QUASICONVEXITY AND AMALGAMS

Ilya Kapovich

ABSTRACT. We obtain a criterion for quasiconvexity of a subgroup of an amalgamated free product of two word hyperbolic groups along a virtually cyclic subgroup. The result provides a method of constructing new word hyperbolic group in class (Q), that is such that all their finitely generated subgroups are quasiconvex. It is known that free groups, hyperbolic surface groups and most 3-dimensional Kleinian groups have property (Q). We also give some applications of our results to one-relator groups and exponential groups.

0.INTRODUCTION

An important notion in the theory of word hyperbolic groups is the concept of a quasiconvex (or rational) subgroup, which, roughly speaking, corresponds to a geometrically finite subgroup of a classical hyperbolic group.

Proposition-Definition. (see [1] for proof) Let G be a word hyperbolic group and A be a subgroup of G. Then the following conditions are equivalent:

- (1) for some finite generating set $X = S \cup S^{-1}$ of G there is an $\epsilon \ge 0$ such that for any $a \in A$ for each d_X -geodesic path p from 1 to a in the Cayley graph $\Gamma(G, X)$ of G for any point x on p there is $a' \in A$ such that $d_X(x, a') \le \epsilon$ (here d_X denotes the word metric on $\Gamma(G, X)$ corresponding to X);
- (2) A is finitely generated and for some finite generating set $Y = T \cup T^{-1}$ of A and for some finite generating set $X = S \cup S^{-1}$ of G there is a constant C > 0 such that for any $a \in A$

$$d_Y(a,1) \le C \cdot d_X(a,1) + C.$$

If any of these conditions is satisfied then A is called a *quasicovex subgroup* of G.

It can be shown that if A is a quasiconvex subgroup of G then conditions (1) and (2) of the previous definition are satisfied for any finite generating set of G and any finite generating set of A. Quasiconvex subgroups of word hyperbolic groups are themselves word hyperbolic and an intersection of a finite number of quasiconvex subgroups is again quasiconvex. Also finite subgroups, subgroups of finite index, virtually cyclic subgroups, free factors and conjugates of quasiconvex subgroups of word hyperbolic groups are quasiconvex (see [1], [8], [9], [10]).

The following class of groups is of considerable interest.

Definition. We say that a word hyperbolic group G has property (Q) if any finitely generated subgroup of G is quasiconvex in G.

We note some good properties of groups with property (Q).

- (1) a finitely generated subgroup of a group with property (Q) also has property (Q) and so it is finitely presented and word hyperbolic;
- (2) a finite extension of a group with property (Q) also has property (Q);

¹⁹⁹¹ Mathematics Subject Classification. Primary 2.2F10; Secondary 2.2F32.

The author is supported by Alfred P. Sloan Foundation Doctoral Dissertation Fellowship

- (3) if G has property (Q) then the intersection of any two finitely generated subgroups of G is finitely generated (that is G has the *Howson property*) (see [19]);
- (4) any infinite finitely generated subgroup H of G has finite index in its virtual normalizer $VN_G(H) = \{g \in G \mid |H : gHg^{-1} \cap H| < \infty, |gHg^{-1} : gHg^{-1} \cap H| < \infty\}$ (see [14]);
- (5) if H_1, H_2 are infinite subgroups of G such that $H = H_1 \cap H_2$ has finite index in both H_1 and H_2 then H is of finite index in $E = gp(H_1, H_2)$ (see [14]);
- (6) if G is a torsion-free group from class (Q) then any maximal cyclic subgroup of G is a Burns subgroup (see [4] and [6] for definitions) of G (see [12] for proof). In particular it implies that if B is a group with the Howson property, $b \in B$ is an element of infinite order, $g \in G$ is an element of infinite order which is not a proper power then the group $G \underset{g=b}{*} B$ has the Howson property.

It seems that most word hyperbolic groups have property (Q) but, nevertheless, there are relatively few examples for which it is proven. A finitely generated free group and the fundamental group of a closed hyperbolic surface belong to class (Q) (see [19], [18] and [20]). Also, if G is a torsion-free geometrically finite Kleinian group without parabolics whose limit set is not the whole S^2 , then G has property (Q) (see [20]).

In this paper we show how to build new groups with property (Q) from existing ones using free constructions.

Theorem A. Suppose $G = A_{-1} *_C A_1$ is word hyperbolic group where C is virtually cyclic and the groups A_1 , A_{-1} have property (Q). Then G has property (Q).

In order to obtain this result we need the following statement which characterizes quasiconvex subgroups of an amalgamated free product of two word hyperbolic groups and is of considerable independent interest.

Theorem B. Let $G = A_{-1} *_C A_1$ be a word hyperbolic group where C is virtually cyclic (this implies, by the results of [11], that A_1 and A_{-1} are quasiconvex in G). Suppose H is a finitely generated subgroup of G. Then the following conditions are equivalent.

- (a) H is quasiconvex in G;
- (b) for any $g \in G$ and for any i = 1, -1 the subgroup $gHg^{-1} \cap A_i$ is quasiconvex in A_i .

Corollary 1. Let A_1 , A_{-1} be groups with property (Q) and suppose that A_1 is torsion-free. Let $x \in A_1$ be an element of infinite order which is not a proper power. Let $y \in A_{-1}$ be an element of infinite order. Then the group $A_1 \underset{x=u}{*} A_{-1}$ is word hyperbolic and has property (Q).

Corollary 2. Let $G = A_1 *_C A_{-1}$ where C is finite and A_1 , A_{-1} have property (Q). Then G is word hyperbolic and also has property (Q).

Corollary 2 is a generalization of the result of [11] where its statement was proved when $C = \{1\}$.

There are also some interesting consequences of these results for so-called *exponential* groups (see [15] for definitions).

Corollary 3. Let G be a torsion-free hyperbolic group with property (Q) (e.g. finitely generated free group, hyperbolic surface group etc). Let $G^{\mathbb{Q}}$ be the tensor \mathbb{Q} -completion of G where \mathbb{Q} is the ring of rational numbers. Then

- (1) $G^{\mathbb{Q}}$ is a locally (Q)-group that is any finitely generated subgroup of $G^{\mathbb{Q}}$ is word hyperbolic and has property (Q);
- (2) $G^{\mathbb{Q}}$ has the Howson property that is the intersection of any two finitely generated subgroups of G is finitely generated
- (3) if H_1 and H_2 are infinite commensurable subgroups of G, that is the intersection $H = H_1 \cap H_2$ has finite index in both H_1 and H_2 , then H has finite index in their join $E = gp(H_1 \cup H_2)$.

As a by-product of our results we also obtain the following statement.

Corollary 4. Let $G = G_1 *_C G_2$ be a word hyperbolic group such that C is finitely generated. Suppose H is a finitely generated subgroup of G such that for any $g \in G$ we have $g^{-1}Hg \cap G_1 = g^{-1}Hg \cap G_2 = \{1\}$. Then H is quasiconvex in G.

Corollary 5. Suppose G is a one-relator group $G = \langle x_1, \ldots, x_k, y_1, \ldots, y_s | vu = 1 \rangle$ where v is a nontrivial freely reduced word in x_1, \ldots, x_k , u is a nontrivial freely reduced word in y_1, \ldots, y_s and u is not a proper power in the free group $F(y_1, \ldots, y_s)$. Then G has property (Q).

Notice that fundamental groups of closed hyperbolic surfaces have one-relator presentations as in Corollary 5. G.Swarup [20] and C.Pittet [18] showed using the techniques of hyperbolic geometry that these groups belong to class (Q). Corollary 5 gives another, more combinatorial, proof of this fact.

It follows from the result of R.Burns [4] that a group G as in Corollary 5 has the Howson property. Since the groups from class (Q) have the Howson property, Corollary 5 may be considered as a generalization of Burns' theorem.

We would like to stress that quasiconvexity of a subgroup is not a question of the isomorphism type of the subgroup but rather that of comparing the word metrics on the subgroup and on the ambient group. This makes the proof of Theorem B rather more difficult than it may seem from the first sight. To illustrate this point, consider the following example. Let M be a closed hyperbolic 3-manifold fibering over a circle with fiber S, where S is a closed hyperbolic surface. We may also think of G as a geometrically finite group of isometries of \mathbb{H}^3 such that $\mathbb{H}^3/G = M$. Let $x_0 \in S$ and $G = \pi_1(G, x_0)$, $H = \pi_1(S, x_0)$. Then there is a short exact sequence

$$1 \to H \to G \to \mathbb{Z} \to 1$$

and therefore H is not quasiconvex in G (see [1]). Consider a simple closed curve γ on S passing through x_0 such that γ separates S into two non-contractible components. Then γ defines a decomposition of H as an amalgamated free product $H = F_1 *_C F_2$ where F_1, F_2 are nonabelian free groups, C is an infinite cyclic group which is malnormal in both F_1 and F_2 . It follows from geometric considerations that both F_1 and F_2 are geometrically finite groups of isometries of \mathbb{H}^3 and therefore (see [20]) both F_1 and F_2 are quasiconvex in G. Thus we see that F_1, F_2 and C are quasiconvex in G and $H = gp(F_1, F_2) \simeq F_1 *_C F_2$ is not quasiconvex in G.

1.SUBGROUP STRUCTURE OF AN AMALGAMATED PRODUCT

Some definitions and notations.

If G is a finitely generated group and X is a finite generating set of G closed under taking inverses, we denote the Cayley graph of G with respect to X by $\Gamma(G, X)$. The word metric on $\Gamma(G, X)$ corresponding to X is denoted d_X . Also, for an element $g \in G$ we put $l_X(g) = d_X(g, 1)$. If w is a word in X, we denote by \overline{w} the element of G represented by w. A word w in X is termed d_X -geodesic if the length l(w) of w is equal to $l_X(\overline{w})$. A word w in X is called λ -quasigeodesic with respect to d_X if for any subword u of w we have $l(u) \leq \lambda \cdot l_X(\overline{u}) + \lambda$.

We also will need some notations regarding graphs of groups. Let \mathbb{A} be a graph of groups and A be the underlying graph of \mathbb{A} . Then VA and EA denote the set of vertices and the set of edges of A respectively. We also denote by $E^+(A)$ the set of positively oriented edges of A. If e is an edge of A then its inverse is denoted by e^{-1} . For any vertex v of A the corresponding vertex group is denoted A_v . Similarly, if e is an oriented edge of A, the edge group corresponding to e is denoted A_e . We also denote the initial vertex of e by $\partial_0(e)$ and the terminal vertex of e by $\partial_1(e)$. The edge monomorphism $A_e \to A_{\partial_0(e)}$ is denoted by α_e . The edge-monomorphism $A_e \to A_{\partial_1(e)}$ is denoted by ω_e . Recall that $\partial_0(e) = \partial_1(e^{-1})$ and $(e^{-1})^{-1} = e$ for any $e \in EA$. We also have $A_e = A_{e^{-1}}$ and $\alpha_e = \omega_{e^{-1}}$ for every $e \in EA$.

The premises.

Suppose G is a finitely generated group and

$$G = A_1 * A_{-1} \tag{1}$$

where A_1, A_{-1} and C are finitely generated.

Let X_{-1} and X_1 be finite generating sets for A_{-1} and A_{-1} closed under inversions and containing a finite generating set C of C. Put $\mathcal{G} = X_{-1} \cup X_1$ to be a finite generating set for G. We denote by d_{X_i} the word

metric corresponding to X_i on A_i , $i = \pm 1$. Denote by $d_{\mathcal{G}}$ the word metric corresponding to \mathcal{G} on G. Also, fix an ordering on the sets X_1, X_{-1} .

Let L_i be the set of all X_i -geodesic words w such that

- (i) \overline{w} is shortest (with respect to d_{X_i}) in the coset $\overline{w}C$;
- (ii) if v is another X_i -geodesic word such that $\overline{v}C = \overline{w}C$ and l(v) = l(w) then w is lexicographically smaller than v.

Then $T_i = \overline{L_i}$ is a left transversal for C in A_i .

An expression of g as an alternating product $g = u_1 \dots u_k$ where $u_j \in A_{-1} \cup A_1$, $u_j \notin C$ for j < k, and $u_{s+1} \notin A_i$ whenever $u_s \in A_i$, $s = 1, \dots, k-1$, is called a *reduced form* of g with respect to presentation (1). The elements u_i are called *syllables* of g. If $g = u_1 \dots u_k$ is a reduced form of g and $u_k \notin C$ then we say that g has syllable length k. If $u_k \in C$ and so $g = u_k$, we say that g has syllable length zero.

If $x = u_1 \dots u_k$ and $y = v_1 \dots v_s$ are reduced forms of x and y, we say that y is a *right segment* of x and that x ends in Cy if $s \leq k$ and $u_{k-s+1} \dots u_k y^{-1} \in C$. Under these circumstances y is said to be a proper right segment of x if s < k or $y \in C$ and $x \notin C$. It is not hard to see that these definitions do not depend on the choices of reduced forms for x and y.

Analogously if $x = u_1 \dots u_k$ and $y = v_1 \dots v_s$ are reduced expressions, we say that y is a *left segment* of x and x begins in yC if $s \leq k$ and $y^{-1}u_1 \dots u_s \in C$. A left segment y of x is said to be *proper* if s < k or $y \in C$, $x \notin C$. Again these definitions do not depend on the choices of reduced forms for x and y. Observe also that y is a (proper) right segment of x if and only if y^{-1} is a (proper) left segment of x^{-1} . If $g = u_1 \dots u_k$ is a reduced expression and $u_k \in A_i - C$, the element g is said to *end* in A_i . Elements of C are said to end in C.

A little bit of Bass-Serre theory.

Let H be a finitely generated subgroup of G which is not elliptic, that is H is not conjugate to a subgroup of A_i .

Let \hat{T} be a Bass-Serre tree associated with the free product decomposition $G = A_{-1} *_C A_1$. The vertices of \hat{T} are just coset classes gA_i of A_1 and A_{-1} in G. There is a distinguished vertex $d_1 = A_1$ which is a basepoint of \hat{T} . Also denote $d_{-1} = A_{-1}$. For any vertex gA_i and $a \in T_i - \{1\}$ there is a positively oriented edge (gA_i, gaA_{-i}) which we label by a. There is also a positively oriented edge $(d_1, d_{-1}) = (A_1, A_{-1})$ labelled by 1. The action of G on \hat{T} is obvious: $g \cdot fA_i = gfA_i, g \cdot (fA_i, faA_{-i}) = (gfA_i, gfaA_{-i})$. The stabilizer in G of a vertex gA_i is clearly gA_ig^{-1} and the stabilizer in G of an edge (gA_i, ga_iA_{-i}) is equal to $gA_ig^{-1} \cap ga_iA_{-i}a_i^{-1}g^{-1} = ga_i(A_i \cap A_{-i})a_i^{-1}g^{-1} = ga_iCa_i^{-1}g^{-1}$. We will say that a vertex $v = gA_i$ of \hat{T} has type A_i . Any edge-path $p = (e_1, \ldots, e_k)$ such that each e_j is positively oriented and labelled by a_j , has a label $a_1 \ldots a_k \in G$. For any vertex v of T there is a unique reduced edge-path p_v from d_1 to v whose label is denoted by s_v . Notice that $s_v = 1$ if and only if $v = d_{\pm 1}$. We say that every vertex w on p_v is less than or equal to v and write $w \leq v$. It is obvious that " \leq " is a partial ordering on $V\hat{T}$. For a vertex $v \neq d_1$ the closest to v vertex on p_v which is different from v is called the preceding vertex for v. In other words, u is a preceding vertex for v if and only if (u, v) is a positively oriented edge of \hat{T} .

Then H acts on T as a subgroup of G and two vertices g_1A_i and g_2A_i lie in the same H-orbit if and only if the double coset classes Hg_1A_i and Hg_2A_{-i} are equal. There is a subtree T of \hat{T} which is H-invariant and does not contain any proper H-invariant subtrees, that is the action of H on T is minimal. If T is a single vertex, say $T = gA_i$ then $H \leq gA_ig^{-1}$ which is impossible since we assumed that H is not conjugate to a subgroup of A_i . Thus T has at lease one edge. There is a finite subtree Y of T which serves as a "fundamental domain" for the action of H, that is any edge of T lies in the H-orbit of a unique edge of Y. Then we can find a subtree Y_1 of Y such that any vertex of T is H-equivalent a unique vertex of Y_1 . Thus Y is a union of Y_1 and a finite number of disjoint edges. By conjugating H we may assume that the edge (d_{-1}, d_1) is in Y_1 where $d_{-1} = A_{-1}, d_1 = A_1$. Notice that for $v \in VY, v \neq d_1$ the preceding vertex for vbelongs to Y_1 .

Observe also that no edge (u, v) of Y, where u precedes v, other than (A_1, A_{-1}) , has label 1. Clearly, if v is of type A_i then $v = s_v A_i$. Now let $v = s_v A_i$ be a vertex of Y. Put $A_v = A_i \cap s_v^{-1} H s_v$. Thus $H \cap s_v A_i s_v^{-1} = s_v A_v s_v^{-1}$. Also, let $e = (s_u A_{-i}, s_v A_i) = (s_u A_{-i}, s_u a_{-i} A_i)$ be a positively oriented edge of Y. Recall that its stabilizer in G is $s_v C s_v^{-1}$. Put $C_e = C \cap s_v^{-1} H s_v$. Then $s_v C_e s_v^{-1} = s_v C s_v^{-1} \cap H = s_v (A_v \cap C) s_v^{-1}$ is a subgroup of $s_u A_u s_u^{-1}$ and $s_v A_v s_v^{-1}$. Notice also that if $v = s_v A_i$ is a vertex of $Y - Y_1$ and $q = s_q A_i$ is the only vertex of Y_1 *H*-equivalent to v then for any $h \in H$ such that hq = v we have $h^{-1}s_vA_vs_v^{-1}h = s_qA_qs_q^{-1}$.

Suppose now that $v = s_v A_i$ is a vertex of $Y - Y_1$ and $s_v = s_u a_{-i}$ where u is the vertex of Y_1 preceding v and $a_{-i} \in T_{-i} - \{1\}$. Let $q = s_q A_i$ be the only vertex of Y_1 which is *H*-equivalent to *v*. Then $Hs_q A_i = Hs_v A_i$, so there is an element $a \in A_i$ such that $s_v^{-1} s_q a \in H$. We fix this element $a \in A_i$ for each $v \in Y - Y_1$ and denote $h_v = s_v a^{-1} s_q^{-1} \in H$. Clearly $s_v a^{-1} s_q^{-1} \cdot q = s_v a^{-1} s_q^{-1} \cdot s_q A_i = s_v A_i = v$, that is $h_v q = v$. Since $h_v q = v$ we have

 $h_v^{-1}s_vA_vs_v^{-1}h_v = s_qA_qs_q^{-1}.$ Then $s_v(A_v \cap C)s_v^{-1}$ is a subgroup of $s_uA_us_u^{-1}$ and $h_v^{-1}s_v(A_v \cap C)s_v^{-1}h_v = s_qas_v^{-1}s_v(A_v \cap C)s_v^{-1}s_qa^{-1} = s_qa(A_v \cap C)a^{-1}s_q^{-1} \le s_qA_qs_q^{-1} = s_qAs_q^{-1} \cap H.$ Thus the element h_v conjugates the subgroup $s_v(A_v \cap C)s_v^{-1}$ of A_u into the subgroup $s_qa(A_v \cap C)a^{-1}s_q^{-1}$ of $s_qA_qs_q^{-1}$.

The quotient graph of groups for the action of H on T can obtained from Y in the following way. Let B be an oriented graph such that

- (1) the vertices of B are the vertices of Y_1 ;
- (2) B has one edge (v, w) for each positive edge (v, w) of Y_1 ;
- (3) for any vertex $v \in V(Y Y_1)$ and a vertex $v \in VY_1$ which is *H*-equivalent to v there is an oriented edge (u, q) in B where u is a preceding vertex for q.

We give B the structure of a graph of groups in the following way. For any $v \in VY_1$ put $B_v = s_v A_v s_v^{-1}$ We give B the structure of a graph of groups in the following way. For any $v \in V Y_1$ put $B_v = s_v A_v s_v^{-1}$ to be the vertex group of v. For each edge e = (u, v) of Y_1 where u precedes v, put the edge group $B_e = s_v (A_v \cap C) s_v^{-1}$ where corresponding edge homomorphisms $\alpha_e : s_v (A_v \cap C) s_v^{-1} \to s_v A_v s_v^{-1}$ and $\partial_1 : s_v (A_v \cap C) s_v^{-1} \to s_u A_u s_u^{-1}$ are just the inclusion maps. For any $v \in V(Y - Y_1)$, $u \in VY_1$ preceding v and $q \in VY_1$ which is H-equivalent to v, put $B_e = s_v (A_u \cap C) s_v^{-1} \leq s_u A_u s_u^{-1}$ where $e = (u, q) \in EB$. The boundary homomorphism $\alpha_e : B_e = s_v (A_u \cap C) s_v^{-1} \to s_q A_q s_q^{-1} =$ $s_u A_u s_u^{-1} = B_u$ is the inclusion map. The boundary homomorphism $\omega_e : B_e = s_v (A_u \cap C) s_v^{-1} \to s_q A_q s_q^{-1} =$

 B_q is conjugation by h_v . That is $\omega_e(g) = h_v^{-1}gh_v$ for any $g \in B_e$. This defines a graph of groups \mathbb{B} . Notice that Y_1 is a maximal subtree of B. The fundamental group of the graph of groups \mathbb{B} with respect to the maximal subtree Y has the presentation

$$\pi_1(\mathbb{B}, Y_1) = (*B_v) * F(E^+B) / \{e = 1, e \in EY_1; \alpha_e(b)e = e\omega_e(b), e \in E^+B, b \in B_e\}$$
(2)

Then by the fundamental result of Bass-Serre theory the map $f: \pi_1(\mathbb{B}, Y_1) \to H$ defined by f(g) = g for any $g \in B_v = s_v A_v s_v^{-1}, v \in VY_1, f(e) = h_v$ where $e = (u, q) \in E^+(B - Y_1), u$ precedes $v \in V(Y - Y_1), v \in VY_1$ $u \in VY_1$ is *H*-equivalent to v, is an isomorphism. We will identify $\pi_1(\mathbb{B}, Y_1)$ with H via this isomorphism and will right

$$H = \pi_1(\mathbb{B}, Y_1) \tag{3}$$

Normal forms for the fundamental group of a graph of groups.

Let A be the graph of groups with underlying graph A and let T_0 be the maximal subtree of A. Let d_0 be a fixed vertex of T_0 . We will describe a set of normal forms for the fundamental group of A

$$G_0 = \pi_1(\mathbb{A}, T_0) \tag{4}$$

which is slightly non-standard but which is more suitable for our purposes.

Definition 1.1. A sequence

$$p = (g_1, e_1, g_2, e_2, \dots, g_k, e_k, g_{k+1})$$

is called a *reduced sequence* if

- (1) $e_1 \ldots e_k$ is an edge-path in A;
- (2) for each $i = 1, \ldots k$ $g_i \in A_{\partial_0(e_i)}$ and $g_{i+1} \in A_{\partial_1(e_i)}$;
- (3) $\partial_0(e_1) = \partial_1(e_k) = d_0;$
- (4) $e, 1, e^{-1}$ is not a subsequence of p;

- (5) for any i = 2, ..., k + 1 either $g_i = 1$ or $g_i \notin \omega_{e_{i-1}}(B_{e_i})$;
- (6) if $i < j, g_i \neq 1, g_j \neq 1, g_s = 1$ for i < s < j then

$$g_i \notin (\alpha_i \omega_i^{-1}) \dots (\alpha_{j-1} \omega_{j-1}^{-1}) (B_{\partial_1(e_{j-1})})$$

In the situation above we say that the number of terms in p which are different from 1 in G_0 is the *syllable* length of p. Any subsequence of p represents an element of G_0 , which is just the product of all terms in this subsequence viewed as elements of G_0 . It is clear from the theory of graphs of groups that no subsequence of p represents an element of a vertex group of A unless this subsequence has at most one term different from 1 in G_0 . If g is the element of G_0 represented by p, we say that p is a reduced form of g with respect to presentation (4). Let $p_1 = (u_1, \ldots, u_n)$ be obtained from p by deleting all terms which are equal to 1 in G_0 . Thus each u_i is either a stable letter or a nontrivial element of a vertex group. We call p_1 a normal form of g with respect to presentation (4).

Some Calculations.

Recall that $H = \pi_1(\mathbb{B}, Y_1) \leq G = A_1 *_C A_{-1}$.

Lemma 1.2.

- (1) If $e = (s_v A_i, s_v a_i A_{-i}) = (s_v A_i, s_x A_{-i})$ is a positively oriented edge of Y, where $a_i \neq 1$, then $a_i C \cap A_v = \emptyset$.
- (2) If $v = s_v A_i$ is a vertex of $Y Y_1$, q is the only vertex of Y_1 which is H-equivalent to v and $h_v = s_v a^{-1} s_a^{-1}$ then $A_q a \cap C = \emptyset$.
- (3) If v_1, v_2 are distinct vertices from $Y Y_1$ which are *H*-equivalent to the same vertex q of Y_1 and $h_{v_j} = s_{v_j} a_j^{-1} s_q^{-1}$, i = 1, 2 then $A_q a_1 C \neq A_q a_2 C$.
- (4) If $q = s_q A_i \in Y_1$ is a preceding vertex for $w = s_q b A_{-i} \in VY$ and for some $v \in V(Y Y_1)$ $h_v = s_v a^{-1} s_q^{-1}$, $a \in A_i$ then $bC \cap A_q a = \emptyset$.
- (5) If $v = s_v A_{-i} = s_u a A_{-i} \in VY$, $w = s_w A_{-i} = s_u b A_{-i} \in VY$, $a \neq b, a, b \in T_i$, $u = s_u A_i$, where the edges (u, v) and (u, w) of Y are positively oriented, then $A_u a \neq A_u b$.

Proof.

(1) Suppose $a_i c \in A_v$, $c \in C$ that is $s_v a_i c s_v^{-1} = h \in H$.

Then $s_v = s_u a_{-i}$, where $u = s_u A_{-i}$ is a preceding vertex for v. Notice that $u, v \in Y_1$. We have

$$hu = s_v a_i c s_v^{-1} u = s_v a_i c a_{-i}^{-1} s_u^{-1} \cdot s_u A_{-i} = s_v a_i A_{-i} = x.$$

On the other hand hv = v. Thus h takes the edge (u, v) into the edge (x, v) what contradicts our assumptions that no two edges of Y are H-equivalent.

(2) Let $v = s_v A_i \in Y - Y_1$ and let $q = s_q A_i$ be the vertex of Y_1 *H*-equivalent to v. Let $s_v = s_u a_{-i}$, where $u = s_u A_{-i}$ is a vertex preceding to v. Suppose $a_q a = c \in C$ for some $a_q \in A_q$. Then $a^{-1}a_q^{-1} = c^{-1}$. As before $h_v = s_v a^{-1}s_q^{-1} \in H$ and $h_v \cdot q = v$. Notice also that $h_0 = s_v a_q^{-1}s_v^{-1} \in H$, $h_0 \cdot q = q$. Therefore $h \cdot q = v$ where $h = h_v h_0$. Now let $y = s_y A_{-i}$ be the vertex preceding q and $s_q = s_y a'_{-i}$. Then $h \cdot y = s_v a^{-1}s_q^{-1} \cdot s_q a_q^{-1}s_q^{-1} \cdot s_y A_{-i} = s_u a_{-i}a^{-1}(a'_{-i})^{-1}s_y^{-1} \cdot s_y a'_{-i}a_q^{-1}(a'_{-i})^{-1}s_y^{-1} \cdot s_y A_{-i} = s_v (a^{-1}a_q^{-1})a'_{-i}{}^{-1}A_{-i} = s_v c^{-1}A_{-i} = s_v A_{-i} = s_u a_{-i}A_{-i} = u$. Thus $h \cdot y = u$ and $h \cdot q = v$ and h takes the edge (y, q) into the edge (u, v). This contradicts our assumptions that no two edges of Y are H-equivalent.

(3) Suppose $A_q a_1 C = A_q a_2 C$ that is $a_1^{-1} a_q a_2 = c \in C$ for some $a_q \in A_q$. Put $h_0 = s_q a_q s_q^{-1} \in H$. Let $v_1 = s_{v_1} A_i, v_2 = s_{v_2} A_i$. Thus we know that $q = s_q A_i$ and $a_{\pm 1} \in A_i - C$. Put $h = h_{v_1} h_0 h_{v_2}^{-1}$. Then $h \cdot v_2 = v_1$ since $h_{v_j}(q) = v_j, j = \pm 1$ and $h_0(q) = q$. Let $s_{v_j} = s_{u_j} b_j$ where $b_j \in T_{-j}$ and $u_j = s_{u_j} A_{-i}$ is the preceding vertex for $v_j, j = 1, 2$.

Then $hu_2 = s_{u_1}b_1a_1^{-1}s_q^{-1} \cdot s_qa_qs_q^{-1} \cdot s_qa_2b_2^{-1}s_{u_2}^{-1} \cdot s_{u_2}A_{-i} = s_{u_1}b_1cb_2^{-1}A_{-i} = s_{u_1}A_{-i} = u_1$. Thus h takes the edge (u_2, v_2) into the edge (u_1, v_1) what contradicts our assumptions that no two distinct edges of Y are H-equivalent.

(4) Suppose $bca^{-1} = a_q$, $a_q \in A_q$, $c \in C$. Consider the preceding v vertex $u = s_u A_{-i} \in VY_1$. Then $s_v = s_u f$ for some $f \in T_{-i}$. We have $h = s_q bca^{-1} s_q^{-1} \in H$ and $h_v^{-1} = s_q a s_v^{-1} \in H$. Thus $h_1 = hh_v^{-1} = hh_v^{-1}$.

 $s_q bcf^{-1}s_u^{-1} \in H$. Clearly $h_1(u) = s_q bcf^{-1}s_u^{-1}s_u A_{-i} = s_q bA_{-i} = w$ and $h_1(v) = s_q bcf^{-1}s_u^{-1}s_u fA_i = s_q A_i = q$. Thus h_1 takes the edge (u, v) into the edge (w, q) what contradicts the fact that no two distinct edges of Y are H-equivalent.

(5) Suppose $a = a_u b$, $a_u \in A_u$. Then $h = s_u a b^{-1} s_u^{-1} \in H$. However h(u) = u and h(w) = v. Thus h takes the edge (u, w) into the edge (u, v) which is impossible.

Lemma 1.3. If v_1 , v are vertices of Y of type A_i and $Hs_vC = Hs_{v_1}C$ then $v = v_1$.

Proof. Suppose $q = s_q A_i \neq v = s_v A_i$ and $Hs_q C = Hs_v C$, that is $s_q cs_v^{-1} = h$ for some $h \in H$, $c \in C$. Let $us_u A_{-i}$ be the preceding vertex for v when $v \neq d_1$ and let $u = d_{-1}$ when $v = d_1$. Similarly, let $y = s_y A_{-i}$ be the preceding vertex for q when $q \neq d_1$ and let $y = d_{-1}$ when $q = d_1$. Then $s_v = s_u b$, $s_q = s_y d$ for some $b, d \in T_{-i}$. We have $hv = s_q cs_v^{-1} s_v A_i = s_q A_i = q$ and $hu = s_y dcb^{-1} s_u^{-1} s_u A_{-i} = s_y A_{-i} = y$. Thus h takes the edge (u, v) into the edge (y, q) what contradicts the fact that no two distinct edges of Y are H-equivalent.

The following statements are obvious corollaries of the properties of amalgamated free products.

Lemma 1.4. If $x, y \in G$ and $x = u_1 u_2 \dots u_k$, $y = v_1 \dots v_s$ are their reduced expressions. Suppose that $v_s \in A_i$. Then xy ends in A_i unless y^{-1} is a right segment of x.

Lemma 1.5. If v, u are vertices of Y, $v \neq d_1$ then s_u is a left segment of s_v if and only if $u \leq v$.

Transversal elements.

Definition 1.6. Define the following functions $\rho_i, \sigma_i : \{s_v | v \in VY\} \to G, i = \pm 1$. If $v = s_v A_i = s_u b A_i \in VY$, where $b \in T_{-i}$ and $u = s_u A_{-i}$ is the preceding vertex of v, put $\sigma_{-i}(s_v) = s_u$. If $v = s_v A_i \in VY_1$, put $\sigma_i(s_v) = s_v$. If $v = s_v A_i \in V(Y - Y_1)$ and $q = s_q A_i \in VY_1$ is *H*-equivalent to v, then put $\sigma_i(s_v) = s_q$. Now for $v \in VY_1$ put $\rho_{\pm 1}(s_v) = s_v$. Suppose $v = s_v A_i \in V(Y - Y_1)$ and $q = s_q A_i \in VY_1$ is *H*-equivalent to v. Let $h_v = s_v a^{-1} s_q^{-1}$. Then put $\rho_{-i}(s_v) = s_v$ and $\rho_i(s_v) = s_q a$.

The elements of the set $im(\rho_1)C \cup im(\rho_{-1})C$ are called *transversal elements*.

This definition is motivated by the work of B.Baumslag [2] who used a similar construction to analyze the subgroup structure of a free product of two groups. We collect some useful facts about the functions ρ_i, σ_i in the following lemma.

Lemma 1.7. Let $v = s_v A_i \in VY$. Then

(i) $\sigma_j(s_v) \in Hs_v A_j, \ j = \pm 1;$

(iii) $\rho_i(s_v) \in Hs_v, \ j = \pm 1;$

(iii) $\rho_j(s_v) = \sigma_j(s_v)a_j, a_j \in A_j, j = \pm 1;$

(iv) $\sigma_i(s_v)$ is either 1 or it ends in A_{-i} ;

(v) if $v = s_v A_i \in V(Y - Y_1)$ then $h_v = \rho_{-i}(s_v)\rho_i(s_v)^{-1}$;

(vi) if $H\sigma_j(s_v)A_j = H\sigma_j(s_w)A_j$ then $\sigma_j(s_v) = \sigma_j(s_w)$, $j = \pm 1$, $v, w \in VY$;

(vii) if $H\rho_i(s_v)C = H\rho_i(s_w)C$ then $\rho_i(s_v) = \rho_i(s_w)$, $j = \pm 1$, $v, w \in VY$.

Proof. Statements (i), (ii), (iii), (iv) and (v) follow immediately from the definitions of s_v , $h_v \rho_i$ and σ_i .

(vi) For any $r \in im(\sigma_i)$ there is $v = s_v A_i \in VY_1$ such that $r = \sigma_i(s_v) = s_v$.

So if $r_1, r_2 \in im(\sigma_i)$ and $Hr_1A_i = Hr_2A_i$, let $v_j = s_{v_j}A_i \in VY_1$ be such that $r_j = \sigma_i(s_{v_j})$, j = 1, 2. Thus $Hs_{v_1}A_i = Hs_{v_2}A_i$. However, no two distinct vertices of Y_1 are *H*-equivalent. Therefore $v_1 = s_{v_1}A_i = s_{v_2}A_i = v_2$ and $r_1 = s_{v_1} = s_{v_2} = r_2$.

(vii) Any element in the image of ρ_i has the form $s_v, v \in VY_1$ or $s_v a_i$ where $v = s_v A_i \in VY_1$, $a_i \in A_i - C$, $s_v a_i A_{-i} \notin VY_1$.

Suppose $r, y \in im(\rho_i)$ and HrC = HyC. Thus there are $h \in H, c \in C$ such that $h = ycr^{-1}$. There are several cases to consider.

Case 1 Assume first $r = s_v, y = s_w$ for some vertices v, w of Y.

Suppose first v and w have the same type A_j . Then by Lemma 1.3 $Hs_vC = Hs_wC$ implies v = w, $r = s_v = s_w = y$.

Suppose now that $r = s_v$, $v = s_v A_j \in VY$ and $y = s_w$, $w = s_w A_{-j} \in VY$. Since both r and y are in the image of ρ_i , one of the vertices v, w, say v, has type A_i and is in Y_1 and the other, say, w has type A_{-i} and

is in $Y - Y_1$. Consider the preceding vertex $u = s_u A_i \in VY_1$ for w. We have $s_w = s_u a_i$ where $a_i \in T_i$. We know that $h^{-1} = s_v c^{-1} a_i^{-1} s_u^{-1}$. It is clear that $h^{-1}u = s_v c^{-1} a_i^{-1} s_u^{-1} s_u A_i = s_v A_i = v$. Since $v, u \in VY_1$, v = u and $s_v = s_u$. Therefore $h = s_u a_i c s_u^{-1} \in H$ and so $a_i c \in A_u$. This contradicts Lemma 1.2(i). If v has type A_{-i} then $Hs_v C = Hs_w C$ implies $Hs_v A_{-i} = Hs_w A_{-i}$ that is Hv = Hw. Thus v is the only vertex of Y_1 H-equivalent to w. Notice that $h(v) = s_w c s_v^{-1} s_v A_{-i} = s_w A_{-i} = w$. Let $h_w = s_w a^{-1} s_v^{-1} \in H$, $a \in A_{-i}$. Recall that by Lemma 1.2(i) $a \notin C$. Then $h^{-1} = s_v c^{-1} s_w^{-1}$ and $h^{-1} h_w = s_v c^{-1} a^{-1} s_v^{-1}$. However $h^{-1}w = v, h_w(v) = w$ and, therefore, $h^{-1}h_w(v) = v$. Thus $s_v a c s_v^{-1}(v) = v$ that is $ac \in A_v$. But this is impossible by Lemma 1.2(2).

Case 2 Suppose now that $r = s_{q_1}a_1, y = s_{q_2}a_2$ where $q_1, q_2 \in VY_1$ and for some $v_1 = s_{v_1}A_i, v_2 = s_{v_2}A_i \in V(Y - Y_1)$. $h_{v_1} = s_{v_1}a_1^{-1}s_{q_1}^{-1}, h_{v_2} = s_{v_2}a_2^{-1}s_{q_2}^{-1}$. Then q_1, q_2 have type $A_i, a_1, a_2 \in A_i$. Since HrC = HyC, $Hs_{q_1}A_i = Hs_{q_2}A_i$ and therefore $q_1 = q_2 = q$. Suppose $r \neq y$. Then $h_{v_1}hh - v_2^{-1} = s_{v_1}cs_{v_2}^{-1} \in H$, and so $Hs_{v_1}C = Hs_{v_2}C$. Lemma 1.3 implies that $s_{v_1} = s_{v_2}$ and therefore r = y.

Case 3 Suppose now that $r = s_q a$, $y = s_w$, $a \in A_i$, $q = s_q A_i \in VY_1$, $w \in VY$, $h_v = s_v a^{-1} s_q^{-1}$, $v \in V(Y - Y_1)$. Assume first that w has type A_i and therefore $w \in VY_1$. We have $h = ycr^{-1} = s_w ca^{-1}s_q^{-1} \in H$. Thus $h(q) = s_w ca^{-1}s_q^{-1}s_q A_i = s_w A_i = w$. Therefore q = w since $q, w \in VY_1$. Hence $h^{-1} = s_q ac^{-1}s_q^{-1} \in H$ and $ac^{-1} \in A_q$. But this contradicts Lemma 1.2(2). Assume now that w has type A_{-i} . Let $u = s_u A_i$ be the preceding vertex of w if $w \neq d_1$ and $u = d_{-1}$ when $w = d_1$. Thus $u \in Y_1$, $s_w = s_u b$, $b \in T_i$. Then $h = s_u bca^{-1}s_q^{-1} = h \in H$ and so hq = u. This implies u = q since $u, q \in Y_1$. Therefore $h = s_q (bca^{-1})s_q^{-1} \in H$ and $bca^{-1} \in A_q$. If $w \neq d_1$ then this contradicts Lemma 1.2(4). If $w = d_1$ then b = 1 and $ac^{-1} \in A_q$ which contradicts Lemma 1.2(2) This completes the proof of Lemma 1.7

Lemma 1.7(4) implies that different elements in $im(\rho_j)$ represent different double coset classes HgC, $j = \pm 1$. This justifies the term *transversal* for the elements of the set $im(\rho_1)C \cup im(\rho_{-1})C$. Notice that a left segment of a transversal element is again transversal.

Definition 1.8. Let g^{-1} be a nontransversal element and let $w = v_1 \dots v_k$, be a reduced form of g with respect to presentation (1). Let $s \leq k$ be the minimal number such that $g = v_1 \dots v_s v$ where v^{-1} is a transversal. We call the expression $v_1 \dots v_s$ the *nerve* of w. The number s is termed the syllable length of the nerve of w. Notice that if $w_1 = u_1 \dots u_k$ is another reduced form of g and $u_1 \dots u_{s_1}$ is the nerve of w_1 then $s = s_1$ and $u_1 \dots u_s C = v_1 \dots v_s C$.

If g^{-1} is transversal and $w = v_1 \dots v_k$ is a reduced form of g with respect to presentation (1), we say that a nerve of w is empty and that it has the syllable length zero.

Remark. Notice that if $w = v_1 \dots v_k$ is a reduced expression with respect to presentation (1), $(v_j \dots v_k)^{-1}$ is a transversal and $(v_{j-1}v_j \dots v_k)^{-1}$ is not a transversal, then $v_1 \dots v_{j-1}$ is the nerve of w. This immediately follows from the fact that an initial segment of a transversal element is again transversal.

Lemma 1.9.

- (i) If $a_v \in A_v C_v$, $v = s_v A_i \in VY_1$ then $s_v a_v$ is not a transversal.
- (ii) If $a_v \in A_v C_v, v = s_v A_i \in VY_1$, $b \in T_i \{1\}$, $w = s_v b A_{-i} \in VY$, v is a preceding vertex for w and $s_v a_v s_v^{-1}$ does not stabilize the edge (v, w) then $s_v a_v b$ is not a transversal.
- (iii) If $w = s_w A_i \in V(Y Y_1)$, $q = s_q A_i \in VY_1$, $a \in A_i$, $b \in A_{-i} C$ and $h_w = s_w a^{-1} s_q^{-1}$ then $s_q ab$ is not a transversal.
- (iv) Suppose $w = s_w A_i \in V(Y Y_1)$, $q = s_q A_i \in VY_1$, $a \in A_i$ and $h_w = s_w a^{-1} s_q^{-1}$. Suppose $u = s_u A_{-i} \in VY_1$ is the vertex preceding w and $s_w = s_u b$, $b \in T_{-i}$. Suppose $a_1 \in A_i C$. Then $s_w a_1$ is not a transversal.
- (v) Suppose $1 \neq \rho_i(t)\rho_{-i}(t) = s_v a_i a_{-i}^{-1} s_w^{-1}$ where $v = s_v A_i, w = s_w A_{-i} \in VY_1$. Suppose $b \in A_i C$. Then $s_w a_{-i} b$ is not a transversal.
- (vi) Suppose $q = s_q A_i \in VY_1$, $a_q \in A_q$ and $a \in A_i$ is such that for some vertex $v \in V(Y Y_1)$ we have $h_v = s_v a^{-1} s_q^{-1}$. Suppose further that $a_q a C \neq a C$. Then $s_q(a_q a)$ is not a transversal.

Proof.

(i) Suppose first that $s_v a_v \in im(\rho_i)C$. There are two possibilities.

Case 1. There is a vertex $w = s_v b A_{-i}$, $b \in T_i - \{1\}$, such that v precedes w and $a_v C = bC$. Thus $bC \cap A_v \neq \emptyset$ what contradicts Lemma 1.2(1).

Case 2. There is a vertex $w = s_w A_i \in Y - Y_1$ *H*-equivalent to v, $h_w = s_w a^{-1} s_v^{-1}$ and $a_v C = aC$. But this contradicts Lemma 1.2(2) which implies $aC \cap A_v = \emptyset$.

Thus $s_v a_v$ is not in $im(\rho_i)C$.

Suppose now that $s_v a_v \in im(\rho_{-i})C$. Since $s_v a_v$ ends in A_i , it means that there is a vertex $w = s_v b A_{-i}$, $b \in T_i$, such that v precedes w and $a_v C = bC$. But this is impossible by Case 1 above.

(ii) Suppose $s_v a_v b$ is a transversal. Assume first that $s_v a_v b \in im(\rho_i)C$. There are two possibilities.

Case 1. There is a vertex $u = s_v a A_{-i}$, $a \in T_i$, such that v precedes v and $aC = a_v bC$. If a = b then $a^{-1}a_v a = c \in C$. Then $h = s_v a_v s_v^{-1} \in H$ and hv = v. Moreover, $hw = s_v a_v s_v^{-1} s_v a A_{-i} = s_v a_v a A_{-i} = s_v a c A_{-i} = s_v a A_{-i} = w$. This contradicts our assumption that h does not stabilize the edge (v, w). If $a \neq b$, $u \neq w$ then by Lemma 1.2(3) $A_v a C \neq A_v b C$. This contradicts $aC = a_v b C$.

Case 2. There is a vertex $y = s_y A_i \in V(Y - Y_1)$ which is *H*-equivalent to v and $h_y = s_y a^{-1} s_v^{-1}$, $a \in A_i$ and $aC = a_v bC$. But Lemma 1.2(4) implies that $A_v a \cap bC = \emptyset$ which gives us a contradiction.

Suppose now that $s_v a_v b \in im(\rho_{-i})C$. By Lemma 1.2(1) $bC \cap A_v = \emptyset$, so $a_v b \notin C$. Since $s_v(a_v b)$ ends in A_i , there is a vertex $u = s_v a A_{-i} \in VY$, $a \in T_i$ such that v precedes u and $aC = a_v bC$. but this is impossible by Case 1 above.

(iii) Suppose that $s_q ab$ is transversal. This necessarily implies that a represents the same C-coset class as the label of some edge of Y_1 emanating from q. But this is impossible by Lemma 1.2(4).

(iv) Suppose $s_w a_1 = s_u b a_1$ is a transversal. Then b is a label of some edge of Y_1 originating from u. This is impossible since the only edge with label b emanating from u is the edge (u, w) and we know that $w \notin Y_1$.

(v) follows from (iii) and (iv).

(vi) Notice that by Lemma 1.2(2) we have $a_q a \notin C$. Assume that $s_q(a_q a)$ is a transversal. There are two possibilities.

Case 1. There is a positive edge of Y originating from q with label $b \in T_i$ such that $bC = a_q a C$. But by Lemma 1.2(4) we have $bC \cap A_q a = \emptyset$ which gives us a contradiction.

Case 2. There is a vertex $w \in V(Y - Y_1)$ which is *H*-equivalent to *q* such that $h_w = s_w a_1^{-1} s_q^{-1}$ and $a_q a C = a_1 C$. Since by assumption $a_q a C \neq a C$, we conclude that $a_1 C \neq a C$. But by Lemma 1.2(3) we have $A_q a C \neq A_q a_1 C$ which is impossible.

Lemma 1.10. Suppose $1 \neq g = \rho_i(t)\rho_{-i}(t) = s_v a_i a_{-i}^{-1} s_q^{-1}$ where $v = s_v A_i, q = s_q A_{-i} \in VY_1$. Suppose $b \in T_i$ is the label of a positive edge of Y_1 originating from v. Then $a_{-i}C \neq bC$.

Proof. There are two cases to consider.

Case 1. Suppose first that $a_i \in T_i$ is the label of an edge $(v, w) \in E^+(Y - Y_1)$, $s_w = s_v a_i$ and $g = h_w = s_w a_{-i}^{-1} s_q^{-1}$. Then $a_{-i}C \neq bC$ since a_i and b are the labels of different edges (one is in Y_1 and the other is in $Y - Y_1$).

Case 2. Suppose now that a_{-i} is the label of an edge $(q, w) \in E^+(Y - Y_1)$, $s_w = s_q a_{-i}$ and $g = h_w^{-1} = (s_w a_i^{-1} s_q^{-1})^{-1}$. Then $bC \cap A_q a_i = \emptyset$ by Lemma 1.2(4) which implies $bC \neq a_i C$.

Controlling the syllable length of elements of H.

Suppose $u = s_v a s_v^{-1} \in H$, $a \in A_v - C_v$. Then $s_v a s_v^{-1}$ is a reduced form of u with respect to presentation (1) and we denote it by w(u). If $1 \neq u = \rho_i(t)\rho_{-i}(t)^{-1} = s_v a_i a_{-i}^{-1} s_w^{-1}$ then $s_v a_i a_{-i}^{-1} s_w^{-1}$ is a reduced form of u and we denote it by w(u).

Recall that each positive edge of Y has a label $a \in T_i$. Denote $l_{X_i}(a)$ by l(a). Let $v = s_v A_i \in VY$ and $s_v = a_1 \dots a_k$ where a_j is the label of the j-th edge of the reduced edge-path from d_{-i} to v in $Y, j = 1, \dots, k$. Then denote $l(a_1) + \dots + l(a_k)$ by $l(s_v)$. Analogously, for a transversal element $1 \neq t = \rho_i(g)\rho_{-i}(g)^{-1} = s_v a_i a_{-i}^{-1} s_w^{-1}$ put $l(t) = l(s_v) + l_{X_i}(a_i) + l_{X_{-i}}(a_{-i}) + l(s_w)$. Let

$$K = 2 \sum_{t \in im(\rho_{\pm 1})} l(t)$$

and

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$$\Sigma = T_1^{\pm 1} \cup T_{-1}^{\pm 1} \cup \{a | \text{ for some } v \in V(Y - Y_1) \text{ we have } h_v = s_v a^{-1} s_q^{-1} \}^{\pm 1}.$$

Proposition A.

Let p be a reduced form of $h \in H - C$ with respect to presentation (3). Let $U = u_1 \dots u_n$ be obtained from p by deleting all those terms which are equal to 1 in H, that is U is a normal form of h with respect to presentation (3). Put $h' = u_1 \dots u_{n-1}$ when n > 1.

There is a reduced form with respect to presentation (1) $W = v_1 \dots v_m$ of h and, when n > 1, a reduced form with respect to presentation (1) W' of h' such that the following holds.

- (i) If $h \notin C$ and $u_n = s_v a s_v^{-1}$ where $a \in A_v$, $v = s_v A_i$, then w ends in $x s_v^{-1}$ where $x \in A_i$ and $s_v x^{-1}$ is not a transversal. If $u_n = h_v^{\pm 1} = \rho_i(g)\rho_{-i}^{-1}(g) = s_v a_i a_{-i}^{-1} s_w^{-1}$ then w ends in $x a_{-i}^{-1} s_w^{-1}$ where $x \in A_i C$ and $s_w a_{-i} x^{-1}$ is not a transversal.
- (ii) The nerve N of w has syllable length not smaller than the nerve N' of w'. The syllable length of N is strictly greater than the syllable length of N' unless $u_{n-1} = s_v a s_v^{-1}$, $a \in A_i$, $v = s_v A_i \in VY_1$ and $u_n = \rho_i(t)\rho_{-i}(t)^{-1} = s_v a_i a_{-i} s_v^{-1}$ (notice that we allow here $s_v = 1$).
- (iii) For each $u_k = s_v a s_v^{-1}$ where $v = s_v A_i \in VY_1$, $a \in A_v$ for some k = 1, ..., n there is a syllable $v_{i_k} \in A_i C$ of w, called a **core element**, such that
- (1) if $k_1 < k_2$ then $i_{k_1} < i_{k_2}$
- (2) if k < n then $v_{i_k} = faf'$ where each of $f, f' \in \Sigma$;
- (3) if k = n then $v_{i_k} = fa$ where $f \in \Sigma$;
- (4) if $u_k = s_v a s_v^{-1}$, $u_{k+1} = s_u b s_u^{-1} = s_u b_1 b b_1^{-1} s_u$, where v precedes $u, s_u = s_v b_1, b_1 \in T_i, b \in A_{-i}$ then $v_{i_k} = f a b_1$ and $v_{i_{k+1}} = b f'$ where $f, f' \in \Sigma$;
- (5) $if u_k = s_v a s_v^{-1}, u_{k+1} = s_u b s_u^{-1}$ where u precedes $v, s_v = s_u b_1, b_1 \in T_{-i}, b \in A_{-i}$ then $v_{i_k} = fa$ and $v_{i_{k+1}} = b_1^{-1} b f'$ where $f, f' \in \Sigma$;
- (6) if $u_k = s_v a s_v^{-1}$, $u_{k+1} = s_v a_i a_{-i}^{-1} s_w^{-1}$ then $v_{i_k} = f a a_i$ where $f \in \Sigma$;
- (7) if $u_k = s_v a_i a_{-i}^{-1} s_w^{-1}$ and $u_{k+1} = s_w b s_w^{-1}$ then $v_{i_{k+1}} = f a_{-i}^{-1} b$ where $f \in \Sigma$;
- (8) if $u_n = s_v a s_v^{-1}$, $v = s_v A_i \in VY_1$, $a \in A_v C_v$ and w_n ends in $x s_v^{-1}$, $x \in A_i$ then $v_{i_n} = x$;
- (9) v_k and v_{k+1} are both core elements if and only if either $v_k = v_{i_s}, v_{k+1} = v_{i_{s+1}}, u_s = s_v a s_v^{-1}, u_{s+1} = s_u b s_u^{-1}$ and either u precedes v or v precedes uor $v_k = v_{i_s}, v_{k+1} = v_{i_{s+2}}, u_s = s_v b_i s_v^{-1}, u_{s+1} = \rho_i(g) \rho_{-i}^{-1}(g)^{-1} = s_v a_i a_{-i}^{-1} s_w^{-1}, u_{s+2} = s_w b_{-i} s_w^{-1};$
- (10) every $v_k \in A_i$, which is not a core element, has $l_{X_i}(v_k) \leq K$.

Proof. We will prove Proposition A by induction on n. Suppose n = 1. Recall that $h \notin C$.

Suppose first $p = (e_1, 1, e_2, 1, ..., 1, e_k, u_1, e_{k+1}, 1, ..., e_{k+s}, 1)$ be a reduced path in \mathbb{B} representing h. Then s = k and $e_{k+i} = e_{k-i+1}^{-1}$, i = 1, ..., k and $e_i \in Y_1$ for i = 1, ..., k. Thus $u_1 = s_v a s_v^{-1}$, $v = s_v A_i \in VY_1$ where e_1, \ldots, e_k is a path in Y_1 ending as v with the label s_v . Notice that $a \notin C$. Indeed, if $k \ge 1$ and $a = c \in C$ then $u_1 = s_v c s_v^{-1}$ stabilizes the edge e_k and p is not a normal form of h. If k = 0 (that is $v = d_1$) and $a = c \in C$ then $h = c \in C$ which, as we assumed, is not the case. Thus $a \notin C$ and therefore $s_v a^{-1}$ is not a transversal by Lemma 1.9(i). We have verified statement (i). Notice that $w = s_v a s_v^{-1}$ is the reduced form of h with respect to presentation (1) and the nerve N of w is equal to $s_v a$. Put $v_{i_1} = a$ to be the only core element. Then the rest of the statements of Proposition A are automatically satisfied.

Suppose now that $p = (e_1, \ldots, e_k, e, e_{k+2}, \ldots, e_s)$ where $e_i \in EY_1$, $i = 1, \ldots, k, k+2, \ldots, s$ and $e \in E(B - Y_1)$. Then $u_1 = e = \rho_i(t)\rho_{-i}(t)^{-1} = s_v a_i a_{-i}^{-1} s_w^{-1}$. Observe that $s_w a_{-i} a_i^{-1}$ is not a transversal by Lemma 1.9(i). Observe that $w = s_v a_i a_{-i}^{-1} s_w^{-1}$ is the reduced form of h with respect to presentation (1). So the nerve N of $w = s_v a_i a_{-i}^{-1} s_w^{-1}$ is $N = s_v a_i$, there are no core elements and Proposition A for the case n = 1 is established. Thus the basis of induction is verified.

Suppose now n > 1 and Proposition A has been established for smaller values of n. There are several cases to consider.

Case 0. Suppose that $h' = c \in C$. Therefore n = 2 and $u_1 = c$. Notice that $u_2 \notin A_{\pm 1}$ since u_1u_2 is a normal form of h with respect to presentation (3).

Subcase 0.A. Suppose that $u_2 = s_v a s_v^{-1}$ where $A_{\pm 1} \neq v = s_v A_i \in VY_1$.

Thus $p = (c, e_1, \ldots, e_k, s_v a s_v^{-1}, e_k^{-1}, \ldots, e_1^{-1})$ where k > 1 and e_1, \ldots, e_k is a reduced edge-path in Y_1 from d_1 to v.

Notice that $a \in A_i - C$ since $s_v a s_v^{-1}$ does not stabilize the edge e_k . Let $s_v = fz$ where $f \in A_j - C$ is the first syllable of s_v . Then $W = (cf)zas_v^{-1}$ is the reduced form of h with respect to presentation (1). The element $s_v a^{-1}$ is not a transversal by Lemma 1.9(i). Thus W ends in $s_v a^{-1}$ and the nerve N = (cf)za of W has greater syllable length than the nerve N' = 1 of W' = c. Put $v_{i_1} = cf$ and $v_{i_2} = a$ to be the core elements of w. Notice that $f \in \Sigma$. All statements of Proposition A are clearly satisfied.

Subcase 0.B. Suppose that $u_2 = \rho_i(g)\rho_{-i}(g)^{-1} = s_v a_i a_{-i}^{-1} s_w^{-1} \neq 1$ where $v = s_v A_i, w = s_w A_{-i}, a_j \in A_j - C, j = \pm 1$.

Assume first that $s_v \neq 1$. Then $s_v = fz$ where $f \in A_j - C$ is the first syllable of s_v . Then $W = (cf)za_ia_{-i}^{-1}s_w^{-1}$ is a reduced form of h with respect to presentation (1). By Lemma 1.9(v) the element $s_wa_{-i}a_i^{-1}$ is not a transversal. Therefore the nerve N of W is equal to $N = (cf)za_i$ and it has reater syllable length than the nerve N' = 1 of W' = c. Put $v_{i_1} = cf$ to be the only core element for W. Notice that $f \in \Sigma$. All statements of Proposition A are clearly satisfied.

Suppose now that $s_v = 1$. Then $W = (ca_i)a_{-i}^{-1}s_w^{-1}$ is the reduced form of h with respect to presentation (1). By Lemma 1.9(v) the element $s_w a_{-i}a_i^{-1}c^{-1}$ is not a transversal. Therefore the nerve N of W is equal to $N = (ca_i)$ and it has reater syllable length than the nerve N' = 1 of W' = c. Put $v_{i_1} = ca_i$ to be the only core element for W. Notice that $a_i \in \Sigma$. All statements of Proposition A are clearly satisfied.

Case 1. Suppose that $h' \notin C$, $u_{n-1} = s_v b_0 s_v^{-1}$, $v = s_v A_i$, $b_0 \in A_v$, $u_n = s_w a s_w^{-1}$, $a \in A_v$, $w = s_w A_j \in VY_1$ and $w \neq = v$.

Thus $p = (g_1, e_1, \ldots, g_k = s_v b_0 s_v^{-1}, e_k, 1, e_{k+1}, 1, \ldots, 1, e_l, g_{l+1} = s_w a s_w^{-1}, e_{l+1}, 1, \ldots, e_r, 1)$ where $e_i \in EY_1$ for $i = k, \ldots, r, e_k, e_{k+1}, \ldots, e_r$ is a path in Y_1 from v to $d_1, e_{l+1}, \ldots, e_r$ is a reduced path in Y_1 from w to d_1 . Let $e_k, \ldots, e_l = \hat{z}^{-1}\hat{u}, e_{l+1}, \ldots, e_r = \hat{u}^{-1}\hat{y}^{-1}$ where \hat{u}^{-1} is the maximal initial segment of e_{l+1}, \ldots, e_r which is cancelled in $e_k, \ldots, e_l, e_{l+1}, \ldots, e_r$. Thus $\hat{z}^{-1}\hat{y}^{-1}$ is a reduced path in Y_1 from v to d_1 . Let $\hat{z}^{-1}\hat{y}^{-1} = e_{k'}, \ldots, e_{t'}$. Then $p' = (g_1, e_1, \ldots, g_k = s_v b_0 s_v^{-1}, e_{k'}, 1, \ldots, 1, e_{t'}, 1)$ is a reduced form of $h' = u_1 \ldots u_{n-1}$ with respect to presentation (3). Therefore by induction $W' = pxs_v^{-1}$ where $x \in A_i, s_v x^{-1}$ is not a transversal and N' = px is the nerve of W'. Also by induction we know that $v'_{i_{n-1}} = x$ is the last core element of p' and that $x = \sigma b_0$ for some $\sigma \in \Sigma$. Let u be the label of \hat{u}, y be the label of \hat{y} and z be the label of \hat{z} . Therefore $s_v = yz$ and $s_w = yu$.

Subcase 1.A. Suppose first that both \hat{z} and \hat{u} are nonempty.

Then $s_v = yq$ and $s_w = yu$ and $u_n = s_w a s_w^{-1} = yua u^{-1} y^{-1}$. Notice that $a \notin C$ since if $a \in C$ then u_n fixes the last edge of \hat{u} which contradicts our assumption that p is a reduced form for h with respect to presentation (3). Thus $h = pxz^{-1}y^{-1}yuas_w^{-1} = pxz^{-1}uas_w^{-1}$. Suppose that \hat{y} ends in a vertex of type A_k . Let $z = f_1 z_1$ where $f_1 \in T_k$ be the label of the first edge of \hat{z} . Let $u = f_2 u_1$ where where $f_2 \in T_k$ be the label of the first edge of \hat{u} and so $f_1^{-1} f_2 \notin C$. Therefore $W = pxz_1^{-1}(f_1^{-1} f_2)u_1as_w^{-1}$ is a normal form for h with respect to presentation (1). It is clear that s_w is a transversal. Besides $s_w a^{-1}$ is not a transversal by Lemma 1.9(i). Thus the nerve N of W is equal to $pxz_1^{-1}(f_1^{-1} f_2)u_1a$ and it has greater syllable length then the nerve N' = px of W'. Now take the set of core element of W', add to it $v_{i_n} = a$ and declare the result to be the set of core elements of w_n . All statements of Proposition A are clearly satisfied by induction.

Subcase 1.B. Suppose that \hat{z} is empty and \hat{u} is nonempty.

Then $s_v = y$ and $s_w = yu$ and so $h = pxs_v^{-1} \cdot s_w as_w^{-1} = pxy^{-1}yuas_w^{-1} = pxuas_w^{-1}$. Let $u = fu_1$ where f is the label of the first edge of u. Thus $x, f \in A_i$ since v is the vertex of type A_i . Notice that $x \cdot f \notin C$. Indeed, if $xf = c \in C$ then $s_v x^{-1} = s_v f c^{-1}$ which is a transversal element. This clearly contradicts the inductive assumption that $s_v x^{-1}$ is not a transversal. Observe also that $a \notin C$ since if $a \in C$ then u_n fixes the last edge of \hat{u} which contradicts our assumption that p is a reduced form for h with respect to presentation (3). Thus $W = p(xf)u_1as_w^{-1}$ is a normal form for h with respect to presentation (1). Again we see that s_w is a transversal and $s_w a^{-1} = yua^{-1}$ is not a transversal by Lemma 1.9(i). Thus the nerve N of W is equal to $p(xf)u_1a$ and it has greater syllable length then the nerve N' = px of W'. Recall that the last core element

of W' is $v'_{i_{n-1}} = x = \sigma b_0$ where $\sigma \in \Sigma$. Now take the set of core element of W', replace $v'_{i_{n-1}} = \sigma b_0$ by $v_{i_{n-1}} = \sigma b_0 f$, add $v_{i_n} = a$ and declare the result to be the set of core elements of W. All statements of Proposition A are clearly satisfied by induction.

Subcase 1.C. Suppose that \hat{z} is nonempty and \hat{u} is empty.

Then $s_v = yz$ and $s_w = y$. In this case $h = pxs_v^{-1}s_was_w^{-1} = pxz^{-1}y^{-1}yay^{-1} = pxz^{-1}ay^{-1}$. Since p is a reduced form for h with respect to presentation (3), the element $s_was_w^{-1}$ does not stabilize the first edge of z. Thus if $f \in T_j$ is the label of this edge and $z = fz_1$, then $f^{-1}a \in A_j - C$ by Lemma 1.2(1) and $s_w^{-1}a^{-1}f = y^{-1}a^{-1}f$ is not a transversal by Lemma 1.9(ii). Therefore $W = pxz_1^{-1}(f^{-1}a)y^{-1}$ is a normal form for h with respect to presentation (1). Since $y = s_w$ is a transversal and $s_w^{-1}a^{-1}f = y^{-1}a^{-1}f$ is not a transversal, we conclude that the nerve N of W is $pxz_1^{-1}(f^{-1}a)$ and it has greater syllable length than N' = px. Now take the set of core element of w', add to it $v_{i_n} = f^{-1}a$ and declare the result to be the set of core elements of w_n . All statements of Proposition A are clearly satisfied by induction.

Subcase 1.D Suppose that both \hat{z} and \hat{u} are empty. Then w = v which contradicts our assumptions.

Case 2. Suppose that $h' \notin C$, $u_{n-1} = s_v b_0 s_v^{-1}$, $v = s_v A_i$, $b_0 \in A_v$, $1 \neq u_n = \rho_j(g) \rho_{-j}(g)^{-1} = s_t b_j b_{-j}^{-1} s_q^{-1}$ where $q = s_q A_{-j}$, $t = s_t A_j \in VY_1$ and $b_{\pm j} \in A_{\pm j} - C$.

Thus $p = (g_1, e_1, \dots, g_k = s_v b_0 s_v^{-1}, e_k, 1, e_{k+1}, 1, \dots, 1, e_l, g_{l+1} = s_t b_j b_{-j}^{-1} s_q^{-1}, e_{l+1}, 1, \dots, e_r, 1)$ where $e_i \in EY_1$ for $i = k, \dots, r, e_k, e_{k+1}, \dots, e_l$ is a path in Y_1 from v to t, e_{l+1}, \dots, e_r is a reduced path in Y_1 from q to d_1 . Let d_1, \dots, d_s be the reduced path in Y_1 from t to d_1 .

Then $e_k, e_{k+1}, \ldots, e_l, d_1, \ldots, d_s$ is a path in Y_1 from v to d_1 . Let $e_k, e_{k+1}, \ldots, e_l = \hat{z}^{-1}\hat{u}, d_1, \ldots, d_s = \hat{u}^{-1}\hat{y}^{-1}$ where \hat{u}^{-1} is the maximal initial segment of d_1, \ldots, d_s which is cancelled in the product $e_k, e_{k+1}, \ldots, e_l, d_1, \ldots, d_s$. Then $\hat{u}^{-1}\hat{y}^{-1} = e'_k \ldots e'_m$ is a reduced path in Y_1 from v to d_1 . Therefore $p' = (g_1, e_1, \ldots, g_k = s_v b_0 s_v^{-1}, e'_k, 1, \ldots, 1, e'_m, 1)$ is a reduced form of $h' = u_1 \ldots u_{n-1}$ with respect to presentation (3).

By induction $W' = pxs_v^{-1}$, where $x \in A_i$, $s_v x^{-1}$ is not a transversal and N' = px is the nerve of W'. Also by induction we know that for some $\sigma \in \Sigma v'_{i_{n-1}} = x = \sigma b_0$ is the last core element of W'. Denote the labels of $\hat{u}, \hat{z}, \hat{y}$ by u, z, y. Therefore $s_v = yz$ and $s_t = yu$.

Subcase 2.A. Suppose that \hat{u} is empty and \hat{z} is non-empty.

Then $s_v = yz$ and $s_t = y$. Therefore $h = pxs_v^{-1}s_tb_jb_{-j}^{-1}s_q^{-1} = pxz^{-1}y^{-1}yb_jb_{-j}^{-1}s_q^{-1} = pxz^{-1}b_jb_{-j}^{-1}s_q^{-1}$. Let $z = f_1z_1$ where f_1 is the label of the first edge of z. Then $f_1, b_j \in A_j$ and either b_j is a label of the edge $(t, w) \in V(Y - Y_1)$ and $h = h_w$ or b_{-j} is a label of the edge $(q, w) \in V(Y - Y_1)$ and $h = h_w^{-1}$. In the first case $f_1C \neq fC$ since the first edge of z is in Y_1 and $(t, w) \in E(Y - Y_1)$. In the second case $f_1C \cap A_tb_j = \emptyset$ by Lemma 1.2(4). Thus $(f_1^{-1}b_j) \notin C$ and $W = pxz_1^{-1}(f_1^{-1}b_j)b_{-j}^{-1}s_q^{-1}$ is the normal form of h with respect to presentation (1). Notice that s_qb_{-j} is transversal and $s_qb_{-j}(b_j^{-1}f_1)$ is not transversal by Lemma 1.9(v). Thus the nerve N of W is $pxz_1^{-1}(f_1^{-1}b_j)$ and it has greater syllable length than the nerve N' = px of W'. Take the core elements of W' and declare them to be the core elements of W. Proposition A now follows from the inductive hypothesis.

Subcase 2.B Suppose that \hat{u}, \hat{z} are empty.

Then v = t, i = j, $s_v = y$ and $s_t = y$. Therefore $h = pxs_v^{-1}s_tb_jb_{-j}^{-1}s_q^{-1} = pxy^{-1}yb_jb_{-j}^{-1}s_q^{-1} = pxb_jb_{-j}^{-1}s_q^{-1}$. Note that $x, b_j \in A_j, b_{-j} \in A_{-j}$. Observe that $xb_j \notin C$ since if $xb_j = c \in C$ then $s_vx^{-1} = s_tx^{-1} = s_tb_jc^{-1}$ is a transversal which contradicts our assumptions. Recall also that $b_{-j} \notin C$. Thus $W = p(xb_j)b_{-j}s_q^{-1}$ is is the normal form of h with respect to presentation (1). Lemma 1.9(v) implies that $g = s_qb_{-j}(b_j^{-1}x^{-1})$ is not a transversal. Thus the nerve N of W is equal to $p(xb_j)$ and it has the same syllable length as the nerve N' = px of W'. By the inductive hypothesis $v'_{i_{n-1}} = x = \sigma b_0$ is the last core element of W' for some $\sigma \in \Sigma$. We take the collection of core elements of W' replace $v'_{i_{n-1}} = \sigma b_0$ by $v_{i_{n-1}} = \sigma b_0 b_j$ and declare this to be the collection of core elements of W. All statements of Proposition A are clearly satisfied by induction.

Subcase 2.C Suppose now that \hat{u} and \hat{z} are nonempty.

Then $s_v = yz$ and $s_t = yu$. Therefore $h = pxs_v^{-1}s_tb_jb_{-j}^{-1}s_q^{-1} = pxz^{-1}y^{-1}yub_jb_{-j}^{-1}s_q^{-1} = pxz^{-1}ub_jb_{-j}^{-1}s_q^{-1}$. Assume that \hat{y} is a path from d_1 to the vertex of type A_k . Let $z = f_1z_1$ where $f_1 \in T_k$ is the label of the first edge of \hat{z} and let $u = f_2u_1$ where $f_2 \in T_k$ is the label of the first edge of \hat{u} . Notice that $f_1C \neq f_2C$ by definition of \hat{u} and \hat{z} . Thus $(f_1^{-1}f_2) \notin C$ and $W = pxz_1^{-1}(f_1^{-1}f_2)u_1b_jb_{-j}^{-1}s_q^{-1}$ is the normal form for h with respect to presentation (1). Again we observe that $s_q b_{-j}$ is a transversal and $s_q b_{-j} b_j^{-1}$ is not a transversal by Lemma 1.9(v). Thus the nerve of W is equal to $N = pxz_1^{-1}(f_1^{-1}f_2)u_1b_j$ and it has greater syllable length than the nerve N' = px of W'. Take the core elements of W' and declare them to be the core elements of W. It is clear that all statements of Proposition A follow from the inductive hypothesis.

Subcase 2.D Suppose that \hat{u} is nonempty and \hat{z} is empty.

Then $s_v = y$, $v = yA_i$ and $s_t = yu$. Therefore $h = pxs_v^{-1}s_tb_jb_{-j}^{-1}s_q^{-1} = pxy^{-1}yub_jb_{-j}^{-1}s_q^{-1} = pxub_jb_{-j}^{-1}s_q^{-1}$. Let $u = f_1u_1$ where $f_1 \in T_i$ is the label of the first edge of u. Then $(xf_1) \notin C$. Indeed, if $xf_1 = c \in C$ then $s_v x^{-1} = s_v f_1 c^{-1}$ is a transversal which contradicts our assumptions. Thus $W = p(xf_1)u_1 b_j b_{-j}^{-1} s_q^{-1}$ is the normal form for h with respect to presentation (1). As in the previous case $s_q b_{-i}$ is a transversal and $s_q b_{-i} b_i^{-1}$ is not a transversal by Lemma 1.9(v). So the nerve N of W is $p(x f_1) u_1 b_i$ and it has greater syllable length than the nerve N' = px of W'.

Recall that by inductive hypothesis $v'_{i_{n-1}} = x = \sigma b_0$ is the last core element of W'. Take the core elements of W' and replace $v'_{i_{n-1}} = \sigma b_0$ by $v_{i_{n-1}} = \sigma b_0 f_1$ to get the collection of core elements of W. Proposition A follows now from the inductive hypothesis.

Case 3. $h' \notin C, \ 1 \neq u_{n-1} = \rho_i(g)\rho_{-i}(g)^{-1} = s_v a_i a_{-i}^{-1} s_w^{-1} \text{ and } 1 \neq u_n = \rho_j(g')\rho_{-j}(g')^{-1} = s_t b_j b_{-j}^{-1} s_q^{-1}$

where $v = s_v A_i, w = s_w A_{-i}, t = s_t A_j, q = s_q^{-j} \in VY_1, a_{\pm i} \in A_{\pm i} - C, b_{\pm j} \in A_{\pm j} - C.$ Thus $p = (g_1, e_1, \dots, g_k = s_v a_i a_{-i}^{-1} s_w^{-1}, e_k, 1, \dots, 1, e_l, g_{l+1} = s_t b_j b_{-j}^{-1} s_q^{-1}, e_{l+1}, \dots, e_r, 1)$ where $e_i \in EY_1$ for $i \ge k, e_k, \dots, e_l$ is a path in Y_1 from w to t and e_{l+1}, \dots, e_r is a reduced path in Y_1 from q to d_1 . Let d_1, \ldots, d_s be the reduced path in Y_1 from t to d_1 . Then $e_k, \ldots, e_l, d_1, \ldots, d_s$ is a path in Y_1 from w to d_1 . Let a_1, \ldots, a_s be the reduced path in T_1 from i to a_1 . Then $e_k, \ldots, e_l, a_1, \ldots, a_s$ is a path in T_1 from w to a_1 . Let $e_k, \ldots, e_l = \hat{z}^{-1}u$ and $d_1, \ldots, d_s = \hat{u}^{-1}\hat{y}^{-1}$ where \hat{u} is the maximal terminal segment of e_k, \ldots, e_l which is cancelled in $e_k, \ldots, e_l, d_1, \ldots, d_s$. Then $\hat{u}^{-1}\hat{y}^{-1} = e'_k \ldots e'_m$ is a reduced path in Y_1 from w to d_1 . Therefore $p' = (g_1, e_1, \ldots, g_k = s_v a_i a_{-i}^{-1} s_w^{-1}, e'_k, 1, \ldots, 1, e'_m, 1)$ is a reduced form of $h' = u_1 \ldots u_{n-1}$ with respect to presentation (3). By induction $W' = pxa_{-i}^{-1}s_w^{-1}$, where $x \in A_i$, $s_wa_{-i}x^{-1}$ is not a transversal and N' = pxis the nerve of W'. Denote the labels of $\hat{u}, \hat{z}, \hat{y}$ by u, z, y. Thus $s_w = yz$ and $s_t = yu$.

Subcase 3.A. Suppose that \hat{z} is nonempty and \hat{u} is empty.

Then $s_w = yz$, $s_t = y$. Therefore $h = pxa_{-i}^{-1}s_w^{-1}s_tb_jb_{-j}^{-1}s_q^{-1} = pxa_{-i}^{-1}z^{-1}y_jb_jb_{-j}^{-1}s_q^{-1} = pxa_{-i}^{-1}z^{-1}b_jb_{-j}^{-1}s_q^{-1}$. Let $z = f_1z_1$ where $f_1 \in T_j$ is the label of the first edge of z. Observe that $f_1^{-1}b_j \notin C$ by Lemma 1.10 and therefore $W = pxa_{-i}^{-1}z_1^{-1}(f_1^{-1}b_j)b_{-j}^{-1}s_q^{-1}$ is the normal form of h with respect to presentation (1). The element s_qb_{-j} is transversal and $s_qb_{-j}(f_1^{-1}b_j)^{-1}$ is not transversal by Lemma 1.9(v). Therefore the nerve N of W is equal to $pxa_{-i}^{-1}z_1^{-1}(f_1^{-1}b_j)$ and it has greater syllable length than the nerve N' = px of W'. We take the collection of core elements of W' and declare them to be the core elements of W. Proposition A follows now from the inductive hypothesis.

Subcase 3.B. Suppose that \hat{u} is nonempty and \hat{z} is empty. Then $s_w = y$ and $s_t = yu$. We have $h = pxa_{-i}^{-1}s_w^{-1}s_tb_jb_{-j}^{-1}s_q^{-1} = pxa_{-i}^{-1}y^{-1}yub_jb_{-j}^{-1}s_q^{-1} = pxa_{-i}^{-1}ub_jb_{-j}^{-1}s_q^{-1}$. Let $u = f_1u_1$ where $f_1 \in T_{-i}$ is the label of the first edge of \hat{u} . Then $(a_{-i}^{-1}f_1) \notin C$ by Lemma 1.10. Thus $W = px(a_{-i}^{-1}f_1)u_1b_jb_{-j}^{-1}s_q^{-1}$ is the normal form of h with respect to presentation (1). The element s_qb_{-j} is transversal and $s_qb_{-j}b_j^{-1}$ is not transversal by Lemma 1.9(v). Therefore the nerve N of W is equal to $px(a_{-i}^{-1}f_1)u_1b_j$ and it has greater syllable length than the nerve N' = px of W'. We take the collection of core elements of W' and declare them to be the core elements of W. Proposition A follows now from the inductive hypothesis.

Subcase 3.C. Suppose that \hat{u} and \hat{z} are nonempty. Then $s_w = yz$ and $s_t = yu$ and $h = pxa_{-i}^{-1}s_w^{-1}s_tb_jb_{-j}^{-1}s_q^{-1} = pxa_{-i}^{-1}z^{-1}yub_jb_{-j}^{-1}s_q^{-1} = pxa_{-i}^{-1}z^{-1}ub_jb_{-j}^{-1}s_q^{-1}$. Assume that \hat{y} ends in a vertex of type A_k . Let $z = f_1z_1$ and $u = f_2u_1$ where $f_1 \in T_k$ is the label of the first edge of \hat{z} and $f_2 \in T_k$ is the label of the first edge of \hat{u} . Clearly $f_1C \neq f_2C$ and so $f_1^{-1}f_2 \notin C$. Thus $W = pxa_{-i}^{-1}z_1^{-1}(f_1^{-1}f_2)u_1b_jb_{-j}^{-1}s_q^{-1}$ is the normal form of h with respect to presentation (1). The element s_qb_{-j} is transversal and $s_qb_{-j}b_j^{-1}$ is not transversal by Lemma 1.9(v). Therefore the nerve N of W is equal to $pxa_{-i}^{-1}z_1^{-1}(f_1^{-1}f_2)u_1b_j$ and it has greater syllable length than the nerve N' = px of W'. We take the collection of core elements of W' and declare them to be the core elements of W. Proposition A follows now from the inductive hypothesis.

Subcase 3.D. Suppose that \hat{u} and \hat{z} are empty.

 $pxa_{-i}^{-1}b_jb_{-j}^{-1}s_q^{-1}$. By Lemma 1.7(vii) either $a_{-i}^{-1}b_j \notin C$ or $u_n = u_{n-1}^{-1}$. The later is impossible since p is the reduced form for h with respect to presentation (3). Thus $a_{-i}^{-1}b_j \notin C$ and $W = px(a_{-i}^{-1}b_j)b_{-i}^{-1}s_q^{-1}$ is the normal form of h with respect to presentation (1). The element $s_q b_{-j}$ is transversal and $s_q b_{-j} (a_{-i}^{-1} b_j)^{-1}$ is not transversal by Lemma 1.9(v). Therefore the nerve N of W is equal to $px(a_{-i}^{-1}b_j)$ and it has greater syllable length than the nerve N' = px of W'. We take the collection of core elements of W' and declare them to be the core elements of W. Proposition A follows now from the inductive hypothesis.

Case 4. Suppose that $h' \notin C$, $1 \neq u_{n-1} = \rho_i(g)\rho_{-i}(g)^{-1} = s_v a_i a_{-i}^{-1} s_w^{-1}$ and $u_n = s_t b s_t^{-1}$ where $v = s_v A_i, w = s_w A_{-i}, t = s_t A_j \in VY_1, a \pm i \in A_{\pm i} - C, b \in A_j.$

Thus $p = (g_1, e_1, \dots, g_k = s_v a_i a_{-i}^{-1} s_w^{-1}, e_k, 1, \dots, 1, e_l, g_{l+1} = s_t b s_t^{-1}, e_{l+1}, \dots, e_r, 1)$ where $e_i \in EY_1$ for $i \geq k, e_k, \ldots, e_l$ is a path in Y_1 from w to t and e_{l+1}, \ldots, e_r is a reduced path in Y_1 from t to d_1 . Then $e_k, \ldots, e_l, e_{l+1}, \ldots, e_r$ is a path in Y_1 from w to d_1 . Let $e_k, \ldots, e_l = \hat{z}^{-1}u$ and $d_1, \ldots, d_s =$ $\hat{u}^{-1}\hat{y}^{-1}$ where \hat{u} is the maximal terminal segment of e_k, \ldots, e_l which is cancelled in $e_k, \ldots, e_l, e_{l+1}, \ldots, e_r$. Then $\hat{u}^{-1}\hat{y}^{-1} = e'_k, \ldots, e'_m$ is a reduced path in Y_1 from w to d_1 . Therefore $p' = (g_1, e_1, \ldots, g_k)$ $s_v a_i a_{-i}^{-1} s_w^{-1}, e'_k, 1, \ldots, 1, e'_m, 1$ is a reduced form of $h' = u_1 \ldots u_{n-1}$ with respect to presentation (3). By induction $W' = pxa_{-i}^{-1}s_w^{-1}$, where $x \in A_i$, $s_wa_{-i}x^{-1}$ is not a transversal and N' = px is the nerve of W'. Denote the labels of $\hat{u}, \hat{z}, \hat{y}$ by u, z, y. Thus $s_w = yz$ and $s_t = yu$.

Subcase 4.A. Suppose that \hat{z} is nonempty and \hat{u} is empty.

Then $s_w = yz$, $s_t = y$, $t = yA_j$. Therefore $h = pxa_{-i}^{-1}s_w^{-1}s_bs_t^{-1} = pxa_{-i}^{-1}z^{-1}y^{-1}ybs_t^{-1} = pxa_{-i}^{-1}z^{-1}bs_t^{-1}$. Notice that \hat{z} starts at $t = yA_j$. Let $z = f_1z_1$, where where $f_1 \in T_j$ is the label of the first edge of \hat{z} . Then $f_1^{-1}b \notin C$ by Lemma 1.2(1). Thus $W = pxa_{-i}^{-1}z_1^{-1}(f_1^{-1}b)s_t^{-1}$ is the normal form of h with respect to presentation (1). The element s_q is transversal and $s_t b s_t^{-1}$ does not stabilize the first edge of \hat{z} since p is a reduced form of h with respect to presentation (3). Therefore $s_q b^{-1} f_1$ is not transversal by Lemma 1.9(ii). Thus the nerve N of W is $pxa_{-i}^{-1}z_1^{-1}(f_1^{-1}b)$ and it has greater syllable length than the nerve N' = px of W'. Take the core elements of W', add to them $v_{i_n} = f_1^{-1}b$ and declare the result the collection of core elements of W. Proposition A follows now from the inductive hypothesis.

Subcase 4.B. Suppose that \hat{u} is nonempty and \hat{z} is empty. Then $s_w = y$, $w = yA_{-i}$, $s_t = yu$. Therefore $h = pxa_{-i}^{-1}s_w^{-1}s_tbs_t^{-1} = pxa_{-i}^{-1}y^{-1}yubs_t^{-1} = pxa_{-i}^{-1}ubs_t^{-1}$. Notice that \hat{u} starts at w and ends at t. Let $u = f_1u_1$, where where $f_1 \in T_{-i}$ is the label of the first edge of \hat{z} . Then $a_{-i}^{-1}f_1 \notin C$ by Lemma 1.9(v). Thus $W = px(a_{-i}^{-1}f_1)u_1bs_t^{-1}$ is the normal form of h with respect to presentation (1). Since \hat{u} is nonempty, the element $s_t b s_t^{-1}$ does not fix the last edge of \hat{u} because p is the reduced form of h. Therefore $b \in A_t - C_v$. This implies that $s_v b^{-1}$ is not a transversal by Lemma 1.9(i). Thus the nerve N of W is $px(a_{-i}^{-1}f_1)u_1b$ and it has greater syllable length than the nerve N' = px of W'. Take the core elements of W', add to them $v_{i_n} = b$ and declare the result the collection of core elements for W. Proposition A follows now from the inductive hypothesis.

Subcase 4.C. Suppose that \hat{z} , \hat{u} are empty.

Then $s_w = y = s_t$, -i = j, $w = yA_{-i} = yA_j = t$. Therefore $h = pxa_{-i}^{-1}s_w^{-1}s_tbs_t^{-1} = pxa_{-i}^{-1}y^{-1}ybs_t^{-1} = t$ $pxa_{-i}^{-1}bs_{t}^{-1}$.

Suppose $a_{-i}^{-1}b \in C$. There are two possibilities. First, it can happen that a_{-i} is the label of an edge originating from t. This is clearly impossible since Lemma 1.2(1) implies $a_{-i}C \cap A_t = \emptyset$. Secondly, it is possible that $u_{n-1} = h_{w'} = s_{w'}a_{-i}^{-1}s_t^{-1}$ where $w' \in V(Y - Y_1)$ is some vertex *H*-equivalent to t = w. Recall that $b \in A_t = A_w$. Then $A_t a_{-i} \cap C = \emptyset$ by Lemma 1.2(2) and so $a_{-i}^{-1} b \notin C$ which gives us a contradiction. Thus $W = px(a_{-i}^{-1}b)s_t^{-1}$ is the normal form of h with respect to presentation (1).

Suppose now that $s_t(b^{-1}a_{-i})$ is a transversal. There are again two possibilities to consider.

First, suppose that $u_{n-1} = h_{w'}^{-1} = s_v a_i s_{w'}^{-1} = s_v a_i a_i s_w^{-1}$ for a vertex $w' \in V(Y - Y_1)$. Then a_{-i} is the label of the edge $(w, w') = (t, w') \in E(Y - Y_1)$. Since $s_t(b^{-1}a_{-i})$ is a transversal, Lemma 1.9(ii) implies that $s_t b^{-1} s_t^{-1}$ (and so $s_t b s_t^{-1}$) stabilizes the edge (t, w') = (w, w'). Recall that $h_{w'}$ conjugates the subgroup $s_{w'}(A_{w'} \cap C)s_{w'}^{-1}$ into a subgroup of $s_vA_vs_v^{-1}$. Therefore $b = a_{-i}ca_{-i}^{-1}$ and $h_{w'}^{-1}s_tbs_t^{-1}h_{w'} = s_va_vs_v^{-1}$ for some

 $a_v \in A_v$. Thus $u_{n-1}u_n = s_v a_v s_v^{-1} u_n$ which contradicts the fact that p is a reduced form for h with respect to presentation (3). Therefore in this case $s_t(b^{-1}a_{-i})$ is not a transversal.

Secondly, suppose that $u_{n-1} = h_{w'} = s_{w'}a_{-i}^{-1}s_t^{-1} = s_v a_i a_{-i}^{-1}s_t^{-1}$ where $w' \in V(Y - Y_1)$ is some vertex H-equivalent to t = w. Then $s_{w'} = s_v a_i$ and a_i is the label of the edge $(v, w') \in E(Y - Y_1)$. Recall that $b \in A_t = A_w$. Since $s_t(b^{-1}a_{-i})$ is a transversal, Lemma 1.9(vi) implies that $b^{-1}a_{-i} = a_{-i}c$ for some $c \in C$. Thus $b \in A_t \cap a_{-i}Ca_{-i}^{-1}$. Recall that in this situation $h_{w'}^{-1}$ conjugates the subgroup $A_t \cap a_{-i}Ca_{-i}^{-1}$ of A_t into the subgroup $s_{w'}(A_{w'} \cap C)s_{w'}^{-1}$. Thus $h_{w'}(s_t bs_t^{-1})h_{w'}^{-1} = s_v(a_i c_1 a_i^{-1})s_v^{-1} = s_v a_v s_v^{-1}$. Consequently, we have $u_{n-1}u_n = s_v a_v s_v^{-1} \cdot u_{n-1}$ which contradicts the fact that p is a reduced form for h with respect to presentation (3). Therefore in this case $s_t(b^{-1}a_{-i})$ is not a transversal.

We have established that that $s_t(b^{-1}a_{-i})$ is not a transversal and that $(b^{-1}a_{-i}) \notin C$. Therefore the nerve N of W is equal to $px(a_{-i}^{-1}b)$ and it has greater syllable length than the nerve N' = px of W'. Take the core elements of W', add to them $v_{i_n} = a_{-i}^{-1}b$ and declare the result the collection of core elements for W. Proposition A follows now from the inductive hypothesis.

Subcase 4.D. Suppose that \hat{z} and \hat{u} are nonempty.

Then $s_w = yz$, $s_t = yu$ and so $h = pxa_{-i}^{-1}s_w^{-1}s_tbs_t^{-1} = pxa_{-i}^{-1}z^{-1}y^{-1}yubs_t^{-1} = pxa_{-i}^{-1}z^{-1}ubs_t^{-1}$. Suppose \hat{y} ends in a vertex of type A_k . Let $z = f_1z_1$ and $u = f_2u_1$ where $f_1 \in T_k$ is the label of the first edge of \hat{z} and $f_2 \in T_k$ is the label of the first edge of u. Then clearly $f_1C \neq f_2C$ and so $f_1^{-1}f_2 \notin C$. Notice also that $b \notin C$ since if $b \in C$ then $s_t bs_t^{-1}$ stabilizes the last edge of \hat{u} which contradicts the fact that p is the reduced form for h with respect to presentation (3). Thus $W = pxa_{-i}^{-1}z_1^{-1}(f_1^{-1}f_2)u_1bs_t^{-1}$ is the normal form for h with respect to presentation (1). Since $b \notin C$, Lemma 1.9(i) implies that $s_t b^{-1}$ is not a transversal. That is why the nerve N of W is equal to $pxa_{-i}^{-1}z_1^{-1}(f_1^{-1}f_2)u_1b$ and it has greater syllable length than the nerve N' = px of W'. Take the core elements of W', add to them $v_{i_n} = b$ and declare the result the collection of core elements of W. Proposition A follows now from the inductive hypothesis.

This completes the proof of Proposition A.

Corollary 1.11 (c.f. Corollary 4 from the Introduction). Suppose $G = A_1 *_C A_{-1}$ where the groups G and C are finitely generated. Suppose H is a finitely generated subgroup of G such that for any $g \in G$ we have $g^{-1}Hg \cap A_1 = g^{-1}Hg \cap A_{-1} = \{1\}$. Then the subgroup H is quasiisometrically embedded in G (in particular, if G is word hyperbolic then H is quasiconvex in G).

Proof. Since G and C are finitely generated, the groups A_1 and A_{-1} are also finitely generated. Fix a finite generating set C of C and a finite generating set X_i containing C of A_i for $i = \pm 1$. Put $\mathcal{G} = X_1 \cup X_{-1}$ to be the finite generating set of G.

Let T, Y, Y_1 and \mathbb{B} be as in Proposition A. Then H is a free group on $\mathcal{H} = E^+(B - Y_1)$ since $A_v = \{1\}$ for each $v \in VY_1$. Suppose $h \in H$ and $U = U_1 \dots U_2$ is a freely reduced word over $\mathcal{H} = E^+(B - Y_1)$, $U_i \in \mathcal{H}^{\pm 1}$. By Proposition A there is a reduced form $W = v_1 \dots v_m$ of h with respect to the presentation $G = A_1 *_C A_{-1}$ such that $n \leq m$. On the other hand m is the syllable length of h with respect to the presentation $G = A_1 *_C A_{-1}$. Therefore $l_{\mathcal{G}}(h) \geq m$.

Thus $l_{\mathcal{H}}(h) = n \leq m \leq l_{\mathcal{G}}(h)$ and so H is quasiisometrically embedded in G.

2. Word metric on fundamental groups of graphs of groups

Some auxiliarily facts.

Lemma 2.1. Let G be a word hyperbolic group generated by a finite set \mathcal{G} . Let $w = w_1 \dots w_t$ be a Kquasigeodesic word over \mathcal{G} where all w_i are nonempty. Suppose for each $i = 1, \dots, t$ the word u_i represents $\overline{w_i}$ and is λ -quasigeodesic. Then for some constant K' > 0 depending only on K, λ the word $w' = u_1 \dots u_t$ is K'-quasigeodesic.

Proof. The statement of Lemma 2.1 is rather transparent and its proof is a standard exercise on quasiconvexity. Nevertheless the fact is of importance here and we will give a detailed argument.

Let $K_1 = max(K, \lambda)$ and suppose any two K_1 -quasigeodesics with the same endpoints in the Cayley graph of G are ϵ -Hausdorff-close. Let u be a subword of w'. There are two possibilities.

Case 1. There is u_i such that u is a subword of u_i .

In this case, obviously,

$$l(u) \le K_1 \cdot l_{\mathcal{G}}(\overline{u}) + K_1$$

Case 2. The word u has the form $u = u'_i u_{i+1} \dots u_{j-1} u'_j$ where i < j, u'_i is a terminal segment (perhaps empty) of u_i and u''_j is an initial segment (perhaps empty) of u_j .

We want to show that for some constant K'

$$l(u'_{i}u_{i+1}\dots u_{j-1}u''_{j}) \le K' l_{\mathcal{G}}(\overline{u'_{i}u_{i+1}\dots u_{j-1}u''_{j}}) + K'$$
(†)

There is a terminal segment w'_i of w_i and an initial segment w''_j of w_j such that $l_{\mathcal{G}}(\overline{w'_i u'_i}^{-1}) \leq \epsilon$ and $l_{\mathcal{G}}(\overline{w''_i}^{-1}u''_j) \leq \epsilon$.

Therefore $l_{\mathcal{G}}(\overline{w'_i w_{i+1} \dots w_{j-1} w''_j}) \leq l_{\mathcal{G}}(\overline{u'_i u_{i+1} \dots u_{j-1} u''_j}) + 2\epsilon$. We have

$$l(u_i') \le K_1 l_{\mathcal{G}}(\overline{u_i'}) + K_1 \le K_1 (l_{\mathcal{G}}(\overline{w_i'}) + \epsilon) + K_1 \le K_1 (l(w_i') + \epsilon) + K_1,$$

$$l(u_j'') \le K_1 l_{\mathcal{G}}(\overline{u_j''}) + K_1 \le K_1 (l_{\mathcal{G}}(\overline{w_j''}) + \epsilon) + K_1 \le K_1 (l(w_j'') + \epsilon) + K_1$$

$$l(u_k) \le K_1 l_{\mathcal{G}}(\overline{u_k}) + K_1 \le K_1 l(w_k) + K_1$$

and

$$j - i \le l(w'_i w_{i+1} \dots w_{j-1} w''_j) + 2.$$

Therefore

$$l(u'_{i}u_{i+1}\dots u_{j-1}u''_{j}) \leq K_{1}l(w'_{i}w_{i+1}\dots w_{j-1}w''_{j}) + K_{1}(i-j) + 2(K_{1}+\epsilon) \leq (K_{1}+1)l(w'_{i}w_{i+1}\dots w_{j-1}w''_{j}) + 2(2K_{1}+\epsilon)$$

Put $K_2 = 2(2K_1 + \epsilon) + 1$. Then

$$\begin{split} l(u'_{i}u_{i+1}\dots u_{j-1}u''_{j}) &\leq K_{2}l(w'_{i}w_{i+1}\dots w_{j-1}w''_{j}) + K_{2} \leq \\ K_{2}K_{1}l_{\mathcal{G}}(\overline{w'_{i}w_{i+1}\dots w_{j-1}w''_{j}}) + K_{2}K_{1} + K_{2} \leq \\ K_{2}K_{1}(l_{\mathcal{G}}(\overline{u'_{i}u_{i+1}\dots u_{j-1}u''_{j}}) + 2\epsilon) + K_{2}K_{1} + K_{2} \leq \\ K'l_{\mathcal{G}}(\overline{u'_{i}u_{i+1}\dots u_{j-1}u''_{j}}) + K' \end{split}$$

where $K' = 2K_2K_1 + 2K_2K_1\epsilon + K_2K_1 + K_2$. Thus (†) is established and Lemma 2.1 is proved.

Lemma 2.2. Suppose G is a word hyperbolic group generated by a finite set \mathcal{G} . Suppose $C_1, C_2 \leq G$ are virtually cyclic subgroups of G such that is $C_1 \cap C_2$ is finite. Let C_i be a finite generating set of C_i . Assume that $C_i \subseteq \mathcal{G}$.

Then

- there is a constant λ > 0 such that whenever U is a d_G-geodesic word such that U is shortest in the coset class U · C₁ and V is a d_{C1}-geodesic word, then the word UV is λ-quasigeodesic with respect to d_G;
- (2) there is a constant K > 0 such that whenever $u \in G$ is shortest in the double coset class $C_1 u C_1$ and $c_1 \in C$, the element $c_1 u$ is at most K-away from any shortest element in the coset class $c_1 u C_1$;
- (3) there is a constant $\lambda_1 > 0$ such that for any d_{C_1} -geodesic words V, V' and any $d_{\mathcal{G}}$ -geodesic word such that \overline{U} is shortest in the double coset class $C_1\overline{U} \cdot C_1$, then the word VUV' is λ_1 -quasigeodesic with respect to $d_{\mathcal{G}}$;
- (4) there is a constant $K_1 > 0$ such that whenever $u \in G$ is shortest in the double coset class $C_1 u C_1$ and $c_1 \in C_1$, the word $c_1 u$ is at most K_1 -away from any shortest element in the coset class $c_1 u C_1$;

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- (5) there is a constant $K_2 > 0$ such that for any $c_1 \in C_1$ there is an element $c_2 \in C_2$ with $l_{\mathcal{G}}(c_2) \leq K_2$ such that the element $u = c_1c_2$ is shortest in the coset class c_1C_2 ;
- (6) there is a constant $K_3 > 0$ such that for any $c_1 \in C_1$ we have $l_{\mathcal{G}}(u) \ge l_{\mathcal{G}}(c_1)/K_3 K_3$ where u is shortest in the coset class c_1C_2 ;
- (7) there is a constant $\lambda_2 > 0$ such that for any $c_1 \in C_1$ and $c_2 \in C_2$ the word V_1V_2 is λ_2 -quasigeodesic in the Cayley graph $\Gamma(G, \mathcal{G})$ of G where V_i is a d_G -geodesic representative of c_i , i = 1, 2.

Proof.

(1), (2), (3) and (4) follow from the proof of Theorem C in [3]

(5) Let $y \in C_1$ be such that the cyclic group $\langle y \rangle$ has finite index in C_1 . Similarly, let $x \in C_2$ be such that the cyclic group $\langle x \rangle$ has finite index in C_2 . Fix a finite subset $T_1 \subseteq C_1$ such that $C_1 = T_1 \langle y \rangle$ and a finite subset $T_2 \subseteq C_2$ such that $C_2 = T_2 \langle x \rangle$. Observe that the statement (5) of Lemma 2.2 is obvious when at least one of the groups C_1, C_2 is finite. From now on assume that they are both infinite. Thus x, y are of infinite order and no nonzero power of x is equal to a nonzero power of y since $C_1 \cap C_2$ is finite. Let $c_1 \in C_1$ be an arbitrary element. Then $c_1 = t_1 y^n$ for some $t_1 \in T_1$.

Let $t_1y^n = uc_2$ where $c_2 \in C_2$ and u is shortest in the coset class $c_1C_2 = t_1y^nC_2$. Thus $t_1y^n = ut_2x^k$ for some $t_2 \in T_2$. Since $\langle x \rangle$ is infinite and quasiconvex in G, there is a constant $K_2 > 0$ independent of kand n such that $u_1 = ut_2$ is K_2 -close to a shortest element in $ut_2 < x >$. Indeed, assume $ut_2 = u_1 = u'x^p$ where u' is shortest in $ut_2 < x > = u' < x >$. It follows from the proof of Theorem C in [3] that there is a constant N > 0 independent of n, k, p such that $l_{CalG}(u') + l_{\mathcal{G}}(x^p) \leq l_{\mathcal{G}}(u_1) + N$. Suppose p is such that $l_{\mathcal{G}}(x^p) > l_{\mathcal{G}}(t_2) + N$. Then $l_{\mathcal{G}}(u) \geq l_{\mathcal{G}}(u_1) - l_{\mathcal{G}}(t_2) \leq l_{CalG}(u') + l_{\mathcal{G}}(x^p) - N - l_{\mathcal{G}}(t_2) > l_{CalG}(u')$. Notice also that $uC_2 = u'C_2$ which contradicts our choice of u. Thus $l_{\mathcal{G}}(x^p) \leq K_2 = l_{\mathcal{G}}(t_2) + N$.

We want to show that |k| is small. Let Q_1, Y, U_1 and X be \mathcal{G} -words representing t_1, y, u_1 and x. It follows from (1) that there is $\lambda > 0$ independent of n and k such that UX^k and Q_1Y^n are λ -quasigeodesics in the Cayley graph $\Gamma(G, \mathcal{G})$ of G. Thus the paths Q_1Y^n and UX^k are ϵ -hausdorff-close for some constant $\epsilon > 0$. If |k| is greater than the number of elements in G of length at most $\epsilon + 2$ then there are numbers $n_1, n_2, k_1, k_2 \neq 0$ such that $zy^{n_2}z^{-1} = x^{k_2}$ where $z = x^{-k_1}y^{n_1}$. Therefore $x^{-k_1}y^{n_1}y^{n_2}y^{-n_1}x^{k_1} = x^{k_2}$ and $y^{n_2} = x^{k_2}$. This contradicts the fact that $\langle x \rangle \cap \langle y \rangle \geq \{1\}$. Thus |k| is bounded by a constant independent of n which implies statement (5) of Lemma 2.2.

(6) follows from (1).

Word metric on fundamental groups of graphs of groups.

Suppose a word hyperbolic group G is the fundamental group of a finite graph of groups A with respect to a maximum subtree T.

$$G = \pi_1(\mathbb{A}, T) \tag{3}.$$

Assume that all edge groups A_e are virtually cyclic. Then G has a presentation

$$G = (*A_v) * F(E^+A) / \{e = 1, e \in ET; \alpha_e(a)e = e\omega_e(a), e \in E^+A, a \in A_e\}$$

For each $e \in EA$ we fix $x_e \in A_e$ such that $\langle x_e \rangle$ has finite index in A_e . Denote $x_{e,\alpha} = \alpha_e(x_e) \in A_{\partial_0(e)}$ and $x_{e,\omega} = \omega_e(x_e) \in A_{\partial_1(e)}$.

For each vertex $v \in VA$ we fix a finite generating set Z_v closed under taking inverses. We may assume that for each edge e of \mathbb{A} originating from v the set Z_v includes the generator $c_{e,\alpha}$ of the subgroup of finite index in $\alpha_e(A_e)$. Put

$$Z = \bigcup_{v \in VA} Z_v \bigcup \{ e | e \in E(A - T) \}$$

Then Z is a finite generating set for A. Put $Z' = Z \cup ET$. It is another finite generating set for G (every $e \in ET$ represents the trivial element of G). Any $d_{Z'}$ -geodesic word w contains no letters $e \in ET$ and so it is a word over Z. Clearly it is d_Z -geodesic. Thus d_Z and $d_{Z'}$ coincide on G.

Lemma 2.3. Suppose W = UeV where $e \in EA$, $v_0 = \partial_0(e)$, $v_1 = \partial_1(e)$, U is a $d_{Z_{v_0}}$ -geodesic word, $v_1 = \partial_1(e)$, V is a $d_{Z_{v_0}}$ -geodesic word. Suppose $\overline{V} = \in \omega_e(A_e)$.

Assume that W is a K-quasigeodesic in the $d_{Z'}$ -metric for some K > 0. Then there is K' > 0 independent of \overline{U} , \overline{V} , such that $W' = U_1 e$ is K'-quasigeodesic where U_1 is a $d_{Z_{v_0}}$ -geodesic word representing $\overline{U}\alpha_e(\omega_e)^{-1}(\overline{V})$.

Proof. The subgroup $A_{\partial_0(e)} = A_{v_0}$ is e quasiconvex in G. Thus there is a constant K_1 such that y_e is K_1 quasigeodesic in the $d_{Z'}$ -metric for any $d_{Z_{v_0}}$ -geodesic word y. Let V_1 be a $d_{Z_{v_0}}$ -geodesic word representing $\alpha_e(\omega_e)^{-1}(\overline{V})$. Then $\overline{V_1e} = \overline{eV}$. Therefore by Lemma 2.1 the word $W_2 = UV_1e$ is K_2 -quasigeodesic in the $d_{Z'}$ -metric for some constant K_2 .

Recall that U_1 is a $d_{G_{v_0}}$ -geodesic representative of $\overline{UV_1}$. Since $A_{G_{v_0}}$ is quasiconvex in G, we know that U_1 is a K_3 -quasigeodesic in the Cayley graph of G. Thus by Lemma 2.1 $W' = U_1 e$ is a K'-quasigeodesic in the $d_{Z'}$ -metric for some constant K'.

Lemma 2.4. Let F be the subgroup of G generated by EA that is F is a free group on $E^+(A-T)$. Then

- (a) The subgroup F is isometrically embedded in G that is any freely reduced word in $E^+(A T)$ is d_Z -geodesic.
- (b) For each K > 0 and M > 0 there is $K_1 > 0$ such that whenever $W = W_1 \dots W_s$ is a K-quasigeodesic in the $d_{Z'}$ -metric and U_0, \dots, U_s are words in E^+T of length at most M, the word

$$W' = U_0 W_1 U_1 \dots U_{s-1} W_s U_s$$

is K_1 -quasigeodesic in the $d_{Z'}$ -metric.

Proof. Statement (a) follows obviously from the properties of HNN-extensions.

Statement (b) is obvious.

Proposition B. There is a constant K > 0 such that for any $g \in G$ there is a K-quasigeodesic with respect to $d_{Z'}$ word W representing g of the form

$$W = W_1 \dots W_n$$

where each W_k is either $e^{\pm 1}$ for some $e \in EA$ or W_k is a d_{Z_v} -geodesic word for some $v \in VA$ and

$$\overline{W_1},\ldots,\overline{W_n}$$

is a reduced form for h with respect to presentation (3).

Proof. Let W be a Z-geodesic word representing g. We will transform W to the required form in several steps.

Step 1 We can write W as $W = Q_1 \dots Q_m$ where for $k = 1, \dots, m$ each Q_k is either an edge of A - T or it represents an element of a vertex group of A and whenever $1 \leq i < j \leq m$, the word $Q_i \dots Q_j$ does not represent an element of a vertex group of A. Notice that we do not claim that each Q_i is a word in generators of some vertex group. Now each A_v is quasiconvex in G since all the edge groups are virtually cyclic [11]. Let $K_1 > 0$ be such that for any $v \in VA$ any d_{Z_v} geodesic word is K_1 -quasigeodesic in the d_Z -metric. For each $k = 1, \dots, m$ such that $\overline{Q_k} \in A_v$ we find a d_{Z_v} -geodesic representative U_k of $\overline{Q_k}$. For other Q_k we put $U_k = Q_k$. Let $W_1 = U_1 \dots U_m$. Then by Lemma 2.1 the word W_1 is K_2 -quasigeodesic in the d_Z -metric for some constant K_2 independent of g.

Step 2 Now between every U_k, U_{k+1} representing elements of vertex groups v_k and v_{k+1} of A we insert the reduced edge-path r_k in T from v_k to v_{k+1} . Between every U_k, U_{k+1} such that $\overline{U_k} \in A_{v_k}$ and $U_{k+1} = e \in E(A - T)$ we insert the reduced edge-path r_k in T from v_k to the initial vertex of e. Between every U_k, U_{k+1} such that $U_k = e \in E(A - T)$ and $\overline{U_{k+1}} \in A_{v_{k+1}}$ we insert the reduced edge-path r_k in T from the terminal vertex of e to v_k . We put r_0 to be the reduced edge-path from d_1 to the initial vertex of U_1 when U_1 is an edge of A - T and we put r_0 to be the reduced edge-path from d_1 to the vertex v_1 when $\overline{U_k} \in A_{v_1}$. Analogously, we put r_m to be the reduced edge-path from the terminal vertex of U_m to d_1 when U_m is an edge of A - T and we put r_m to be the reduced edge-path from v_m to d_1 when $\overline{U_m} \in A_{v_m}$. Then

$$r = r_0, \overline{U_1}, r_1, \dots, r_{m-1}, \overline{U_m}, r_m$$

is a loop at d_1 in the graph of groups \mathcal{A} which represents g. Observe that each edge-path r_i has length at most N_0 where N_0 is the number of oriented edges of T since T is a tree and r_i has no backtrackings. Therefore by Lemma 2.4 the word

$$W_2 = r_0 U_1 r_2 \dots r_{m-1} U_m r_m$$

is a K_3 -quasigeodesic with respect to $d_{Z'}$ -metric where K_3 is some constant independent of g.

Step 3 For each k = 1, ..., m we find the maximal initial segment r'_k of r_k such that ${r'_k}^{-1} = {r''_{k-1}}$ is a terminal segment of r_{k-1} and ${r'_k}^{-1}U_kr_k$ represents an element u'_k of a vertex group of A. Notice that r'_k and r''_k are disjoint subwords of r_k since otherwise there is a subword $U_i \ldots U_j$ of W_1 , i < j, representing an element of a vertex group.

Replace in r each loop $r'_k^{-1}, \overline{U_k}, r'_k$ by u'_k .

This gives us the path

$$r' = r'_0 u'_1 r'_2 \dots r'_{m-1} u'_m r'_0$$

such that

- (1) each u'_k is either an edge of A T or a nontrivial element of a vertex group of \mathcal{A} ,
- (2) whenever i < j the element $u'_i \dots u'_j$ is not in a vertex group of \mathcal{A}
- (3) whenever the last edge e of r'_i is inverse to the first edge of r'_{i+1} , then $\overline{u'_i} \notin \omega_e(A_e)$.

Let $U'_i = e$ whenever $u'_i = e \in E(A - T)$ and let U'_i be a d_{Z_v} -geodesic representative of u'_i whenever $u'_i \in A_v$. For each $i = 1, \ldots, m$ replace the subword $r''_{k-1}U_kr'_k$ of W_2 by the word U'_k . This produces a new word

$$W_3 = r'_0 U'_1 r'_2 \dots r'_{m-1} U'_m r'_0.$$

Since the vertex groups are quasiconvex in G, Lemma 2.1 implies that W_3 is a K_3 -quasigeodesic in the $d_{Z'}$ -metric where K_3 is a constant independent of g. Notice that r' is close to being a normal form of g with respect to presentation (3). The only conditions of Definition 1.1 which are possibly not satisfied are conditions 5) and 6).

Step 4 The sequence r' can be broken into maximal pieces $r' = P_1 \dots P_{s'}$ where each P_k has the form

$$g_{i_k}, e_{i_k}, \omega_{e_{i_k}}(c_{i_k})\alpha_{e_{i_k+1}}(c_{i_k+1}^{-1}), e_{i_k+1}, \omega_{e_{i_k+1}}(c_{i_k+1})\alpha_{e_{i_k+2}}(c_{i_k+2}^{-1}), e_{i_k+2}, \dots$$

$$\dots, e_{j_k-1}, \omega_{e_{j_k-1}}(c_{j_k-1})\alpha_{e_{j_k}}(c_{j_k}^{-1}), e_{j_k}, \omega_{e_{j_k}}(c_{j_k})$$
(5)

where $c_s \in A_{e_s}$ for $s = i_k, \ldots, j_k$, $g_{i_k} \in A_{\partial_0(e_{i_k})}$ and $g_{i_k} \alpha_{e_{i_k}}(c_{i_k})$ cannot be "pulled to the left" that is either $i_k = 1$ and $e_{i_k} = e_1$ is the first edge of r' or $i_k > 1$ and $g_{i_k} \alpha_{e_{i_k}}(c_{i_k}) \notin \omega_{e_{i_k-1}}(A_{e_{i_k-1}})$. To each P_k there is a corresponding subword \hat{P}_k of W_2 of the form

$$\hat{P}_{k} = V_{i_{k}} e_{i_{k}} V_{i_{k}+1} e_{i_{k}+1} \dots e_{j_{k}} V_{j_{k}+1}$$
(6)

Observe that for each k the path e_{i_k}, \ldots, e_{j_k} has no backtrackings. Indeed, suppose $e_{s+1} = e_s^{-1}$. Then by construction of r' we have $1 \neq \omega_{e_s}(c_s)\alpha_{e_{s+1}}(c_{s+1}^{-1}) = \overline{U_t}$ for some t. On the other hand, equation (5) implies that $\omega_{e_s}(c_s)\alpha_{e_{s+1}}(c_{s+1}^{-1}) = \omega_{e_s}(c_s)\omega_{e_s}(c_{s+1}^{-1}) = \omega_{e_s}(c_s c_{s+1}^{-1})$ and therefore $e_s\omega_{e_s}(c_s)\alpha_{e_{s+1}}(c_{s+1}^{-1})e_{s+1} = \alpha_{e_s}(c_s c_{s+1}^{-1})$. This contradicts property 3) of r'.

For each k let l_k be the maximal among $\{i_k, \ldots, j_k - 1\}$ such that for $s = i_k, \ldots, l_k - 1$ the groups $\omega_{e_s}(A_{e_s})$ and $\alpha_{e_{s+1}}(A_{e_{s+1}})$ have infinite intersection. If there are no such indices among $\{i_k, \ldots, j_k - 1\}$, we put $l_k = i_k$.

Claim 1. We have $l_k - i_k < N + 1$ where N is the number of oriented edges in the graph A.

Indeed, suppose not and $l_k - i_k \ge N + 1$ Then there is a subpath e_{s_1}, \ldots, e_{t_1} of $e_{i_k}, \ldots, e_{l_k}, s_1 \ne t_1$, such that $e_{s_1} = e_{t_1}$. We know that for each $s = i_k, \ldots, l_k$ the groups $\langle x_{s,\omega} \rangle$ and $\langle x_{s+1,\alpha} \rangle$ are infinite and commensurable (i.e. they are finite extension of a common infinite cyclic subgroup). Thus we can find $M \neq 0$ and $M_1 \neq 0$ such that $x_{s_1,\alpha}^M e_{s_1} \dots e_{t_1-1} = e_{s_1} \dots e_{t_1-1} x_{t_1,\alpha}^{M_1} = e_{s_1} \dots e_{t_1-1} x_{s_1,\alpha}^{M_1}$ in G. Notice that the edge-path e_{s_1}, \dots, e_{t_1} contains an edge which is not equal to the trivial element in G. Indeed, if they are all trivial then, since $e_{s_1} = e_{t_1}$, the path e_{s_1}, \ldots, e_{t_1} contains a backtracking e, e^{-1} which is impossible. Thus $e_{s_1} \ldots e_{t_1-1}$ represents a non-trivial element $f \in F$. Therefore $x^M f = f x^{M_1}$ where $x = x_{s_1,\alpha} = x_{t_1,\alpha}$. Since G is word hyperbolic, this implies $\langle f \rangle \cap \langle x \rangle \neq \{1\}$ which is impossible by standard properties of graphs of groups. Thus we have established that $l_k - i_k < N + 1$ and Claim 1 is proved.

Observe that P_k represents the same element of G as

$$P'_{k} = g_{i_{k}}, e_{i_{k}}, \omega_{e_{i_{k}}}(c_{i_{k}})\alpha_{e_{i_{k}+1}}(c_{i_{k}+1}^{-1}), e_{i_{k}+1}, \omega_{e_{i_{k}+1}}(c_{i_{k}+1})\alpha_{e_{i_{k}+2}}(c_{i_{k}+2}^{-1}), e_{i_{k}+2}, \dots$$
$$\dots, e_{l_{k}-1}, \omega_{e_{l_{k}-1}}(c_{l_{k}-1})\alpha_{e_{l_{k}}}(c_{l_{k}}^{-1}), e_{l_{k}}, \omega_{e_{l_{k}}}(c_{l_{k}}), e_{l_{k}+1}, e_{l_{k}+2}, \dots, e_{j_{k}}$$

By definition of l_k the groups $\omega_{e_{l_k}}(A_{e_{l_k}})$ and $\alpha_{e_{l_k+1}}(A_{e_{l_k+1}})$ have finite intersection.

Thus by Lemma 2.2 the word $V'_{l_k}V''_{l_k}$ is λ -quasigeodesic in the Cayley graph of G where V'_{l_k} is a d_{Z_v} -geodesic

representative of $\omega_{e_{l_k}}(c_{l_k})$ and V_{l_k}'' is a d_{Z_v} -geodesic representative of $\alpha_{e_{l_k+1}}(c_{l_k}^{-1}), v = \partial_1(e_{l_k}) = \partial_0(e_{l_k+1})$. In the word $W_3 = r'_0 U'_1 r'_1 \dots r'_{m-1} U'_m r'_m$ for each $k = 1, \dots, s'$ we substitute the word V_{l_k+1} representing the element $\omega_{e_{l_k}}(c_{l_k}) \cdot \alpha_{e_{l_k+1}}(c_{l_k+1})$ by the word $V'_{l_k} V''_{l_k}$. This produces a new word $W_4 = U''_{l_k} U''_{l_k} U''_{l_k} U''_{l_k} U''_{l_k} U''_{l_k} U''_{l_k} U''_{l_k}$. $r_0'' U_1'' r_1'' \dots r_{m-1}'' U_m'' r_m''$ which is λ_1 -quasigeodesic by Lemma 2.1.

Besides, we know that the edge-path e_{i_k}, \ldots, e_{j_k} has no backtrackings and therefore the lengths of the segments, into which elements of E(A - T) divide it, do not exceed N where N is the number of edges in A. Thus by Lemma 2.4 the word $e_{k_0+1}, e_{k_0+2}, \ldots, e_j$ is λ_4 -quasigeodesic in $d_{Z'}$ -metric where λ_4 is some constant independent of q.

For each $k = 1, \ldots, s'$ let V'_{i_k} be a $d_{Z_{\partial_0(e_i)}}$ -geodesic representative of $g_{i_k} \alpha_{e_{i_k}}(c_{i_k})$. Now for each $k = 1, \ldots, s'$ we replace the subword $V_{l_k}''e_{l_k+1}V_{l_k+1}e_{l_k+2}V_{l_k+2}\dots V_{j_k}e_{j_k}V_{j_k+1}$ of W_4 by $e_{l_k+1}e_{l_k+2}\dots e_{j_k}$ and the subword $V_{i_k}e_{i_k}V_{i_k+1}e_{i_k+1}V_{i_k+2}\dots V_{l_k}e_{l_k}V'_{l_k}$ of W_4 by $V'_{i_k}e_{i_k}e_{i_k+1}\dots e_{l_k}$ to get a word W_4 . Since $l_k - i_k \le N + 1$, Lemma 2.1 implies that the new word W_4 is λ_5 -quasigeodesic in the $d_{Z'}$ -metric where λ_5 is some constant independent of g. It also follows from the construction that W_4 satisfies all requirements of Definition 1.1 except, possibly, condition 6).

Step 5 Let $r^{(3)}$ be the path in A corresponding to the word W_4 , that is $r^{(3)}$ is obtained from W_4 by "barring" every letter.

Then $r^{(3)}$ can be broken into maximal pieces

 $r^{(3)} = R_1 \dots R_{s''}$ where each piece R_k has the form

$$g_{l_0(k)}, r_{1,k}, g_{l_1(k)}, r_{2,k} \dots r_{j,k} g_{l_j(k)}$$

where each $g_{l_i(k)} \in A_{v_{i(k)}} - \{1\}$ is a nontrivial vertex group element, each $r_{i,k}$ is an edge-path with trivial vertex group elements inserted between the consecutive edges and for each $i = 0, \ldots, j-1$ the element $g_{l_i(k)}$ can be "pulled through" the edge-path $r_{i+1,k}r_{i+2,k}\ldots r_{j,k}$ to the element of $A_{v_{j(k)}}$.

Recall that each edge-path $r_{i,k}$ is without backtracks by construction of W_4 . Moreover $r_{1,k}r_{2,k}\ldots r_{j,k}$ also does not have backtracks. Indeed, if the last edge e of $r_{i,k}$ is inverse to the first edge of $r_{i+1,k}$ then the element $g_{l_i(k)}$ can be "pulled to the left" trough the edge e which contradicts the properties of W_4 .

Find minimal i (if any) such that $g_{l_i(k)}$ is of infinite order and denote it i_0 . If there are no such i, put $i_0 = l_i(k).$

Claim 2 The length of $r_{i_0+1,k}r_{i_0+2,k}\ldots r_{j_k}$ is at most N where N is the number of oriented edges in the graph A.

The proof is exactly the same as that of Claim 1.

Notice that there is a uniform bound on the lengths of elements of finite order in vertex groups which come are images of elements of finite order in edge groups. For each R_k there is a corresponding subword $\hat{R}_{k} = V_{l_{0}(k)}r_{1,k}V_{l_{1}(k)}\dots V_{l_{j}(k)} \text{ of } W_{4}. \text{ Let } V'_{l_{j}(k)} \text{ be a } Z_{v_{j}(k)}\text{-geodesic word such that } r_{1,k}r_{2,k}\dots r_{j,k}V'_{l_{j}(k)}$ represents the same element of G as R_{k} . Notice that $\overline{V'_{l_{j}(k)}} = c\overline{V_{l_{j}(k)}}$ for some $c \in \omega_{e}(A_{e})$ where e is the last edge of $r_{j,k}$. Thus $\overline{V'_{l_{j}(k)}} \notin \omega_{e}(A_{e})$ since $\overline{V_{l_{j}(k)}} \notin \omega_{e}(A_{e})$.

Substitute every \hat{R}_k in W_4 by $r_{1,k}r_{2,k} \dots r_{j,k}V'_{l_j(k)}$ to get a new word W_5 . Applying Lemma 2.3 at most N times we conclude that W_5 is λ_6 -quasigeodesic with respect to $d_{Z'}$ where λ_6 does not depend on g. It follows from the construction that W_5 corresponds to a reduced form of g with respect to presentation (3). This completes the proof of Proposition B.

3. Proofs of Theorem A and Theorem B

Theorem B. Let

$$G = A_1 *_C A_{-1} \tag{7}$$

be a word hyperbolic group where the group C is virtually cyclic (and therefore A_1 and A_{-1} are quasiconvex in G). Let H be a finitely generated subgroup of G.

Then H is quasiconvex in G if and only if for each $g \in G$ and $i = \pm 1$ the subgroup $g^{-1}Hg \cap A_i$ is quasiconvex in A_i .

Before proceeding with the proof of Theorem B we need the following

Lemma 3.1. Let G, A_1 , A_{-1} and C be as above. Let X_i be a finite generating set of A_i containing a finite generating set C of C. Put $\mathcal{G} = X_1 \cup X_{-1}$ to be the finite generating set for G. Then the following holds.

(1) There exists K > 0 such that each $g \in G$ has a K-quasigeodesic (with respect to $d_{\mathcal{G}}$) representative of the form

$$u = u_0 \dots u_s$$

where

- (i) each u_k is a d_{X_j} -geodesic word for some $j \in \{\pm 1\}$ such that when k > 0 the element $\overline{u_k}$ does not belong to C and it is shortest (with respect to d_{X_j}) in the coset class $C\overline{u_k}$;
- (ii) If $\overline{u_k} \in A_i C$ then $\overline{u_{k-1}} \in A_{-i} C$, $k = 1, \ldots, s$.
- (iii) If s = 0 and $g = c \in C$ then w_0 is a d_{X_1} -geodesic representative of c.
- (2) Suppose N > 0 is a fixed number. Then there is a constant $K_1 > 0$ such that the following holds. Suppose $g \in G$ and $w = w_0 \dots w_s$ is a word such that
- (i) $\overline{w} = g;$
- (ii) each w_k is a d_{X_j} -geodesic word for some $j \in \{\pm 1\}$ such that for k > 0 the element $\overline{u_k}$ does not belong to C;
- (iii) If $\overline{w_k} \in A_j C$ then $\overline{w_{k-1}} \in A_{-j} C$, $k = 1, \ldots, s$;
- (iv) If k > 0, $\overline{w_k} = c'v_k$, $\overline{w_{k-1}} = v''_{k-1}c''$ where $c', c'' \in C$, v'_k is shortest (with respect to d_{X_j} in the coset class $C\overline{w_k}$ and v''_{k-1} is shortest (with respect to $d_{X_{-j}}$) in the coset class $\overline{w_{k-1}}C$ then

$$l_{\mathcal{C}}(c''c') \ge l_{\mathcal{C}}(c'') + l_{\mathcal{C}}(c') - N$$

Then $l_{\mathcal{G}}(g) \geq K_1 l(w) + K_1$.

Proof.

(1) This is a more or less immediate corollary of Proposition B applied to the group G. If $g = c \in C$, that is w is a $d_{\mathcal{C}}$ -geodesic representative of c, the statement of Lemma 3.1(1) is obvious. Assume from now on that $\overline{w} \notin C$. By Proposition B there exists K > 0 such that every element $g \in G$ has a $d_{\mathcal{G}}$ -quasigeodesic representative of the form

 $w = w_0 \dots w_s$ where

- (1) each w_k is a d_{X_j} -geodesic word for some $j \in \{\pm 1\}$ such that the element $\overline{w_k}$ does not belong to C;
- (2) if $\overline{w_k} \in A_j C$ then $\overline{w_{k-1}} \in A_{-j} C$, $k = 1, \dots, s$.

We will transform w to the required form in several steps. **Step 1.**

For each $k = 0, \ldots, s$ express $\overline{w_k} \in A_j - C$ as $\overline{w_k} = \overline{x_k v_k z_k}$ where x_k, z_k are $d_{\mathcal{C}}$ -geodesic words, v_k is d_{X_j} geodesic word and $\overline{v_k}$ is shortest in the double coset class $C\overline{w_k}C$. It follows from Lemma 2.2(3) that there
is a constant $K_1 > 0$ independent of g such that each $x_k v_k z_k$ is a K_1 -quasigeodesic in the d_{X_j} metric. Since A_1 and A_{-1} are quasiconvex in G, there is $K_2 > 0$ independent of g such that $x_k v_k z_k$ is K_2 -quasigeodesic
in $d_{\mathcal{G}}$ -metric. Replace every w_k by $x_k v_k z_k$ to get a word

$$w' = x_0 v_0 z_0 x_1 v_1 z_1 x_2 \dots x_{s-1} v_{s-1} z_{s-1} x_s v_s z_s.$$

By Lemma 2.1 there is $\lambda > 0$ independent of g such that w' is λ -quasigeodesic in the $d_{\mathcal{G}}$ -metric.

Step 2. For each k = 0, ..., s - 1 we find a $d_{\mathcal{C}}$ -geodesic word y_k representing the element $\overline{z_k x_{k+1}}$. Since C is quasiconvex in G, there is a constant $K_2 > 0$ such that each y_k is K_2 -quasigeodesic with respect to $d_{\mathcal{G}}$. Replace for each k = 0, ..., s - 1 the word $z_k x_{k+1}$ by y_k to get a new word

$$w'' = x_0 v_0 y_0 v_1 y_1 v_2 \dots y_{s-2} v_{s-1} y_{s-1} v_s z_s$$

By Lemma 2.1 there is $\lambda_1 > 0$ independent of g such that w'' is λ_1 -quasigeodesic in the $d_{\mathcal{G}}$ -metric.

Step 3. Express $\overline{v_s z_s} \in A_j - C$ as $\overline{v_s z_s} = \overline{q_s u_s}$ where u_s is a d_{X_j} -geodesic word, q_s is a $d_{\mathcal{C}}$ -geodesic word and $\overline{q_s}$ is shortest (with respect to d_{X_j}) in the coset class $C\overline{v_s z_s}$. Then express $\overline{v_{s-1}y_{s-1}q_s} \in A_{-j} - C$ as $\overline{v_{s-1}y_{s-1}q_s} = \overline{q_{s-1}u_{s-1}}$ where u_{s-1} is a $d_{X_{-j}}$ -geodesic word, q_{s-1} is a $d_{\mathcal{C}}$ -geodesic word and $\overline{q_{s-1}}$ is shortest (with respect to $d_{X_{-j}}$) in the coset class $C\overline{v_{s-1}y_{s-1}q_s}$. And so on. Finally, express $\overline{x_0v_0y_0q_1} \in A_i - C$ as $\overline{x_0v_0y_0q_1} = \overline{u_0}$ where u_0 is a d_{X_i} -geodesic word.

By Lemma 2.2(2) there is a constant $K_3 > 0$ independent of g such that $l_{\mathcal{C}}(\overline{q_k}) \leq K_3$, $k = 1, \ldots, s$. Between each y_{k-1} and v_k is w'' we insert a word $q_k q_k^{-1}$ to get a word

$$w''' = x_0 v_0 q_1 q_1^{-1} v_1 y_1 q_2 q_2^{-1} \dots v_{s-1} y_{s-1} q_s q_s^{-1} v_s z_s.$$

Since $l_{\mathcal{C}}(\overline{q_k}) \leq K_3$, the word w''' is λ_2 -quasigeodesic with respect to $d_{\mathcal{G}}$ where λ_2 is a constant independent of g.

Step 4. Recall that A_1 and A_{-1} are quasiconvex in G. Therefore there is a constant $K_4 > 0$ such that any d_{X_i} -geodesic word is K_4 -quasigeodesic with respect to $d_{\mathcal{G}}$. So there is $K_5 > 0$ independent of g such that the word and $q_k u_k$ is K_5 -quasigeodesic with respect to $d_{\mathcal{G}}$ for $k = 1, \ldots, s$.

Replace in w''' each $v_k y_k q_{k+1}$ by $q_k u_k$ for k = 1, ..., s - 1, replace $x_0 v_0 q_1$ by u_0 and replace $v_s z_s$ by $q_s u_s$ to get the word

$$w^{(4)} = u_0 q_1^{-1} q_1 u_1 q_2^{-1} q_2 u_2 \dots q_{s-1} u_{s-1} q_s^{-1} q_s u_s.$$

By Lemma 2.1 $w^{(4)}$ is λ_3 -quasigeodesic for some constant $\lambda_3 > 0$ independent of g.

Step 5. Finally we replace each $q_k^{-1}q_k$ by the empty word to get the word

$$u = u_0 u_1 \dots u_s$$

By Lemma 2.1 $w^{(5)}$ is λ_4 -quasigeodesic for some constant $\lambda_4 > 0$ independent of g. It follows from the construction that $\overline{u} = g$ and that u satisfies all the requirements of Lemma 3.1 (1). This completes the proof of part (1) of Lemma 3.1.

(2) Let $w = w_0 \dots w_s$ be as in Lemma 3.1 (2). We will show that it can be transformed to a quasigeodesic $u = u_0 \dots u_s$ as in Lemma 3.1(1) without loosing too much length. Assume $g = \overline{w} \notin C$.

For each $k = 0, \ldots, s$ express $\overline{w_k} \in A_j - C$ as $\overline{x_k v_k z_k}$ where x_k, z_k are $d_{\mathcal{C}}$ -geodesic words, v_k is a d_{X_j} -geodesic word such that $\overline{v_k}$ is shortest in the double coset class $C\overline{w_k}C$. Lemma 2.2 implies that $x_k v_k z_k$ is λ -quasigeodesic with respect to d_{X_j} where $\lambda > 0$ does not depend on g. Obviously, $l(w_k) \leq l(x_k v_k z_k)$ since w_k is d_{X_j} -geodesic. Put

$$w' = x_0 v_0 z_0 x_1 v_1 z_1 \dots x_s v_s z_s.$$

Then $l(w) \leq l(w')$.

Notice that by Lemma 2.2 $\overline{x_k v_k}$ is at most K_1 -away from the shortest element in the coset class $\overline{w_k}C$ and $\overline{v_k z_k}^{-1}$ is at most K_1 away from the inverse of the shortest element in $C\overline{w_k}$ (here $K_1 > 0$ is a constant independent of g). Therefore there is $K_2 > 0$, depending on N but independent of g, such that

$$l_{\mathcal{C}}(\overline{z_{k-1}x_k}) \ge l_{\mathcal{C}}(\overline{z_{k-1}}) + l_{\mathcal{C}}(\overline{x_k}) - K_2.$$

Take y_k to be a $d_{\mathcal{C}}$ -geodesic representative of $\overline{z_{k-1}x_k}$. Then $l(z_{k-1}x_k) \leq l(y_k) + K_2$. Replace each $z_{k-1}x_k$ by y_k in w' to get

$$w'' = x_0 v_0 y_1 v_1 y_2 \dots y_s v_s z_s.$$

Then $l(w') \leq l(w'') + sK_2 \leq l(w'') + l(w'')K_2 = (K_2+1)l(w'')$ and therefore $l(w) \leq l(w') \leq (K_2+1)l(w'')$. Express $\overline{v_s z_s} \in A_j - C$ as $\overline{q_s u_s}$ where q_s is $d_{\mathcal{C}}$ -geodesic, u_s is X_j -geodesic and $\overline{u_s}$ is shortest with respect to d_{X_j} in the coset class $C\overline{v_s z_s}$. Then express $\overline{v_{s-1}y_s q_s}$ as $\overline{q_{s-1}u_{s-1}}$ where q_{s-1} is $d_{\mathcal{C}}$ -geodesic, u_{s-1} is X_j -geodesic and $\overline{u_{s-1}}$ is shortest with respect to $d_{X_{-j}}$ in the coset class $C\overline{v_{s-1}y_s q_s}$. And so on. Finally, we rewrite $\overline{x_0v_0y_1q_1} \in A_i - C$ as $\overline{u_0}$ where u_0 is d_{X_i} -geodesic. Recall that $l_{\mathcal{C}}(\overline{q_k}) \leq K_1$. Therefore there is $K_3 > 0$ independent of g such that

$$l(v_k y_{k+1}) \le K_3 l(u_k), \quad k = 1, \dots, s-1$$

 $l(v_s z_s) \le K_3 l(u_s)$
 $l(x_0 v_0 y_1) \le K_3 l(u_0).$

Put $u = u_0 \dots u_s$. Then $l(w'') \leq K_3 l(u)$ and therefore $l(w) \leq K_3 (K_2 + 1) l(u)$. It is clear from the construction that $\overline{u} = g$ and that u satisfies the requirements of Lemma 3.1(1) and therefore it is K-quasigeodesic with respect to $d_{\mathcal{G}}$. Thus $l(w) \leq K_3 (K_2 + 1) l(u) \leq K K_3 (K_2 + 1) l_{\mathcal{G}}(g) + K K_3 (K_2 + 1)$ This completes the proof of Lemma 3.1.

Proof of Theorem B.

Suppose H is quasiconvex in G. Then for every $g \in G$ and $i = \pm 1$ the subgroups $g^{-1}Hg$ and A_i are quasiconvex in G. Therefore their intersection $g^{-1}Hg \cap A_i$ is quasiconvex in G. Since A_i is quasiconvex in G and $g^{-1}Hg \cap A_i \leq A_i$ this implies that $g^{-1}Hg \cap A_i$ is quasiconvex in A_i .

From now on we assume that H is a finitely generated subgroup of G such that for any $g \in G$ and $i = \pm 1$ the subgroup $g^{-1}Hg \cap A_i$ is quasiconvex in A_i . If H is conjugate in G to a subgroup of A_i then H is quasiconvex in a conjugate of A_i and so in G. Thus we may assume that H is not conjugate to a subgroup of A_i .

We recall some notations from section 1 which will be used here. For $i = \pm 1$ we have a finite generating set X_i of A_i which contains a finite generating set C of C. Then $\mathcal{G} = X_1 \cup X_{-1}$ is a finite generating set for G. In section 1 we constructed a language L_i over X_i such that $T_i = \overline{L_i}$ is a right transversal for C in A_i . We also denoted by \hat{T} the Bass-Serre tree corresponding to presentation (7). There are two distinguished vertices $d_1 = A_1$ and $d_{-1} = A_{-1}$ in \hat{T} . Every positive edge of \hat{T} is labelled by an element of T_i . Each vertex v of \hat{T} has the form $s_v A_i$ where s_v is the label of a reduced path in \hat{T} from d_1 to v.

There is a minimal *H*-invariant subtree *T* of \hat{T} . Since *H* is not conjugate to a subgroup of A_i , the tree *T* has at least one edge. We constructed the "fundamental domain" *Y* for the action of *H* on *T* and a finite subtree Y_1 of *Y* which define the algebraic structure of *H* as the fundamental group of a graph of groups.

By conjugating H we may assume that (d_1, d_{-1}) is an edge of Y_1 . (Notice that a conjugate H_1 of H is quasiconvex in G if and only if H is quasiconvex in G. Besides, H_1 still satisfies the property that $g^{-1}H_1g \cap A_i$ is quasiconvex in A_i for each $g \in G$ and $i = \pm 1$.)

In section 1 we defined a finite graph of groups \mathbb{B} such that Y_1 is the maximal subtree of B and H has the structure of the fundamental group of a graph of groups:

$$H = \pi_1(\mathbb{B}, Y_1) \tag{8}$$

We make several immediate observations about H.

Lemma 3.2. The group H is word hyperbolic.

Proof. Notice first that each vertex group B_v of \mathbb{B} is word hyperbolic. Indeed, for $v = s_v A_i \in V\mathbb{B} = VY_1$ we have $B_v = s_v A_i s_v^{-1} \cap H \cong A_i \cap s_v^{-1} H s_v = A_v$. We know that $A_i \cap s_v^{-1} H s_v$ is quasiconvex in A_i and thus word hyperbolic. Thus we know that H is the fundamental group of a finite graph of groups with virtually cyclic edge groups and word hyperbolic vertex groups. By the results of M.Bestvina and M.Feign [5] and O.Kharlampovich and A.Myasnikov [13], such H is word hyperbolic if and only if it does not contain Baumslag-Solitar subgroups. But H is a subgroup of G which is word hyperbolic and so does not contain Baumslag-Solitar subgroups. Therefore H is word hyperbolic.

We now return to the proof of Theorem B. For each $v = s_v A_i \in VY_1 = VB$ we fix a finite generating set R_v for A_v and a finite generating set $Z_v = s_v R_v s_v^{-1}$ for $B_v = s_v A_v s_v^{-1}$. Recall that each edge $e \in E(B-T)$ is identified with an element $\rho_i(t)\rho_{-i}^{-1}(t) \in H$. Thus $Z = \bigcup_{v \in VY_1} Z_v \bigcup E(B-T)$ is a finite generated set for H.

Each $e \in EY_1$ represents the trivial element of H so that $Z' = Z \cup EB$ is also a finite generating set for H. Let $h \in H \cap C$. Then $h \in B_{d_1} = A_1 \cap H$. Since B_{d_1} is quasiconvex in H and A_1 is quasiconvex in G, there is a constant $\lambda_0 > 0$ such that for any $g \in B_{d_1}$ we have $l_Z(g) \leq \lambda_0 l_G(g) + \lambda_0$. In particular, $l_Z(h) \leq \lambda_0 l_G(h) + \lambda_0$. We want to establish a similar inequality for the case when $h \notin C$.

So assume $h \in H-C$. By Proposition B there is a K'-quasigeodesic word W with respect to $d_{Z'}$ such that W corresponds to the normal form of h with respect to presentation (8) and K' is a constant independent of h. Thus W has the form

$$W = e_1 \dots e_k U_1 e_{k+1} \dots e_{t-1} U_q e_t \dots e_s$$

where

$$p = e_1, \dots, e_k, \overline{U_1}, e_{k+1}, \dots, e_{t-1}, \overline{U}_q, e_t, \dots, e_s$$

is the normal form for h with respect to presentation (8).

Recall that each $\overline{U_i}$ is either an edge of $B - Y_1$ or a nontrivial element of a vertex group of $B, e_i \in EY_1$ and p is a loop at the basepoint d_1 in the graph of groups \mathbb{B} representing H. Recall also that p contains no backtrackings and that no nontrivial element of a vertex group can be "pulled to the left".

Then there is a normal form of h with respect to presentation (7)

$$h = v_1 \dots v_r$$

satisfying the requirements of Proposition A.

Thus by Proposition A(ii) the syllable length r of h is at least q_1 where q_1 in the number of those U_i which represent edges of $B - Y_1$. We also know that for each core element $v_{i_k} \in A_i$ there is a corresponding $U_{j_k} \in B_{v_k}$ such that $l_{Z_{v_k}}(U_{j_k}) \leq K_0 l_{X_i}(\overline{v_{i_k}})$ where $K_0 > 0$ is some constant independent of h. For each $v_k \in A_i, k = 1, \ldots, r$ take a d_{X_i} -geodesic word \hat{v}_k representing v_k . Put $\hat{v} = \hat{v}_1 \ldots \hat{v}_r$.

The word $U = U_1 \dots U_q$ is obtained from W by deleting some pieces representing identity whose length is bounded by the number of edges in Y_1 . Thus U is a K_1 -quasigeodesic with respect to d_Z for some $K_1 > 0$ independent of h. Then

$$l(U) = m_1 + \sum l_{Z_{v_k}}(U_{j_k}) \le n + K_0 l(\hat{v}) \le l(\hat{v}) + K_0 l(\hat{v}) = (K_0 + 1)l(\hat{v})$$
(9)

Calim. We claim that there is a number N > 0 independent of h such that the following holds. Suppose $1 \le k < n-1$ and $v_k \in A_j - C$, $v_{k+1} \in A_{-j} - C$. Express v_k as $p_k z_k$ where $z_k \in C$ and p_k is shortest with respect to d_{X_j} in the coset class $v_k C$. Express v_{k+1} as $x_k t_k$ where $x_k \in C$ and t_k is shortest with respect to $d_{X_{-j}}$ in the coset class Cv_{k+1} . Then

$$l_{\mathcal{C}}(z_k x_k) \ge l_{\mathcal{C}}(z_k) + l_{\mathcal{C}}(x_k) - N \tag{10}$$

It is obvious that (10) is satisfied when C is finite. So assume C is infinite. Since C is virtually cyclic and infinite, there is a element $c \in C$ of infinite order and a constant P > 0 such that for any $c' \in C$ there are integers n, m and elements $c_1, c_2 \in C$ such that $l_{\mathcal{C}}(c_1), l_{\mathcal{C}}(c_2) \leq P$ and $c' = c^n c_1 = c_2 c^m$.

It is clear that $l_{\mathcal{C}}(z_k) + l_{\mathcal{C}}(x_k) - l_{\mathcal{C}}(z_k x_k)$ is bounded when at least one of v_k , v_{k+1} is not a core element because the lengths of syllables which are not core elements are bounded by 2K (see Proposition A(iii)). Assume now that they are both core elements. Thus both v_k and v_{k+1} correspond to the vertex group elements in U. Proposition A shows that it is only possible in the following three cases.

Case 1. There is $U_s = s_v b s_v^{-1}$ and $U_{s+1} = s_w a s_w^{-1}$ where $v = s_v A_i \in VY_1$, $w = s_v b_1 A_{-i}$, $b_1 \in T_i$, $b \in A_i$, $a \in A_{-i}$ and $v_k = v_{i_s} = f b b_1$, $v_{k+1} = v_{i_{s+1}} = a f'$ where f, f' have length at most K.

Case 2. There is $U_s = s_w b s_w^{-1} = s_v b_1 b b_1^{-1} s_v^{-1}$ and $U_{s+1} = s_v a s_v^{-1}$, where $w = s_w A_{-i} = s_v b_1 A_i$, $v = s_v A_i$, $b_1 \in T_i$, $b \in A_{-i} \cap A_w$, $a \in A_i \cap A_v$ and $v_k = v_{i_s} = fb$, $v_{k+1} = v_{i_{s+1}} = b_1 a f'$ where f, f' have length at most K.

We will treat Case 1 and it will be clear that Case 2 is exactly analogous. So assume Case 1 takes place. Recall that $b \in A_v$, $a \in A_w$ and that the edge group of \mathbb{B} corresponding to the edge (v, w) is $s_v b_1(A_w \cap C)b_1^{-1}s_v^{-1}$.

Recall that $v_k = v_{i_s} = fbb_1$, $v_{k+1} = v_{i_{s+1}} = af_1$, $U_s = s_v bs_v^{-1}$ and $U_{s+1} = s_w as_w^{-1} = s_v b_1 ab_1^{-1} s_v^{-1}$. Besides we know that the lengths of f and f_1 are bounded by K. We have $v_k = p_k z_k$ where $z_k \in C$ and p_k is shortest with respect to d_{X_j} in the coset class $v_k C$. Likewise, $v_{k+1} = x_k t_k$ where $x_k \in C$ and t_k is shortest with respect to $d_{X_{-j}}$ in the coset class Cv_{k+1} .

Recall that $U = U_1 \dots U_n$ is a K_1 -quasigeodesic representative of h in the d_Z -metric where K_1 does not depend on h. In particular $U_s U_{s+1}$ is K_1 -quasigeodesic. Observe that U_s is a word over $Z_v = s_v R_v s_v^{-1}$ and U_{s+1} is a word over $Z_w = s_v b_1 R_w b_1^{-1} s_v^{-1}$. Recall also that $\overline{U_s} = s_v b s_v^{-1}$ and $\overline{U_{s+1}} = s_w a s_w^{-1}$. Let α be a d_{R_w} -geodesic representative of b.

Express x_k as $x_k = c^n c_a$ and $z_k = c_b c^m$ where $l_{\mathcal{C}}(c_a), l_{\mathcal{C}}(c_b) \leq P$. Find a d_{X_i} -geodesic representative u_b of $p_k c_b$ and a $d_{X_{-i}}$ -geodesic representative u_a of $c_a t_k$. Then $v_k = fbb_1 = \overline{u_b c^m}$ and $v_{k+1} = af_1 = \overline{c^n u_a}$. Moreover, since the lengths of c_a and c_b are bounded, Lemma 2.2 implies that there is $\lambda > 0$ independent of h such that $u_b c^m$ and $c^n u_a$ are λ -quasigeodesic in d_{X_i} and $d_{X_{-i}}$ respectively.

We may also assume that λ is big enough so that any R_v -geodesic word defines a λ -quasigeodesic in d_{X_i} and any R_w -geodesic word defines a λ -quasigeodesic in $d_{X_{-i}}$.

Thus there is $\epsilon > 0$ independent of h such that

(i) for any terminal segment $c^{m'}$ of $u_b c^m$ there is a terminal segment β' of β with $l_{X_i}(\overline{\beta'} b_1 c^{-m'}) \leq \epsilon$ and

(ii) for any initial segment $c^{n'}$ of $c^n u_a$ there is an initial segment α' of α such that $l_{X_{-i}}(\overline{\alpha'}^{-1}c^{n'}) \leq \epsilon$.

Let N_1 be the maximal l_Z -length of those elements of H whose \mathcal{G} -length is at most $2l_{\mathcal{G}}(s_v) + 2\epsilon$. Let $K_2 > 0$ be such that for any integer $j \ l_{X_1}(c^j) \ge K_2|j|$ and $l_{X_{-1}}(c^j) \ge K_2|j|$.

Suppose

$$|n| + |m| - |n + m| > \frac{\lambda(K_1 N_1 + K_1 + 2\lambda)}{K_2}.$$

Then there is a terminal segment c^{l} of $u_{b}c^{m}$ and an initial segment c^{-l} of $c^{n}u_{a}$ such that

$$|l| > \frac{\lambda(K_1N_1 + K_1 + 2\lambda)}{2K_2}$$

Thus there is a terminal segment β_1 of β and an initial segment α_1 of α such that $l_{X_i}(\overline{\beta_1}b_1c^{-l}) \leq \epsilon$ and $l_{X_{-i}}(\overline{\alpha_1}^{-1}c^{-l}) \leq \epsilon$.

Therefore $l_{\mathcal{G}}(s_v\overline{\beta_1}s_v^{-1} \cdot s_v b_1\overline{\alpha_1}b_1^{-1}s_v^{-1}) \leq 2l_{\mathcal{G}}(s_v) + l_{\mathcal{G}}(\overline{\beta_1}b_1\overline{\alpha_1}) \leq 2l_{\mathcal{G}}(s_v) + 2\epsilon + l_{\mathcal{G}}(c^lc^{-l}) = 2l_{\mathcal{G}}(s_v) + 2\epsilon$. Thus $l(\beta_1\alpha_1) \leq K_1 l_Z(s_v\overline{\beta_1}s_v^{-1} \cdot s_v b_1\overline{\alpha_1}b_1^{-1}s_v^{-1}) + K_1 \leq K_1 N_1 + K_1$ since U is a K_1 -quasigeodesic with respect to d_Z .

On the other hand $l(\alpha_1\beta_1) = l(\alpha_1) + l(\beta_1) \ge (1/\lambda)(l_{X_i}(c^l) + \epsilon) - \lambda + (1/\lambda)(l_{X_{-i}}(c^{-l}) + \epsilon) - \lambda \ge (2K_2|l|/\lambda) - (2\epsilon/\lambda) > K_1N_1 + K_1$ by the choice of l. This gives us a contradiction. So $|n| + |m| - |n + m| \le \frac{\lambda(K_1N_1 + K_1 + 2\lambda)}{K_2}$ and (10) follows.

Case 3. There is $U_s = s_v b s_v^{-1}$, $U_{s+1} = \rho_i(g) \rho_{-i}(g)^{-1} = s_v a_i a_{-i} s_w^{-1}$ and $U_{s+2} = s_w a s_w^{-1}$ such that $v_k = v_{i_s} = f b a_i$ and $v_{k+1} = v_{i_{s+2}} = a_{-i} a f'$ where $f, b, a_i \in A_i, b_{-i}, b, f' \in A_{-i}$ and $f, f' \in \Sigma$.

We have $v_k = p_k z_k$ where $z_k \in C$ and p_k is shortest with respect to d_{X_i} in the coset class $v_k C$. Likewise, $v_{k+1} = x_k t_k$ where $x_k \in C$ and t_k is shortest with respect to $d_{X_{-i}}$ in the coset class Cv_{k+1} .

Recall that $U = U_1 \dots U_n$ is a K_1 -quasigeodesic representative of h with respect to d_Z . In particular $U_s U_{s+1} U_{s+2}$ is K_1 -quasigeodesic. Observe that U_s is a word over $Z_v = s_v R_v s_v^{-1}$ and U_{s+2} is a word over $Z_w = s_w R_w s_w^{-1}$. Recall also that $\overline{U_s} = s_v b s_v^{-1}$ and $\overline{U_{s+2}} = s_w a s_w^{-1}$. Let β be a d_{R_v} -geodesic representative of b and let α be a d_{R_w} -geodesic representative of a.

Let $x_k = c^n c_a$ and $z_k = c_b c^m$ where $l_{\mathcal{C}}(c_a), l_{\mathcal{C}}(c_b) \leq P$. Find a d_{X_i} -geodesic representative u_b of $p_k c_b$ and a $d_{X_{-i}}$ -geodesic representative u_a of $c_a t_k$. Then $v_k = f b a_i = \overline{u_b c^m}$ and $v_{k+1} = a_{-i}^{-1} a f_1 = \overline{c^n u_a}$. Moreover, there is $\lambda > 0$ independent of h such that $u_b c^m$ and $c^n u_a$ are λ -quasigeodesic in d_{X_i} and $d_{X_{-i}}$ respectively.

We may also assume that λ is big enough so that any R_v -geodesic word defines a λ -quasigeodesic in d_{X_i} and any R_w -geodesic word defines a λ -quasigeodesic in $d_{X_{-i}}$.

Thus there is $\epsilon > 0$ independent of h such that

- (i) for any terminal segment $c^{m'}$ of $u_b c^m$ there is a terminal segment β' of β with $l_{X_i}(\overline{\beta'}a_i c^{-m'}) \leq \epsilon$ and
- (ii) for any initial segment $c^{n'}$ of $c^n u_a$ there is an initial segment α' of α such that $l_{X_{-i}}(\overline{\alpha'}^{-1}a_{-i}^{-1}c^{n'}) \leq \epsilon$.

Let N_1 be the maximal l_Z -length of those elements of H whose \mathcal{G} -length is at most $2l_{\mathcal{G}}(s_v) + 2\epsilon$. Let $K_2 > 0$ be such that for any integer j we have $l_{X_1}(c^j) \ge K_2|j|$ and $l_{X_{-1}}(c^j) \ge K_2|j|$. Suppose

$$|n| + |m| - |n + m| > \frac{\lambda(K_1N_1 + K_1 + 2\lambda)}{K_2}$$

Then there is a terminal segment c^{l} of $u_{b}c^{m}$ and an initial segment c^{-l} of $c^{n}u_{a}$ such that

$$|l| > \frac{\lambda(K_1N_1 + K_1 + 2\lambda)}{2K_2}.$$

Thus there is a terminal segment β_1 of β and an initial segment α_1 of α such that $l_{X_i}(\overline{\beta_1}a_ic^{-l}) \leq \epsilon$ and $l_{X_{-i}}(\overline{\alpha_1}^{-1}a_{-i}^{-1}c^{-l}) \leq \epsilon$.

Therefore $l_{\mathcal{G}}(s_v\overline{\beta_1}s_v^{-1} \cdot s_va_ia_{-i}^{-1}s_w^{-1} \cdot s_v\overline{\alpha_1}s_v^{-1}) \leq 2l_{\mathcal{G}}(s_v) + l_{\mathcal{G}}(\overline{\beta_1}a_ia_{-i}^{-1}\overline{\alpha_1}) \leq 2l_{\mathcal{G}}(s_v) + 2\epsilon + l_{\mathcal{G}}(c^lc^{-l}) = 2l_{\mathcal{G}}(s_v) + 2\epsilon$. Thus $l(\beta_1) + 1 + l(\alpha_1) \leq K_1 l_Z(s_v\overline{\beta_1}s_v^{-1} \cdot s_va_ia_{-i}^{-1}s_w^{-1} \cdot s_w\overline{\alpha_1}bs_w^{-1}) + K_1 \leq K_1 N_1 + K_1$ since U is a K_1 -quasigeodesic with respect to d_Z .

On the other hand $l(\alpha_1) + 1 + l(\beta_1) \ge (1\lambda)(l_{X_i}(c^l) + \epsilon) - \lambda + 1 + (1\lambda)(l_{X_{-i}}(c^{-l}) + \epsilon) - \lambda \ge (2K_2|l|/\lambda) + (2\epsilon/\lambda) - 2\lambda + 1 > K_1N_1 + K_1$ by the choice of l. This gives us a contradiction. So $|n| + |m| - |n + m| \le \frac{\lambda(K_1N_1 + K_1 + 2\lambda)}{K_2}$ and (10) follows.

Thus we have verified (10) and established the Claim.

Therefore by Lemma 3.1(2) there is a constant $K_3 > 0$ independent of h such that

$$l_{\mathcal{G}}(h) \ge K_2 l(\hat{v}) - K_2 \tag{11}$$

Recall that U is K_1 -quasigeodesic with respect to d_Z . Thus (9) and (11) imply that

$$l_{\mathcal{G}}(h) \ge K_2 l(\hat{v}) - K_2 \ge (K_2/(K_0 + 1))l(U) - K_2 \ge (K_2/(K_0 + 1))K_1 l_Z(h) - K_2 - (K_2/(K_0 + 1))K_1$$

which implies that H is quasiconvex in G.

This completes the proof of Theorem B.

Theorem A. Suppose $G = A_1 *_C A_{-1}$ is a word hyperbolic group where C is virtually cyclic and A_1 , A_{-1} have property (Q). Then G has property (Q)

Proof. Let H be a finitely generated subgroup of G. Since C is virtually cyclic, for any $g \in G$, $i = \pm 1$ the group $g^{-1}Hg \cap A_i$ is finitely generated. Since A_i has property (Q), the subgroup $g^{-1}Hg \cap A_i$ is quasiconvex in A_i . Therefore by Theorem A the subgroup H is quasiconvex in G.

Corollary 3.3 (c.f. Corollary 1 from the Introduction). Let $G = A_1 *_C A_{-1}$ where the groups A_1, A_{-1} belong to the class $(Q), A_1$ is torsion-free and C is a maximal cyclic subgroup of A_1 . Then G has property (Q).

Proof. By the results of [8], the subgroup C is malnormal in A_1 . Moreover, since C is cyclic, it is quasiconvex in A_1 and in A_{-1} . Therefore by the combination theorem for word hyperbolic groups (see [5], [13], [17]) the group G is word hyperbolic. Theorem A implies that G has property (Q).

Corollary 3.4 (c.f. Corollary 2 from the Introduction). Let $G = A_1 *_C A_{-1}$ where C is finite and A_1 , A_{-1} have property (Q). Then G has property (Q).

Proof. It is easy to show (see [9]) that G is word hyperbolic. The group C is finite and therefore it is virtually cyclic. Theorem A implies that G has property (Q).

Corollary 3.5 (c.f. Corollary 3 from the Introduction). Let G be a torsion-free hyperbolic group with property (Q) (e.g. finitely generated free group, hyperbolic surface group etc). Let $G^{\mathbb{Q}}$ be the tensor \mathbb{Q} -completion of G where \mathbb{Q} is the ring of rational numbers. Then

- (1) $G^{\mathbb{Q}}$ is a locally (Q)-group that is any finitely generated subgroup of $G^{\mathbb{Q}}$ is word hyperbolic and has property (Q);
- (2) $G^{\mathbb{Q}}$ has the Howson property that is the intersection of any two finitely generated subgroups of G is finitely generated
- (3) if H_1 and H_2 are infinite commensurable subgroups of G, that is the intersection $H = H_1 \cap H_2$ has finite index in both H_1 and H_2 , then H has finite index in their join $E = gp(H_1 \cup H_2)$.

Proof.

All maximal abelian subgroups of a torsion free word hyperbolic group G are infinite cyclic and malnormal [13]. Therefore, by the results of A.Myasnikov and V.Remeslennikov [15], there is a sequence of groups $G = G_0 \leq G_1 \leq \cdots \leq G_n \leq \ldots$ such that

- (a) $G^{\mathbb{Q}} = \underset{n \ge 0}{\cup} G_n$
- (b) for each $n \ge 0$ we have $G_{n+1} = G_n \underset{g_n = x^{k_n}}{*} < x >$ where $g_n \in G_n$ is nontrivial and not a proper power.

We claim that each G_n is torsion-free, word hyperbolic and has property (Q). Indeed, it is true for $G = G_0$. Suppose the claim has been proven for G_n , $n \ge 0$. Then the cyclic subgroup $\langle g_n \rangle$ is malnormal in G_n since G_n is torsion-free word hyperbolic and g_n is not a proper power (see [13]). The infinite cyclic group $\langle x \rangle$ is word hyperbolic and has property (Q). The group G_n has property (Q) by inductive hypothesis. Thus by Corollary 3.3 the group G_{n+1} is word hyperbolic and has property (Q). Obviously, G_{n+1} is torsion-free. This concludes the inductive step and the claim is proved.

Any finitely generated subgroup H of $G^{\mathbb{Q}}$ is contained in some G_n , $n \ge 0$ and therefore H has property Q. This proves (1). Moreover, if H_1 and H_2 are finitely generated subgroups of $G^{\mathbb{Q}}$ then there is $n \ge 0$ such that $H_1, H_2 \le G_n$. The group G_n has the Howson property since it belongs to class (Q). Thus $H_1 \cap H_2$ is finitely generated. Moreover, if the subgroups H_1 and H_2 of G_n are commensurable then by the result of [14] their intersection has finite index in their join. This proves (2) and (3).

Corollary 3.6 (c.f. Corollary 5 from the Introduction). Suppose G is a one-relator group $G = \langle x_1, \ldots, x_k, y_1, \ldots, y_s | vu = 1 \rangle$ where v is a nontrivial freely reduced word in x_1, \ldots, x_k and u is a nontrivial freely reduced word in y_1, \ldots, y_s which is not a proper power in the free group $F(y_1, \ldots, y_s)$. Then G has property (Q).

Proof. Finitely generated free groups $F(x_1, \ldots, x_k)$ and $F(y_1, \ldots, y_s)$ have property (Q) (see [19]). The group G is an amalgamated free product $G = F(x_1, \ldots, x_k) \underset{u^{-1}=v}{*} F(y_1, \ldots, y_s)$. Therefore by Corollary 3.3 the group G has property (Q).

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL AND UNIVERSITY CENTER OF THE CITY UNIVERSITY OF NEW YORK, 33 WEST 42-ND STREET, NEW YORK, NY10036

E-mail address: ilya@groups.sci.ccny.cuny.edu