strongly non-split, then there is $v \in V(\Gamma) \backslash \operatorname{supp}(g)$ with $v \rightleftharpoons g$, hence $\|g\|_{*}=1$. In either case, $\|g\|_{*} \leqslant 2$.
(ii) Suppose that $g$ is strongly non-split with $|\operatorname{supp}(g)| \geqslant 2$. Then $g$ is non-split by definition. Assume $\|g\|_{*}=1$. Then there exists $v \in V(\Gamma)$ with $\operatorname{supp}(g) \subset Z(v)$. Since $g$ is strongly non-split, $v \in \operatorname{supp}(g)$. Since $|\operatorname{supp}(g)| \geqslant 2$ and $\operatorname{supp}(g) \subset Z(v), g=v^{n} g_{1}$ is geodesic for some $n \neq 0$ and $g_{1} \in A(\Gamma) \backslash\{1\}$ with $g_{1} \rightleftharpoons v$. Namely, $g$ is split, which is a contradiction. Therefore $\|g\|_{*} \geqslant 2$.

Conversely, suppose that $g$ is non-split with $\|g\|_{*} \geqslant 2$. Then $|\operatorname{supp}(g)| \geqslant 2$ and there does not exit $v \in V(\Gamma) \backslash \operatorname{supp}(g)$ with $v \rightleftharpoons g$. Therefore $g$ is strongly non-split.

## 5. Prefixes of powers of cyclically Reduced elements

In this section, we study prefixes of powers of cyclically reduced elements. The main result is Theorem 5.3, which plays important roles in the study of the asymptotic translation length and the acylindricity of the action of $A(\Gamma)$ on $\Gamma^{e}$.

Lemma 5.1. Let $u, g_{1}, g_{2}, \ldots, g_{m} \in A(\Gamma)$. If $g_{1} g_{2} \cdots g_{m}$ is geodesic, then for each $1 \leqslant k \leqslant m$ there exists a geodesic decomposition $g_{k}=a_{k} b_{k}$ such that
(i) $u \wedge_{L}\left(g_{1} \cdots g_{k}\right)=a_{1} \cdots a_{k}$;
(ii) $a_{k} \rightleftharpoons b_{j}$ for all $1 \leqslant j \leqslant k-1$;
(iii) $a_{1} \cdots a_{k} b_{1} \ldots b_{k}$ is a geodesic decomposition of $g_{1} \cdots g_{k}$.

Proof. The relation $u \wedge_{L}\left(g_{1} \cdots g_{k}\right)=a_{1} \cdots a_{k}$ determines the elements $a_{k}$ inductively for $k=1, \ldots, m$. Then the relation $g_{k}=a_{k} b_{k}$ determines the elements $b_{k}$ for all $1 \leqslant k \leqslant m$. Therefore we get elements $a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{m}$ such that $u \wedge_{L}\left(g_{1} \cdots g_{k}\right)=a_{1} \cdots a_{k}$ and $g_{k}=a_{k} b_{k}$ for all $1 \leqslant k \leqslant m$.

Since $g_{1} \cdots g_{m}$ is geodesic, $g_{1} \cdots g_{k} \leqslant L g_{1} \cdots g_{k+1}$ for each $1 \leqslant k \leqslant m-1$ (by Lemma 2.6), hence

$$
a_{1} \cdots a_{k}=u \wedge_{L}\left(g_{1} \cdots g_{k}\right) \leqslant_{L} u \wedge_{L}\left(g_{1} \cdots g_{k+1}\right)=a_{1} \cdots a_{k+1}
$$

Therefore $a_{1} \cdots a_{m}$ and hence each $a_{1} \cdots a_{k}$ are geodesic (by Lemma 2.6 again).
For each $1 \leqslant k \leqslant m$, let $u_{k} \in A(\Gamma)$ be the element such that $u=a_{1} \cdots a_{k} u_{k}$. Then each $a_{1} \cdots a_{k} u_{k}$ is geodesic because $a_{1} \cdots a_{k} \leqslant_{L} u$.

Claim. For each $1 \leqslant k \leqslant m$,
(a) $a_{k} \leqslant L g_{k}$, hence $g_{k}=a_{k} b_{k}$ is geodesic;
(b) $a_{k} \rightleftharpoons b_{j}$ for all $1 \leqslant j \leqslant k-1$;
(c) $a_{1} \cdots a_{k} b_{1} \cdots b_{k}$ is a geodesic decomposition of $g_{1} \cdots g_{k}$.

Proof of Claim. We use induction on $k$.
For $k=1$, (a) and (c) hold because $a_{1}=u \wedge_{L} g_{1} \leqslant_{L} g_{1}$ and $g_{1}=a_{1} b_{1}$, and (b) is vacuously true.
Assume that the claim holds for some $1 \leqslant k<m$. We now have the following geodesic decompositions at hand:

$$
\begin{aligned}
u & =\left(a_{1} \cdots a_{k}\right) u_{k} \\
g_{1} \cdots g_{k} & =\left(a_{1} \cdots a_{k}\right)\left(b_{1} \cdots b_{k}\right) \\
g_{1} \cdots g_{k+1} & =\left(a_{1} \cdots a_{k}\right)\left(b_{1} \cdots b_{k}\right) g_{k+1}
\end{aligned}
$$

Since $u \wedge_{L}\left(g_{1} \cdots g_{k}\right)=a_{1} \cdots a_{k}$, we have $u_{k} \wedge_{L}\left(b_{1} \cdots b_{k}\right)=1$ (by Lemma 2.14).
Since $u \wedge_{L}\left(g_{1} \cdots g_{k+1}\right)=a_{1} \cdots a_{k+1}$, we have $a_{k+1}=u_{k} \wedge_{L}\left(b_{1} \cdots b_{k} g_{k+1}\right)$, hence

$$
a_{k+1} \leqslant L \quad u_{k} \quad \text { and } \quad a_{k+1} \leqslant L\left(b_{1} \cdots b_{k}\right) g_{k+1}
$$

Since $a_{k+1} \leqslant_{L} u_{k}$, we have $a_{k+1} \wedge_{L}\left(b_{1} \cdots b_{k}\right) \leqslant_{L} u_{k} \wedge_{L}\left(b_{1} \cdots b_{k}\right)=1$ (by Lemma 2.14).
Since $a_{k+1} \leqslant_{L}\left(b_{1} \cdots b_{k}\right) g_{k+1}$ and $a_{k+1} \wedge_{L}\left(b_{1} \cdots b_{k}\right)=1$, we have

$$
a_{k+1} \rightleftharpoons b_{1} \cdots b_{k} \quad \text { and } \quad a_{k+1} \leqslant_{L} g_{k+1}
$$

(by Lemma 2.11 (ii)). In particular, $a_{k+1} \rightleftharpoons b_{j}$ for all $1 \leqslant j \leqslant k$. Therefore (a) and (b) hold for $k+1$. Since $a_{k+1} \rightleftharpoons b_{j}$ for all $1 \leqslant j \leqslant k$, we have

$$
\begin{aligned}
g_{1} \cdots g_{k} g_{k+1} & =\left(a_{1} \cdots a_{k}\right)\left(b_{1} \cdots b_{k}\right)\left(a_{k+1} b_{k+1}\right) \\
& =\left(a_{1} \cdots a_{k+1}\right)\left(b_{1} \cdots b_{k+1}\right)
\end{aligned}
$$

The above three decompositions are all geodesic because $g_{1} \cdots g_{k} g_{k+1}, g_{k+1}=a_{k+1} b_{k+1}$ and $g_{1} \cdots g_{k}=$ $a_{1} \cdots a_{k} b_{1} \cdots b_{k}$ are all geodesic. Therefore (c) holds for $k+1$.

The above claim completes the proof.
In the following, we frequently use the notation $\operatorname{St}_{\bar{\Gamma}[g]}(X)$, for $X \subset \operatorname{supp}(g)$, which denotes the star of $X$ in $\bar{\Gamma}[g]=\bar{\Gamma}[\operatorname{supp}(g)]$. Hence, $v \in \operatorname{St}_{\bar{\Gamma}}[g](X)$ if and only if either $v \in X$ or $v \in \operatorname{supp}(g)$ and $\left\{v, v_{1}\right\}$ is an edge in $\bar{\Gamma}$ for some $v_{1} \in X$. Therefore

$$
\left.\operatorname{St}_{\bar{\Gamma}}^{[g]} \text { ( } X\right)=\operatorname{St}_{\bar{\Gamma}}(X) \cap \operatorname{supp}(g) .
$$

When $g_{1}=\cdots=g_{m}$ in Lemma 5.1, we have the following.
Corollary 5.2. Let $m \geqslant 1$ and $g, u \in A(\Gamma)$ with $g$ cyclically reduced. Then for each $1 \leqslant k \leqslant m$ there exists a geodesic decomposition $g=a_{k} b_{k}$ such that
(i) $u \wedge_{L} g^{k}=a_{1} a_{2} \cdots a_{k}, a_{k} \rightleftharpoons b_{j}$ for all $1 \leqslant j<k$ and $a_{1} \cdots a_{k} b_{1} \ldots b_{k}$ is a geodesic decomposition of $g^{k}$;
(ii) $\left\{a_{k}\right\}_{k=1}^{m}$ is descending with respect to $\leqslant_{L}$ such that

$$
\begin{gathered}
1 \leqslant_{L} a_{m} \leqslant_{L} \cdots \leqslant_{L} a_{2} \leqslant_{L} a_{1} \leqslant_{L} g \\
\operatorname{St}_{\bar{\Gamma}[g]}\left(\operatorname{supp}\left(a_{k+1}\right)\right) \subset \operatorname{supp}\left(a_{k}\right)
\end{gathered}
$$

(iii) $\left\{b_{k}\right\}_{k=1}^{m}$ is ascending with respect to $\leqslant_{R}$ such that

$$
\begin{gathered}
1 \leqslant_{R} b_{1} \leqslant_{R} b_{2} \leqslant_{R} \cdots \leqslant_{R} b_{m} \leqslant_{R} g, \\
\operatorname{St}_{\bar{\Gamma}[g]}\left(\operatorname{supp}\left(b_{k}\right)\right) \subset \operatorname{supp}\left(b_{k+1}\right) .
\end{gathered}
$$

In (ii) and (iii), we let $a_{m+1}=1$ and $b_{m+1}=g$ for notational convenience.

Proof. Since $g$ is cyclically reduced, $g g \cdots g$ is geodesic. By Lemma 5.1, there exists a geodesic decomposition $g=a_{k} b_{k}$ for $1 \leqslant k \leqslant m$ satisfying (i).

Since $g=a_{k} b_{k}=a_{k+1} b_{k+1}$, we have $a_{k+1} \leqslant_{L} a_{k} b_{k}$. Since $a_{k+1} \rightleftharpoons b_{k}$, we have $a_{k+1} \leqslant_{L} a_{k}$ (by Lemma 2.15(iv)), hence $\left\{a_{k}\right\}_{k=1}^{m}$ is descending with respect to $\leqslant_{L}$. Since $a_{1} \leqslant_{L} g$ and $1 \leqslant_{L} a_{m}$,

$$
1 \leqslant_{L} a_{m} \leqslant_{L} \cdots \leqslant_{L} a_{2} \leqslant_{L} a_{1} \leqslant_{L} g .
$$

Since $g=a_{k} b_{k}$, it follows immediately from the above inequalities that the sequence $\left\{b_{k}\right\}_{k=1}^{m}$ is ascending with respect to $\leqslant_{R}$ such that

$$
1 \leqslant_{R} b_{1} \leqslant_{R} b_{2} \leqslant_{R} \cdots \leqslant_{R} b_{m} \leqslant_{R} g .
$$

Since $\operatorname{supp}(g)=\operatorname{supp}\left(a_{j}\right) \cup \operatorname{supp}\left(b_{j}\right)$ for $j=k, k+1$,

$$
\begin{aligned}
\operatorname{supp}(g)-\operatorname{supp}\left(a_{k+1}\right) & \subset \operatorname{supp}\left(b_{k+1}\right) \\
\operatorname{supp}(g)-\operatorname{supp}\left(b_{k}\right) & \subset \operatorname{supp}\left(a_{k}\right)
\end{aligned}
$$

Since $a_{k+1} \rightleftharpoons b_{k}$, by Lemma 2.3

$$
\begin{aligned}
& \operatorname{supp}\left(b_{k}\right) \cap \operatorname{St}_{\bar{\Gamma}}\left(\operatorname{supp}\left(a_{k+1}\right)\right)=\emptyset, \\
& \operatorname{supp}\left(a_{k+1}\right) \cap \operatorname{St}_{\bar{\Gamma}}\left(\operatorname{supp}\left(b_{k}\right)\right)=\emptyset .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathrm{St}_{\bar{\Gamma}[g]}\left(\operatorname{supp}\left(a_{k+1}\right)\right) & =\operatorname{supp}(g) \cap \mathrm{St}_{\bar{\Gamma}}\left(\operatorname{supp}\left(a_{k+1}\right)\right) \\
& \subset \operatorname{supp}(g)-\operatorname{supp}\left(b_{k}\right) \subset \operatorname{supp}\left(a_{k}\right) \\
\mathrm{St}_{\bar{\Gamma}[g]}\left(\operatorname{supp}\left(b_{k}\right)\right) & =\operatorname{supp}(g) \cap \mathrm{St}_{\bar{\Gamma}}\left(\operatorname{supp}\left(b_{k}\right)\right) \\
& \subset \operatorname{supp}(g)-\operatorname{supp}\left(a_{k+1}\right) \subset \operatorname{supp}\left(b_{k+1}\right) .
\end{aligned}
$$

Therefore (ii) and (iii) are proved.
Theorem 5.3. Let $m \geqslant 2$ and $g, u \in A(\Gamma)$ with $g$ cyclically reduced and non-split. If

$$
g \not{ }_{L} u \nless_{L} g^{m-1} \quad \text { and } \quad u \leqslant_{L} g^{m},
$$

then the following hold.
(i) $m \leqslant \operatorname{diam}(\bar{\Gamma}[g])$. In particular, $m \leqslant|\operatorname{supp}(g)|-1 \leqslant|V(\Gamma)|-1$.
(ii) There is a geodesic decomposition $g=g_{m} g_{m-1} \cdots g_{1} g_{0}$ such that
(a) $g_{k} \neq 1$ for all $0 \leqslant k \leqslant m$;
(b) $g_{i} \rightleftharpoons g_{j}$ whenever $|i-j| \geqslant 2$;
(c) $u \wedge_{L} g^{k}=\left(g_{m} \cdots g_{1}\right)\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m} \cdots g_{k}\right)$ for all $1 \leqslant k \leqslant m$.

In particular, $u=u \wedge_{L} g^{m}=\left(g_{m} \cdots g_{1}\right)\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m} g_{m-1}\right)\left(g_{m}\right)$.
(iii) $\|u\|_{*} \leqslant\|g\|_{*}+1$.
(iv) If $\|g\|_{*} \geqslant 3$, then $m=2$. (Equivalently, if $m \geqslant 3$, then $\|g\|_{*} \leqslant 2$.)

Proof. For each $k \geqslant 1$, let $g=a_{k} b_{k}$ be the geodesic decomposition given by Corollary 5.2. Then

- $u \wedge_{L} g^{k}=a_{1} \cdots a_{k}$ is geodesic and $a_{k} \rightleftharpoons b_{j}$ for all $1 \leqslant j<k ;$
- $\left\{a_{k}\right\}_{k=1}^{\infty}$ is descending with respect to $\leqslant_{L}$ such that $1 \leqslant_{L} \cdots \leqslant_{L} a_{2} \leqslant_{L} a_{1} \leqslant_{L} g$;
- $\left\{b_{k}\right\}_{k=1}^{\infty}$ is ascending with respect to $\leqslant_{R}$ such that $1 \leqslant_{R} b_{1} \leqslant_{R} b_{2} \leqslant_{R} \cdots \leqslant_{R} g$.

The following claim is a result of the hypothesis that $g \not ষ_{L} u \not ڭ_{L} g^{m-1}$ and $u \leqslant_{L} g^{m}$.
Claim 1. For each $1 \leqslant k \leqslant m, a_{k} \notin\left\{1, g, a_{k+1}\right\}$ and hence $b_{k} \notin\left\{1, g, b_{k+1}\right\}$. For each $k>m, a_{k}=1$ and hence $b_{k}=g$.

Proof of Claim 1. For each $k \geqslant 1, a_{1} \cdots a_{k} \leqslant_{L} u$ and $a_{1} \cdots a_{k} \leqslant_{L} g^{k}$ (because $a_{1} \cdots a_{k}=u \wedge_{L} g^{k}$ ). Furthermore, $u=u \wedge_{L} g^{m}=a_{1} \cdots a_{m}$ (from the hypothesis $u \leqslant_{L} g^{m}$ ). Therefore

$$
a_{1} \leqslant_{L} u, \quad a_{1} \cdots a_{m-1} \leqslant_{L} g^{m-1}, \quad a_{1} \cdots a_{m}=u
$$

If $a_{1}=g$, then $g \leqslant_{L} u$, which contradicts the hypothesis $g \not k_{L} u$. If $a_{m}=1$, then $u=$ $a_{1} \cdots a_{m-1} a_{m}=a_{1} \cdots a_{m-1} \leqslant_{L} g^{m-1}$, which contradicts the hypothesis $u \not k_{L} g^{m-1}$. Thus $a_{1} \neq g$ and $a_{m} \neq 1$. Therefore, for each $1 \leqslant k \leqslant m$, we get $a_{k} \notin\{1, g\}$ (because $1 \leqslant_{L} a_{m} \leqslant_{L} a_{k} \leqslant L a_{1} \leqslant L g$ ) and hence $b_{k} \notin\{1, g\}$ (because $g=a_{k} b_{k}$ ).

Assume that $a_{k}=a_{k+1}$ for some $1 \leqslant k \leqslant m$. Then $a_{k} \rightleftharpoons b_{k}$ because $a_{k+1} \rightleftharpoons b_{k}$. Since $g=a_{k} b_{k}$ and both $a_{k}$ and $b_{k}$ are nontrivial, this contradicts that $g$ is non-split. Therefore $a_{k} \neq a_{k+1}$ and hence $b_{k} \neq b_{k+1}$ for all $1 \leqslant k \leqslant m$.

Let $j \geqslant 1$. Since $u \leqslant_{L} g^{m}$ and $g$ is cyclically reduced, we have $u \leqslant_{L} g^{m+j}$, hence $u \wedge_{L} g^{m}=$ $u=u \wedge_{L} g^{m+j}$. Therefore $a_{1} \cdots a_{m}=a_{1} \cdots a_{m} a_{m+1} \cdots a_{m+j}$, hence $a_{m+1} \cdots a_{m+j}=1$. Since the decomposition $a_{m+1} \cdots a_{m+j}$ is geodesic, we have $a_{m+j}=1$. Namely, for all $k>m, a_{k}=1$ and hence $b_{k}=g$.

Define $\left\{g_{k}\right\}_{k=0}^{m}$ by $g_{0}=b_{1}$ and $g_{k}=a_{k+1}^{-1} a_{k}$ (hence $a_{k}=a_{k+1} g_{k}$ ) for $1 \leqslant k \leqslant m$. Then $a_{k}=a_{k+1} g_{k}$ is geodesic for all $1 \leqslant k \leqslant m$ because $a_{k+1} \leqslant L a_{k}$.

Claim 2. The decomposition $g=g_{m} g_{m-1} \cdots g_{1} g_{0}$ is geodesic such that
(a) $g_{k} \neq 1$ for all $0 \leqslant k \leqslant m$;
(b) $g_{i} \rightleftharpoons g_{j}$ whenever $|i-j| \geqslant 2$;
(c) $u \wedge_{L} g^{k}=\left(g_{m} \cdots g_{1}\right)\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m} \cdots g_{k}\right)$ for all $1 \leqslant k \leqslant m$;
(d) $a_{k}=g_{m} g_{m-1} \cdots g_{k}$ and $b_{k}=g_{k-1} g_{k-2} \cdots g_{0}$ for all $1 \leqslant k \leqslant m$.

Proof of Claim 2. Since $a_{k}=a_{k+1} g_{k}$ is geodesic, $\left\|g_{k}\right\|=\left\|a_{k}\right\|-\left\|a_{k+1}\right\|$ for all $1 \leqslant k \leqslant m$. Since $g_{0}=b_{1}, a_{m+1}=1$ and $g=a_{1} b_{1}$ is geodesic,

$$
\begin{aligned}
\left\|g_{0}\right\| & +\left\|g_{1}\right\|+\cdots+\left\|g_{m}\right\| \\
& =\left\|b_{1}\right\|+\left(\left\|a_{1}\right\|-\left\|a_{2}\right\|\right)+\cdots+\left(\left\|a_{m}\right\|-\left\|a_{m+1}\right\|\right) \\
& =\left\|b_{1}\right\|+\left\|a_{1}\right\|-\left\|a_{m+1}\right\|=\|g\| .
\end{aligned}
$$

Consequently, $\|g\|=\left\|g_{0}\right\|+\left\|g_{1}\right\|+\cdots+\left\|g_{m}\right\|$.
For $1 \leqslant k \leqslant m, a_{k}=a_{k+1} g_{k}=a_{k+2} g_{k+1} g_{k}=\cdots=a_{m+1} g_{m} \cdots g_{k}=g_{m} \cdots g_{k}$ because $a_{m+1}=1$.
Therefore we have the following decompositions:

$$
\begin{aligned}
a_{k} & =g_{m} \cdots g_{k} \quad \text { for all } 1 \leqslant k \leqslant m, \\
g & =a_{1} b_{1}=\left(g_{m} \cdots g_{1}\right) g_{0}=g_{m} \cdots g_{0}, \\
b_{k} & =a_{k}^{-1} g=g_{k-1} \cdots g_{0} \quad \text { for all } 1 \leqslant k \leqslant m .
\end{aligned}
$$

Observe that $g=g_{m} \cdots g_{0}$ is geodesic because $\|g\|=\left\|g_{0}\right\|+\left\|g_{1}\right\|+\cdots+\left\|g_{m}\right\|$.
The decompositions for $a_{k}$ and $b_{k}$ in the above prove (d).
For each $1 \leqslant k \leqslant m, u \wedge_{L} g^{k}=a_{1} a_{2} \cdots a_{k}=\left(g_{m} \cdots g_{1}\right)\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m} \cdots g_{k}\right)$. This proves (c).
By Claim $1, g_{0}=b_{1} \neq 1$ and $g_{k}=a_{k+1}^{-1} a_{k} \neq 1$ for all $1 \leqslant k \leqslant m$. This proves (a).
For each $(i, j)$ with $0 \leqslant j<j+2 \leqslant i \leqslant m$, we know that $a_{i} \rightleftharpoons b_{j+1}$. Since $a_{i}=g_{m} \cdots g_{i}$ and $b_{j+1}=g_{j} \cdots g_{0}$ are geodesic, we have $g_{i} \leqslant_{R} a_{i}$ and $g_{j} \leqslant_{L} b_{j+1}$, hence $g_{i} \rightleftharpoons g_{j}$. This proves (b).

Recall from Claim 2 that both $g_{0}=b_{1}$ and $g_{m}=a_{m}$ are nontrivial.
Claim 3. For any path $\left(v_{0}, v_{1}, \ldots, v_{r-1}, v_{r}\right)$ in $\bar{\Gamma}[g]$ such that $v_{0} \in \operatorname{supp}\left(g_{0}\right)=\operatorname{supp}\left(b_{1}\right)$ and $v_{r} \in$ $\operatorname{supp}\left(g_{m}\right)=\operatorname{supp}\left(a_{m}\right)$, we have $m \leqslant r$. In particular, $m \leqslant \operatorname{diam}(\bar{\Gamma}[g])$.

Proof of Claim 3. Using induction on $k$, we first show that

$$
v_{k} \in \operatorname{supp}\left(b_{k+1}\right)
$$

for all $0 \leqslant k \leqslant \min \{m-1, r-1\}$. By the hypothesis of the claim, $v_{0} \in \operatorname{supp}\left(b_{1}\right)$. Assume that $v_{k} \in \operatorname{supp}\left(b_{k+1}\right)$ for some $0 \leqslant k \leqslant \min \{m-2, r-2\}$. Since $\left\{v_{k}, v_{k+1}\right\}$ is an edge in $\bar{\Gamma}[g]$, we have $v_{k+1} \in \operatorname{St}_{\bar{\Gamma}[g]}\left(v_{k}\right)$. Since $v_{k} \in \operatorname{supp}\left(b_{k+1}\right)$ by induction hypothesis, $\operatorname{St}_{\bar{\Gamma}[g]}\left(v_{k}\right) \subset \operatorname{St}_{\bar{\Gamma}[g]}\left(\operatorname{supp}\left(b_{k+1}\right)\right)$, hence $v_{k+1} \in \operatorname{St}_{\bar{\Gamma}[g]}\left(\operatorname{supp}\left(b_{k+1}\right)\right)$. By Corollary [5.2. $\mathrm{St}_{\overline{\mathrm{T}}[g]}\left(\operatorname{supp}\left(b_{k+1}\right)\right) \subset \operatorname{supp}\left(b_{k+2}\right)$, hence $v_{k+1} \in$ $\operatorname{supp}\left(b_{k+2}\right)$.

If $m>r$, then $a_{m} \rightleftharpoons b_{r}$ (by Corollary 5.2). Since $v_{r} \in \operatorname{supp}\left(a_{m}\right)$ and $v_{r-1} \in \operatorname{supp}\left(b_{r}\right)$, we have $v_{r} \rightleftharpoons v_{r-1}$, which contradicts that $\left\{v_{r-1}, v_{r}\right\}$ is an edge in $\bar{\Gamma}$. Therefore $m \leqslant r$.

Since $\bar{\Gamma}[g]$ is connected and both $g_{0}$ and $g_{m}$ are nontrivial (by Claim 2), we may assume that $\left(v_{0}, \ldots, v_{r}\right)$ is a shortest path from $v_{0} \in \operatorname{supp}\left(g_{0}\right)$ to $v_{r} \in \operatorname{supp}\left(g_{m}\right)$ in $\bar{\Gamma}[g]$, hence $r \leqslant \operatorname{diam}(\bar{\Gamma}[g])$. Therefore $m \leqslant r \leqslant \operatorname{diam}(\bar{\Gamma}[g])$.

Claim 3 proves (i) and Claim 2 proves (ii).
Since $g_{0} \neq 1$ and $g_{j} \rightleftharpoons g_{0}$ for all $j \geqslant 2$, we have $\left\|\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m} g_{m-1}\right) g_{m}\right\|_{*} \leqslant 1$. Since $g_{m} \cdots g_{1} \leqslant L$ $g$, we have $\left\|g_{m} \cdots g_{1}\right\|_{*} \leqslant\|g\|_{*}$. Therefore

$$
\|u\|_{*} \leqslant\left\|g_{m} \cdots g_{1}\right\|_{*}+\left\|\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m} g_{m-1}\right) g_{m}\right\|_{*} \leqslant\|g\|_{*}+1
$$

This proves (iii).
Assume $m \geqslant 3$. Since $g_{0} \neq 1, g_{m} \neq 1, g_{m} \cdots g_{2} \rightleftharpoons g_{0}$ and $g_{1} g_{0} \rightleftharpoons g_{m}$, we have $\left\|g_{m} \cdots g_{2}\right\|_{*} \leqslant 1$ and $\left\|g_{1} g_{0}\right\|_{*} \leqslant 1$. Therefore $\|g\|_{*} \leqslant\left\|g_{m} \cdots g_{2}\right\|_{*}+\left\|g_{1} g_{0}\right\|_{*} \leqslant 2$. This proves (iv).

Remark 5.4. From the disjoint commutativity $g_{i} \rightleftharpoons g_{j}$ for $|i-j| \geqslant 2$, the following decompositions are geodesic for all $1 \leqslant k \leqslant m$.

$$
\begin{aligned}
g^{k} & =\left(g_{m} \cdots g_{0}\right)\left(g_{m} \cdots g_{0}\right) \cdots\left(g_{m} \cdots g_{0}\right) \\
& =\left(\left(g_{m} \cdots g_{1}\right) \cdot\left(g_{0}\right)\right)\left(\left(g_{m} \cdots g_{2}\right) \cdot\left(g_{1} g_{0}\right)\right) \cdots\left(\left(g_{m} \cdots g_{k}\right) \cdot\left(g_{k-1} \cdots g_{0}\right)\right) \\
& =\left(g_{m} \cdots g_{1}\right)\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m} \cdots g_{k}\right) \cdot\left(g_{0}\right)\left(g_{1} g_{0}\right) \cdots\left(g_{k-1} \cdots g_{0}\right) \\
& =\left(u \wedge_{L} g^{k}\right)\left(g_{0}\right)\left(g_{1} g_{0}\right) \cdots\left(g_{k-1} \cdots g_{0}\right)
\end{aligned}
$$

In particular, $g^{m}=u u^{\prime}$ is geodesic, where

$$
\begin{aligned}
u & =\left(g_{m} \cdots g_{1}\right)\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m}\right), \\
u^{\prime} & =\left(g_{0}\right)\left(g_{1} g_{0}\right) \cdots\left(g_{m-1} \cdots g_{0}\right) .
\end{aligned}
$$

The following example shows that the upper bounds $m \leqslant \operatorname{diam}(\bar{\Gamma}[g])$ and $m \leqslant|V(\Gamma)|-1$ in Theorem 5.3(i) are sharp.

Example 5.5. Let $\Gamma=\bar{P}_{4}$, where $P_{4}=\left(v_{1}, \ldots, v_{4}\right)$ is a path graph, and let the underlying rightangled Artin group here be $A(\Gamma)$. Let $g=v_{1}^{2} v_{2} v_{3} v_{4}$ and $u=v_{1}^{2} v_{2} v_{3} v_{1}^{2} v_{2} v_{1}$. Then $g$ is clearly cyclically reduced and non-split. It is easy to see that $g \not \mathbb{Z}_{L} u$ and $u \not \mathbb{K}_{L} g^{2}$. (By Lemma [2.6] if $g \leqslant_{L} u$ then $v_{4} \leqslant_{L} v_{1}^{2} v_{2} v_{1}$, and if $u \leqslant_{L} g^{2}$ then $v_{1} \leqslant_{L} v_{4} v_{3} v_{4}$.) On the other hand, $u \leqslant_{L} g^{3}$ because

$$
\begin{aligned}
g^{3} & =\left(v_{1}^{2} v_{2} v_{3} v_{4}\right)\left(v_{1}^{2} v_{2} v_{3} v_{4}\right)\left(v_{1}^{2} v_{2} v_{3} v_{4}\right) \\
& =\left(v_{1}^{2} v_{2} v_{3} \cdot v_{4}\right)\left(v_{1}^{2} v_{2} \cdot v_{3} v_{4}\right)\left(v_{1} \cdot v_{1} v_{2} v_{3} v_{4}\right) \\
& =\left(v_{1}^{2} v_{2} v_{3} \cdot v_{1}^{2} v_{2} \cdot v_{1}\right)\left(v_{4} \cdot v_{3} v_{4} \cdot v_{1} v_{2} v_{3} v_{4}\right) \\
& =u\left(v_{4} \cdot v_{3} v_{4} \cdot v_{1} v_{2} v_{3} v_{4}\right) .
\end{aligned}
$$

In the notation of Theorem 5.3,

$$
m=3=\operatorname{diam}(\bar{\Gamma})=\operatorname{diam}(\bar{\Gamma}[g])=|\operatorname{supp}(g)|-1=|V(\Gamma)|-1 .
$$

Thus the bounds of $m$ in Theorem 5.3(i) are sharp.
Corollary 5.6. Let $g, u \in A(\Gamma)$ with $g$ cyclically reduced and non-split. If $u \leqslant_{L} g^{m}$ for some $m \geqslant 1$, then $u=g^{k} a$ is geodesic for some $0 \leqslant k \leqslant m$ and $a \in A(\Gamma)$ with $\|a\|_{*} \leqslant\|g\|_{*}+1$.

Proof. We may assume that $u \nless L g^{m-1}$. Let $k=\max \left\{l \geqslant 0: g^{l} \leqslant L u\right\}$. Then $0 \leqslant k \leqslant m$ and $u=g^{k} a$ is geodesic for some $a \in A(\Gamma)$ with $g \not \mathbb{K}_{L} a \not \mathbb{K}_{L} g^{m-k-1}$ and $a \leqslant_{L} g^{m-k}$.

If $m-k \leqslant 1$, then it is obvious that $\|a\|_{*} \leqslant\|g\|_{*} \leqslant\|g\|_{*}+1$.
If $m-k \geqslant 2$, then $\|a\|_{*} \leqslant\|g\|_{*}+1$ by Theorem 5.3.

For a cyclically reduced element $g \in A(\Gamma)$, we have seen in Corollary 4.10 that the sequence $\left\{\left\|g^{n}\right\|_{*}\right\}_{n=0}^{\infty}$ is increasing. In particular, if $\|g\|_{*} \geqslant 3$, then $\left\|g^{n+1}\right\|_{*} \geqslant\left\|g^{n}\right\|_{*}+1$ for all $n \geqslant 0$. However, if $\|g\|_{*}=2$, then it may happen that $\|g\|_{*}=\left\|g^{2}\right\|_{*}=\cdots=\left\|g^{n}\right\|_{*}=2$ for some $n \geqslant 2$. The following proposition finds $m$ with $\left\|g^{m}\right\|_{*} \geqslant 3$ when $\|g\|_{*}=2$.

Proposition 5.7. Let $g \in A(\Gamma)$ be cyclically reduced and non-split with $\|g\|_{*}=2$. Then the following hold.
(i) Let $m \geqslant 2$. If either $m \geqslant|V(\Gamma)|-2$ or $m \geqslant \operatorname{diam}(\bar{\Gamma}[g])+1$, then $\left\|g^{m}\right\|_{*} \geqslant 3$.
(ii) Let $m \geqslant 2$. If $\left\|g^{m}\right\|_{*}=2$, then $m \leqslant|V(\Gamma)|-3$ and $m \leqslant \operatorname{diam}(\bar{\Gamma}[g])$.
(iii) If $|V(\Gamma)| \leqslant 4$, then $\left\|g^{2}\right\|_{*} \geqslant 3$.

Proof. The statements (i) and (ii) are equivalent, and (iii) follows from (i). Therefore we prove only (ii).

Since $g$ is non-split, $\bar{\Gamma}[g]$ is connected. Suppose that $\left\|g^{m}\right\|_{*}=2$ for some $m \geqslant 2$. Then there is a geodesic decomposition

$$
g^{m}=u u^{\prime}
$$

for some $u, u^{\prime} \in A(\Gamma)$ with $\|u\|_{*}=\left\|u^{\prime}\right\|_{*}=1$.
Claim 1. $g \not \mathbb{K}_{L} u \not \nless L_{L} g^{m-1}$ and $u \leqslant_{L} g^{m}$.
Proof of Claim 1. Since $g^{m}=u u^{\prime}$ is geodesic, $u \leqslant L g^{m}$. Since $\|u\|_{*}=1$ and $\|g\|_{*}=2, g \not{ }_{k} L u$. If $u \leqslant_{L} g^{m-1}$, then $g \leqslant_{R} u^{\prime}$ (by Lemma [2.6(iv)), which is impossible because $\left\|u^{\prime}\right\|_{*}=1$ and $\|g\|_{*}=2$. Therefore $u \not{ }_{L} g^{m-1}$.

By Claim 1, we can apply Theorem 5.3 , hence $m \leqslant \operatorname{diam}(\bar{\Gamma}[g])$, which is the second inequality of (ii).

By Theorem 5.3 and Remark 5.4, there is a geodesic decomposition $g=g_{m} g_{m-1} \cdots g_{0}$ such that $g_{i} \neq 1$ for all $0 \leqslant i \leqslant m, g_{i} \rightleftharpoons g_{j}$ whenever $|i-j| \geqslant 2$ and

$$
\begin{aligned}
u & =\left(g_{m} \cdots g_{1}\right)\left(g_{m} \cdots g_{2}\right) \cdots\left(g_{m}\right), \\
u^{\prime} & =\left(g_{0}\right)\left(g_{1} g_{0}\right) \cdots\left(g_{m-1} \cdots g_{0}\right) .
\end{aligned}
$$

Since $\|u\|_{*}=\left\|u^{\prime}\right\|_{*}=1$, there exist vertices $x, y \in V(\Gamma)$ such that $u \in Z(x)$ and $u^{\prime} \in Z(y)$, where $Z(\cdot)$ denotes the centralizer. Since $\left\|u u^{\prime}\right\|_{*}=2, x \neq y$. Notice that

$$
\begin{aligned}
& \operatorname{supp}\left(g_{1}\right), \ldots, \operatorname{supp}\left(g_{m}\right) \subset Z(x) \\
& \operatorname{supp}\left(g_{0}\right), \ldots, \operatorname{supp}\left(g_{m-1}\right) \subset Z(y)
\end{aligned}
$$

Claim 2. There is a path $\left(x, v_{0}, v_{1}, v_{2}, \ldots, v_{r-1}, v_{r}, y\right)$ in $\bar{\Gamma}$ such that
(a) $v_{0} \in \operatorname{supp}\left(g_{0}\right)$ and $v_{r} \in \operatorname{supp}\left(g_{m}\right)$;
(b) the subpath $\left(v_{0}, \ldots, v_{r}\right)$ is a shortest path from $v_{0}$ to $v_{r}$ in $\bar{\Gamma}[g]$;
(c) all the vertices on the path are mutually distinct.

Proof of Claim 2. If either $\operatorname{supp}(g) \subset Z(x)$ or $\operatorname{supp}(g) \subset Z(y)$, then $\|g\|_{*}=1$, hence $\operatorname{supp}(g) \backslash Z(x) \neq \emptyset$ and $\operatorname{supp}(g) \backslash Z(y) \neq \emptyset$.

Choose any vertices $v_{x} \in \operatorname{supp}(g) \backslash Z(x)$ and $v_{y} \in \operatorname{supp}(g) \backslash Z(y)$, equivalently, $v_{x}, v_{y} \in \operatorname{supp}(g)$ such that $\left\{v_{x}, x\right\},\left\{v_{y}, y\right\} \in E(\bar{\Gamma})$. Since $v_{x} \in \operatorname{supp}(g) \backslash Z(x)=\left(\bigcup_{k=0}^{m} \operatorname{supp}\left(g_{k}\right)\right) \backslash Z(x)$ and $\bigcup_{k=1}^{m} \operatorname{supp}\left(g_{k}\right) \subset$ $Z(x)$, we have $v_{x} \in \operatorname{supp}\left(g_{0}\right)$. Similarly, $v_{y} \in \operatorname{supp}\left(g_{m}\right)$. Furthermore, $v_{x} \neq v_{y}$ and $\left\{v_{x}, v_{y}\right\} \notin E(\bar{\Gamma})$ because $g_{0} \rightleftharpoons g_{m}$.

Since $\left\{v_{x}, x\right\},\left\{v_{y}, y\right\} \in E(\bar{\Gamma})$ and $\bar{\Gamma}[g]$ is connected, there is a path in $\bar{\Gamma}$ from $x$ to $y$

$$
\left(x, v_{0}=v_{x}, v_{1}, v_{2}, \ldots, v_{r-1}, v_{r}=v_{y}, y\right)
$$

such that $v_{k} \in \operatorname{supp}(g)$ for all $0 \leqslant k \leqslant r$. Observe that $v_{0}=v_{x} \in \operatorname{supp}\left(g_{0}\right)$ and $v_{r}=v_{y} \in \operatorname{supp}\left(g_{m}\right)$.
We may assume that it is a shortest path among all the paths from $x$ to $y$ such that $v_{k} \in \operatorname{supp}(g)$ for all $0 \leqslant k \leqslant r$. Then the subpath $\left(v_{0}, \cdots, v_{r}\right)$ must be a shortest path from $v_{0}$ to $v_{r}$ in $\bar{\Gamma}[g]$, hence $v_{0}, \ldots, v_{r}$ are mutually distinct.

If $x=v_{j}$ for some $0 \leqslant j \leqslant r-1$, then the path $\left(x, v_{j+1}, \ldots, v_{r}, y\right)$ is shorter than the original one, all of whose middle vertices belong to $\operatorname{supp}(g)$. This contradicts that $\left(x, v_{0}, \cdots, v_{r}, y\right)$ is a shortest path among such paths. If $x=v_{r}$ then $\left\{x, v_{x}\right\} \notin E(\bar{\Gamma})$ (because $x=v_{r} \in \operatorname{supp}\left(g_{m}\right), v_{x}=v_{0} \in \operatorname{supp}\left(g_{0}\right)$ and $g_{0} \rightleftharpoons g_{m}$ ). This is a contradiction. Therefore $x \neq v_{j}$ for any $0 \leqslant j \leqslant r$. Similarly, $y \neq v_{j}$ for any $0 \leqslant j \leqslant r$. Since $x \neq y$, all the vertices on the path $\left(x, v_{0}, v_{1}, v_{2}, \ldots, v_{r-1}, v_{r}, y\right)$ are mutually distinct.

Since the $(r+3)$ points on the path in Claim 2 are mutually distinct, $|V(\Gamma)| \geqslant r+3$. By Claim 3 in the proof of Theorem [5.3, we have $m \leqslant r$. Therefore $|V(\Gamma)| \geqslant r+3 \geqslant m+3$. This proves the first inequality of (ii), hence (ii) is proved.

The following example illustrates that the upper bounds $m \leqslant|V(\Gamma)|-3$ and $m \leqslant \operatorname{diam}(\bar{\Gamma}[g])$ in Proposition 5.7(ii) are sharp.

Example 5.8. Let $\Gamma=\bar{P}_{6}$, where $P_{6}=\left(v_{0}, v_{1}, \ldots, v_{5}\right)$ is a path graph, and let the underlying right-angled Artin group here be $A(\Gamma)$. Let $g=v_{1} v_{2} v_{3} v_{4}$. It is easy to see that $\|g\|_{*}=2$. Since $\operatorname{supp}(g)=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}, \operatorname{diam}(\bar{\Gamma}[g])=3$. Observe

$$
\begin{aligned}
g^{3} & =\left(v_{1} v_{2} v_{3} v_{4}\right) \cdot\left(v_{1} v_{2} v_{3} v_{4}\right) \cdot\left(v_{1} v_{2} v_{3} v_{4}\right) \\
& =\left(v_{1} v_{2} v_{3} \cdot v_{4}\right) \cdot\left(v_{1} v_{2} \cdot v_{3} v_{4}\right) \cdot\left(v_{1} \cdot v_{2} v_{3} v_{4}\right) \\
& =\left(v_{1} v_{2} v_{3} \cdot v_{1} v_{2} \cdot v_{1}\right)\left(v_{4} \cdot v_{3} v_{4} \cdot v_{2} v_{3} v_{4}\right) .
\end{aligned}
$$

Let $u=v_{1} v_{2} v_{3} v_{1} v_{2} v_{1}$ and $u^{\prime}=v_{4} v_{3} v_{4} v_{2} v_{3} v_{4}$. Then $g^{3}=u u^{\prime}$ is geodesic. Since $u \in Z\left(v_{5}\right)$ and $u^{\prime} \in Z\left(v_{0}\right)$, we have $\|u\|_{*}=\left\|u^{\prime}\right\|_{*}=1$, hence $\left\|g^{3}\right\|_{*}=2$. Notice that $3=|V(\Gamma)|-3=\operatorname{diam}(\bar{\Gamma}[g])$.

## 6. Asymptotic translation length

In this section, we study asymptotic translation lengths of elements of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$ and on ( $\Gamma^{e}, d$ ), and then find a lower bound of the minimal asymptotic translation length of $A(\Gamma)$ on $\Gamma^{e}$.

Proposition 6.1. If $g \in A(\Gamma)$ is cyclically reduced and non-split with $\|g\|_{*} \geqslant 2$, then

$$
\tau_{\left(A(\Gamma), d_{*}\right)}(g) \geqslant \frac{1}{\max \{2,|V(\Gamma)|-2\}}
$$

Proof. Let $\tau_{*}$ denote $\tau_{\left(A(\Gamma), d_{*}\right)}$, and let $V=|V(\Gamma)|$.
Notice that if $\|g\|_{*} \geqslant 3$, then $\left\|g^{n}\right\|_{*} \geqslant n+2$ for all $n \geqslant 1$ (by Corollary 4.10), hence

$$
\tau_{*}(g)=\lim _{n \rightarrow \infty} \frac{\left\|g^{n}\right\|_{*}}{n} \geqslant \lim _{n \rightarrow \infty} \frac{n+2}{n} \geqslant 1 .
$$

Suppose $\|g\|_{*}=2$. By Proposition 5.7, if $V \leqslant 4$ then $\left\|g^{2}\right\|_{*} \geqslant 3$, and if $V \geqslant 5$ then $\left\|g^{V-2}\right\|_{*} \geqslant 3$. Therefore $\left\|g^{\max \{2, V-2\}}\right\|_{*} \geqslant 3$. From the above discussion, $\tau_{*}\left(g^{\max \{2, V-2\}}\right) \geqslant 1$ and hence $\tau_{*}(g) \geqslant$ $\frac{1}{\max \{2, V-2\}}$.

Remark 6.2. When we study the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$, we will assume that " $|V(\Gamma)| \geqslant 2$ and $\bar{\Gamma}$ is connected" because otherwise $\|g\|_{*} \leqslant 2$ for all $g \in A(\Gamma)$ and hence $\left(A(\Gamma), d_{*}\right)$ has diameter at most 2: if $|V(\Gamma)|=1$, then $\|g\|_{*} \leqslant 1$ for all $g \in A(\Gamma)$; if $\bar{\Gamma}$ is disconnected (i.e. $\Gamma$ is a join), then $\|g\|_{*} \leqslant 2$ for all $g \in A(\Gamma)$.

Lemma 6.3. Suppose that $|V(\Gamma)| \geqslant 2$ and $\bar{\Gamma}$ is connected. The following are equivalent for a cyclically reduced element $g \in A(\Gamma)$.
(i) $g$ is strongly non-split and $|\operatorname{supp}(g)| \geqslant 2$.
(ii) $g$ is non-split and $\|g\|_{*} \geqslant 2$.
(iii) $\left\|g^{n}\right\|_{*} \geqslant 3$ for some $n \geqslant 1$.
(iv) $g$ is loxodromic on $\left(A(\Gamma), d_{*}\right)$, i.e. $\tau_{\left(A(\Gamma), d_{*}\right)}(g)>0$.
(v) $\tau_{\left(A(\Gamma), d_{*}\right)}(g) \geqslant \frac{1}{\max \{2,|V(\Gamma)|-2\}}$.

Proof of Lemma 6.3. (i) $\Leftrightarrow$ (ii) follows from Lemma 4.15,
(ii) $\Rightarrow$ (v) follows from Proposition 6.1.
$(\mathrm{v}) \Rightarrow$ (iv) and (iv) $\Rightarrow$ (iii) are obvious.
(iii) $\Rightarrow$ (i): Since $\left\|g^{n}\right\|_{*} \geqslant 3, g^{n}$ is strongly non-split (by Lemma 4.15), hence $g$ is also strongly non-split (see Remark 4.14). It is obvious that $|\operatorname{supp}(g)| \geqslant 2$.
Remark 6.4. Suppose that $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected. Then the condition $|\operatorname{supp}(g)| \geqslant 2$ of Lemma 6.3(i) is not necessary because all strongly non-split elements $g$ must have $|\operatorname{supp}(g)| \geqslant 2$. Moreover, $g$ is loxodromic on $\left(A(\Gamma), d_{*}\right)$ if and only if it is loxodromic on $\left(\Gamma^{e}, d\right)$ by Corollary 4.4. Therefore, if $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected, then (i) and (iv) in the above lemma are equivalent to the following ( $\mathrm{i}^{\prime}$ ) and ( $\mathrm{iv}^{\prime}$ ) respectively.
( $\mathrm{i}^{\prime}$ ) $g$ is strongly non-split.
$\left(\mathrm{iv}^{\prime}\right) g$ is loxodromic on $\left(\Gamma^{e}, d\right)$, i.e. $\tau_{\left(\Gamma^{e}, d\right)}(g)>0$.
Kim and Koberda [18, Lemma 33] showed that if $g \in A(\Gamma)$ is cyclically reduced and strongly non-split, then $\left\|g^{2 n|V(\Gamma)|^{2}}\right\|_{*} \geqslant n$ for all $n \geqslant 1$. Therefore (by Corollary 4.4)

$$
\tau_{\left(\Gamma^{e}, d\right)}(g) \geqslant \tau_{\left(A(\Gamma), d_{*}\right)}(g) \geqslant \frac{1}{2|V(\Gamma)|^{2}}
$$

From this, a lower bound of the minimal asymptotic translation length of $A(\Gamma)$ on $\Gamma^{e}$ follows:

$$
\mathcal{L}_{\left(\Gamma^{e}, d\right)}(A(\Gamma)) \geqslant \frac{1}{2|V(\Gamma)|^{2}}
$$

We improve the denominator of the lower bound from a quadratic function to a linear function of $|V(\Gamma)|$ as follows.

Theorem 6.5. Let $\Gamma$ be a finite simplicial graph such that $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected. Then

$$
\mathcal{L}_{\left(\Gamma^{e}, d\right)}(A(\Gamma)) \geqslant \frac{1}{|V(\Gamma)|-2}
$$

Proof. Since $|V(\Gamma)| \geqslant 4,|V(\Gamma)|-2=\max \{2,|V(\Gamma)|-2\}$. Let $g \in A(\Gamma)$ be loxodromic on $\left(\Gamma^{e}, d\right)$ and hence on $\left(A(\Gamma), d_{*}\right)$ (by Remark 6.4). We may assume that $g$ is cyclically reduced because asymptotic translation lengths are invariant under conjugation. By Corollary 4.4 and Lemma 6.3,

$$
\tau_{\left(\Gamma^{e}, d\right)}(g) \geqslant \tau_{\left(A(\Gamma), d_{*}\right)}(g) \geqslant \frac{1}{|V(\Gamma)|-2}
$$

Therefore $\mathcal{L}_{\left(\Gamma^{e}, d\right)}(A(\Gamma)) \geqslant \frac{1}{|V(\Gamma)|-2}$.

## 7. UniQUENESS OF QUASI-ROOTS

The notion of quasi-roots in $A(\Gamma)$ was introduced in [25], where the quasi-roots are defined using word length. The uniqueness up to conjugacy was established by using the normal form of elements introduced by Crisp, Godelle and Wiest [7]. In this section, we extend the uniqueness of quasi-roots from word length to star length.

Definition 7.1. (quasi-root) An element $g \in A(\Gamma) \backslash\{1\}$ is called a quasi-root of $h \in A(\Gamma)$ if there is a decomposition

$$
h=a g^{n} b
$$

for some $n \geqslant 1$ and $a, b \in A(\Gamma)$ such that $\|h\|=\|a\|+\|b\|+n\|g\|$. The decomposition is called a quasi-root decomposition of $h$. The conjugates $a g a^{-1}$ and $b^{-1} g b$ are called the leftward-extraction and the rightward-extraction of the quasi-root $g$, respectively. We consider the following two cases.
(i) $g$ is called an $(A, B, r)$-quasi-root of $h$ if $\|a\| \leqslant A,\|b\| \leqslant B$ and $\|g\| \leqslant r$.
(ii) $g$ is called an $(A, B, r)^{*}$-quasi-root of $h$ if $\|a\|_{*} \leqslant A,\|b\|_{*} \leqslant B$ and $\|g\|_{*} \leqslant r$.

In the above definition, the condition $\|h\|=\|a\|+\|b\|+n\|g\|$ implies $\left\|g^{n}\right\|=n\|g\|$, hence $g$ is cyclically reduced when $n \geqslant 2$.

Notice that if $g_{1}=a g a^{-1}$ and $g_{2}=b^{-1} g b$ are respectively the leftward- and the rightward-extractions of $g$, then we have decompositions $h=g_{1}^{n} a b=a b g_{2}^{n}$, which are not necessarily geodesic.

Definition 7.2 (primitive). An element $g \in A(\Gamma) \backslash\{1\}$ is called primitive if $g$ is not a nontrivial power of another element, i.e. $g=h^{n}$ never holds for any $n \geqslant 2$ and $h \in A(\Gamma)$.

The following proposition is Proposition 3.5 in [25] written in the setting of this paper. It shows a kind of uniqueness property of quasi-roots in right-angled Artin groups.

Proposition 7.3 ([25, Proposition 3.5]). Let $h \in A(\Gamma), A, B \geqslant 0$ and $r \geqslant 1$. If

$$
\|h\| \geqslant A+B+(2 V+1) r
$$

where $V=|V(\Gamma)|$, then strongly non-split and primitive $(A, B, r)$-quasi-roots of $h$ are conjugate to each other, and moreover, their leftward- and rightward-extractions are unique.

In other words, Proposition 7.3 shows that if

$$
h=a_{1} g_{1}^{n_{1}} b_{1}=a_{2} g_{2}^{n_{2}} b_{2}
$$

are two quasi-root decompositions of $h$ such that for each $i=1,2, g_{i}$ is strongly non-split and primitive,

$$
\begin{gathered}
\left\|a_{i}\right\| \leqslant A, \quad\left\|b_{i}\right\| \leqslant B, \quad\left\|g_{i}\right\| \leqslant r \\
\|h\| \geqslant A+B+(2 V+1) r
\end{gathered}
$$

then $g_{1}$ and $g_{2}$ are conjugate, and moreover, $a_{1} g_{1} a_{1}^{-1}=a_{2} g_{2} a_{2}^{-1}$ and $b_{1}^{-1} g_{1} b_{1}=b_{2}^{-1} g_{2} b_{2}$.
The following theorem is the main result of this section. It is a star length version of Proposition 7.3, which plays an important role in the proof of Theorem 8.2. We remark that the word length and the star length are quite independent, hence the word length version does not naively extend to a star length version. We exploit lattice structures developed in $\% 2$,

We also remark that in the following theorem since $\|h\|_{*} \geqslant 2 A+2 B+(2 V+3) r+2 \geqslant 3 r+2 \geqslant 5$, the existence of such an element $h$ implies that $|V(\Gamma)| \geqslant 2$ and $\bar{\Gamma}$ is connected (as explained in Remark 6.2).

Theorem 7.4. Let $h \in A(\Gamma), A, B \geqslant 0$ and $r \geqslant 1$. If

$$
\|h\|_{*} \geqslant 2 A+2 B+(2 V+3) r+2
$$

where $V=|V(\Gamma)|$, then primitive $(A, B, r)^{*}$-quasi-roots of $h$ are conjugate to each other, and moreover, their leftward- and rightward-extractions are unique. In other words, if

$$
h=a_{1} g_{1}^{n_{1}} b_{1}=a_{2} g_{2}^{n_{2}} b_{2}
$$

are two quasi-root decompositions of $h$ such that for each $i=1,2, g_{i}$ is primitive and

$$
\begin{gathered}
\left\|a_{i}\right\|_{*} \leqslant A, \quad\left\|b_{i}\right\|_{*} \leqslant B, \quad\left\|g_{i}\right\|_{*} \leqslant r \\
\|h\|_{*} \geqslant 2 A+2 B+(2 V+3) r+2
\end{gathered}
$$

then $g_{1}$ and $g_{2}$ are conjugate to each other such that

$$
a_{1} g_{1} a_{1}^{-1}=a_{2} g_{2} a_{2}^{-1} \quad \text { and } \quad b_{1}^{-1} g_{1} b_{1}=b_{2}^{-1} g_{2} b_{2}
$$

Proof. Let $i=1,2$.
Claim 1. $n_{i} \geqslant 4$ and $g_{i}$ is cyclically reduced and strongly non-split with $\left\|g_{i}\right\|_{*} \geqslant 2$.
Proof of Claim 1. If $n_{i} \leqslant 3$, then

$$
\|h\|_{*}=\left\|a_{i} g_{i}^{n_{i}} b_{i}\right\|_{*} \leqslant\left\|a_{i}\right\|_{*}+n_{i}\left\|g_{i}\right\|_{*}+\left\|b_{i}\right\|_{*} \leqslant A+3 r+B
$$

This contradicts the hypothesis $\|h\|_{*} \geqslant 2 A+2 B+(2 V+3) r+2$. Therefore $n_{i} \geqslant 4$ and hence $g_{i}$ is cyclically reduced (see the paragraph following Definition (7.1).

Observe that $\left\|g_{i}^{n_{i}}\right\|_{*} \geqslant 5$ because

$$
\left\|g_{i}^{n_{i}}\right\|_{*} \geqslant\|h\|_{*}-\left\|a_{i}\right\|_{*}-\left\|b_{i}\right\|_{*} \geqslant A+B+(2 V+3) r+2 \geqslant 3 r+2 \geqslant 5 .
$$

Therefore $g_{i}$ is strongly non-split and $\left\|g_{i}\right\|_{*} \geqslant 2$ (by Lemma 6.3).
Let $\alpha_{i}$ and $\beta_{i}$ be integers defined by

$$
\begin{aligned}
\alpha_{i} & =\min \left\{k \geqslant 1:\left\|g_{i}^{k}\right\|_{*} \geqslant A+2\right\}, \\
\beta_{i} & =\min \left\{k \geqslant 1:\left\|g_{i}^{k}\right\|_{*} \geqslant B+2\right\} .
\end{aligned}
$$

The numbers $\alpha_{i}$ and $\beta_{i}$ are well-defined because the sequence $\left\{\left\|g_{i}^{k}\right\|_{*}\right\}_{k=1}^{\infty}$ is increasing such that $\lim _{k \rightarrow \infty}\left\|g_{i}^{k}\right\|_{*}=\infty$ (by Claim 1, Lemmas (4.5 and 6.3). Since $\left\|g_{i}^{k}\right\|_{*}-\left\|g_{i}^{k-1}\right\|_{*} \leqslant\left\|g_{i}\right\|_{*} \leqslant r$ for all $k \geqslant 1$, we get

$$
\begin{aligned}
& A+2 \leqslant\left\|g_{i}^{\alpha_{i}}\right\|_{*} \leqslant A+1+r, \\
& B+2 \leqslant\left\|g_{i}^{\beta_{i}}\right\|_{*} \leqslant B+1+r .
\end{aligned}
$$

Claim 2. $n_{i}-\alpha_{i}-\beta_{i} \geqslant 2 V+1$.
Proof of Claim 2. Observe that

$$
\begin{aligned}
\left\|g_{i}^{n_{i}}\right\|_{*} & \geqslant\|h\|_{*}-\left\|a_{i}\right\|_{*}-\left\|b_{i}\right\|_{*} \geqslant\|h\|_{*}-A-B, \\
\left\|g_{i}^{n_{i}}\right\|_{*}-\left\|g_{i}^{\alpha_{i}+\beta_{i}}\right\|_{*} & \geqslant\left\|g_{i}^{n_{i}}\right\|_{*}-\left\|g_{i}^{\alpha_{i}}\right\|_{*}-\left\|g_{i}^{\beta_{i}}\right\|_{*} \\
& \geqslant\left(\|h\|_{*}-A-B\right)-(A+1+r)-(B+1+r) \\
& =\|h\|_{*}-2 A-2 B-2 r-2 \geqslant(2 V+1) r .
\end{aligned}
$$

Since $\left\{\left\|g_{i}^{k}\right\|_{*}\right\}_{k=1}^{\infty}$ is increasing and $\left\|g_{i}^{n_{i}}\right\|_{*}-\left\|g_{i}^{\alpha_{i}+\beta_{i}}\right\|_{*} \geqslant(2 V+1) r>0$, we have $n_{i}>\alpha_{i}+\beta_{i}$. Since

$$
\begin{aligned}
\left(n_{i}-\alpha_{i}-\beta_{i}\right)\left\|g_{i}\right\|_{*} & \geqslant\left\|g_{i}^{n_{i}-\alpha_{i}-\beta_{i}}\right\|_{*} \geqslant\left\|g_{i}^{n_{i}}\right\|_{*}-\left\|g_{i}^{\alpha_{i}+\beta_{i}}\right\|_{*} \\
& \geqslant(2 V+1) r \geqslant(2 V+1)\left\|g_{i}\right\|_{*},
\end{aligned}
$$

we get $n_{i}-\alpha_{i}-\beta_{i} \geqslant 2 V+1$ as desired.
Let $a_{0}=a_{1} \wedge_{L} a_{2}$ and $b_{0}=b_{1} \wedge_{R} b_{2}$. Then we have geodesic decompositions

$$
\left\{\begin{array} { l } 
{ a _ { 1 } = a _ { 0 } a _ { 1 } ^ { \prime } , } \\
{ a _ { 2 } = a _ { 0 } a _ { 2 } ^ { \prime } , }
\end{array} \quad \left\{\begin{array}{l}
b_{1}=b_{1}^{\prime} b_{0}, \\
b_{2}=b_{2}^{\prime} b_{0}
\end{array}\right.\right.
$$

for some $a_{1}^{\prime}, a_{2}^{\prime}, b_{1}^{\prime}, b_{2}^{\prime} \in A(\Gamma)$ with $a_{1}^{\prime} \wedge_{L} a_{2}^{\prime}=1$ and $b_{1}^{\prime} \wedge_{R} b_{2}^{\prime}=1$. Observe that

$$
\begin{aligned}
& \left\|a_{i}^{\prime}\right\|_{*} \leqslant\left\|a_{i}\right\|_{*} \leqslant A \\
& \left\|b_{i}^{\prime}\right\|_{*} \leqslant\left\|b_{i}\right\|_{*} \leqslant B .
\end{aligned}
$$

Since $h=a_{1} g_{1}^{n_{1}} b_{1}=a_{2} g_{2}^{n_{2}} b_{2}$, we have $h=a_{0}\left(a_{1}^{\prime} g_{1}^{n_{1}} b_{1}^{\prime}\right) b_{0}=a_{0}\left(a_{2}^{\prime} g_{2}^{n_{2}} b_{2}^{\prime}\right) b_{0}$.
Let $h_{0}=a_{0}^{-1} h b_{0}^{-1}$. Then $h_{0}$ has the following two geodesic decompositions.

$$
\begin{equation*}
h_{0}=a_{1}^{\prime} g_{1}^{n_{1}} b_{1}^{\prime}=a_{2}^{\prime} g_{2}^{n_{2}} b_{2}^{\prime} \tag{4}
\end{equation*}
$$

On the other hand, since $a_{1}$ and $a_{2}$ have a common right multiple, say $h$, we have $a_{1}^{\prime} \rightleftharpoons a_{2}^{\prime}$ (by Theorem (2.12). Since $a_{1}^{\prime} \leqslant L a_{2}^{\prime} g_{2}^{n_{2}} b_{2}^{\prime}$ and $a_{2}^{\prime} \leqslant L a_{1}^{\prime} g_{1}^{n_{1}} b_{1}^{\prime}$, we have (by Lemma (2.15)

$$
a_{1}^{\prime} \leqslant L g_{2}^{n_{2}} b_{2}^{\prime} \quad \text { and } \quad a_{2}^{\prime} \leqslant L g_{1}^{n_{1}} b_{1}^{\prime} .
$$

Let $A^{\prime}=\left\|a_{1}^{\prime}\right\|+\left\|a_{2}^{\prime}\right\|$ and $B^{\prime}=\left\|b_{1}^{\prime}\right\|+\left\|b_{2}^{\prime}\right\|$.
Claim 3. $\left\|h_{0}\right\| \geqslant A^{\prime}+B^{\prime}+(2 V+1)\left\|g_{i}\right\|$.
Proof of Claim 3. We know that $n_{1}-\alpha_{1}>0$ (by Claim 2) and $a_{2}^{\prime} \leqslant L g_{1}^{n_{1}} b_{1}^{\prime}=g_{1}^{\alpha_{1}} \cdot g_{1}^{n_{1}-\alpha_{1}} b_{1}^{\prime}$. Since $g_{1}^{\alpha_{1}} \cdot g_{1}^{n_{1}-\alpha_{1}} b_{1}^{\prime}$ is geodesic and $\left\|g_{1}^{\alpha_{1}}\right\|_{*} \geqslant A+2 \geqslant\left\|a_{2}^{\prime}\right\|_{*}+2$, we have $a_{2}^{\prime} \leqslant L g_{1}^{\alpha_{1}}$ (by Corollary 4.7). Similarly, $b_{2}^{\prime} \leqslant R g_{1}^{\beta_{1}}$. (In other words, $g_{1}^{n_{1}}$ has a geodesic decomposition $g_{1}^{n_{1}}=g_{1}^{\alpha_{1}} \cdot g_{1}^{n_{1}-\alpha_{1}-\beta_{1}} \cdot g_{1}^{\beta_{1}}$ such that $a_{2}^{\prime} \leqslant_{L} g_{1}^{\alpha_{1}}$ and $b_{2}^{\prime} \leqslant_{R} g_{1}^{\beta_{1}}$.) Therefore

$$
\begin{aligned}
& \left\|a_{2}^{\prime}\right\| \leqslant\left\|g_{1}^{\alpha_{1}}\right\|=\alpha_{1}\left\|g_{1}\right\|, \\
& \left\|b_{2}^{\prime}\right\| \leqslant\left\|g_{1}^{\beta_{1}}\right\|=\beta_{1}\left\|g_{1}\right\| .
\end{aligned}
$$

Since $h_{0}=a_{1}^{\prime} g_{1}^{n_{1}} b_{1}^{\prime}=a_{2}^{\prime} g_{2}^{n_{2}} b_{2}^{\prime}$, we get

$$
\begin{aligned}
\left\|h_{0}\right\|-\left(A^{\prime}+B^{\prime}\right) & =\left(\left\|a_{1}^{\prime}\right\|+n_{1}\left\|g_{1}\right\|+\left\|b_{1}^{\prime}\right\|\right)-\left(\left\|a_{1}^{\prime}\right\|+\left\|a_{2}^{\prime}\right\|\right)-\left(\left\|b_{1}^{\prime}\right\|+\left\|b_{2}^{\prime}\right\|\right) \\
& =n_{1}\left\|g_{1}\right\|-\left\|a_{2}^{\prime}\right\|-\left\|b_{2}^{\prime}\right\| \geqslant n_{1}\left\|g_{1}\right\|-\alpha_{1}\left\|g_{1}\right\|-\beta_{1}\left\|g_{1}\right\| \\
& =\left(n_{1}-\alpha_{1}-\beta_{1}\right)\left\|g_{1}\right\| \geqslant(2 V+1)\left\|g_{1}\right\| .
\end{aligned}
$$

In the same way, we get $\left\|h_{0}\right\|-\left(A^{\prime}+B^{\prime}\right) \geqslant(2 V+1)\left\|g_{2}\right\|$.
Notice that $\left\|a_{i}^{\prime}\right\| \leqslant A^{\prime}$ and $\left\|b_{i}^{\prime}\right\| \leqslant B^{\prime}$. Let $r^{\prime}=\max \left\{\left\|g_{1}\right\|,\left\|g_{2}\right\|\right\}$. Then each $a_{i}^{\prime} g_{i}^{n_{i}} b_{i}^{\prime}$ in (4) is a $\left(A^{\prime}, B^{\prime}, r^{\prime}\right)$-quasi-root decomposition of $h_{0}$ such that $\left\|h_{0}\right\| \geqslant A^{\prime}+B^{\prime}+(2 V+1) r^{\prime}$.

Applying Proposition [7.3] to (4) yields $a_{1}^{\prime} g_{1} a_{1}^{\prime-1}=a_{2}^{\prime} g_{2} a_{2}^{\prime-1}$ and $b_{1}^{\prime-1} g_{1} b_{1}^{\prime}=b_{2}^{\prime-1} g_{2} b_{2}^{\prime}$. Consequently

$$
\begin{aligned}
& a_{1} g_{1} a_{1}^{-1}=a_{0}\left(a_{1}^{\prime} g_{1} a_{1}^{\prime-1}\right) a_{0}^{-1}=a_{0}\left(a_{2}^{\prime} g_{2} a_{2}^{\prime-1}\right) a_{0}^{-1}=a_{2} g_{2} a_{2}^{-1}, \\
& b_{1}^{-1} g_{1} b_{1}=b_{0}^{-1}\left(b_{1}^{\prime-1} g_{1} b_{1}^{\prime}\right) b_{0}=b_{0}^{-1}\left(b_{2}^{\prime-1} g_{2} b_{2}^{\prime}\right) b_{0}=b_{2}^{-1} g_{2} b_{2} .
\end{aligned}
$$

## 8. Acylindricity of the action of $A(\Gamma)$ on $\Gamma^{e}$

In this section, we prove the following two theorems.
Theorem 8.1. Let $\Gamma$ be a finite simplicial graph such that $|V(\Gamma)| \geqslant 2$ and $\bar{\Gamma}$ is connected. Then the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$ is $(R, N)$-acylindrical with

$$
\begin{aligned}
& R=R(r)=(2 V+7) r+8 \\
& N=N(r)=2(V-2)(r-1)-1
\end{aligned}
$$

where $V=\max \{4,|V(\Gamma)|\}$. Moreover, for any $x, y \in A(\Gamma)$ with $d_{*}(x, y) \geqslant R$, if $\xi(x, y ; r) \neq\{1\}$, then there exists a loxodromic element $g \in A(\Gamma)$ such that
(i) $\xi(x, y ; r)=\left\{1, g^{ \pm 1}, g^{ \pm 2}, \ldots, g^{ \pm k}\right\}$ for some $1 \leqslant k \leqslant(V-2)(r-1)-1$;
(ii) the Hausdorff distance between the $\langle g\rangle$-orbit of $x$ and that of $y$ is at most $2 r+3$.

Theorem 8.2. Let $\Gamma$ be a finite simplicial graph such that $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected. Then the action of $A(\Gamma)$ on $\Gamma^{e}$ is $(R, N)$-acylindrical with

$$
\begin{aligned}
& R=R(r)=D(2 V+7)(r+1)+10 D \\
& N=N(r)=2(V-2) r-1
\end{aligned}
$$

where $D=\operatorname{diam}(\Gamma)$ and $V=|V(\Gamma)|$. Moreover, for any $x, y \in V\left(\Gamma^{e}\right)$ with $d(x, y) \geqslant R$, if $\xi(x, y ; r) \neq$ $\{1\}$, then there exists a loxodromic element $g \in A(\Gamma)$ such that
(i) $\xi(x, y ; r) \subset\left\{1, g^{ \pm 1}, g^{ \pm 2}, \ldots, g^{ \pm k}\right\}$ for some $1 \leqslant k \leqslant(V-2) r-1$;
(ii) the Hausdorff distance between the $\langle g\rangle$-orbit of $x$ and that of $y$ is at most $D(2 r+7)$.

The following lemma connects the acylindricities of the actions of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$ and on $\left(\Gamma^{e}, d\right)$. It is an improvement of the argument of Kim and Koberda in the proof of Theorem 30 in [18].

Lemma 8.3. Suppose that $|V(\Gamma)| \geqslant 4$ and both $\Gamma$ and $\bar{\Gamma}$ are connected. Let $D=\operatorname{diam}(\Gamma)$. If the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$ is $\left(R_{1}(r), N_{1}(r)\right)$-acylindrical, then the action of $A(\Gamma)$ on $\left(\Gamma^{e}, d\right)$ is $\left(R_{2}(r), N_{2}(r)\right)$-acylindrical with

$$
\begin{aligned}
& R_{2}(r)=D \cdot R_{1}(r+1)+2 D \\
& N_{2}(r)=N_{1}(r+1)
\end{aligned}
$$

More precisely, for any $v_{1}^{w_{1}}, v_{2}^{w_{2}} \in V\left(\Gamma^{e}\right)$, where $v_{1}, v_{2} \in V(\Gamma)$ and $w_{1}, w_{2} \in A(\Gamma)$, if $d\left(v_{1}^{w_{1}}, v_{2}^{w_{2}}\right) \geqslant$ $R_{2}(r)$, then
(i) $d_{*}\left(w_{1}, w_{2}\right) \geqslant R_{1}(r+1)$;
(ii) $\xi_{\left(\Gamma^{e}, d\right)}\left(v_{1}^{w_{1}}, v_{2}^{w_{2}} ; r\right)$ is contained in $\xi_{\left(A(\Gamma), d_{*}\right)}\left(w_{1}, w_{2} ; r+1\right)$.

Proof. Note that $D=\operatorname{diam}(\Gamma) \neq 0$. Let $d\left(v_{1}^{w_{1}}, v_{2}^{w_{2}}\right) \geqslant R_{2}(r)$ for $v_{1}^{w_{1}}, v_{2}^{w_{2}} \in V\left(\Gamma^{e}\right)$. Since $\Gamma$ is connected, we can apply Lemma 4.3 and obtain

$$
\begin{aligned}
& d_{*}\left(w_{1}, w_{2}\right)=\left\|w_{2} w_{1}^{-1}\right\|_{*} \\
& \quad \geqslant \frac{d\left(v_{2}, v_{2}^{w_{2} w_{1}^{-1}}\right)}{D}-1 \geqslant \frac{d\left(v_{1}, v_{2}^{w_{2} w_{1}^{-1}}\right)-d\left(v_{1}, v_{2}\right)}{D}-1 \\
& \quad \geqslant \frac{d\left(v_{1}^{w_{1}}, v_{2}^{w_{2}}\right)-D}{D}-1 \geqslant \frac{R_{2}(r)-2 D}{D}=R_{1}(r+1)
\end{aligned}
$$

which proves (i).

Let $g \in \xi_{\left(\Gamma^{e}, d\right)}\left(v_{1}^{w_{1}}, v_{2}^{w_{2}} ; r\right)$. Then $d\left(v_{i}^{w_{i} g}, v_{i}^{w_{i}}\right) \leqslant r$ for $i=1,2$. By Lemma 4.3 again,

$$
\begin{aligned}
d_{*}\left(w_{i} g, w_{i}\right) & =\left\|w_{i} g w_{i}^{-1}\right\|_{*} \leqslant d\left(v_{i}^{w_{i} g w_{i}^{-1}}, v_{i}\right)+1 \\
& =d\left(v_{i}^{w_{i} g}, v_{i}^{w_{i}}\right)+1 \leqslant r+1
\end{aligned}
$$

for $i=1,2$, hence $g \in \xi_{\left(A(\Gamma), d_{*}\right)}\left(w_{1}, w_{2} ; r+1\right)$. This shows that the set $\xi_{\left(\Gamma^{e}, d\right)}\left(v_{1}^{w_{1}}, v_{2}^{w_{2}} ; r\right)$ is contained in $\xi_{\left(A(\Gamma), d_{*}\right)}\left(w_{1}, w_{2} ; r+1\right)$, hence (ii) is proved.

Since $\xi_{\left(\Gamma^{e}, d\right)}\left(v_{1}^{w_{1}}, v_{2}^{w_{2}} ; r\right) \subset \xi_{\left(A(\Gamma), d_{*}\right)}\left(w_{1}, w_{2} ; r+1\right)$ and $d_{*}\left(w_{1}, w_{2}\right) \geqslant R_{1}(r+1)$, the $\left(R_{1}, N_{1}\right)$ acylindricity of the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$ implies that

$$
\left|\xi_{\left(\Gamma^{e}, d\right)}\left(v_{1}^{w_{1}}, v_{2}^{w_{2}} ; r\right)\right| \leqslant\left|\xi_{\left(A(\Gamma), d_{*}\right)}\left(w_{1}, w_{2} ; r+1\right)\right| \leqslant N_{1}(r+1)=N_{2}(r) .
$$

Therefore the action of $A(\Gamma)$ on $\left(\Gamma^{e}, d\right)$ is $\left(R_{2}(r), N_{2}(r)\right)$-acylindrical.
Proposition 8.4. Let $g, w \in A(\Gamma) \backslash\{1\}$ and $r, R \geqslant 1$ such that

$$
\|g\|_{*} \leqslant r, \quad\left\|w^{-1} g w\right\|_{*} \leqslant r, \quad\|w\|_{*} \geqslant R, \quad R \geqslant 3 r+7 .
$$

Then there exists a quasi-root decomposition

$$
w=a\left(g_{1}^{\epsilon}\right)^{n} b,
$$

where $a, b, g_{1} \in A(\Gamma), \epsilon \in\{ \pm 1\}$ and $n \geqslant 2$ such that
(i) $\|a\|_{*} \leqslant \frac{1}{2} r+1$ and $\|b\|_{*} \leqslant \frac{3}{2} r+2$;
(ii) $g_{1}$ is cyclically reduced and $g=a g_{1} a^{-1}$ is geodesic.

Notice that $\|w\|_{*} \geqslant R \geqslant 3 r+7 \geqslant 7$, hence the existence of such an element $w$ implies that $|V(\Gamma)| \geqslant 2$ and $\bar{\Gamma}$ is connected (as explained in Remark 6.2).
Proof. Let $g=a g_{1} a^{-1}$ be the geodesic decomposition such that $g_{1}$ is cyclically reduced. Let $h=$ $w^{-1} g w$. Then

$$
h=w^{-1} a g_{1} a^{-1} w=\left(a^{-1} w\right)^{-1} g_{1}\left(a^{-1} w\right) .
$$

By Theorem 3.9, there exists a geodesic decomposition of $a^{-1} w$

$$
\begin{equation*}
a^{-1} w=w_{1} w_{2} w_{3} \tag{5}
\end{equation*}
$$

such that (i) $w_{1} \rightleftharpoons g_{1}$; (ii) $g_{1}^{w_{2}}$ is a cyclic conjugation; (iii) $h=w_{3}^{-1} \cdot g_{1}^{w_{2}} \cdot w_{3}$ is geodesic.
Claim 1. The following hold.
(i) $\left\|w_{2}\right\|_{*} \geqslant 3$, hence $w_{2}$ is strongly non-split.
(ii) $g_{1}^{w_{2}}$ is either a left cyclic conjugation or a right cyclic conjugation.

Proof of Claim 1. Since both $g=a g_{1} a^{-1}$ and $h=w_{3}^{-1} g_{1}^{w_{2}} w_{3}$ are geodesic decompositions,

$$
\begin{gathered}
\left\|g_{1}\right\|_{*}+2\|a\|_{*}-4 \leqslant\|g\|_{*} \leqslant\left\|g_{1}\right\|_{*}+2\|a\|_{*}, \\
\left\|g_{1}^{w_{2}}\right\|_{*}+2\left\|w_{3}\right\|_{*}-4 \leqslant\|h\|_{*} \leqslant\left\|g_{1}^{w_{2}}\right\|_{*}+2\left\|w_{3}\right\|_{*}
\end{gathered}
$$

(by Corollary 4.8), whence

$$
\begin{gathered}
\frac{\|g\|_{*}-\left\|g_{1}\right\|_{*}}{2} \leqslant\|a\|_{*} \leqslant \frac{\|g\|_{*}-\left\|g_{1}\right\|_{*}}{2}+2, \\
\frac{\|h\|_{*}-\left\|g_{1}^{w_{2}}\right\|_{*}}{2} \leqslant\left\|w_{3}\right\|_{*} \leqslant \frac{\|h\|_{*}-\left\|g_{1}^{w_{2}}\right\|_{*}}{2}+2 .
\end{gathered}
$$

Since $w_{1} \rightleftharpoons g_{1} \neq 1$, we have $\left\|w_{1}\right\|_{*} \leqslant 1$. Since $g_{1} \neq 1$ and both $g=a g_{1} a^{-1}$ and $h=w_{3}^{-1} g_{1}^{w_{2}} w_{3}$ are geodesic, we have $1 \leqslant\left\|g_{1}\right\|_{*} \leqslant\|g\|_{*} \leqslant r$ and $1 \leqslant\left\|g_{1}^{w_{2}}\right\|_{*} \leqslant\|h\|_{*} \leqslant r$. Since $a^{-1} w=w_{1} w_{2} w_{3}$,

$$
\begin{aligned}
\|w\|_{*} & \leqslant\|a\|_{*}+\left\|w_{1}\right\|_{*}+\left\|w_{2}\right\|_{*}+\left\|w_{3}\right\|_{*} \\
& \leqslant\left(\frac{\|g\|_{*}-\left\|g_{1}\right\|_{*}}{2}+2\right)+1+\left\|w_{2}\right\|_{*}+\left(\frac{\|h\|_{*}-\left\|g_{1}^{w_{2}}\right\|_{*}}{2}+2\right) \\
& \leqslant\left(\frac{r-1}{2}+2\right)+1+\left\|w_{2}\right\|_{*}+\left(\frac{r-1}{2}+2\right) \\
& =\left\|w_{2}\right\|_{*}+r+4 .
\end{aligned}
$$

Therefore $\left\|w_{2}\right\|_{*} \geqslant\|w\|_{*}-r-4 \geqslant R-r-4 \geqslant 2 r+3 \geqslant 3$, hence $w_{2}$ is strongly non-split (by Lemma 4.15). This proves (i).

Assume that the cyclic conjugation $g_{1}^{w_{2}}$ is neither a left cyclic conjugation nor a right cyclic conjugation. Then, by Proposition 3.7(v), $w_{2}=w_{2}^{\prime} w_{2}^{\prime \prime}$ is geodesic for some $w_{2}^{\prime}, w_{2}^{\prime \prime} \in A(\Gamma) \backslash\{1\}$ such that $g_{1}^{w_{2}^{\prime}}$ (resp. $g_{1}^{w_{2}^{\prime \prime}}$ ) is a left (resp. right) cyclic conjugation and $w_{2}^{\prime} \rightleftharpoons w_{2}^{\prime \prime}$. Since both $w_{2}^{\prime}$ and $w_{2}^{\prime \prime}$ are nontrivial, we have $\left\|w_{2}^{\prime}\right\|_{*}=\left\|w_{2}^{\prime \prime}\right\|_{*}=1$, hence $\left\|w_{2}\right\|_{*}=\left\|w_{2}^{\prime} w_{2}^{\prime \prime}\right\|_{*} \leqslant\left\|w_{2}^{\prime}\right\|_{*}+\left\|w_{2}^{\prime \prime}\right\|_{*}=2$, which contradicts $\left\|w_{2}\right\|_{*} \geqslant 3$. Therefore $g_{1}^{w_{2}}$ is either a left cyclic conjugation or a right cyclic conjugation. This proves (ii).

Claim 2. The following hold.
(i) $g_{1}$ is strongly non-split with $\left|\operatorname{supp}\left(g_{1}\right)\right| \geqslant 2$ and $2 \leqslant\left\|g_{1}\right\|_{*} \leqslant r$.
(ii) $\|a\|_{*} \leqslant \frac{1}{2} r+1, w_{1}=1,\left\|w_{2}\right\|_{*} \geqslant R-r-2$ and $\left\|w_{3}\right\|_{*} \leqslant \frac{1}{2} r+1$.

Proof of Claim 2. (i) Since $g_{1}^{w_{2}}$ is either a left or a right cyclic conjugation (by Claim 1),

$$
w_{2} \leqslant_{L} g_{1}^{n} \quad \text { or } \quad w_{2}^{-1} \leqslant_{R} g_{1}^{n}
$$

for some $n \geqslant 1$ (by Proposition [3.8). Since $\left\|w_{2}\right\|_{*} \geqslant 3$, we have $\left\|g_{1}^{n}\right\|_{*} \geqslant 3$. By Lemma 6.3, $g_{1}$ is strongly non-split with $\left|\operatorname{supp}\left(g_{1}\right)\right| \geqslant 2$ and $\left\|g_{1}\right\|_{*} \geqslant 2$. On the other hand, $\left\|g_{1}\right\|_{*} \leqslant\|g\|_{*} \leqslant r$ because $g=a g_{1} a^{-1}$ is geodesic.
(ii) If $w_{1} \neq 1$, then $\left\|g_{1}\right\|_{*} \leqslant 1$ because $g_{1} \rightleftharpoons w_{1}$, which contradicts $\left\|g_{1}\right\|_{*} \geqslant 2$. Therefore $w_{1}=1$.

Since $g_{1}$ is strongly non-split and $g_{1}^{w_{2}}$ is a cyclic conjugation, $g_{1}^{w_{2}}$ is also strongly non-split and $\left|\operatorname{supp}\left(g_{1}^{w_{2}}\right)\right|=\left|\operatorname{supp}\left(g_{1}\right)\right| \geqslant 2$. Therefore $\left\|g_{1}^{w_{2}}\right\|_{*} \geqslant 2$ (by Lemma 4.15).

Since $w_{1}=1,\|g\|_{*} \leqslant r,\|h\|_{*} \leqslant r,\left\|g_{1}\right\|_{*} \geqslant 2$ and $\left\|g_{1}^{w_{2}}\right\|_{*} \geqslant 2$, using the inequalities in the proof of Claim 1, we have

$$
\begin{aligned}
\|a\|_{*} & \leqslant \frac{\|g\|_{*}-\left\|g_{1}\right\|_{*}}{2}+2 \leqslant \frac{r-2}{2}+2=\frac{r}{2}+1 \\
\left\|w_{3}\right\|_{*} & \leqslant \frac{\|h\|_{*}-\left\|g_{1}^{w_{2}}\right\|_{*}}{2}+2 \leqslant \frac{r-2}{2}+2=\frac{r}{2}+1 \\
\|w\|_{*} & \leqslant\|a\|_{*}+\left\|w_{1}\right\|_{*}+\left\|w_{2}\right\|_{*}+\left\|w_{3}\right\|_{*} \\
& \leqslant\left(\frac{r}{2}+1\right)+0+\left\|w_{2}\right\|_{*}+\left(\frac{r}{2}+1\right)=\left\|w_{2}\right\|_{*}+r+2 .
\end{aligned}
$$

Therefore $\|a\|_{*} \leqslant \frac{1}{2} r+1,\left\|w_{3}\right\|_{*} \leqslant \frac{1}{2} r+1$ and $\left\|w_{2}\right\|_{*} \geqslant\|w\|_{*}-r-2 \geqslant R-r-2$.
Claim 3. Let $\epsilon=1$ (resp. $\epsilon=-1$ ) if $g_{1}^{w_{2}}$ is a left (resp. right) cyclic conjugation. Then there exists a quasi-root decomposition

$$
w=a\left(g_{1}^{\epsilon}\right)^{n} b
$$

such that $n \geqslant 2$ and $\|b\|_{*} \leqslant \frac{3}{2} r+2$.
Proof of Claim 3. Suppose that $g_{1}^{w_{2}}$ is a left cyclic conjugation. Then $w_{2} \leqslant L g_{1}^{k}$ for some $k \geqslant 1$ (by Proposition (3.8). Hence

$$
w_{2}=g_{1}^{n} d
$$

is geodesic for some $0 \leqslant n \leqslant k$ and $d \in A(\Gamma)$ with $\|d\|_{*} \leqslant\left\|g_{1}\right\|_{*}+1$ (by Corollary 5.6). Notice that $n \geqslant 2$ because $\left\|g_{1}\right\|_{*} \leqslant r$ whereas

$$
\left\|g_{1}^{n}\right\|_{*} \geqslant\left\|w_{2}\right\|_{*}-\|d\|_{*} \geqslant(R-r-2)-(r+1)=R-2 r-3 \geqslant r+4 .
$$

Since $a^{-1} w=w_{1} w_{2} w_{3}$ and $w_{1}=1$, we have

$$
w=a w_{2} w_{3}=a g_{1}^{n} d w_{3} .
$$

We will now prove that the decomposition $w=a g_{1}^{n} d w_{3}$ is geodesic. Since $g=a g_{1} a^{-1}$ is geodesic and $g_{1}$ is cyclically reduced, $a g_{1}^{n}$ is geodesic (by Lemmas 2.8 and 3.4). Since both $w_{2} w_{3}$ and $w_{2}=g_{1}^{n} d$ are geodesic, $g_{1}^{n} d w_{3}$ is also geodesic. Recall $\left\|g_{1}^{n}\right\|_{*} \geqslant r+4 \geqslant 2$. Therefore $w=a g_{1}^{n} d w_{3}$ is geodesic (by Lemma 4.12).

Let $b=d w_{3}$. Then $w=a g_{1}^{n} b$ is geodesic and

$$
\|b\|_{*} \leqslant\|d\|_{*}+\left\|w_{3}\right\|_{*} \leqslant\left(\left\|g_{1}\right\|_{*}+1\right)+\left(\frac{r}{2}+1\right) \leqslant r+1+\frac{r}{2}+1=\frac{3}{2} r+2 .
$$

Therefore $w=a g_{1}^{n} b$ is a quasi-root decomposition with the desired properties.
Now suppose that $g_{1}^{w_{2}}$ is a right cyclic conjugation. Then $\left(g_{1}^{-1}\right)^{w_{2}}$ is a left cyclic conjugation (by Proposition (3.8). From the above argument, $w=a\left(g_{1}^{-1}\right)^{n} b$ is a quasi-root decomposition with the desired properties.

The proof is now completed.
Remark 8.5. In Proposition 8.4, notice that

$$
\begin{aligned}
& w=a\left(g_{1}^{\epsilon}\right)^{n} b=\left(a\left(g_{1}^{\epsilon}\right)^{n} a^{-1}\right) a b=\left(g^{\epsilon}\right)^{n} a b, \\
& \|a b\|_{*} \leqslant\|a\|_{*}+\|b\|_{*} \leqslant\left(\frac{1}{2} r+1\right)+\left(\frac{3}{2} r+2\right)=2 r+3
\end{aligned}
$$

Thus one could understand Proposition 8.4 as follows: if $\|w\|_{*}$ is large but both $\|g\|_{*}$ and $\left\|w^{-1} g w\right\|_{*}$ are small, then $w=g^{n} c$ for some integer $n$ and $c \in A(\Gamma)$ with $\|c\|_{*}$ small.

If $\left\|w^{-1} g w\right\|_{*} \leqslant r$ and $g=a g_{1} a^{-1}$ in the statement of Proposition 8.4 are respectively replaced with $\left\|w g w^{-1}\right\|_{*} \leqslant r$ and $g=a^{-1} g_{1} a$, then we have the following corollary.

Corollary 8.6. Let $g, w \in A(\Gamma) \backslash\{1\}$ and $r, R \geqslant 1$ such that

$$
\|g\|_{*} \leqslant r, \quad\left\|w g w^{-1}\right\|_{*} \leqslant r, \quad\|w\|_{*} \geqslant R, \quad R \geqslant 3 r+7 .
$$

Then there exists a quasi-root decomposition

$$
w=b\left(g_{1}^{\epsilon}\right)^{n} a,
$$

where $a, b, g_{1} \in A(\Gamma), \epsilon \in\{ \pm 1\}$ and $n \geqslant 2$ such that
(i) $\|a\|_{*} \leqslant \frac{1}{2} r+1$ and $\|b\|_{*} \leqslant \frac{3}{2} r+2$;
(ii) $g_{1}$ is cyclically reduced and $g=a^{-1} g_{1} a$ is geodesic.

We will now prove Theorem 8.1 .

Proof of Theorem 8.1. Choose $x, y \in A(\Gamma)$ with $d_{*}(x, y) \geqslant R$. Let $w=y x^{-1}$, hence $\|w\|_{*}=d_{*}(x, y) \geqslant$ $R$.

We may assume $\xi(1, w ; r) \neq\{1\}$ because otherwise $\xi(x, y ; r)=x^{-1} \xi(1, w ; r) x=\{1\}$ and there is nothing to prove.

By Lemma 4.5(ii), the set $\xi(1, w ; r)$ is closed under taking a root, i.e. if $h^{k} \in \xi(1, w ; r)$ for some $h \in A(\Gamma)$ and $k \geqslant 1$, then $h \in \xi(1, w ; r)$. Therefore there exists a primitive element $g_{0}$ in $\xi(1, w ; r) \backslash\{1\}$, hence $\left\|g_{0}\right\|_{*}=d_{*}\left(g_{0}, 1\right) \leqslant r$ and $\left\|w g_{0} w^{-1}\right\|_{*}=d_{*}\left(w g_{0}, w\right) \leqslant r$.

We will now show that $g_{0}^{ \pm 1}$ is uniquely determined from $w=y x^{-1}$. Let

$$
g_{0}=a^{-1} g_{1} a
$$

be the geodesic decomposition such that $g_{1}$ is cyclically reduced. Then $g_{1}$ is also primitive and $\left\|g_{1}\right\|_{*} \leqslant\left\|g_{0}\right\|_{*} \leqslant r$. Since $R=(2 V+7) r+8 \geqslant 3 r+7,\left(w, g_{0}, g_{1}, a, R, r\right)$ satisfies the conditions on ( $w, g, g_{1}, a, R, r$ ) in Corollary 8.6, hence there exists a quasi-root decomposition

$$
w=b\left(g_{1}^{\epsilon}\right)^{n} a,
$$

where $b \in A(\Gamma), \epsilon \in\{ \pm 1\}, n \geqslant 2,\|a\|_{*} \leqslant \frac{1}{2} r+1$ and $\|b\|_{*} \leqslant \frac{3}{2} r+2$.
Let $A=\frac{1}{2} r+1$ and $B=\frac{3}{2} r+2$. Then $g_{1}^{\epsilon}$ is a primitive $(B, A, r)^{*}$-quasi-root of $w$. Observe that $2 A+2 B+(2 V+3) r+2=(r+2)+(3 r+4)+(2 V+3) r+2=(2 V+7) r+8=R$, hence

$$
\|w\|_{*} \geqslant R=2 A+2 B+(2 V+3) r+2
$$

The tuple ( $w, g_{1}^{\epsilon}, b, a, B, A, r$ ) now satisfies the conditions on ( $h, g_{1}, a_{1}, b_{1}, A, B, r$ ) in Theorem 7.4, Therefore the primitive element $g_{0}^{\epsilon}$ is uniquely determined from $w$ because $g_{0}^{\epsilon}=a^{-1} g_{1}^{\epsilon} a$ is the rightward-extraction of the $(B, A, r)^{*}$-quasi-root $g_{1}^{\epsilon}$. This means that each element of $\xi(1, w ; r)$ is a power of $g_{0}$, hence $\xi(1, w ; r) \subset\left\langle g_{0}\right\rangle$. Since

$$
\begin{aligned}
\left\|g_{1}^{n}\right\|_{*} & =\left\|\left(g_{1}^{\epsilon}\right)^{n}\right\|_{*} \geqslant\|w\|_{*}-\|b\|_{*}-\|a\|_{*} \geqslant R-B-A \\
& \geqslant A+B+(2 V+3) r+2 \geqslant 3
\end{aligned}
$$

the cyclically reduced element $g_{1}$ is loxodromic (by Lemma 6.3), hence $\left\|g_{1}^{(V-2) j}\right\|_{*} \geqslant j+2$ for all $j \geqslant 1$ (by Lemma 6.3, Proposition 5.7 and Corollary 4.10). Since $g_{0}$ is conjugate to $g_{1}, g_{0}$ is also loxodromic. Since $g_{0}^{(V-2) j}=a^{-1} g_{1}^{(V-2) j} a$ is a geodesic decomposition (by Lemma [2.8(iii)), $\left\|g_{0}^{(V-2) j}\right\|_{*} \geqslant\left\|g_{1}^{(V-2) j}\right\|_{*} \geqslant j+2$ for all $j \geqslant 1$.

If $k \geqslant(V-2)(r-1)$, then $\left\|g_{0}^{k}\right\|_{*} \geqslant\left\|g_{0}^{(V-2)(r-1)}\right\|_{*} \geqslant(r-1)+2=r+1$, hence $g_{0}^{k} \notin \xi(1, w ; r)$. From this fact and Lemma 4.5, it follows that

$$
\xi(1, w ; r)=\left\{1, g_{0}^{ \pm 1}, \ldots, g_{0}^{ \pm k}\right\}
$$

for some $1 \leqslant k \leqslant(V-2)(r-1)-1$.
Let $g=x^{-1} g_{0} x$. Then $g$ is also loxodromic. Since $\xi(x, y ; r)=x^{-1} \cdot \xi(1, w ; r) \cdot x$,

$$
\xi(x, y ; r)=\left\{1, g^{ \pm 1}, \ldots, g^{ \pm k}\right\}
$$

hence (i) is proved.
Let $N(r)=2(V-2)(r-1)-1$. Since $|\xi(x, y ; r)|=2 k+1 \leqslant 2(V-2)(r-1)-1=N(r)$, the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$ is $(R(r), N(r))$-acylindrical.

Since $g_{0}=a^{-1} g_{1} a, g=x^{-1} g_{0} x$ and $y x^{-1}=w=b\left(g_{1}^{\epsilon}\right)^{n} a$, we get

$$
y=w x=b\left(g_{1}^{\epsilon}\right)^{n} a x=b a x \cdot x^{-1} a^{-1}\left(g_{1}^{\epsilon}\right)^{n} a x=\operatorname{bax}\left(g^{\epsilon}\right)^{n} .
$$

Hence $d_{*}\left(y, x\left(g^{\epsilon}\right)^{n}\right)=d_{*}\left(b a x\left(g^{\epsilon}\right)^{n}, x\left(g^{\epsilon}\right)^{n}\right)=\|b a\|_{*} \leqslant\|b\|_{*}+\|a\|_{*} \leqslant 2 r+3$. Therefore the Hausdorff distance between the $\langle g\rangle$-orbits $x\langle g\rangle$ and $y\langle g\rangle$ is at most $2 r+3$, hence (ii) is proved.

Remark 8.7. The above proof shows that $g_{1}^{\epsilon}$ is a primitive $\left(\frac{3}{2} r+2, \frac{1}{2} r+1, r\right)^{*}$-quasi-root of $w=$ $b\left(g_{1}^{\epsilon}\right)^{n} a$. Notice that the rightward-extraction of $g_{1}^{\epsilon}$ is $a^{-1} g_{1}^{\epsilon} a$, and that $x g x^{-1}=a^{-1} g_{1} a$. Therefore either $x g x^{-1}$ or $x g^{-1} x^{-1}$ is the rightward-extraction of a primitive $\left(\frac{3}{2} r+2, \frac{1}{2} r+1, r\right)^{*}$-quasi-root of $y x^{-1}$.

We are now ready to prove Theorem 8.2 .
Proof of Theorem 8.2. By Theorem8.1, the action of $A(\Gamma)$ on $\left(A(\Gamma), d_{*}\right)$ is $\left(R_{1}(r), N_{1}(r)\right)$-acylindrical with

$$
\begin{aligned}
& R_{1}(r)=(2 V+7) r+8, \\
& N_{1}(r)=2(V-2)(r-1)-1 .
\end{aligned}
$$

Applying Lemma 8.3 to the above, the action of $A(\Gamma)$ on $\left(\Gamma^{e}, d\right)$ is $(R(r), N(r))$-acylindrical with

$$
\begin{aligned}
& R(r)=D \cdot R_{1}(r+1)+2 D=D(2 V+7)(r+1)+10 D, \\
& N(r)=N_{1}(r+1)=2(V-2) r-1 .
\end{aligned}
$$

Choose $x, y \in V\left(\Gamma^{e}\right)$ with $d(x, y) \geqslant R(r)$ and $\xi_{\left(\Gamma^{e}, d\right)}(x, y ; r) \neq\{1\}$. Then there exist $v_{1}, v_{2} \in V(\Gamma)$ and $w_{1}, w_{2} \in A(\Gamma)$ such that $x=v_{1}^{w_{1}}$ and $y=v_{2}^{w_{2}}$. By Lemma 8.3,

$$
\begin{gathered}
d_{*}\left(w_{1}, w_{2}\right) \geqslant R_{1}(r+1), \\
\xi_{\left(\Gamma^{e}, d\right)}\left(v_{1}^{w_{1}}, v_{2}^{w_{2}} ; r\right) \subset \xi_{\left(A(\Gamma), d_{*}\right)}\left(w_{1}, w_{2} ; r+1\right) .
\end{gathered}
$$

Since $\xi_{\left(\Gamma^{e}, d\right)}\left(v_{1}^{w_{1}}, v_{2}^{w_{2}} ; r\right) \neq\{1\}$, we have $\xi_{\left(A(\Gamma), d_{*}\right)}\left(w_{1}, w_{2} ; r+1\right) \neq\{1\}$. Hence $\left(w_{1}, w_{2}\right)$ satisfies all the conditions on $(x, y)$ in Theorem 8.1. Therefore, by Theorem 8.1(i),

$$
\xi_{\left(\Gamma^{e}, d\right)}(x, y ; r) \subset \xi_{\left(A(\Gamma), d_{*}\right)}\left(w_{1}, w_{2} ; r+1\right)=\left\{1, g^{ \pm 1}, g^{ \pm 2}, \ldots, g^{ \pm k}\right\}
$$

for some loxodromic element $g \in A(\Gamma)$ and $1 \leqslant k \leqslant(V-2) r-1$, hence (i) is proved.
Since the Hausdorff distance between the $\langle g\rangle$-orbits of $w_{1}$ and $w_{2}$ is at most $2(r+1)+3=2 r+5$ (by Theorem 8.1(ii)), $w_{2}=c w_{1} g^{n}$ for some $n \in \mathbb{Z}$ and $c \in A(\Gamma)$ with $\|c\|_{*} \leqslant 2 r+5$. Hence we get (by Lemma 4.3)

$$
\begin{aligned}
d\left(x^{g^{n}}, y\right) & =d\left(v_{1}^{w_{1} g^{n}}, v_{2}^{w_{2}}\right)=d\left(v_{1}, v_{2}^{w_{2} g^{-n} w_{1}^{-1}}\right)=d\left(v_{1}, v_{2}^{c}\right) \\
& \leqslant d\left(v_{1}, v_{2}\right)+d\left(v_{2}, v_{2}^{c}\right) \leqslant D+D\left(\|c\|_{*}+1\right) \\
& =D\left(\|c\|_{*}+2\right) \leqslant D(2 r+7) .
\end{aligned}
$$

Therefore the Hausdorff distance between $x^{\langle g\rangle}$ and $y^{\langle g\rangle}$ is at most $D(2 r+7$ ), hence (ii) is proved.

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