A census of small Schurian association schemes

Jesse Lansdown

ABSTRACT. Using the classification of transitive groups of degree n, for $2 \le n \le 48$, we classify the Schurian association schemes of order n, and as a consequence, the transitive groups of degree n that are 2-closed. In addition, we compute the character table of each association scheme and provide a census of important properties. Finally, we compute the 2-closure of each transitive group of degree n, for $2 \le n \le 48$. The results of this classification are made available as a supplementary database.

1. Introduction

Association schemes are some of the most important objects in algebraic combinatorics, with applications to coding theory, finite geometry, group theory, and even statistics. They generalise the concepts of strongly-regular and distance-regular graphs to describe structures with high degrees of regularity. As a result they arise naturally in many settings. Moreover, Delsarte showed that many combinatorial objects can be described by the eigenspaces of an association scheme [7], providing a powerful tool for studying geometric objects, cliques of graphs, designs, codes, and more.

The (not necessarily commutative) association schemes of order n have been classified for $1 \le n \le 34$ and also for n = 38 by Nomiyama [29], Hirasaka [15], Hirasaka and Suga [16], Hanaki and Miyamoto [11, 12, 13], and Hanaki, Kharaghani, Mohammadian, and Tayfeh-Rezaie [9]. With the exception of very small values of n, this classification necessitated the use of a computer and further classification is extremely difficult due to combinatorial explosion. Hanaki maintains a website with these association schemes at [10]. This database has been used for forming and testing conjectures on association schemes.

The regularity properties of association schemes may be thought of as capturing combinatorial symmetry. Indeed, their connection to group actions was a crucial motivator in their development [14]. Association schemes with the strongest connection to groups are called *Schurian*; their relations correspond to the orbitals of a transitive group. Schurian association schemes remain one of the strongest tools for studying group actions and combinatorial objects with high degrees of symmetry. For example, they have found recent use in exploring the synchronisation hierarchy of permutation groups (e.g. [1]).

The transitive groups of degree n have been classified for $1 \le n \le 48$ by Miller [28, 27], Royle [30], Hulpke [19], Cannon and Holt [6], Holt and Royle [17], and Holt, Royle,

School of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand

Centre for the Mathematics of Symmetry and Computation, Department of Mathematics and Statistics, The University of Western Australia, Crawley, WA 6009, Australia.

E-mail address: jesse.lansdown@canterbury.ac.nz.

The author was supported by the Australian Research Council Discovery Grant DP200101951. This work was supported by resources provided by the Pawsey Supercomputing Centre with funding from the Australian Government and the Government of Western Australia.

and Tracey [18], mostly by computer, with the exception of some small values of n. For n=48 alone, there are 195,826,352 conjugacy classes of transitive groups, making their classification a particular milestone in the classification of transitive permutation groups of low degree. This recent work on classifying transitive groups makes the main result of this paper possible.

Theorem 1.1. There are 24678 Schurian association schemes of order 2 to 48 up to isomorphism.

There is a bijection between the Schurian association schemes and the 2-closed transitive groups, which yields the following:

COROLLARY 1.2. There are 24678 2-closed groups of degree 2 to 48 up to isomorphism.

We provide the association schemes corresponding to Theorem 1.1 as a supplement [22] to this paper. It is hoped that this database will prove as useful for studying Schurian association schemes and 2-closed groups as [10] has for general association schemes. We also compute the character table of each association scheme since this is a computationally intensive, and potentially prohibitive, task. A census of some of the important properties is provided in Table 1, where the total number of association schemes of each order is given, along with the number which are stratifiable, commutative, symmetric, primitive, metric, cometric, and thin, respectively. The rows that are highlighted indicate previously unclassified Schurian association schemes, which are not available in [10] or elsewhere.

We also identify and provide as an additional supplement [23] containing the 2-closures for all transitive groups of degree 2 to 48. This supplementary data is likely to be of special interest in the case n = 48 since transitive identification is not computable in Magma [4] or GAP [8] for this degree.

The computations required by the classification are described in Section 3. In particular, the fast practical computation of automorphisms and isomorphisms is made possible by representing association schemes as suitable digraphs. Hence the 2-closures of the corresponding groups may also be computed quickly, which is otherwise difficult and slow.

2. Association schemes

The term association scheme is used in different ways throughout the literature¹. The definition used in this paper is also known as a homogeneous coherent configuration. We use this definition because it is the most general and is used for the corresponding objects in the catalogue of association schemes at [10]. We refer to [3] for greater detail on association schemes, but provide some of the relevant definitions and results (without proof) in this section. The connection between association schemes and permutation groups is explored in greater detail in [5].

Let Ω be a finite set of cardinality n and let $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ be subsets of $\Omega \times \Omega$. We shall call $R_i \in \mathcal{R}$ a relation and refer to $R_i^{\top} = \{(y, x) : (x, y) \in R_i\}$ as its converse relation. Then (Ω, \mathcal{R}) is an association scheme with d classes and order n if the following hold:

- (1) \mathcal{R} is a partition of $\Omega \times \Omega$,
- (2) $R_0 = \{(x, x) : x \in \Omega\},\$

 $^{^{1}}$ Peter Cameron discusses the differing "association scheme" terminology on his blog: https://cameroncounts.wordpress.com/2014/06/08/terminology-association-scheme-or-coherent-configuration/

- (3) $R_i^{\top} \in \mathcal{R}$ for all $R_i \in \mathcal{R}$, (4) there exist constants p_{ij}^k (called *intersection numbers*) for $0 \leq i, j, k \leq d$ such that for any $(x,y) \in R_k$,

$$p_{ij}^k = |\{z \in \Omega : (x, z) \in R_i, (z, y) \in R_j\}|.$$

Moreover an association scheme is symmetric if $R_i^{\top} = R_i$ for all $0 \leq i \leq d$, commutative if $p_{ij}^k = p_{ji}^k$ for all $0 \le i, j, k \le d$, and *stratifiable* if an association scheme is formed by replacing each relation and its converse by their union. A scheme is called *thin* if d = n - 1, in which case it is equivalent to a group. Two association schemes (Ω, \mathcal{R}) and (Δ, \mathcal{S}) are isomorphic if they have the same number of classes, d, and there exist bijections $f:\Omega\to\Delta$ and $g: \{0, \ldots d\} \to \{0, \ldots d\}$ such that $(x, y) \in R_i$ if and only if $(f(x), f(y)) \in S_{q(i)}$.

Let G be a transitive permutation group acting on Ω . Then the orbits of G on $\Omega \times \Omega$ form the relations of an association scheme, $\mathcal{K}(G)$. An association scheme (Ω, \mathcal{R}) is called Schurian if $(\Omega, \mathcal{R}) = \mathcal{K}(G)$ for some transitive permutation group G. An automorphism of an association scheme (Ω, \mathcal{R}) is a permutation of Ω which fixes every $R_i \in \mathcal{R}$. The set of all automorphisms forms the automorphism group, $\operatorname{Aut}((\Omega, \mathcal{R}))$. Note that $G \leq \operatorname{Aut}(\mathcal{K}(G))$. Moreover, G is 2-closed in the case of equality².

The adjacency matrix with respect to R_i is the $n \times n$ matrix A_i indexed by the elements of Ω , where

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } (x,y) \in R_i; \\ 0, & \text{otherwise.} \end{cases}$$

If the digraph defined by the adjacency matrix A_i is connected for all $1 \leq i \leq d$, then we call (Ω, \mathcal{R}) primitive. To store an association scheme compactly, we define the relation matrix,

$$M := \sum_{i=0}^{d} iA_i.$$

Note that since the relations partition $\Omega \times \Omega$, we can recover each A_i and hence the relations \mathcal{R} for (Ω, \mathcal{R}) from the relation matrix M.

For an association scheme (Ω, \mathcal{R}) , the adjacency matrices span a semisimple \mathbb{C} -algebra called the adjacency algebra or the Bose-Mesner algebra, denoted \mathbb{CR} . When (Ω, \mathcal{R}) is commutative, there exists a second basis $\{E_0, \ldots, E_d\}$ for $\mathbb{C}\mathcal{R}$ consisting of minimal idempotents (ie. $E_i E_j = 0$ for $i \neq j$, and $E_i E_i = E_i$). A representation φ of an association scheme (Ω, \mathcal{R}) is an algebra homomorphism from the adjacency algebra $\mathbb{C}\mathcal{R}$ to the full matrix algebra over \mathbb{C} and we define the *character* χ afforded by φ by $\chi(A_i) = \operatorname{trace}(\varphi(A_i))$. The set of irreducible characters of \mathbb{CR} is denoted by $Irr(\mathcal{R})$. The standard representation $\Gamma_{\mathcal{R}}$ is the representation which sends each adjacency matrix to itself, that is, $\Gamma_{\mathcal{R}}(A_i) = A_i$ for all $0 \leqslant i \leqslant d$. The standard character $\gamma_{\mathcal{R}}$ is the character afforded by the standard representation $\Gamma_{\mathcal{R}}$ and satisfies

$$\gamma_{\mathcal{R}}(A_i) = \begin{cases} |\Omega| 1_{\mathbb{C}}, & i = 0; \\ 0, & \text{otherwise,} \end{cases}$$

²The 2-closure of a group G is the largest subgroup of $\operatorname{Sym}(\Omega)$ preserving the orbits of G on $\Omega \times \Omega$ and G is called 2-closed if it equals its 2-closure.

and has the irreducible decomposition

$$\gamma_{\mathcal{R}} = \sum_{\chi \in \operatorname{Irr}(\mathcal{R})} m_{\chi} \cdot \chi,$$

where m_{χ} is called the multiplicity of χ . Let T be the $|\operatorname{Irr}(\mathcal{R})| \times (d+1)$ matrix with rows indexed by $\operatorname{Irr}(\mathcal{R})$ and columns by \mathcal{R} such that the $T_{\chi,R_i} = \chi(A_i)$. Then T is called the character table of (Ω, \mathcal{R}) .

Finally, let (Ω, \mathcal{R}) be symmetric. If there is an ordering $\{A_i\}_{i=0}^d$ such that there exist polynomials v_i of degree i for $i \in \{0, \ldots, d\}$ with the property $v_i(A_1) = A_i$, then we call (Ω, \mathcal{R}) P-polynomial or metric. Similarly, if there is an ordering $\{E_i\}_{i=0}^d$ such that there exist polynomials q_i of degree i for $i \in \{0, \ldots, d\}$ with the property $q_i(E_1) = E_i$, then we call (Ω, \mathcal{R}) Q-polynomial or cometric. Metric and cometric association schemes have interesting algebraic and combinatorial properties and are often studied in their own right.

3. Computation

By definition, an association scheme (Ω, \mathcal{R}) is Schurian precisely when there exists a group G acting transitively on Ω such that $(\Omega, \mathcal{R}) = \mathcal{K}(G)$. Let \mathcal{G} be a set of transitive permutation groups of degree n such that every conjugacy class of transitive subgroups of S_n has precisely one representative in \mathcal{G} . Then G is conjugate in S_n to some group in \mathcal{G} , and without loss of generality we may assume that $G \in \mathcal{G}$.

Note that for two non-isomorphic groups G_1 and G_2 in \mathcal{G} it is possible that $\mathcal{K}(G_1) \cong \mathcal{K}(G_2)$. However, we may use the fact that $G \leq \operatorname{Aut}(\mathcal{K}(G))$ to determine non-isomorphic representatives of all Schurian association schemes of order n by only keeping $\mathcal{K}(G)$ when G is the full automorphism group. Hence we derive the classification of Schurian association schemes of order n from the classification of transitive permutation groups of degree n by the following procedure:

```
\begin{array}{l} L \leftarrow \{\} \\ \textbf{for } G \in \mathcal{G} \ \textbf{do} \\ \textbf{if } G = \operatorname{Aut}(\mathcal{K}(G)) \ \textbf{then} \\ L \leftarrow L \cup \{\mathcal{K}(G)\} \\ \textbf{end if} \\ \textbf{end for} \end{array}
```

The transitive permutation groups of degree at most 48 are available in both MAGMA [4] and GAP [8] (via the TransGrp package [20]). In both cases, the groups of degree 32 and 48 must be downloaded³ separately due to their size. The computations for this paper were done in GAP, making use of the author's GAP package, AssociationSchemes [2], which among other things can construct $\mathcal{K}(G)$, find the automorphism group of an association scheme, and compute isomorphisms between association schemes.

The 2-closure of a group G is precisely $Aut(\mathcal{K}(G))$, and so classifying the Schurian association schemes is equivalent to classifying the 2-closed transitive permutation groups. In GAP the 2-closure of a group can be computed using the GAP package GRAPE [31], however

 $^{^3}$ Transitive groups of degree 32 and 48 in Magma: http://magma.maths.usyd.edu.au/magma/download/db/, Transitive groups of degree 32 in GAP: https://www.math.colostate.edu/~hulpke/transgrp/trans32.tgz, Transitive groups of degree 48 in GAP: https://zenodo.org/record/5935751

this is significantly slower than computing the automorphism group of $\mathcal{K}(G)$ using AssociationSchemes directly, although it does provide a means of verification. In practice, it may be faster to check that $\operatorname{Aut}(\mathcal{K}(G)) \leq G$ by testing containment of generators of $\operatorname{Aut}(\mathcal{K}(G))$ in G. To accelerate the computation, the groups in G were divided and multiple jobs were run in parallel. The character table and other properties of the Schurian association schemes were also computed using the AssociationSchemes package. In fact, some improvements have been made to the package in the process. However, manual intervention was required in many difficult cases.

The number of association schemes of each type with a given order are provided in Table 1. The number of Schurian association schemes which are *stratifiable*, *commutative*, symmetric, primitive, metric, cometric, and non-Schurian are abbreviated as strat., com., sym., prim., met., comet., and NS respectively. Up to order 30, the values in the total, stratifiable, symmetric, and primitive columns agree with the values given in Table 2 of [5] where they correspond to the number of 2-closed permutation groups which are transitive, stratifiable, generously transitive, and primitive, respectively. The highlighted rows indicate that the enumeration of the corresponding Schurian association schemes is new, and is not available at [10]. The number of non-Schurian association schemes is given, as found at [10], in column NS for completeness. They indicate that the total number of association schemes may be significantly larger than the number which are Schurian, particularly as the order grows. As a result, further enumeration of association schemes is likely to be increasingly difficult. For example, there are 32730 strongly regular graphs (which are equivalent to 2class association schemes) with 36 vertices [26], already far exceeding the total number of Schurian association schemes of this order.

Having computed the 2-closed groups we were able to further compute the 2-closure (up to conjugacy in S_n) of every transitive group of degree n, for $2 \le n \le 48$. This was achieved by utilising the following observation: if G and G' are groups such that G' is 2-closed, then G' is conjugate in S_n to the 2-closure of G if and only if $\mathcal{K}(G) \cong \mathcal{K}(G')$. Identifying the 2-closure of a group G is then a matter of testing isomorphism of $\mathcal{K}(G)$ against the elements of G. These isomorphism checks were also performed using AssociationSchemes. Recall that an isomorphism between two association schemes G0, G0 and G1 and G2 with G3 classes is given by bijections G3 and G4 and G5. In AssociationSchemes an isomorphism is given by G5, G7 where G7 and G9 are permutations corresponding to G9. Indeed, if G9, and hence the 2-closure of G9, is given by G6, then the automorphism group of G6, and hence the 2-closure of G9, is given by G6.

The results of this paper exploit the speed at which AssociationSchemes is able to compute the automorphism group of an association scheme. Finding the 2-closure of a related group is typically much slower and more difficult. We achieve this by representing an association scheme as an edge-coloured digraph where (a, b) is a directed edge with colour i if and only if $(a, b) \in R_i$. Such a digraph is complete and satisfies the following property: the number of coloured triangles on a given directed edge depends only on the choice of colours. The automorphism group of the association scheme is simply the automorphism group of the corresponding directed graph. Following [25, §14], an edge-coloured digraph may be represented as a vertex-coloured graph with $\log(d+1)$ layers. The automorphism group of the original digraph is then the found by taking the action of the automorphism group of this layered digraph on its first layer. This method is used by AssociationSchemes to compute

Order	Total	Strat.	Com.	Sym.	Prim.	Met.	Comet.	Thin	NS
2	1	1	1	1	1	1	1	1	0
3	2	2	2	1	2	1	1	1	0
4	4	4	4	3	1	2	2	2	0
5	3	3	3	2	3	2	2	1	0
6	8	7	7	4	1	4	4	2	0
7	4	4	4	2	4	2	2	1	0
8	21	20	19	10	1	5	5	5	0
9	12	12	12	6	2	4	4	2	0
10	13	11	11	8	2	6	6	2	0
11	4	4	4	2	4	2	2	1	0
12	59	47	47	21	1	8	8	5	0
13	6	6	6	4	6 1	3	3	$\frac{1}{2}$	0
14	16	14	14	8	$\frac{1}{2}$	6 6	6 5	1	0
15 16	24 206	23 171	$\begin{array}{c} 23 \\ 158 \end{array}$	10 56	$\frac{2}{4}$	9	9		1 16
17	200 5	5	158 5	4	5	3	3	$\begin{array}{c} 14 \\ 1 \end{array}$	$\begin{array}{c} 16 \\ 0 \end{array}$
18	93	71	71	32	1	8	3 7	5	$\frac{0}{2}$
19	6	6	6	3	6	$\frac{3}{2}$	$\frac{7}{2}$	1	1
20	95	73	73	41	1	10	8	5	0
21	$\frac{35}{32}$	29	29	11	3	6	5	$\frac{3}{2}$	0
22	16	14	14	8	1	6	6	2	0
23	4	4	4	$\frac{\circ}{2}$	4	$\frac{\circ}{2}$	$\frac{\circ}{2}$	1	18
24	669	454	438	136	1	9	10	15	81
25	32	32	32	20	9	5	5	2	13
26	24	20	20	$\frac{14}{14}$	1	6	6	$\overline{2}$	10
27	122	112	112	38	5	7	6	5	380
28	124	103	103	47	4	10	9	4	61
29	6	6	6	4	6	3	3	1	20
30	228	166	166	73	1	11	10	4	15
31	8	8	8	4	8	2	2	1	98299
32	4261	2579	2264	413	1	13	11	51	13949
33	27	27	27	9	1	4	4	1	0
34	20	16	16	13	1	5	5	2	0
35	43	43	43	17	3	6	6	1	
36	1274	806	804	276	9	17	15	14	
37	9	9	9	6	9	3	3	1	
38	22	19	19	10	1	5	5	2	11
39	44	41	41	15	1	4	4	2	
40	1095	712	687	262	3	11	11	14	
41	8	8	8	6	8	3	3	1	
42	298	210	210	81	1	11	10	6	
43	8	8	8	4	8	2	2	1	
44	112	93	93	40	1	7	7	4	
45	286	270	270	93	5	10	8	2	
46	15	13	13	$7 \\ 2$	$\frac{1}{4}$	$\frac{5}{2}$	$\frac{5}{2}$	2 1	
47	4 15305	7800	7220			12			
48		7890 Sebur		1394			13	32	

Table 1. Schurian association schemes and their properties.

the automorphism group of an association scheme. Since it calls *bliss* [21] or *nauty* [24] to find the automorphism group, it is able to do this very quickly in practice. This method avoids computing intersections of groups or stabilisers of sets and so avoids many significant computational bottlenecks.

AssociationSchemes is also able to compute canonical forms and isomorphisms of association schemes by considering isomorphisms of the corresponding graphs. A key difficulty in this process is accounting for the edge colourings (the order of the relations), since these may be relabelled to give isomorphic association schemes without the corresponding edge-coloured graphs being isomorphic. Prior to Version 3.0.0 of AssociationSchemes, we overcame this by first computing all the algebraic automorphisms (reordering of relations preserving the intersection numbers) and applying the ismorphism/canonisation process to each of the edge-coloured digraphs that resulted. From Version 3.0.0, we instead allow colours to be exchanged within the digraph, combining the ideas in [25, §14] for edge-colouring and interchangable vertex-colouring. This requires a digraph with d layers to be constructed, with an additional vertex added for each colour present. Calling bliss [21] on this digraph establishes an ordering on the relations. Accounting for this ordering on relations, a second call to bliss then determines an ordering on the vertices. The result is a fast, practical method for computing canonical forms of association schemes and isomorphism between association schemes.

4. Database

The complete list of Schurian association schemes of order $2 \le n \le 48$ is made available as a supplementary database at [22]. The database consists of files with the name *SchurianSchemesN* where N is the order of the association schemes. Each line of each file has the form

$$[M,S,x,T,L],$$

where:

- M is the relation matrix of the association scheme.
- S is a list of generators (as permutations) for the automorphism group of the association scheme; note that if $G = \langle S \rangle$ then $\mathcal{K}(G)$ is the association scheme defined by M.
- The integer x is the transitive identification of the automorphism group G in the Magma and GAP libraries⁴.
- T is the character table of the association scheme. Note that some cyclotomic numbers are described as sums of roots of unity with rational coefficients, where the primitive n-th root of unity $e^{2\pi/n}$ is written in the GAP notation E(n).
- The *i*-th entry of L is the multiplicity of the character corresponding to the *i*-th row of T.

⁴The transitive groups use the same ordering in both Magma and GAP, so the transitive identification numbers match.

Consider, for example, the 3-class association scheme with order 8 defined by the relation matrix

$$M = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 0 & 1 & 2 & 2 & 3 & 2 \\ 1 & 2 & 2 & 0 & 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 & 0 & 2 & 1 \\ 2 & 2 & 3 & 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 3 & 1 & 2 & 2 & 0 \end{bmatrix}.$$

It is the Schurian scheme $\mathcal{K}(G)$, where

$$G = \langle (1,3,5,7)(2,4,6,8), (1,3,8)(4,5,7) \rangle,$$

with transitive identification 12. Its character table is

$$T = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & \sqrt{3}i & -\sqrt{3}i & -1 \\ 1 & -\sqrt{3}i & \sqrt{3}i & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

with multiplicities [1,2,2,3]. Observe, $\sqrt{3}i=e^{2\pi/3}-e^{4\pi/3}$ which is E(3) - E(3)^2 in GAP notation. The corresponding entry in the database can be found on line 12 of the file "SchurianSchemes8" where it appears as:

[[[0, 1, 2, 2, 3, 2, 1, 1], [2, 0, 2, 1, 1, 3, 1, 2], [1, 1, 0, 1, 2, 2, 3, 2], [1, 2, 2, 0, 2, 1, 1, 3], [3, 2, 1, 1, 0, 1, 2, 2], [1, 3, 1, 2, 2, 0, 2, 1], [2, 2, 3, 2, 1, 1, 0, 1], [2, 1, 1, 3, 1, 2, 2, 0]], [(1,3,5,7)(2,4,6,8), (1,3,8)(4,5,7)], 12, [[1, 3, 3, 1], [1, E(3)-E(3)^2, -E(3)+E(3)^2, -1], [1, -E(3)+E(3)^2, E(3)-E(3)^2, -1], [1, -1, -1, 1]], [1, 2, 2, 3]].

The properties given in Table 1 are not given in the database because they are easily computable from the data provided. The character tables are included in the database, however, since they are incredibly useful in applications but can be very slow to compute. Note that it is significantly easier to verify a character table than to compute it.

The 2-closures of the transitive permutation groups of degree $2 \le n \le 48$ are made available as a supplementary database at [23]. The database consists of files with the name TwoClosuresN where N is the degree of the permutation groups. Each line of each file has the form

where:

- The integer x is the transitive identification of the group.
- The integer y is the transitive identification of the group isomorphic to the 2-closure of the group with transitive identification x.

For example, line 5491 of the file "TwoClosures48" appears as:

[5491,271829]

This says that the 2-closure of the group accessed by TransitiveGroup(48, 5491) is conjugate in S_{48} to the group accessed by TransitiveGroup(48, 271829). Note that a group is 2-closed if x = y. This data is likely to be particularly useful in the case where n = 48, since there is no means of determining the transitive identification of a group in the GAP or Magma libraries for this degree.

References

- [1] J. Bamberg, M. Giudici, J. Lansdown, and G. F. Royle. Synchronising primitive groups of diagonal type exist. *Bulletin of the London Mathematical Society*, 2022.
- [2] J. Bamberg, A. Hanaki, and J. Lansdown. AssociationSchemes, A GAP package for working with association schemes and homogeneous coherent configurations, Version 2.1.0. http://www.jesselansdown.com/AssociationSchemes, June 2022. GAP package.
- [3] E. Bannai and T. Ito. *Algebraic combinatorics*. *I*. The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984. Association schemes.
- [4] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).
- [5] P. J. Cameron. Coherent configurations, association schemes and permutation groups. In Groups, combinatorics & geometry (Durham, 2001), pages 55-71. World Sci. Publ., River Edge, NJ, 2003.
- [6] J. J. Cannon and D. F. Holt. The transitive permutation groups of degree 32. Experimental Mathematics, 17(3):307–314, 2008.
- [7] P. Delsarte. An algebraic approach to the association schemes of coding theory. PhD thesis, Universite Catholique de Louvain, Eindhoven, Netherlands, 1973.
- [8] The GAP Group. GAP Groups, Algorithms, and Programming, Version 4.11.1, 2021.
- [9] A. Hanaki, H. Kharaghani, A. Mohammadian, and B. Tayfeh-Rezaie. Classification of skew-Hadamard matrices of order 32 and association schemes of order 31. J. Combin. Des., 28(6):421–427, 2020.
- [10] A. Hanaki and I. Miyamoto. Classification of association schemes with small vertices. Available at http://math.shinshu-u.ac.jp/~hanaki/as/.
- [11] A. Hanaki and I. Miyamoto. Classification of association schemes with 16 and 17 vertices. *Kyushu J. Math.*, 52(2):383–395, 1998.
- [12] A. Hanaki and I. Miyamoto. Classification of association schemes with 18 and 19 vertices. *Korean J. Comput. Appl. Math.*, 5(3):543–551, 1998.
- [13] A. Hanaki and I. Miyamoto. Classification of association schemes of small order. volume 264, pages 75–80. 2003. The 2000 Com²MaC Conference on Association Schemes, Codes and Designs (Pohang).
- [14] D. G. Higman. Coherent algebras. Linear Algebra Appl., 93:209–239, 1987.
- [15] M. Hirasaka. The classification of association schemes with 11 or 12 vertices. Kyushu J. Math., 51(2):413–428, 1997.
- [16] M. Hirasaka and Y. Suga. The classification of association schemes with 13 or 15 points. Number 962, pages 71–80. 1996. Algebraic combinatorics (Japanese) (Kyoto, 1995).
- [17] D. Holt and G. Royle. A census of small transitive groups and vertex-transitive graphs. *J. Symbolic Comput.*, 101:51–60, 2020.
- [18] D. Holt, G. Royle, and G. Tracey. The transitive groups of degree 48 and some applications. *Journal of Algebra*, 2021.
- [19] A. Hulpke. Constructing transitive permutation groups. Journal of Symbolic Computation, 39(1):1–30, 2005.
- [20] A. Hulpke. TransGrp, transitive groups library, Version 3.6.2. https://www.math.colostate.edu/~hulpke/transgrp, May 2022. GAP package.
- [21] T. Junttila and P. Kaski. Engineering an efficient canonical labeling tool for large and sparse graphs. In D. Applegate, G. S. Brodal, D. Panario, and R. Sedgewick, editors, Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments and the Fourth Workshop on Analytic Algorithms and Combinatorics, pages 135–149. SIAM, 2007.

- [22] J. Lansdown. Database of small schurian association schemes. Available at http://doi.org/10.5281/zenodo.8025982.
- [23] J. Lansdown. Two-closures of transitive groups of small degree. Available at http://doi.org/10.5281/zenodo.8026133.
- [24] B. D. McKay and A. Piperno. Practical graph isomorphism, II. J. Symbolic Comput., 60:94–112, 2014.
- [25] B. D. McKay and A. Piperno. Nauty and traces user's guide (version 2.8.6). Available at https://pallini.di.uniromal.it/Guide.html, 2022.
- [26] B. D. McKay and E. Spence. Classification of regular two-graphs on 36 and 38 vertices. *Australas. J. Combin.*, 24:293–300, 2001.
- [27] G. A. Miller. List of transitive substitution groups of degree twelve. Q. J. Math, 28:193–231, 1896.
- [28] G. A. Miller. The collected works of George Abram Miller, 5 volumes. University of Illinois, Urbana, Ill., 1935–1959.
- [29] E. Nomiyama. Classification of association schemes with at most ten vertices. Kyushu J. Math., 49(1):163–195, 1995.
- [30] G. F. Royle. The transitive groups of degree twelve. *Journal of Symbolic Computation*, 4(2):255–268, 1987.
- [31] L. H. Soicher. GRAPE, Graph algorithms using permutation groups, Version 4.8.3. https://gap-packages.github.io/grape, Dec 2019. Refereed GAP package.