ON THE STRUCTURE OF FINITELY PRESENTED BESTVINA-BRADY GROUPS

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ABSTRACT. Right-angled Artin groups and their subgroups are of great interest because of their geometric, combinatorial and algorithmic properties. It is convenient to define these groups using finite simplicial graphs. The isomorphism type of the group is uniquely determined by the graph. Moreover, many structural properties of right angled Artin groups can be expressed in terms of their defining graph.

In this article we address the question of understanding the structure of a class of subgroups of right-angled Artin groups in terms of the graph. Bestvina and Brady, in their seminal work, studied these subgroups (now called Bestvina-Brady groups or Artin kernels) from a finiteness conditions viewpoint. Unlike the right-angled Artin groups the isomorphism type of Bestvina-Brady groups is not uniquely determined by the defining graph. We prove that certain finitely presented Bestvina-Brady groups can be expressed as an iterated amalgamated product. Moreover, we show that this amalgamated product can be read off from the graph defining the ambient right-angled Artin group.

1. INTRODUCTION

A right-angled Artin group (a RAAG, for short) is a finitely presented group such that the commuting relations are the only relations. It is perhaps easier to describe this group using finite simplicial graphs. Let Γ be such a graph; then the associated RAAG, denoted by A_{Γ} , has generators corresponding to vertices of Γ and two generators commute whenever the corresponding vertices are connected by an edge. We refer the reader to [8] for an encyclopedic introduction to RAAGs. They have become central in group theory, their study interweaves geometric group theory with other areas of mathematics. This class interpolates between two of the most classical families of groups, free and free abelian groups, and its study provides uniform approaches and proofs, as well as rich generalisations of the results for free and free abelian groups. The study of this class from different perspectives has contributed to the development of new, rich theories such as the theory of CAT(0) cube complexes and has been an essential ingredient in Ian Agol's solution to Thurston's virtual fibering and virtual Haken conjectures. RAAGs are important in geometric group theory for many reasons, including the fact that they have interesting subgroups; for example, Bestvina-Brady groups.

A Bestvina-Brady group (a B-B group, for short) is the kernel of the group homomorphism $A_{\Gamma} \to \mathbb{Z}$ which takes all the generators of A_{Γ} to 1. One of the reasons they

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are interesting is that they provide an example of a group that satisfies the finiteness property $\mathbf{FP_n}$ but not $\mathbf{FP_{n+1}}$. They also provide counterexamples either to the Eilenberg–Ganea Conjecture or the Whitehead Conjecture; see [2] for details.

Though RAAGs interpolate between free groups and free abelian groups their structure is not always straightforward. However, many of their structural properties can be read off from the underlying graph. For example, a RAAG A_{Γ} is a free product of two of its sub-RAAGs if and only if Γ is disconnected; by a theorem of Clay [9] all nontrivial splittings of A_{Γ} over \mathbb{Z} correspond to cut vertices of Γ , further it was proved by Groves and Hull [12] that A_{Γ} splits over an abelian group if and only if Γ is disconnected, or complete or contains a separating clique.

It is easy to observe that two non-isomorphic graphs can give rise to the same B-B group H_{Γ} (see 3.7). Moreover, though the finiteness properties are completely determined by the (topology of the) clique complex of Γ , not much work has been done to understand structural properties of H_{Γ} in terms of graph theoretic input. We should mention here the recent work of Chang [6] on abelian splittings of B-B groups and the work of Barquinero-Ruffoni-Ye [1] decomposing Artin kernels (a class of subgroups generalizing B-B groups) as graphs groups. In both the works the decomposition of H_{Γ} is expressed in terms of the underlying Γ ; hence we consider them as a motivation for our paper.

The aim of this article is to show that a certain class of finitely presented B-B groups can be decomposed as an (iterated) amalgamated product of a RAAG and finitely many copies of \mathbb{Z}^2 . We express this amalgamated product completely in terms of the underlying graph. To be precise, we decompose Γ as a union of Γ' and some triangles (i.e., 3-cliques); this union is with the help of a suitable spanning tree. The subgraph Γ' is selected on the basis that the corresponding B-B group $H_{\Gamma'}$ is isomorphic to a RAAG; the triangles correspond to copies of \mathbb{Z}^2 . The classes of graphs for which this decomposition works include the 1-skeleta of certain (extra)-special triangulations of the 2-disk and connected graphs with a separating clique K_n , $n \ge 3$. Our main theorem implies that such B-B groups can be decomposed as an iterated amalgamation of RAAGs. We should note here that Papadima and Suciu [14, Proposition 9.4] showed that the B-B group corresponding to an extra-special triangulation is not isomorphic to any RAAG. The existence of a separating clique also implies that the corresponding B-B group splits over an abelian subgroup, see Chang [6, Theorem 3.9]; we expand on this aspect in Section 4.

2. Preliminaries and notations

Below we present the necessary preliminaries on graph theory and B-B groups.

2.1. Graph theory. We recall and set up some basic notations and terminologies in graph theory. Throughout this article, we assume finite graphs which have no loops and multi-edges, i.e., all the graphs are finite simplicial. Given a graph Γ , we denote the set of its vertices and edges by $V(\Gamma)$ and $E(\Gamma)$, respectively. We denote e = (v, w)

to be an edge connecting vertices v and w. The initial and terminal vertices of the edge e are denoted by $\iota(e)$ and $\tau(e)$, respectively, such that $e = (\iota(e), \tau(e))$. Two vertices are adjacent if they are connected by an edge. A *spanning tree* of Γ is a subgraph of Γ which is a tree and contains every vertex of Γ . As we are dealing with finite graphs it is straightforward to see that, for any spanning tree T of Γ , $|E(T)| = |V(\Gamma)| - 1$. Given any subset V' of $V(\Gamma)$, the *induced subgraph* (in some literature, it is also called as full subgraph) on V' is a graph Γ' whose vertex set is V', and two vertices are adjacent in Γ' if and only if they are adjacent in Γ .

The star graph S_n of order n, sometimes simply known as an *n*-star, is a tree on n vertices with one vertex having degree n-1 and the other n-1 vertices having degree 1.

Given two graphs $\Gamma_1 = (V(\Gamma_1), E(\Gamma_1))$ and $\Gamma_2 = (V(\Gamma_2), E(\Gamma_2))$, the union of Γ_1 and Γ_2 is $\Gamma_1 \cup \Gamma_2 = (V(\Gamma_1) \cup V(\Gamma_2), E(\Gamma_1) \cup E(\Gamma_2))$. We denote the disjoint union of two graphs Γ_1 and Γ_2 as $\Gamma_1 \sqcup \Gamma_2$, i.e., Γ_1 and Γ_2 share no vertices. The join of two graphs Γ_1 and Γ_2 , denoted by $\Gamma_1 \vee \Gamma_2$, is defined to be the graph union $\Gamma_1 \cup \Gamma_2$ and with every pair of vertices $(v, w) \in V(\Gamma_1) \times V(\Gamma_2)$ being adjacent. The join operation is commutative, that is, $\Gamma_1 \vee \Gamma_2 = \Gamma_2 \vee \Gamma_1$. When a graph Γ decomposes as a join of a vertex v and another graph Γ' , the vertex v is called a *dominating vertex*, and Γ is called a *cone graph* or the *cone on* Γ' .

Two graphs Γ_1 and Γ_2 are said to be isomorphic, denoted by $\Gamma_1 \cong \Gamma_2$, if there is a bijection $\varphi : V(\Gamma_1) \to V(\Gamma_2)$ such that two vertices v, w are adjacent in Γ_1 if and only if $\varphi(v), \varphi(w)$ are adjacent in Γ_2 . The star graph S_n is isomorphic to the complete bipartite graph $K_{(1,n-1)}$. Also let Γ_1, Γ_2 be cone graphs on Γ'_1 and Γ'_2 respectively, then $\Gamma_1 \cong \Gamma_2$ if and only if $\Gamma'_1 \cong \Gamma'_2$.

Given a graph Γ we construct a simplicial complex Δ_{Γ} , called the *flag complex*, as follows: the vertex set is the ground vertex set $V(\Gamma)$ and a subset of cardinality k is a (k-1)-simplex if and only if the induced subgraph is a k-clique. In the literature, the term clique complex is also used for the flag complex. Note that, we do not differentiate between an abstract simplicial complex and its geometric realization; any topological statement about the flag complex (equivalently, the clique complex) is about its geometric realization. Our main focus is on those graphs whose flag complex is simply connected. Such graphs form a fairly large class, for example, a connected chordal graph has the contractible flag complex.

2.2. Bestvina-Brady groups.

Definition 2.1. Let Γ be a finite simplicial graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. The right-angled Artin group A_{Γ} associated to Γ has the following finite presentation:

$$A_{\Gamma} = \langle V(\Gamma) \mid [v, w] = 1 \text{ for each edge } (v, w) \in E(\Gamma) \rangle.$$

Let $\varphi \colon A_{\Gamma} \to \mathbb{Z}$ be the group homomorphism sending all generators of A_{Γ} to 1. The *Bestvina–Brady group* H_{Γ} associated to Γ is the kernel, ker φ .

We already mentioned in Section 1 that the B-B groups were first introduced in the influential work of Bestvina and Brady [2] as an answer to a long standing open question regarding the existence of non-finitely presented groups of type **FP**—a result based on homological group theory.

Theorem 2.2 ([2], Main Theorem). *Let* Γ *be a finite simplicial graph.*

- (1) H_{Γ} is finitely generated if and only if Γ is connected.
- (2) H_{Γ} is finitely presented if and only if \triangle_{Γ} is simply-connected.
- (3) H_{Γ} is of type **FP**_{**n**+1} if and only if Γ is *n*-acyclic.

This result includes the Stallings' group [15] — H_{Γ} associated to the RAAG $F_2 \times F_2 \times F_2 - F_2$ an example of finitely presented but not of type **FP**₃ and R. Bieri's group [3] of type **FP**_n but not of type **FP**_{n+1}, which is H_{Γ} corresponding to the Γ , a join of (n + 1) pairs of points.

The presentation of B-B groups was described by Dicks-Leary in [10]:

Theorem 2.3 ([10], Theorem 1). Let Γ be connected. The group H_{Γ} has a presentation with generators the set of directed edges of Γ , and relators all words of the form $e_1^n e_2^n \cdots e_{\ell}^n$, where $\ell, n \in \mathbb{Z}, n \ge 0, \ell \ge 2$, and (e_1, \ldots, e_{ℓ}) is a directed cycle in Γ . In terms of the given generators for $A_{\Gamma}, e = \iota e(\tau e)^{-1}$.



FIGURE 1. A directed triangle.

Let us fix a linear order on the vertices, and orient the edges increasingly. A triplet of edges (e, f, g) forms a directed triangle if e = (u, v), f = (v, w), g = (u, w), and u < v < w; see Figure 1.

When Γ is connected and Δ_{Γ} is simply connected, we can write down a presentation for H_{Γ} , called the *Dicks–Leary presentation* [10, Theorem 1, Corollary 3]. In other words, the aforementioned theorem could be simplified as:

Theorem 2.4. Suppose the flag complex \triangle_{Γ} is simply connected. Then H_{Γ} has presentation

$$H_{\Gamma} = \langle e \in E(\Gamma) \mid ef = fe, ef = g \text{ if } \triangle(e, f, g) \text{ is a directed triangle } \rangle.$$

Moreover, the inclusion $\iota \colon H_{\Gamma} \hookrightarrow A_{\Gamma}$ is given by $\iota(e) = uv^{-1}$ for every edge e = (u, v) of Γ .

The Dicks-Leary presentation is not necessarily a minimal presentation, i.e., there are some redundant generators. Dicks-Leary considered all the edges of Γ as the generators. The simpler presentation was given by Papadima-Suciu in [14]. The authors proved that for the generators of H_{Γ} it is enough to consider the edges of a spanning tree of Γ . **Theorem 2.5** ([14, Corollary 2.3]). If \triangle_{Γ} is simply-connected, then H_{Γ} has a presentation $H_{\Gamma} = F/R$, where F is the free group generated by the edges of a spanning tree of Γ , and R is a finitely generated normal subgroup of the commutator group [F, F].

Here are some examples of B-B groups.

Example 2.6. If Γ is a complete graph on n vertices, then any spanning tree has n-1 edges. In fact, we can choose the spanning tree as a star graph. Moreover, any two edges in the spanning tree form the two sides of a triangle in Γ . The corresponding Bestvina–Brady group has n-1 generators and any two of them commute, hence it is \mathbb{Z}^{n-1} .

Example 2.7. Now let Γ be a tree on n vertices. The spanning tree is the graph Γ itself and there are no triangles. The corresponding Bestvina–Brady group has n - 1 generators and none of them commute, hence it is F_{n-1} the free group on n - 1 generators.

Example 2.8. Let Γ be the graph in Figure 2. Choosing the spanning tree $T = \{e_1, \ldots, e_5\}$ as indicated, the presentation of the B-B group reads as follows:

$$H_{\Gamma} = \langle e_1, \dots, e_5 \mid [e_1, e_2], [e_2, e_3], [e_3, e_4], e_5e_2^{-1}e_3 = e_2^{-1}e_3e_5 \rangle.$$

Unlike the previous two examples, this particular H_{Γ} is not isomorphic to any RAAG (see [14, Proposition 9.4] for details).



FIGURE 2. A graph whose corresponding B-B group is *not* a RAAG.

Here we also recall the definition of an amalgamated product.

Definition 2.9. Let G_1 and G_2 be groups with distinguished isomorphic subgroups $H \leq G_1$ and $K \leq G_2$. Fix an isomorphism $\varphi \colon H \to K$. The free product of G_1 and G_2 with amalgamation of H and K by the isomorphism φ is the quotient of $G_1 * G_2$ by the normal closure of the set $\{\varphi(h)h^{-1} \mid h \in H\}$. We will refer to this factor group briefly as the *amalgamated product* and have the following notations:

$$\langle G_1 * G_2 \mid h = \varphi(h), h \in H \rangle, \quad G_1 *_{H=K} G_2, \quad G_1 *_H G_2.$$

3. On the structure of B-B groups

As stated earlier the aim of this article is to describe the structure of Bestvina-Brady groups and we do this, up to some extent, in this section. However, we start by focusing at a class of B-B groups that are in fact isomorphic to some RAAG. These type of B-B groups can be recognized from the graph defining the ambient RAAG. Then we move to the *iterated amalgamated product* structure having the very first factor group isomorphic to an arbitrary RAAG and other factors isomorphic to \mathbb{Z}^2 (which are also RAAGs). Most importantly, the iterated amalgamated product structure of the B-B group can also be derived from the graph defining its ambient RAAG.

3.1. Isomorphism between RAAGs and B-B groups. Since the only relations in B-B groups are the commuting relations, for a given Γ it is natural to ask whether H_{Γ} isomorphic to $A_{\Gamma'}$ for some finite simplicial graph Γ' . This question was first considered by Papadima and Suciu in [14]. They also constructed a family of B-B groups not isomorphic to any RAAGs.

We briefly review their result; first we recall the definition of *special* and *extra-special triangulation* (Figure 3).

Definition 3.1. A triangulation of the 2-disk D^2 is said to be *special* if it can be obtained from a triangle by adding one triangle at a time, along a unique boundary edge. A triangulation of D^2 is called *extra-special* if it is obtained from a special triangulation, by adding one triangle along each boundary edge (see Figure 3).



FIGURE 3. Building an extra-special triangulation of the disk.

Proposition 3.2 ([14, Proposition 9.4]). Let Γ be the 1-skeleton of an extra-special triangulation of D^2 . Then the corresponding Bestvina–Brady group H_{Γ} is not isomorphic to any Artin group.

Papadima and Suciu [14] also describe an explicit presentation of H_{Γ} , where Γ is the 1-skeleton of a special triangulation of D^2 .

Lemma 3.3. Let $\Gamma = (V, E)$ be the 1-skeleton of a special triangulation of D^2 , then:

(i) 2|V| - |E| = 3,

(ii) H_{Γ} admits a presentation with |V| - 1 generators and |V| - 2 commutator relators.

Now we identify a class of graphs such that the corresponding B-B group is isomorphic to some RAAG. The complete classification of graphs such that the corresponding (finitely presented) B-B group is a RAAG can be found in the recent paper of Chang and Ruffoni. In particular, they prove that such graphs admit a *tree 2-spanner* [7, Theorem A]. Since we do not need the full extent of their result we provide a proof, for the benefit of the reader, of the sufficiency condition without introducing any more technical definitions. The authors sincerely thank Chang and Ruffoni for bringing their work to our notice.

Theorem 3.4. Let Γ be a finite simplicial graph such that Δ_{Γ} is simply-connected and Γ has a spanning tree T such that each triangle of Γ has 2 edges in E(T). Then $H_{\Gamma} \cong A_{\Gamma'}$ for some finite simplicial graph Γ' .

Proof. Let T be a spanning tree of Γ such that each triangle of Γ has exactly 2 edges in E(T). Let $E(T) = \{e_1, e_2, \dots, e_n\}$. From Theorem 2.5, E(T) corresponds to a generating set of H_{Γ} . Since H_{Γ} is finitely presented and that each triangle has 2 edges in E(T), each relator is of the form $[e_i, e_j]$.

We construct the graph Γ' as follows: the vertex set $V(\Gamma')$ is E(T); and two vertices in $V(\Gamma')$ are adjacent whenever the corresponding edges form a triangle in Γ , i.e., they commute in H_{Γ} . Let us denote $V(\Gamma') = \{v'_1, v'_2, \ldots, v'_n\}$, and let $\varphi \colon E(T) \to V(\Gamma')$ be a map sending each e_i to v_i for $i = 1, 2, \ldots, n$. Clearly, φ is a bijection. By the construction of Γ' , we have $\varphi([e_i, e_j]) = [v'_i, v'_j]$ and $\varphi^{-1}([v'_i, v'_j]) = [e_i, e_j]$. Thus, φ is our desired isomorphism between H_{Γ} and $A_{\Gamma'}$.

Example 3.5. Let $\Gamma = v \vee \Gamma'$ be a cone graph with v as its dominating vertex. A spanning tree can be chosen to be the star graph consisting of consists in v, $V(\Gamma')$ and all the edges joining v to each vertex of Γ' . Every edge in Γ' and the two edges that connect each of its boundary vertices to v form a triangle. Hence, $H_{\Gamma} \cong A_{\Gamma'}$. This example has also appeared in [14, Example 2.5].

Example 3.6. Let Γ be the graph as shown in Figure 4. Let T be the spanning tree with edges e_1 , e_2 , e_3 , e_4 and e_5 . By Theorem 3.4, H_{Γ} is isomorphic to $A_{\Gamma'}$, where Γ' is a line graph with vertices w_1, w_2, w_3, w_4 and w_5 as shown on the right hand side of Figure 4. Note that, Γ is not a cone graph, however, H_{Γ} is still isomorphic to $A_{\Gamma'}$.



FIGURE 4. A graph whose corresponding B-B group is a RAAG.

Note 3.7. Recall that two RAAGs A_{Γ_1} and A_{Γ_2} are isomorphic if and only if their underlying finite simplicial graphs Γ_1 and Γ_2 are isomorphic. However, this is not the case for B-B groups. For example, consider any two non-isomorphic trees on n vertices for $n \ge 3$. In both the cases the corresponding B-B group is F_{n-1} . Let Γ_1 , Γ_2 be cone graphs on Γ'_1 and Γ'_2 respectively, then $\Gamma_1 \cong \Gamma_2$ if and only if $\Gamma'_1 \cong \Gamma'_2$. So if we restrict ourselves to cone graphs, then $\Gamma_1 \cong \Gamma_2$ if and only if $H_{\Gamma_1} \cong H_{\Gamma_2}$. See [7, Corollary 1] for the general result.

3.2. **B-B** groups as an iterated amalgamated product. In this subsection we will mainly focus on the triangles of Γ . More precisely, we will concentrate on how a particular triangle \triangle of Γ intersects its edge-set complement (see Def. 3.8). We introduce the notions of *favourable* and *unfavourable* triangles with respect to the chosen spanning tree *T*. Accordingly we also introduce the notions of *favourable* and *unfavourable* graphs.

Definition 3.8. Let Γ be a connected graph and \triangle be a triangle of Γ . Let Γ' be the graph with $V(\Gamma') = V(\Gamma)$ and $E(\Gamma') = E(\Gamma) \setminus E(\triangle)$. Let S be the isolated vertices of Γ' and $V^c = V(\Gamma) \setminus S$. The *edge-set complement* of \triangle is the induced subgraph of Γ generated by the set of vertices V^c . We will denote the edge-set complement of \triangle in Γ by \triangle_{Γ}^c .

Definition 3.9. A triangle \triangle of Γ is said to be an *internal triangle* if its intersection with the edge-set complement \triangle_{Γ}^{c} is neither one vertex nor one edge. A triangle \triangle of Γ is said to be a *strictly internal triangle* if it is contained in an induced K_n , $n \ge 4$.

In Figure 5 the triangles $\triangle(v_2, v_4, v_5)$ and $\triangle(v_2, v_3, v_4)$ are the internal triangles.



FIGURE 5. A graph with internal triangles.

It is straightforward to note that all strictly internal triangles are internal.

Definition 3.10. Let T be a spanning tree of Γ . A triangle \triangle of Γ is a *favourable* triangle with respect to T if, either it has exactly 2 edges in E(T) or it is strictly internal. Otherwise, we say that \triangle is unfavourable.

Note 3.11. A graph Γ can have several spanning trees. So, it may very well happen that a triangle \triangle is *favourable* in one spanning tree T_1 of Γ but not favourable in another spanning tree T_2 of Γ . Hence to talk about favourable triangles we have to fix a spanning tree T of Γ . We will choose a spanning tree with the *maximal* number of *favourable* triangles or equivalently, the number of *unfavourable* triangles is *minimal*. Since by definition every strictly internal triangle of Γ is favourable, strictly internal triangles of Γ are always favourable with respect to any choice of spanning tree of Γ . Hence, we define *favourable graphs* as follows.

Definition 3.12. Let Γ be a finite simplicial graph such that the flag complex on Γ is simply-connected. Γ is said to be a *favourable graph* if there is a spanning tree T such that each triangle of Γ is favourable with respect to T. On the other hand, if for every spanning tree T of Γ , there exists at least one unfavourable triangle then Γ is called an *unfavourable graph*.

The graph in Figure 4 is favourable and the graph in Figure 2 is unfavourable. In light of Theorem 3.4, if Γ is a favourable graph, then the corresponding B-B group is isomorphic to a RAAG. However, it is important to note that an unfavourable graph doesn't mean that the corresponding B-B group is not isomorphic to any RAAG.

Observation 3.13. Any non-internal triangle can have 1 edge or 2 edges in any spanning tree. On the other hand, we know that if the triangle has 2 edges in the spanning tree, then it is favourable with respect to that particular spanning tree.

Let T be a chosen spanning tree of Γ and \triangle be a non-internal triangle. If the triangle \triangle is unfavourable with respect to T, then it has exactly one edge in T.

Now some preparatory results.

Lemma 3.14. If a triangle \triangle of Γ intersects the corresponding edge-set complement \triangle_{Γ}^{c} in one vertex, then it is a favourable triangle irrespective of the choice of a spanning tree.

Proof. Let the triangle \triangle intersect \triangle_{Γ}^{c} in one vertex, say v. Then v is a cut vertex. Hence, for any spanning tree T, \triangle has exactly two edges in E(T).

The following result is a direct consequence of Dicks-Leary[10] presentation. We provide a proof for the benefit of the reader.

Lemma 3.15. Let Γ be a finite simplicial graph with simply-connected flag complex. Then there is a group element in H_{Γ} for each $e \in E(\Gamma)$.

Proof. By abusing the notation we let e denote the edge as well as the corresponding group element. Let us first fix a spanning tree T of Γ , and $\iota(e) = u$ and $\tau(e) = v$. If $e \in E(T)$, then there is nothing to prove. Let us suppose that e is an edge not in T. Also, let A_{Γ} be the RAAG defined on Γ . Picking a path $e_1^{\epsilon_1}, \ldots, e_r^{\epsilon_r}$ in T connecting u to v, we see that $\iota(e) = \iota(e_1^{\epsilon_1} \ldots e_r^{\epsilon_r}) \in A_{\Gamma}$, $\epsilon_i = \pm 1$. Clearly, $e = e_1^{\epsilon_1} \ldots e_r^{\epsilon_r} \in H_{\Gamma}$. \Box

Proposition 3.16. Let T be a spanning tree of Γ and let \triangle be an unfavourable triangle of Γ with respect to T such that it intersects the edge-set complement \triangle_{Γ}^{c} exactly in one edge. Then $H_{\Gamma} \cong H_{\triangle} *_{\mathbb{Z}} H_{\triangle_{\Gamma}^{c}}$.

Proof. The Bestvina–Brady group H_{Γ} is finitely presented and its generators are the edges of the spanning tree T. Let $E(\triangle) = \{e_1, e_2, e_3\}$. Since \triangle intersects \triangle_{Γ}^c exactly in one edge, \triangle is non-internal by definition. On the other hand, \triangle is unfavourable.

Hence, by Observation 3.13 we may assume that $e_1 \in E(T)$ and $e_2, e_3 \notin E(T)$. We know that H_{\triangle} , the Bestvina-Brady group associated to \triangle is a copy of \mathbb{Z}^2 with the following presentation:

$$H_{\triangle} = \langle e_1, e_2 \mid e_1 e_2 = e_2 e_1 \rangle.$$

Let e_2 be the edge common to both \triangle and \triangle_{Γ}^c . To avoid any confusion, we will denote e_2 by $\overline{e_2}$ when seen as an edge of \triangle_{Γ}^c . Note that the restriction of T to \triangle_{Γ}^c denoted by T' is a spanning tree of \triangle_{Γ}^c . As, $\overline{e_2} \notin E(T')$, it is not a generator of $H_{\triangle_{\Gamma}^c}$. However by Lemma 3.15, $\langle \overline{e_2} \rangle$ is a subgroup of $H_{\triangle_{\Gamma}^c}$. On the other hand e_2 is a generator of H_{\triangle} and of course, $\langle e_2 \rangle$ is a subgroup of H_{\triangle} .

Note that all generators and relators of H_{Γ} are present in $H_{\triangle_{\Gamma}^c}$, except the generator e_1 and the relations involving e_1 . So our first step is to include e_1 in some extension of $H_{\triangle_{\Gamma}^c}$; which can be done by taking the free product of $H_{\triangle_{\Gamma}^c}$ and H_{\triangle} . Thus $H_{\triangle} * H_{\triangle_{\Gamma}^c}$ is generated by all the generators of H_{Γ} and e_2 . So our next step is to get rid of e_2 . Let us consider the following amalgamated product:

$$H_{\Delta} \ast_{\langle e_2 \rangle = \langle \overline{e_2} \rangle} H_{\Delta_{\Gamma}^c} = H_{\Delta} \ast_{\langle e_2 \rangle} H_{\Delta_{\Gamma}^c}$$

Thus we include e_1 in the generating set and at the same time we discard the generator e_2 . In the amalgamated free product one of the relations is $e_2 = \overline{e_2}$ and $\overline{e_2}$ can be expressed as a word in terms of the other generators of $H_{\triangle_{\Gamma}^c}$; leaving e_2 redundant. Now comparing the presentations of H_{Γ} and $H_{\triangle} *_{\langle e_2 \rangle} H_{\triangle_{\Gamma}^c}$, it is clear that, $H_{\Gamma} \cong H_{\triangle} *_{\langle e_2 \rangle} H_{\triangle_{\Gamma}^c} \cong H_{\triangle} *_{\mathbb{Z}} H_{\triangle_{\Gamma}^c}$. This completes the proof.

We note that, if $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are finite simplicial graphs and their flag complexes are simply-connected; moreover, if $\Gamma_3 = \Gamma_1 \cap \Gamma_2$ is a connected induced subgraph of Γ , then $H_{\Gamma} = H_{\Gamma_1} *_{H_{\Gamma_3}} H_{\Gamma_2}$. This is proved in [4, Proposition 3.5] and discussed in [5].

For the *iterated amalgamated product* structure, we are mainly interested in *unfavourable* graphs. We want to have an iterated amalgamation structure such that the factor groups are RAAGs. Such a decomposition is helpful in understanding various properties that are known for RAAGs and can pass through the iteration of amalgamations. The class of graphs for which such an iterated amalgamation exists is the following:

Definition 3.17. Let \mathcal{G} be a family of finite, simplicial, unfavourable graphs Γ with simply connected flag complex which, in addition, have a spanning tree T such that all the internal triangles are favourable with respect to T.

Now we describe the structure of B-B groups associated to graphs in \mathcal{G} .

Theorem 3.18. Let $\Gamma \in \mathcal{G}$. Then H_{Γ} splits as an iterated amalgamated product of a rightangled Artin group and finitely many copies of \mathbb{Z}^2 and in each iteration the amalgamation is over an infinite cyclic group. *Proof.* Since Γ is in \mathcal{G} , Γ has spanning trees with respect to which none of the internal triangles is unfavourable. Among those we can choose a spanning tree T with respect to which Γ has minimal number of unfavourable triangles. By Observation 3.13, each of the unfavourable triangles has exactly one edge in the spanning tree. Let $\{\triangle_1, \triangle_2, \dots, \triangle_n\}$ be the set of unfavorable triangles and denote by e_i that particular edge of \triangle_i which is in E(T). We denote the other two edges of \triangle_i by f_i and f'_i ; note that, these edges are not in E(T). It follows from Lemma 3.14 that Δ_i intersects the edge-set complement $(\Delta_i)^c_{\Gamma}$ in exactly one edge, say f_i for each i = 1, 2, ..., n.

We already know that the B-B group H_{\triangle_i} associated to \triangle_i is finitely presented with the following presentation:

$$H_{\triangle_i} = \left\langle e_i, f_i | e_i f_i = f'_i = f_i e_i \right\rangle \cong \mathbb{Z}^2; \qquad i = 1, 2, \dots, n.$$
(3.1)

Consider the subgraphs defined as follows:

$$\Gamma_1 := (\Delta_1)_{\Gamma}^c \text{ and } \Gamma_i := (\Delta_i)_{\Gamma_{i-1}}^c, \quad i = 2, 3, \dots, n.$$
(3.2)

Also note that, Γ_1 is an induced subgraph of Γ and Γ_i is an induced subgraph of Γ_{i-1} .

By Proposition 3.16 we have,

$$H_{\Gamma} = H_{\Gamma_1} *_{\langle f_1 \rangle} H_{\triangle_1} \tag{3.3}$$

and

$$H_{\Gamma_{i-1}} = H_{\Gamma_i} *_{\langle f_i \rangle} H_{\Delta_i} \quad \text{for } i = 2, 3, \dots, n.$$
(3.4)

Note that we choose the restriction of T to Γ_i as the spanning tree for Γ_i and denote it by T_i , for i = 1, 2, 3, ..., n. From the construction and Equation (3.2) it is clear that the graph $\Gamma_n := (\Delta_n)_{\Gamma_{n-1}}^c$ is a *favourable graph*. So, H_{Γ_n} is isomorphic to $A_{\overline{\Gamma}}$ for some finite simplicial graph $\overline{\Gamma}$, see Theorem 3.4. Recall that, the edges of T_n are the vertices of $\overline{\Gamma}$ and two such vertices are adjacent whenever the corresponding edges form a triangle in Γ_n .

We can now express the B-B group H_{Γ} as the following (iterated) amalgamated product.

$$H_{\Gamma} = \left(\left(\left(A_{\overline{\Gamma}} \ast_{\langle f_n \rangle} H_{\triangle_n} \right) \ast_{\langle f_{n-1} \rangle} H_{\triangle_{n-1}} \right) \ast_{\langle f_{n-2} \rangle} H_{\triangle_{n-2}} \right) \cdots \ast_{\langle f_{n-i} \rangle} H_{\triangle_{n-i}} \cdots \right) \ast_{\langle f_1 \rangle} H_{\triangle_1}.$$

This completes the proof.

This completes the proof.

To characterize the class of graphs which belong to the family \mathcal{G} we use the notion of *separating clique* from the literature (see [12] for more details).

Definition 3.19. Let Γ_0 be an induced subgraph of Γ . Γ_0 is said to be *separating* if the induced subgraph spanned by the vertices $V(\Gamma) \setminus V(\Gamma_0)$ has more connected components than Γ . If Γ_0 is a complete graph on *n* vertices, then Γ_0 is called a *separating clique* of Γ and is denoted by K_n .

Here is a partial characterization of graphs in \mathcal{G} .

Theorem 3.20. Let Γ be a finite simplicial connected unfavourable graph such that Δ_{Γ} , the associated flag complex, is simply connected. If Γ has a separating clique K_n , $n \ge 3$, then $\Gamma \in \mathcal{G}$.

Proof. Let us suppose that Γ has $\ell \geq 1$ many internal triangles, say $\Delta_1, \Delta_2, \ldots, \Delta_\ell$. Our claim is that each Δ_i is a favourable triangle, for $i = 1, 2, \ldots, \ell$. We will prove our claim by induction on ℓ .

First consider the case $\ell = 1$, i.e., only one internal triangle, say \triangle . We can choose a tree T' by taking any two edges of \triangle and then we expand T' to a spanning tree of Γ . This makes \triangle a favourable triangle with respect to T.

Assume that the claim holds for all graphs which have fewer than ℓ internal triangles. As Γ has a separating clique K_n , then $\Gamma \setminus K_n = \Gamma_1 \sqcup \Gamma_2$ where Γ_1 and Γ_2 are each nonempty and share no vertices. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. There exists an u_i which is adjacent to a vertex of Γ_1 and the same thing holds for Γ_2 . Without loss of generality we can assume that u_1 is adjacent to Γ_1 and u_2 is adjacent to a vertex of Γ_2 . Now let Γ'_1 be the induced subgraph spanned by $V(\Gamma_1) \cup \{u_1\}$ and Γ'_2 be the induced subgraph spanned by $V(\Gamma_2) \cup \{u_2\}$. Then both Γ'_1 and Γ'_2 have less than ℓ internal triangles. By the induction hypothesis, Γ'_1 and Γ'_2 have spanning trees T'_1 and T'_2 respectively for which all the internal triangles are favourable. Since K_n is a complete graph, we can choose the spanning tree of K_n as a star graph. Let T_{K_n} be the spanning tree of K_n . Hence for T_{K_n} every triangle of K_n is favourable. Let us consider $T = T'_1 \cup T_{K_n} \cup T'_2$. Then T is the desired spanning tree of Γ for which all the internal triangles are favourable. \Box

Observation 3.21. If Γ is a 1-skeleton of an extra-special triangulation of the 2-disk such that the underlying special triangulation is favourable, then $\Gamma \in \mathcal{G}$. The fact holds because of the following reason: the internal triangles of an extra-special triangulation of D^2 are the triangles of the special triangulation on which the extra-special triangulation is built by attaching one triangle along each boundary edge. According to our hypothesis, this special triangulation is favourable, hence it possesses a spanning tree T'containing 2 edges from each triangle of the special triangulation. Now if we construct the spanning tree T of Γ by expanding the tree T', then our observation follows.

We now look at two examples. First we consider the graph in Fig. 2 which is an extra-special triangulation. Recall that, from [14, Proposition 9.4] it follows that the corresponding B-B group is not isomorphic to a RAAG.

Example 3.22. Our choice of spanning tree is $T = \{e_1, \ldots, e_5\}$ for the graph in Fig. 2. Hence the presentation for the B-B groups is

$$H_{\Gamma} = \langle e_1, \dots, e_5 \mid [e_1, e_2], [e_2, e_3], [e_3, e_4], e_5e_2^{-1}e_3 = e_2^{-1}e_3e_5 \rangle.$$

Note that the triangle \triangle_1 spanned by $\{v_4, v_5, v_6\}$ is unfavourable with respect to T and let $E(\triangle_1) = \{e_5, f_1, f'_1\}$ where f_1, f'_1 denote the respective edges (v_5, v_4) and (v_5, v_6) . Let Γ_1 be the edge-set complement of \triangle_1 . It is not hard to see that Γ_1 is a favourable

graph; in fact, $\{e_1, e_2, e_3, e_4\}$ is a spanning tree with respect to which all the three triangles are favourable. Consequently,

$$H_{\Gamma} = H_{\Gamma_1} *_{\langle f_1 \rangle} H_{\triangle_1} \cong A_{\overline{\Gamma}} *_{\mathbb{Z}} \mathbb{Z}^2.$$

Now we consider an example where the underlying graph is not an extra-special triangulation and the minimal separating clique is K_2 .

Example 3.23. Let us consider Γ in Figure 6. We choose the following spanning tree $T = \{e_1, e_2, \ldots, e_{11}\}$. Note that there are two non-favourable triangles $\triangle(v_1, v_2, v_5) = \triangle_1$ and $\triangle(v_{10}, v_{11}, v_9) = \triangle_2$. In fact, the reader can verify that for any choice of spanning tree, there will be at least two unfavourable triangles.

We have, $E(\Delta_1) = \{e_1, f_1, f'_1\}$ where $e_1 \in E(T)$ and $f_1, f'_1 \notin E(T)$ and $E(\Delta_2) = \{e_{10}, f_2, f'_2\}$ where $e_{10} \in E(T)$ and $f_2, f'_2 \notin E(T)$. Let us denote $\Gamma_1 := \Delta_{1\Gamma}^c$ (see Figure 7) and $\Gamma_2 := \Delta_{2\Gamma_1}^c$ (see Figure 8). Hence, Γ_1 is the induced subgraph of Γ and Γ_2 is the induced subgraph of Γ_1 . We have:

$$H_{\triangle_1} = \langle e_1, f_1 | e_1 f_1 = f_1' = f_1 e_1 \rangle \cong \mathbb{Z}^2.$$

$$H_{\triangle_2} = \langle e_{10}, f_2 | e_{10} f_2 = f_2' = f_2 e_{10} \rangle \cong \mathbb{Z}^2.$$

Thus

$$H_{\Gamma} = H_{\Gamma_1} *_{\langle f_1 \rangle} H_{\Delta_1}, \tag{3.5}$$

and,

$$H_{\Gamma_1} = H_{\Gamma_2} *_{\langle f_2 \rangle} H_{\triangle_2}. \tag{3.6}$$

Note that Γ_2 is a favourable graph. So, $H_{\Gamma_2} \cong A_{\overline{\Gamma}}$ where

 $A_{\overline{\Gamma}} = \langle w_2, \dots, w_9, w_{11} \mid [w_2, w_3], [w_2, w_4], [w_4, w_5], [w_7, w_8], [w_8, w_9], [w_9, w_{11}] \rangle.$

Thus finally we have the desired iterated amalgamation structure of H_{Γ} :

$$H_{\Gamma} = (A_{\overline{\Gamma}} *_{\langle f_2 \rangle} H_{\triangle_2}) *_{\langle f_1 \rangle} H_{\triangle_1} \cong (A_{\overline{\Gamma}} *_{\mathbb{Z}} \mathbb{Z}^2) *_{\mathbb{Z}} \mathbb{Z}^2.$$



FIGURE 6. The graph Γ .



FIGURE 7. The graph Γ_1 .



FIGURE 8. The graph Γ_2 .

4. Concluding remarks

We end the article with a non-example and some open questions.

Example 4.1. Consider the graph in Fig. 9, which is the 3-fold join of two isolated vertices. The RAAG corresponding to this graph is the direct product of three copies of F_2 and the B-B group is the same group that appeared in the seminal paper of Stallings. It is a finitely presented group which is not of type \mathbf{FP}_3 . The reader can easily verify that no matter which spanning tree is chosen there are at least two internal triangles that are unfavourable. In fact, any *k*-fold join of two isolated vertices has this property; so these graphs do not belong to the class \mathcal{G} .



FIGURE 9. A graph not in the class G.

Question 4.2. What is the complete characterization of the family \mathcal{G} ? Moreover, if a Bestvina-Brady group H_{Γ} decomposes as an iterated amalgamation (say, like the one specified in Theorem 3.18) what can one say about the structure of Γ ?

A group G is called *coherent* if each finitely generated subgroup of G is finitely presented. Droms [11, Theorem 1] characterised coherent RAAGs in terms of the defining graph. A right-angled Artin group A_{Γ} is *coherent* if and only if Γ does not have any cycle of length $n \ge 4$ as an induced subgraph (i.e., Γ is chordal). Equivalently, Γ has a separating clique and each component is either a clique or has a separating clique and so on.

RAAGs corresponding to trees are obvious examples of coherent RAAGs. For example, $A_{P_4} \cong \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$, the RAAG corresponding to the path on 4 vertices is an important group. Kim and Koberda [13, Theorem 7] proved that any 2-dimensional coherent RAAG embeds in A_{P_4} .

Suppose A_{Γ} is a coherent right-angled Artin group with connected defining graph Γ . Then the corresponding Bestvina-Brady group H_{Γ} is also expressible as either an amalgamated product or a free product of free abelian (and infinite cyclic) groups. This follows directly from the definition of coherent RAAGs and [4, Proposition 3.5]. We recall (see Example 2.7) that when Γ is a finite tree, then H_{Γ} is a free group of finite rank, i.e., a free product of finitely many copies of the infinite cyclic group. Most importantly, this decomposition also relies completely on the underlying graph structure. Note that the underlying graph of a coherent RAAG need not belong to the family \mathcal{G} .

We describe an example of a Bestvina-Brady group H_{Γ} which decomposes as an amalgamated product over free-abelian groups but not covered by Theorem 3.18.



FIGURE 10. Defining graph of a coherent RAAG.

Example 4.3. A_{Γ} defined on the graph depicted in Figure 10 is coherenrt. It is clear that the induced subgraph Γ_0 on the set of vertices $\{v_1, v_2, v_3\}$ is a separating clique. Thus, $A_{\Gamma} = A_{\Gamma_1} *_{A_{\Gamma_0}} A_{\Gamma_2}$, where Γ_1 is the induced subgraph on the vertices $\{v_1, v_2, v_3, v_4, v_5\}$ and Γ_2 is the induced subgraph on the vertices $\{v_1, v_2, v_3, v_6\}$. Similarly, we can apply the same method of decomposition on A_{Γ_1} . As Γ_1 has the separating clique generated by the vertices $\{v_1, v_3, v_4\}$ and Γ_2 is itself a clique. Finally (see [6, Corollary 3.12]), we have $H_{\Gamma} \cong (\mathbb{Z}^3 *_{\mathbb{Z}^2} \mathbb{Z}^3) *_{\mathbb{Z}^2} \mathbb{Z}^3$.

This stimulates the ensuing question:

Question 4.4. For which graphs can the corresponding B-B group be expressed as an iterated amalgamation of RAAGs? In other words, what will be the possible characterization of the family of graphs \mathcal{F} so that H_{Γ} can be written as an iterated amalgamated product of RAAGs if and only if $\Gamma \in \mathcal{F}$?

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