EXPLICIT BRACKET IN THE EXCEPTIONAL SIMPLE LIE SUPERALGEBRA $\mathfrak{cvect}(0|3)_*$

IRINA SHCHEPOCHKINA AND GERHARD POST

ABSTRACT. This note is devoted to a more detailed description of one of the five simple exceptional Lie superalgebras of vector fields, $\operatorname{cvect}(0|3)_*$, a subalgebra of $\operatorname{vect}(4|3)$. We derive differential equations for its elements, and solve these equations. Hence we get an exact form for the elements of $\operatorname{cvect}(0|3)_*$. Moreover we realize $\operatorname{cvect}(0|3)_*$ by "glued" pairs of generating functions on a (3|3)-dimensional periplectic (odd symplectic) supermanifold and describe the bracket explicitly.

1991 Mathematics Subject Classification: 17A70. Keywords: Lie superalgebra, Cartan prolongation.

March 3, 1997

Introduction

V. Kac [3] classified simple finite-dimensional Lie superalgebras over \mathbb{C} . Kac further conjectured [3] that passing to infinite-dimensional simple Lie superalgebras of vector fields with polynomial coefficients we only acquire the straightforward analogues of the four well-known Cartan series: $\mathfrak{vect}(n)$, $\mathfrak{svect}(n)$, $\mathfrak{h}(2n)$ and $\mathfrak{k}(2n+1)$ (of all, divergence-free, Hamiltonian and contact vector fields, respectively, realized on the space of dimension indicated).

It soon became clear [4], [1], [5], [6] that the actual list of simple vectoral Lie superalgebras is much larger. Several new series were found.

Next, exceptional vectoral algebras were discovered [8], [9]; for their detailed description see [10], [2]. All of them are obtained with the help of a Cartan prolongation or a generalized prolongation, cf. [8]. This description is, however, not always satisfactory; a more succinct presentation (similar to the one via generating functions for the elements of \mathfrak{h} and \mathfrak{k}) and a more explicit formula for their brackets is desirable.

The purpose of this note is to give a more lucid description of one of these exceptions, $\mathfrak{cvect}(0|3)_*$. In particular we offer a multiplication table for $\mathfrak{cvect}(0|3)_*$ that is simpler than previous descriptions, by use of "glued" pairs of generating functions for the elements of $\mathfrak{cvect}(0|3)_*$.

This note can be seen as a supplement to [10]. To be self-contained and to fix notations we introduce some basic notions in section 0.

Throughout, the ground field is \mathbb{C} .

I.Shch. expresses her thanks: to D. Leites for rising the problem and help; to RFBR grant 95-01-01187 and NFR (Sweden) for part of financial support; University of Twente and Stockholm University for hospitality; to P. Grozman whose computer experiments encouraged her to carry on with unbearable calculations.

§0. Background

0.1. We recall that a superspace V is a $\mathbb{Z}/2$ -graded space; $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The elements of $V_{\bar{0}}$ are called *even*, those of $V_{\bar{1}}$ odd. When considering an element $x \in V$, we will always assume that x is homogeneous, i.e. $x \in V_{\bar{0}}$ or $x \in V_{\bar{1}}$. We write $p(x) = \bar{i}$ if $x \in V_{\bar{i}}$. The superdimension of V is (n|m), where $n = \dim(V_{\bar{0}})$ and $m = \dim(V_{\bar{1}})$.

For a superspace V, we denote by $\Pi(V)$ the same superspace with the shifted parity, i.e., $\Pi(V_{\bar{i}}) = V_{\bar{i}+\bar{1}}$.

0.2. Let $x = (u_1, \ldots, u_n, \xi_1, \ldots, \xi_m)$, where u_1, \ldots, u_n are even indeterminates and ξ_1, \ldots, ξ_m odd indeterminates. In the associative algebra $\mathbb{C}[x]$ we have that $x \cdot y = (-1)^{p(x)p(y)}y \cdot x$ (by definition) and hence $\xi_i^2 = 0$ for all i. The derivations $\mathfrak{der}(\mathbb{C}[x])$ of $\mathbb{C}[x]$ form a Lie superalgebra; its elements are vector fields. These polynomial vector fields are denoted by $\mathfrak{vect}(n|m)$. Its elements are represented as

$$D = \sum_{i} f_{i} \frac{\partial}{\partial u_{i}} + \sum_{j} g_{j} \frac{\partial}{\partial \xi_{j}}$$

where $f_i \in \mathbb{C}[x]$ and $g_j \in \mathbb{C}[x]$ for all i, j = 1..n. We have $p(D) = p(f_i) = p(g_j) + \bar{1}$ and the Lie product is given by the commutator

$$[D_1, D_2] = D_1 D_2 - (-1)^{p(D_1)p(D_2)} D_2 D_1.$$

On the vector fields we have a map, div : $\mathfrak{vect}(n|m) \to \mathbb{C}[x]$, defined by

$$\operatorname{div} D = \operatorname{div} \left(\sum_{i=1}^n f_i \frac{\partial}{\partial u_i} + \sum_{j=1}^n g \frac{\partial}{\partial \xi_j} \right) = \sum_{i=1}^n \frac{\partial f_i}{\partial u_i} - (-1)^{p(D)} \sum_{j=1}^n \frac{\partial g_j}{\partial \xi_j}.$$

A vector field D that satisfies $\operatorname{div} D = 0$ is called *special*. The linear space of special vector fields in $\operatorname{\mathfrak{vect}}(n|m)$ forms a Lie superalgebra, denoted by $\operatorname{\mathfrak{svect}}(n|m)$.

0.3. Next we discuss the Lie superalgebra of Leitesian vector fields $\mathfrak{le}(n)$. It consists of the elements $D \in \mathfrak{vect}(n|n)$ that annihilate the 2-form $\omega = \sum_i du_i d\xi_i$. Hence $\mathfrak{le}(n)$ is an odd superanalogon of the Hamiltonian vector fields (in which case $\omega = \sum_i dp_i dq_i$). Similar to the Hamiltonian case, there is a map $\mathrm{Le}: \mathbb{C}[x] \to \mathfrak{le}(n)$, with $x = (u_1, \ldots, u_n, \xi_1, \ldots, \xi_n)$:

$$Le_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial u_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial u_i}\right)$$

Note that Le maps odd elements of $\mathbb{C}[x]$ to even elements of $\mathfrak{le}(n)$ and vice versa. Moreover $\operatorname{Ker}(\operatorname{Le}) = \mathbb{C}$. We turn $\mathbb{C}[x]$ (with shifted parity) into a Lie superalgebra with (Buttin) bracket $\{f,g\}$ defined by

$$\mathrm{Le}_{\{f,g\}} = [\mathrm{Le}_f, \mathrm{Le}_g]$$

A straightforward calculation shows that

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial u_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial u_i} \right).$$

This way $\Pi\mathbb{C}[x]/\mathbb{C} \cdot 1$ is a Lie superalgebra isomorphic to $\mathfrak{le}(n)$. We call f the generating function of Le_f. Here and throughout p(f) will denote the

parity in $\mathbb{C}[x]$, not in $\Pi\mathbb{C}[x]$. So p(f) is the parity of the number of ξ in a term of f.

0.4. The algebra $\mathfrak{le}(n)$ contains certain important subalgebras. First of all there is $\mathfrak{sle}(n)$, the space of special Leitesian vector fields:

$$\mathfrak{sle}(n) = \mathfrak{le}(n) \cap \mathfrak{svect}(n|n).$$

We have seen that if $D \in \mathfrak{le}(n)$ then $D = \operatorname{Le}_f$ for some $f \in \mathbb{C}[x]$. Now $D \in \mathfrak{sle}(n)$ iff f is harmonic in the following sense

$$\Delta(f) := \sum_{i=1}^{n} \frac{\partial^{2} f}{\partial u_{i} \partial \xi_{i}} = 0$$

Usually we simply say $f \in \mathfrak{sle}(n)$, identifying f and Le_f . This Δ satisfies the condition $\Delta^2 = 0$ and hence $\Delta : \mathfrak{le}(n) \to \mathfrak{sle}(n)$. The image $\Delta(\mathfrak{le}(n)) =: \mathfrak{sle}^{\circ}(n)$ is an ideal of codimension 1 on $\mathfrak{sle}(n)$. This ideal, $\mathfrak{sle}^{\circ}(n)$, can also be defined by the exact sequence

$$0 \longrightarrow \mathfrak{sle}^{\circ}(n) \longrightarrow \mathfrak{sle}(n) \longrightarrow \mathbb{C} \cdot \operatorname{Le}_{\xi_1 \dots \xi_n} \longrightarrow 0.$$

Note that if $\Phi = \sum u_i \xi_i$ and $f \in \mathfrak{sle}(n)$, then

$$\Delta(\Phi f) = (n + \deg_u f - \deg_{\xi} f) \cdot f$$

Let $\nu(f) = n + \deg_u f - \deg_{\xi} f$. Then $\nu(f) \neq 0$ iff $f \in \mathfrak{sle}^{\circ}(n)$. So on $\mathfrak{sle}^{\circ}(n)$ we can define the right inverse Δ^{-1} to Δ by the formula

$$\Delta^{-1}f = \frac{1}{\nu(f)}(\Phi f).$$

0.5. Cartan prolongs. We will repeatedly use Cartan prolongation. So let us recall the definition. Let \mathfrak{g} be a Lie superalgebra and V a \mathfrak{g} -module. Set $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_0 = \mathfrak{g}$ and for i > 0 define the i-th Cartan prolong \mathfrak{g}_i as the space of all $X \in \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1})$ such that

$$X(w_0)(w_1, w_2, \dots, w_i) = (-1)^{p(w_0)p(w_1)}X(w_1)(w_0, w_2, \dots, w_i)$$

for all $w_0, \ldots, w_i \in \mathfrak{g}_{-1}$.

The Cartan prolong (the result of Cartan's prolongation) of the pair (V, \mathfrak{g}) is $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \bigoplus_{i \ge -1} \mathfrak{g}_i$.

Suppose that the \mathfrak{g}_0 -module \mathfrak{g}_{-1} is faithful. Then

 $(\mathfrak{g}_{-1},\mathfrak{g}_0)_* \subset \mathfrak{vect}(n|m) = \mathfrak{der}(\mathbb{C}[x]), \text{ where } n = \dim(V_{\bar{0}}) \text{ and } m = \dim(V_{\bar{1}})$ and $x = (u_1, \dots, u_n, \xi_1, \dots, \xi_m)$. We have for $i \geq 1$

$$\mathfrak{g}_i = \{D \in \mathfrak{vect}(n|m) : \deg D = i, [D, X] \in \mathfrak{g}_{i-1} \text{ for any } X \in \mathfrak{g}_{-1}\}.$$

The Lie superalgebra structure on $\mathfrak{vect}(n|m)$ induces one on $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$. This way the commutator of vector fields [g,v], corresponds to the action $g \cdot v$, $g \in \mathfrak{g}$ and $v \in V$.

We give some examples of Cartan prolongations. Let $\mathfrak{g}_{-1} = V$ be an (n|m)-dimensional superspace and $\mathfrak{g}_0 = \mathfrak{gl}(n|m)$ the space of all endomorphisms of V. Then $(\mathfrak{g}_{-1},\mathfrak{g}_0)_* = \mathfrak{vect}(n|m)$. If one takes for \mathfrak{g}_0 only the supertraceless elements $\mathfrak{sl}(n|m)$, then $(\mathfrak{g}_{-1},\mathfrak{g}_0)_* = \mathfrak{svect}(n|m)$, the algebra of vector fields with divergence 0.

§1. The structure of $\mathfrak{vect}(0|3)_*$

1.1. In this note our primary interest is in a certain Cartan prolongation (denoted by $\mathfrak{vect}(0|3)_*$) and the extension $\mathfrak{cvect}(0|3)_*$ thereof. Here we will discuss $\mathfrak{vect}(0|3)_*$. Now $\mathfrak{vect}(0|3)_*$ is a short-hand notation for the Cartan prolongation with

$$V = \mathfrak{g}_{-1} = \Pi \Lambda(\eta_1, \eta_2, \eta_3) / \mathbb{C}$$
 and $\mathfrak{g}_0 = \mathfrak{der} V$

So V is a superspace of dimension (4|3), with

$$V_{\bar{0}} = \langle \eta_1 \eta_2 \eta_3, \eta_1, \eta_2, \eta_3 \rangle; \qquad V_{\bar{1}} = \langle \eta_2 \eta_3, \eta_3 \eta_1, \eta_1 \eta_2 \rangle$$

and dim $g_0 = (12|12)$.

The elements of \mathfrak{g}_{-1} and \mathfrak{g}_0 can be expressed as vector fields in $\mathfrak{vect}(4|3)$. Choosing

$$\eta_1 \eta_2 \eta_3 \simeq -\partial_y; \quad \eta_i \simeq -\partial_{u_i}; \quad \frac{\partial \eta_1 \eta_2 \eta_3}{\partial \eta_i} \simeq -\partial_{\xi_i}.$$

it is subject to straightforward verification that the elements of \mathfrak{g}_0 , expressed as elements of $\mathfrak{vect}(4|3)$ are of the form:

$$\begin{array}{ll} \partial_{\eta_1} \simeq -y \partial_{\xi_1} - \xi_2 \partial_{u_3} + \xi_3 \partial_{u_2} & -\eta_1 \partial_{\eta_1} \simeq u_1 \partial_{u_1} + \xi_2 \partial_{\xi_2} + \xi_3 \partial_{\xi_3} + y \partial_y \\ \partial_{\eta_2} \simeq -y \partial_{\xi_2} - \xi_3 \partial_{u_1} + \xi_1 \partial_{u_3} & -\eta_2 \partial_{\eta_2} \simeq u_2 \partial_{u_2} + \xi_1 \partial_{\xi_1} + \xi_3 \partial_{\xi_3} + y \partial_y \\ \partial_{\eta_3} \simeq -y \partial_{\xi_3} - \xi_1 \partial_{u_2} + \xi_2 \partial_{u_1} & -\eta_3 \partial_{\eta_3} \simeq u_3 \partial_{u_3} + \xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2} + y \partial_y \end{array}$$

$$\begin{array}{ll} \eta_1\partial_{\eta_2}\simeq -u_2\partial_{u_1}+\xi_1\partial_{\xi_2} & \eta_2\partial_{\eta_1}\simeq -u_1\partial_{u_2}+\xi_2\partial_{\xi_1} & \eta_1\eta_2\eta_3\partial_{\eta_1}\simeq -u_1\partial_y\\ \eta_2\partial_{\eta_3}\simeq -u_3\partial_{u_2}+\xi_2\partial_{\xi_3} & \eta_3\partial_{\eta_2}\simeq -u_2\partial_{u_3}+\xi_3\partial_{\xi_2} & \eta_1\eta_2\eta_3\partial_{\eta_2}\simeq -u_2\partial_y\\ \eta_3\partial_{\eta_1}\simeq -u_1\partial_{u_3}+\xi_3\partial_{\xi_1} & \eta_1\partial_{\eta_3}\simeq -u_3\partial_{u_1}+\xi_1\partial_{\xi_3} & \eta_1\eta_2\eta_3\partial_{\eta_3}\simeq -u_3\partial_y \end{array}$$

$$\eta_1 \eta_2 \partial_{\eta_3} \simeq -u_3 \partial_{\xi_3} \qquad \eta_1 \eta_2 \partial_{\eta_1} \simeq -u_1 \partial_{\xi_3} - \xi_2 \partial_y \qquad \eta_1 \eta_2 \partial_{\eta_2} \simeq -u_2 \partial_{\xi_3} + \xi_1 \partial_y \\
\eta_2 \eta_3 \partial_{\eta_1} \simeq -u_1 \partial_{\xi_1} \qquad \eta_2 \eta_3 \partial_{\eta_2} \simeq -u_2 \partial_{\xi_1} - \xi_3 \partial_y \qquad \eta_2 \eta_3 \partial_{\eta_3} \simeq -u_3 \partial_{\xi_1} + \xi_2 \partial_y \\
\eta_3 \eta_1 \partial_{\eta_2} \simeq -u_2 \partial_{\xi_2} \qquad \eta_3 \eta_1 \partial_{\eta_3} \simeq -u_3 \partial_{\xi_2} - \xi_1 \partial_y \qquad \eta_3 \eta_1 \partial_{\eta_1} \simeq -u_1 \partial_{\xi_2} + \xi_3 \partial_y$$

1.2. Now we will give a more explicit description of $\mathfrak{vect}(0|3)_*$. It will turn out that $\mathfrak{vect}(0|3)_*$ is isomorphic to $\mathfrak{le}(3)$ as Lie superalgebra; however considered as \mathbb{Z} -graded algebras we have to define a different grading. The \mathbb{Z} -graded Lie superalgebra $\mathfrak{le}(3;3)$ is $\mathfrak{le}(3)$ as Lie superalgebra with \mathbb{Z} -degree of D

$$D = \sum_{i} f_{i} \frac{\partial}{\partial u_{i}} + \sum_{j} g_{j} \frac{\partial}{\partial \xi_{j}}$$

the *u*-degree of f_i minus 1 (or the *u*-degree of g_i), i.e. deg $\xi_i = 0$.

Consider the map $i_1: \mathfrak{le}(3;3) \to \mathfrak{vect}(4|3)$ given by

a.) If f = f(u) then

$$i_1(\operatorname{Le}_f) = \operatorname{Le}_{\sum \frac{\partial f}{\partial u_i} \xi_j \xi_k - yf}$$

where y is treated as a parameter and $(i, j, k) \in A_3$ (even permutations of $\{1, 2, 3\}$).

b.) If $f = \sum f_i(u)\xi_i$ then

$$i_1(\operatorname{Le}_f) = \operatorname{Le}_f - \varphi(u) \sum \xi_i \partial_{\xi_i} + (-\varphi(u)y + \Delta(\varphi(u)\xi_1\xi_2\xi_3)) \partial_y$$

where $\varphi(u) = \Delta(f)$ and Δ as given in section 0.4.

c.) If $f = \psi_1(u)\xi_2\xi_3 + \psi_2(u)\xi_3\xi_1 + \psi_3(u)\xi_1\xi_2$ then

$$i_1(\operatorname{Le}_f) = -\Delta(f)\partial_y - \sum_{i=1}^3 \psi_i(u)\frac{\partial}{\partial \xi_i}.$$

d.) If $f = \psi(u)\xi_1\xi_2\xi_3$ then

$$i_1(\operatorname{Le}_f) = -\psi(u)\partial_y.$$

Note that i_1 preserves the \mathbb{Z} -degree. We have the following lemma.

1.3. Lemma. The map i_1 is an isomorphism of \mathbb{Z} -graded Lie superalgebras between $\mathfrak{le}(3;3)$ and $\mathfrak{vect}(0|3)_* \subset \mathfrak{vect}(4|3)$.

Proof. That i_1 is an embedding can be verified by direct computation. To prove that the image of i_1 is in $\mathfrak{vect}(0|3)_*$ it is enough to show that this is the case on the components $\mathfrak{le}(3;3)_{-1} \oplus \mathfrak{le}(3;3)_{0}$, i.e. on functions $f(u,\xi)$ of degree ≤ 1 with respect to u, as the Cartan prolongation is the biggest subalgebra \mathfrak{g} of $\mathfrak{vect}(4|3)$, with given \mathfrak{g}_{-1} and \mathfrak{g}_{0} . The proof that i_1 is surjective onto $\mathfrak{vect}(0|3)_*$ is given in corollary 4.6.

A generalized version of Lemma 1.3 can be found in [10] and [7]. It states that $\mathfrak{le}(n;n)$ and $\mathfrak{vect}(0|n)_*$ are isomorphic for all $n \geq 1$.

§2. The construction of $\mathfrak{cvect}(0|3)_*$

2.1. Let us describe a general construction, which leads to several new simple Lie superalgebras. Let $\mathfrak{u} = \mathfrak{vect}(m|n)$, let $\mathfrak{g} = (\mathfrak{u}_{-1}, \mathfrak{g}_0)_*$ be a simple Lie subsuperalgebra of \mathfrak{u} . Moreover suppose there exists an element $d \in \mathfrak{u}_0$ that determines an exterior derivation of \mathfrak{g} and has no kernel on \mathfrak{u}_+ . Let us study the prolong $\tilde{\mathfrak{g}} = (\mathfrak{g}_{-1}, \mathfrak{g}_0 \oplus \mathbb{C}d)_*$.

Lemma. Either $\tilde{\mathfrak{g}}$ is simple or $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}d$.

Proof. Let I be a nonzero graded ideal of $\tilde{\mathfrak{g}}$. The subsuperspace $(\operatorname{ad}\mathfrak{u}_{-1})^{k+1}a$ of \mathfrak{u}_{-1} is nonzero for any nonzero homogeneous element $a \in \mathfrak{u}_k$ and $k \geq 0$. Since $\mathfrak{g}_{-1} = \mathfrak{u}_{-1}$, the ideal I contains nonzero elements from \mathfrak{g}_{-1} ; by simplicity of \mathfrak{g} the ideal I contains the whole \mathfrak{g} . If, moreover, $[\mathfrak{g}_{-1}, \tilde{\mathfrak{g}}_1] = \mathfrak{g}_0$, then by definition of the Cartan prolongation $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}d$.

If, instead, $[\mathfrak{g}_{-1}, \tilde{\mathfrak{g}}_1] = \mathfrak{g}_0 \oplus \mathbb{C}d$, then $d \in I$ and since $[d, \mathfrak{u}_+] = \mathfrak{u}_+$, we derive that $I = \tilde{\mathfrak{g}}$. In other words, $\tilde{\mathfrak{g}}$ is simple.

As an example, take $\mathfrak{g} = \mathfrak{svect}(m|n)$; $\mathfrak{g}_0 = \mathfrak{sl}(m|n)$, $d = 1_{m|n}$. Then $(\mathfrak{g}_{-1}, \mathfrak{g}_0 \oplus \mathbb{C}d)_* = \mathfrak{vect}(m|n)$.

2.2. Definition. The Lie superalgebra $\operatorname{cvect}(0|3)_* \subset \operatorname{vect}(4|3)$ is the Cartan prolongation with $\operatorname{cvect}(0|3)_{-1} = \operatorname{vect}(0|3)_{-1}$ and $\operatorname{cvect}(0|3)_0 = \operatorname{vect}(0|3)_0 \oplus \mathbb{C}d$, with

$$d = \sum u_i \partial_{u_i} + \sum \xi_i \partial_{\xi_i} + y \partial_y.$$

If now

$$f = \sum_{i=1}^{3} \xi_i \partial_{\xi_i} + 2y \partial_y,$$

then it is clear that $f \in \mathfrak{vect}(0|3) \oplus \mathbb{C}d$, but $f \notin \mathfrak{vect}(0|3)$.

2.3. Theorem. The Lie superalgebra $\mathfrak{cvect}(0|3)_*$ is simple.

Proof. We know that $\mathfrak{vect}(0|3)_* \cong \mathfrak{le}(3;3)$ is simple. According to Lemma 2.1 it is sufficient to find an element $F \in \mathfrak{cvect}(0|3)_1$, which is not in $\mathfrak{vect}(0|3)_1$. For F one can take

$$F = y\xi_1\partial_{\xi_1} + y\xi_2\partial_{\xi_2} + y\xi_3\partial_{\xi_3} + y^2\partial_y - \xi_1\xi_2\partial_{u_3} - \xi_3\xi_1\partial_{u_2} - \xi_2\xi_3\partial_{u_1}$$

Indeed, one easily checks that $\partial_y F = f$, while

$$[\partial_{\xi_i}, F] = -\partial_{\eta_i} \qquad (i = 1, 2, 3),$$

and moreover $[\partial_{u_i}, F] = 0$. This proves the claim.

Similar constructions are possible for general n. For n=2 we obtain $\operatorname{cvect}(0|2)_* \cong \operatorname{vect}(2|1)$, while for n>3 one can prove that $\operatorname{cvect}(0|n)_*$ is not simple. For details, we refer to [10].

2.4. Lemma. A vector field

$$D = \sum_{i=1}^{3} (P_i \partial_{\xi_i} + Q_i \partial_{u_i}) + R \partial_y$$

in $\operatorname{\mathfrak{vect}}(4|3)$ belongs to $\operatorname{\mathfrak{cvect}}(0|3)_*$ if and only if it satisfies the following system of equations:

$$\frac{\partial Q_i}{\partial u_j} + (-1)^{p(D)} \frac{\partial P_j}{\partial \xi_i} = 0 \text{ for any } i \neq j;$$
 (2.1)

$$\frac{\partial Q_i}{\partial u_i} + (-1)^{p(D)} \frac{\partial P_i}{\partial \xi_i} = \frac{1}{2} \left(\sum_{1 \le j \le 3} \frac{\partial Q_j}{\partial u_j} + \frac{\partial R}{\partial y} \right) \text{ for } i = 1, 2, 3; \qquad (2.2)$$

$$\frac{\partial Q_i}{\partial \xi_j} + \frac{\partial Q_j}{\partial \xi_i} = 0 \text{ for any } i, j; \text{ in particular } \frac{\partial Q_i}{\partial \xi_i} = 0; \tag{2.3}$$

$$\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = -(-1)^{p(D)} \frac{\partial R}{\partial \xi_k}$$
 (2.4)

for any k and any even permutation $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$.

$$\frac{\partial Q_i}{\partial y} = 0 \text{ for } i = 1, 2, 3; \tag{2.5}$$

$$\frac{\partial P_k}{\partial y} = (-1)^{p(D)} \frac{1}{2} \left(\frac{\partial Q_i}{\partial \xi_i} - \frac{\partial Q_j}{\partial \xi_i} \right) \tag{2.6}$$

for any k and for any even permutation $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$.

Proof. Denote by $\mathfrak{g} = \bigoplus_{i \geq -1} \mathfrak{g}_i$ the superspace of solutions of the system (2.1)–(2.6). Clearly, $\mathfrak{g}_{-1} \cong \mathfrak{vect}(4|3)_{-1}$. We directly verify that the images of the elements from $\mathfrak{vect}(0|3) \oplus \mathbb{C}d$ satisfy (2.1)–(2.6). Actually, we composed the system of equations (2.1)–(2.6) by looking at these images.

The isomorphism $\mathfrak{g}_0 = \mathfrak{vect}(0|3) \oplus \mathbb{C}d$ follows from dimension considerations.

Set

$$D_{u_{j}}(D) = \sum_{i \leq 3} \left(\frac{\partial P_{i}}{\partial u_{j}} \frac{\partial}{\partial \xi_{i}} + \frac{\partial Q_{i}}{\partial u_{j}} \frac{\partial}{\partial u_{i}}\right) + \frac{\partial R}{\partial u_{j}} \frac{\partial}{\partial y};$$

$$D_{y}(D) = \sum_{i \leq 3} \left(\frac{\partial P_{i}}{\partial y} \frac{\partial}{\partial \xi_{i}} + \frac{\partial Q_{i}}{\partial y} \frac{\partial}{\partial u_{i}}\right) + \frac{\partial R}{\partial y} \frac{\partial}{\partial y};$$

$$\tilde{D}_{\xi_{j}}(D) = (-1)^{p(D)} \sum_{i \leq 3} \left(\frac{\partial P_{i}}{\partial \xi_{j}} \frac{\partial}{\partial \xi_{i}} + \frac{\partial Q_{i}}{\partial \xi_{j}} \frac{\partial}{\partial u_{i}}\right) + (-1)^{p(D)} \frac{\partial R}{\partial \xi_{j}} \frac{\partial}{\partial y}.$$

The operators D_{u_j} , D_y and \tilde{D}_{ξ_j} , clearly, commute with the \mathfrak{g}_{-1} -action. Observe: the operators commute, not supercommute.

Since the operators in the equations (2.1)–(2.6) are linear combinations of only these operators D_{u_j} , D_y and \tilde{D}_{ξ_j} , the definition of Cartan prolongation itself ensures isomorphism of \mathfrak{g} with $\mathfrak{cvect}(0|3)_*$.

- **2.5.** Remark. The left hand sides of eqs. (2.1)–(2.6) determine coefficients of the 2-form $L_D\omega$, where L_D is the Lie derivative and $\omega = \sum_{1 \leq i \leq 3} du_i d\xi_i$. It would be interesting to interpret the right-hand side of these equations in geometrical terms as well.
- **2.6. Remark.** Lemma 2.4 illustrates how $\mathfrak{cvect}(0|3)_*$ can be characterized by a set of first order, constant coefficient, differential operators. This is a general fact of Cartan prolongations; one just replaces the linear constraints on \mathfrak{g}_0 by such operators. For example, for $\mathfrak{vect}(0|3)_*$ we have the equations (2.1)–(2.6) and

$$\frac{\partial R}{\partial y} - \sum_{i=1}^{3} \frac{\partial Q_i}{\partial u_i} = 0 \tag{2.7}$$

Indeed, this equation is satisfied by all elements of $\mathfrak{vect}(0|3)_0$, see section 1.1, but not by d.

§3. Solution of differential equations (2.1) - (2.6)

Set
$$D_{\xi}^3 = \frac{\partial^3}{\partial \xi_1 \partial \xi_2 \partial \xi_3}$$
.

3.1. Theorem. Every solution of the system (2.1) - (2.6) is of the form:

$$D = \text{Le}_f + yA_f - (-1)^{p(f)} \left(y\Delta(f) + y^2 D_{\xi}^3 f \right) \partial_y + A_g - (-1)^{p(g)} \left(\Delta(g) + 2y D_{\xi}^3 g \right) \partial_y,$$
(3.1)

where $f, g \in \mathbb{C}[u, \xi]$ are arbitrary and the operator A_f is given by the formula:

$$A_f = \frac{\partial^2 f}{\partial \xi_2 \partial \xi_3} \frac{\partial}{\partial \xi_1} + \frac{\partial^2 f}{\partial \xi_3 \partial \xi_1} \frac{\partial}{\partial \xi_2} + \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2} \frac{\partial}{\partial \xi_3}.$$
 (3.2)

Proof. First, let us find all solutions of system (2.1)–(2.6) for which $Q_1 = Q_2 = Q_3 = 0$. In this case the system takes the form

$$\frac{\partial P_j}{\partial \mathcal{E}_i} = 0 \text{ for } i \neq j \tag{2.1'}$$

$$(-1)^{p(D)} \frac{\partial P_i}{\partial \xi_i} = \frac{1}{2} \frac{\partial R}{\partial y} \text{ for } i = 1, 2, 3$$
 (2.2')

$$\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = -(-1)^{p(D)} \frac{\partial R}{\partial \xi_k} \text{ for } (i, j, k) \in A_3$$
 (2.4')

$$\frac{\partial P_k}{\partial y} = 0 \text{ for } k = 1, 2, 3 \tag{2.6'}$$

From (2.1'), (2.2') and (2.6') it follows that

$$P_i = \Psi_i(u_1, u_2, u_3) + \xi_i \varphi(u_1, u_2, u_3),$$

where $\varphi = \frac{1}{2}(-1)^{p(D)}\frac{\partial R}{\partial y}$. For brevity we will write $\Psi_i(u)$ and $\varphi(u)$. Then $R = (-1)^{p(D)} \cdot 2\varphi(u)y + R_0(u,\xi)$.

Let us expand the 3 equations of type (2.4'); their explicit form is:

$$\frac{\partial R_0}{\partial \xi_1} = -(-1)^{p(D)} \left(\frac{\partial \Psi_2}{\partial u_3} - \frac{\partial \Psi_3}{\partial u_2} \right) + (-1)^{p(D)} \left(\frac{\partial \varphi}{\partial u_2} \xi_3 - \frac{\partial \varphi}{\partial u_3} \xi_2 \right),
\frac{\partial R_0}{\partial \xi_2} = -(-1)^{p(D)} \left(\frac{\partial \Psi_3}{\partial u_1} - \frac{\partial \Psi_1}{\partial u_3} \right) + (-1)^{p(D)} \left(\frac{\partial \varphi}{\partial u_3} \xi_1 - \frac{\partial \varphi}{\partial u_1} \xi_3 \right),
\frac{\partial R_0}{\partial \xi_3} = -(-1)^{p(D)} \left(\frac{\partial \Psi_1}{\partial u_2} - \frac{\partial \Psi_2}{\partial u_1} \right) + (-1)^{p(D)} \left(\frac{\partial \varphi}{\partial u_1} \xi_2 - \frac{\partial \varphi}{\partial u_2} \xi_1 \right).$$

The integration of these equations yields

$$R_{0} = (-1)^{p(D)} (\Psi_{0}(u) - (\frac{\partial \Psi_{2}}{\partial u_{3}} - \frac{\partial \Psi_{3}}{\partial u_{2}}) \xi_{1} - (\frac{\partial \Psi_{3}}{\partial u_{1}} - \frac{\partial \Psi_{1}}{\partial u_{3}}) \xi_{2} - (\frac{\partial \Psi_{1}}{\partial u_{2}} - \frac{\partial \Psi_{2}}{\partial u_{3}}) \xi_{3} - (\frac{\partial \varphi}{\partial u_{2}} \xi_{3} \xi_{1} + \frac{\partial \varphi}{\partial u_{1}} \xi_{2} \xi_{3} + \frac{\partial \varphi}{\partial u_{3}} \xi_{1} \xi_{2}))$$

$$= (-1)^{p(D)} (\Psi_{0}(u) + \Delta(-\Psi_{1} \xi_{2} \xi_{3} - \Psi_{2} \xi_{3} \xi_{1} - \Psi_{3} \xi_{1} \xi_{2} - \varphi \xi_{1} \xi_{2} \xi_{3})).$$

Therefore, any vector field D with $Q_1=Q_2=Q_3=0$ satisfying (2.1) – (2.6) is of the form

$$D = \sum_{i=1}^{3} \Psi_{i}(u)\partial_{\xi_{i}} + \varphi(u)\sum_{i=1}^{3} \xi_{i}\partial_{\xi_{i}} + (-1)^{p(D)} \cdot (\Psi_{0}(u) + \Delta(-\Psi_{1}\xi_{2}\xi_{3} - \Psi_{2}\xi_{3}\xi_{1} - \Psi_{3}\xi_{1}\xi_{2} - \varphi\xi_{1}\xi_{2}\xi_{3}) + 2\varphi(u)y)\partial_{y}.$$

where, as before,

$$\Delta = \sum_{i=1}^{3} \frac{\partial}{\partial u_i} \frac{\partial}{\partial \xi_i}.$$

Set

$$g(u,\xi) = g_0(u,\xi) - \Psi_1 \xi_2 \xi_3 - \Psi_2 \xi_3 \xi_1 - \Psi_3 \xi_1 \xi_2 - \varphi \xi_1 \xi_2 \xi_3,$$

with $\Delta g_0 = \Psi_0$ and $\deg_{\mathcal{E}}(g_0) \leq 1$. Then

$$A_g = \sum_{i=1}^3 \Psi_i \partial_{\xi_i} + \varphi \sum_{i=1}^3 \xi_i \partial_{\xi_i}; \quad D_{\xi}^3 g = \varphi \text{ and } (-1)^{p(D)} = (-1)^{p(g)+1}$$

for functions g homogeneous with respect to parity. In the end we get:

$$D = A_g + (-1)^{p(D)} (\Delta(g) + 2y D_{\xi}^3 g) \partial_y = A_g - (-1)^{p(g)} (\Delta(g) + 2y D_{\xi}^3 g) \partial_y.$$
 (3.3)

Let us return now to the system (2.1) – (2.6). Equations (2.3), (2.5), (2.6) imply that there exists a function $f(u,\xi)$ (independent of y!) such that

$$Q_i = -(-1)^{p(D)} \frac{\partial f}{\partial \xi_i}$$
 for $i = 1, 2, 3$.

Then (2.1) implies that

$$P_i = \frac{\partial f}{\partial u_i} + f_i(u, \xi_i, y).$$

From (2.6) it follows that

$$\frac{\partial f_i}{\partial y} = \partial_{\xi_j} \partial_{\xi_k} f$$
 for even permutations (i, j, k)

or

$$f_i = y(\partial_{\xi_j}\partial_{\xi_k}f) + \tilde{P}_i(u,\xi_i).$$

Observe that \tilde{P}_i satisfy (2.1') and (2.6'); hence, in view of (2.2), $\frac{\partial \tilde{P}_i}{\partial \xi_i}$ does not depend on i. Therefore, we can choose \tilde{R} so that (\tilde{P}_i, \tilde{R}) satisfy eqs. (2.1'), (2.2'), (2.4'), (2.6'). Thanks to the linearity of system (2.1) – (2.6) the vector field D is then of the form

$$D = D_f + \tilde{D},\tag{3.4}$$

where D_f and \tilde{D} are solutions of (2.1) – (2.6) such that $\tilde{D} = \sum \tilde{P}_i \partial_{\xi_i} + \tilde{R} \partial_y$ (i.e., \tilde{D} is of the form (3.3)) and

$$D_f = \sum (-(-1)^{p(D)} \frac{\partial f}{\partial \xi_i} \partial_{u_i} + \frac{\partial f}{\partial u_i} \partial_{\xi_i}) + \sum y(\partial_{\xi_j} \partial_{\xi_k} f) \partial_{\xi_i}) + R_f \cdot \partial_y$$

= Le_f + yA_f + R_f\partial_y.

It remains to find R_f . Equation (2.2) takes the form

$$(-1)^{p(D)}yD_{\xi}^{3}f = \frac{1}{2}(-(-1)^{p(D)}(\Delta f) + \frac{\partial R_{f}}{\partial y}).$$

Hence,

$$R_f = (-1)^{p(D)} (y^2 D_{\xi}^3 f + y \cdot (\Delta f) + R_0(u, \xi)).$$

Then, we can rewrite (2.4) as

$$-y\frac{\partial \Delta f}{\partial \xi_k} + \frac{\partial R_0}{\partial \xi_k} = y\partial_{u_j}\partial_{\xi_j}\partial_{\xi_k}f - y\partial_{u_i}\partial_{\xi_k}\partial_{\xi_i}f.$$

Observe that the right hand side of the last equation is equal to $-y\frac{\partial \Delta f}{\partial \xi_k}$. This means that $\frac{\partial R_0}{\partial \xi_k} = 0$ or $R_0 = R_0(u)$. Therefore, replacing \tilde{R} with $\tilde{R} + R_0$ we may assume that $R_0 = 0$. Then

$$D_f = \text{Le}_f + yA_f + (-1)^{p(D)}(y(\Delta f) + y^2 D_{\xi}^3 f)\partial_y.$$
 (3.5)

By uniting (3.3) - (3.5) we get (3.1).

§4 How to generate $\operatorname{cvect}(0|3)_*$ by pairs of functions

We constructed $\mathfrak{cvect}(0|3)_*$ as an extension of $\mathfrak{vect}(0|3)_* \cong \mathfrak{le}(3;3)$, see lemma 1.3. Using the results of section 3, we obtain another embedding $i_2: \mathfrak{le}(3) \to \mathfrak{vect}(0|3)_*$.

4.1. Lemma. The map

$$i_2: \operatorname{Le}_f \to \operatorname{Le}_f + yA_f - (-1)^{p(f)} \left(y\Delta(f) + y^2 D_{\varepsilon}^3 f \right) \partial_y$$
 (4.1)

determines an embedding of $\mathfrak{le}(3)$ into $\mathfrak{cvect}(0|3)_*$. This embedding preserves the standard grading of $\mathfrak{le}(3)$.

Proof. We have to verify the equality

$$i_2(\operatorname{Le}_{\{f,g\}}) = [i_2(\operatorname{Le}_f), i_2(\operatorname{Le}_g)].$$

Comparison of coefficients of different powers of y shows that the above equation is equivalent to the following system:

$$Le_{\{f,g\}} = [Le_f, Le_g]. \tag{4.2}$$

$$A_{\{f,g\}} = [\text{Le}_f, A_g] + [A_f, \text{Le}_g] - (-1)^{p(f)} (\Delta(f) \cdot A_g + (-1)^{p(f)p(g)} \Delta(g) A_f).$$
(4.3)

$$[A_f, A_g] = (-1)^{p(f)} \left(D_{\xi}^3 f \cdot A_g + (-1)^{p(f)p(g)} D_{\xi}^3 g A_f \right). \tag{4.4}$$

$$\Delta(\{f,g\}) = \{\Delta f, g\} - (-1)^{p(f)} \{f, \Delta g\}. \tag{4.5}$$

$$\begin{split} D_{\xi}^{3}\{f,g\} &= \{D_{\xi}^{3}f,g\} - (-1)^{p(f)}\{f,D_{\xi}^{3}g\} - (-1)^{p(f)}(A_{f}(\Delta g) \\ &+ (-1)^{p(f)p(g)}A_{g}(\Delta f)) + \Delta f D_{\xi}^{3}g - D_{\xi}^{3}f\Delta g. \end{split} \tag{4.6}$$

Equation (4.2) is known, see section 0.3. The equalities (4.3)–(4.6) are subject to direct verification. \Box

We found two embeddings $i_1: \mathfrak{le}(3;3) \to \mathfrak{vect}(0|3)_*$ and $i_2: \mathfrak{le}(3) \to \mathfrak{cvect}(0|3)_*$. Let us denote

$$\alpha_q = A_q - (-1)^{p(g)} (\Delta g + 2y D_{\varepsilon}^3 g) \partial_y.$$

We want to prove that the sum of the images of i_1 and i_2 cover the whole $\operatorname{cvect}(0|3)_*$. According to Theorem 3.1, it is sufficient to represent α_g in the form $\alpha_g = i_1 g_1 + i_2 g_2$. For convenience we simply write f instead of Le_f .

4.2. Lemma. For α_q we have:

$$\alpha_g = \begin{cases} 0 & \text{if } \deg_{\xi} g = 0\\ i_1(-(\Delta g)\xi_1\xi_2\xi_3) & \text{if } \deg_{\xi} g = 1\\ i_1(g) & \text{if } \deg_{\xi} g = 2\\ i_1(-\Delta^{-1}(D_{\xi}^3g)) + i_2(\Delta^{-1}(D_{\xi}^3g)) & \text{if } \deg_{\xi} g = 3. \end{cases}$$

The right inverse Δ^{-1} of Δ is given in section 0.4. The proof of Lemma 4.2 is a direct calculation.

- **4.3.** A wonderful property of $\mathfrak{sle}^{\circ}(3)$. In the standard grading of $\mathfrak{g} = \mathfrak{sle}^{\circ}(3)$ we have: $\dim \mathfrak{g}_{-1} = (3|3), \, \mathfrak{g}_0 \cong \mathfrak{spe}(3)$. For the regraded superalgebra $R\mathfrak{g} = \mathfrak{sle}^{\circ}(3;3) \subset \mathfrak{le}(3;3)$ we have: $\dim R\mathfrak{g}_{-1} = (3|3), \, R\mathfrak{g}_0 = \mathfrak{svect}(0|3) \cong \mathfrak{spe}(3)$. For the definition of $\mathfrak{spe}(3)$ we refer to [3] or [10]. Therefore, for $\mathfrak{sle}^{\circ}(3)$ and only for it among the $\mathfrak{sle}^{\circ}(n)$, the regrading R determines a nontrivial automorphism. In terms of generating functions the regrading is determined by the formulas:
 - 1) $\deg_{\xi}(f) = 0$: $R(f) = \Delta(f\xi_1\xi_2\xi_3)$;
 - 2) $\deg_{\xi}(f) = 1$: R(f) = f;
 - 3) $\deg_{\xi}(f) = 2$: $R(f) = D_{\xi}^{3}(\Delta^{-1}f)$.

Note that $R^2(f) = (-1)^{p(f)+1}f$. Now we can formulate the following proposition.

4.4. Proposition. The nondirect sum of the images of i_1 and i_2 covers the whole $\operatorname{cvect}(0|3)_*$, i.e.,

$$i_1(\mathfrak{le}(3;3)) + i_2(\mathfrak{le}(3)) = (\mathfrak{cvect}(0|3))_*.$$

We also have

$$i_1(\mathfrak{le}(3;3)) \cap i_2(\mathfrak{le}(3)) \cong \mathfrak{sle}^{\circ}(3;3) \cong \mathfrak{sle}^{\circ}(3).$$

Proof. The first part follows from Lemma 4.2. The second part follows by direct calculation from solving $i_2(\text{Le}_f) = i_1(\text{Le}_g)$. Note that $\text{Le}_f \in \mathfrak{sle}^{\circ}(3)$ iff $\Delta(f) = 0$ and $D_{\xi}^3 f = 0$, and similar for $\text{Le}_g \in \mathfrak{sle}^{\circ}(3;3)$. The equation $i_2(\text{Le}_f) = i_1(\text{Le}_g)$ is only solvable if $f \in \mathfrak{sle}^{\circ}(3)$ and $g \in \mathfrak{sle}^{\circ}(3;3)$, and in this case we obtain $g = (-1)^{p(f)+1}Rf$.

Therefore, we can identify the space of the Lie superalgebra $\mathfrak{cvect}(0|3)_*$ with the quotient space of $\mathfrak{le}(3;3) \oplus \mathfrak{le}(3)$ modulo

$$\{(-1)^{p(g)+1}Rg \oplus (-g), g \in \mathfrak{sle}^{\circ}(3)\}.$$

In other words, we can represent the elements of $\mathfrak{cvect}(0|3)_*$ in the form of the pairs of functions

$$(f,g)$$
, where $f,g \in \Pi \mathbb{C}[u,\xi]/\mathbb{C} \cdot 1$ (4.7)

subject to identifications

$$(-1)^{p(g)+1}(Rg,0)\sim (0,g)\quad \text{for any}\quad g\in\mathfrak{sle}^\circ(3).$$

4.5. Corollary. The map φ defined by the formula

$$\varphi|_{i_1(\mathfrak{le}(3;3))} = \text{sign } i_2 i_1^{-1}; \qquad \varphi|_{i_2(\mathfrak{le}(3))} = i_1 i_2^{-1}$$

is an automorphism of $\operatorname{cvect}(0|3)_*$. Here $\operatorname{sign}(D) = (-1)^{p(D)}D$.

The map φ may be represented in inner coordinates of $\mathfrak{vect}(4|3)$ as a regrading by setting deg y = -1; deg $u_i = 1$; deg $\xi_i = 0$.

In the representation (4.7) we have

$$\varphi(f,g) = (g,(-1)^{p(f)+1}f).$$

Now we can complete the proof of Lemma 1.3.

4.6. Corollary. The embedding $i_1 : \mathfrak{le}(3) \to \mathfrak{cvect}(0|3)_*$ is a surjection onto $\mathfrak{vect}(0|3)_*$.

Proof. By Proposition 4.4 we merely have to prove that $i_2(\text{Le}_f) \in \mathfrak{vect}(0|3)_*$ iff $\Delta f = 0$ and $D_{\xi}^3 f = 0$. Applying equation (2.7) to $i_2(\text{Le}_f)$, this follows immediately.

§5 THE BRACKET IN
$$\operatorname{cvect}(0|3)_*$$

Now we can determine the bracket in $\operatorname{cvect}(0|3)_*$ in terms of representation (f,g) as stated in formula (4.7).

We do this via α_g . By Theorem 3.1 any $D \in \mathfrak{cvect}(0|3)_*$ is of the form $D = i_2(f) + \alpha_g$ for some generating functions f and g. To determine the bracket $[i_2(f), i_1(h)]$, we

- 1. Compute the brackets $[i_2f, \alpha_g]$ for any $f, g \in \mathbb{C}[u, \xi]/\mathbb{C} \cdot 1$;
- 2. Represent $i_1(h)$ in the form

$$i_1(h) = i_2 a(h) + \alpha_{b(h)} \text{ for any } h \in \mathbb{C}[u, \xi]/\mathbb{C} \cdot 1;$$
 (5.1)

In Lemma 4.2 we expressed α_q in i_1 and i_2 .

Remark. The functions a(h) and b(h) above are not uniquely defined. Any representation will do.

5.1. Lemma. For any functions $f, g \in \mathbb{C}[u, \xi]/\mathbb{C} \cdot 1$ the bracket $[i_2 f, \alpha_g]$ is of the form

$$[i_2 f, \alpha_q] = i_2 F + \alpha_G, \tag{5.2}$$

where

$$F = f \cdot D_{\varepsilon}^{3}g - (-1)^{(p(f)+1)(p(g)+1)}A_{g}f \quad and \quad G = -f\Delta g$$

Proof. Direct calculation gives that

$$\begin{split} [i_2f,\alpha_g] &= [\operatorname{Le}_f,A_g] + (-1)^{p(f)p(g)+p(f)+1} \Delta g \cdot A_f \\ &+ y \left([A_f,A_g] + (-1)^{p(f)p(g)+p(f)+1} \cdot 2 \cdot D_\xi^3 g \cdot A_f \right) \\ &+ (-1)^{p(g)+1} \left(\{f,\Delta g\} + (-1)^{p(f)} \Delta f \cdot \Delta g \right) \partial_y \\ &+ \left((-1)^{p(g)+1} A_f (\Delta g) + (-1)^{p(f)p(g)+p(g)+1} A_g (\Delta f) \right. \\ &+ 2 \cdot (-1)^{p(g)+1} \{f,D_\xi^3 g\} + 2 \cdot (-1)^{p(f)+p(g)+1} D_\xi^3 f \cdot \Delta g \right) y \partial_y \\ &+ (-1)^{p(f)+p(g)+1} 2 \cdot D_\varepsilon^3 f \cdot D_\varepsilon^3 g \cdot y^2 \partial_y. \end{split}$$

In order to find the functions F and G, it suffices to observe that the coefficient of ∂_y , non-divisible by y, should be equal to $(-1)^{p(G)+1}\Delta G$. This implies the equations:

$$(-1)^{p(G)+1}\Delta G = (-1)^{p(g)+1} \left(\{f, \Delta g\} + (-1)^{p(f)} \Delta f \cdot \Delta g \right)$$

or

$$(-1)^{p(G)+1}\Delta G = (-1)^{p(f)+p(g)+1}\Delta (f \cdot \Delta g).$$

Here $p(G) = p(f \cdot \Delta g) = p(f) + p(g) + 1$. Hence, $\Delta G = \Delta(-f\Delta g)$. Since G is defined up to elements from $\mathfrak{sle}^{\circ}(3)$, we can take $G = -f\Delta g$.

The function F to be found is determined from the equation

$$i_2 F = [i_2 f, \alpha_a] - \alpha_G. \tag{5.3}$$

By comparing the coefficients of $y\partial_y$ in the left and right hand sides of (5.3) we get

$$\begin{split} (-1)^{p(F)+1}\Delta F &= (-1)^{p(g)+1}A_f(\Delta g) + (-1)^{p(f)p(g)+p(g)+1}A_g(\Delta f) \\ &+ 2(-1)^{p(g)+1}\{f,D_{\xi}^3g\} + (-1)^{p(f)+p(g)+1}2\cdot D_{\xi}^3f\cdot \Delta g \\ &- 2\cdot (-1)^{p(f)+p(g)}D_{\xi}^3(-f\Delta g). \end{split}$$

Observe that

$$D_{\xi}^{3}(f\Delta g) = (D_{\xi}^{3}f)\Delta g + (-1)^{p(f)}A_{f}(\Delta g) + \sum_{i=1}^{3} \frac{\partial f}{\partial \xi_{i}} \frac{\partial}{\partial u_{i}}(D_{\xi}^{3}g)$$
$$= (D_{\xi}^{3}f) \cdot \Delta g + (-1)^{p(f)}A_{f}(\Delta g) + (-1)^{p(f)}\{f, D_{\xi}^{3}g\}.$$

Then

$$(-1)^{p(F)+1}(\Delta F) = (-1)^{p(g)}A_f(\Delta g) + (-1)^{p(f)p(g)+p(g)+1}A_g(\Delta f).$$

By comparing parities we derive that

$$p(F) + 1 = p(A_f(\Delta g)) = p(f) + 1 + p(g) + 1 = p(f) + p(g).$$

It follows that

$$\Delta F = (-1)^{p(f)} A_f(\Delta g) + (-1)^{p(f)p(g) + p(f) + 1} A_g(\Delta f).$$

Let us transform the right hand side of the equality obtained. The sums over i, j, k are over $(i, j, k) \in A_3$:

$$\begin{split} &(-1)^{p(f)}A_f(\Delta g) + (-1)^{p(f)p(g)+p(f)+1}A_g(\Delta f) \\ &= \sum (-1)^{p(f)}\partial_{\xi_j}\partial_{\xi_k}f \cdot \partial_{\xi_i}(\sum_{s=1}^3 \partial_{u_s}\partial_{\xi_s}g) \\ &+ (-1)^{p(f)p(g)+p(f)+1}\sum \partial_{\xi_j}\partial_{\xi_k}g\partial_{\xi_i}(\sum_{s=1}^3 \partial_{u_s}\partial_{\xi_s}f) \\ &= (-1)^{p(f)p(g)+p(f)} \cdot \sum ((\partial_{u_j}\partial_{\xi_i}\partial_{\xi_j}g + \partial_{u_k}\partial_{\xi_i}\partial_{\xi_k}g) \cdot \partial_{\xi_j}\partial_{\xi_k}f) \\ &- (-1)^{p(f)p(g)+p(f)} \cdot \sum (\partial_{\xi_j}\partial_{\xi_k}g(\partial_{u_j}\partial_{\xi_i}\partial_{\xi_j}f + \partial_{u_k}\partial_{\xi_i}\partial_{\xi_k}f)) \\ &= (-1)^{p(f)p(g)+p(f)} \sum \partial_{u_k}(\partial_{\xi_i}\partial_{\xi_k}g \cdot \partial_{\xi_j}\partial_{\xi_k}f + \partial_{\xi_j}\partial_{\xi_k}g \cdot \partial_{\xi_k}\partial_{\xi_i}f) \\ &= (-1)^{p(f)p(g)+p(f)} \sum \partial_{u_k}(\partial_{\xi_i}\partial_{\xi_k}g \cdot \partial_{\xi_j}\partial_{\xi_k}f + \partial_{\xi_j}\partial_{\xi_k}g \cdot \partial_{\xi_k}\partial_{\xi_i}f) \\ &= (-1)^{p(f)p(g)+p(f)+p(g)} \sum_{k=1}^3 (\partial_{u_k}\partial_{\xi_k}(A_gf) - \partial_{u_k}D_{\xi}^3g \cdot \partial_{\xi_k}f)) \\ &= - (-1)^{(p(f)+1)(p(g)+1)} \Delta(A_gf) + (-1)^{p(f)p(g)+p(f)} \Delta(D_{\xi}^3g \cdot f) \\ &= \Delta(f \cdot D_{\xi}^3g) - (-1)^{(p(f)+1)(p(g)+1)} \Delta(A_gf). \end{split}$$

Then

$$F = f \cdot D_{\xi}^{3}g - (-1)^{(p(f)+1)(p(g)+1)}A_{g}f + F_{0}, \text{ where } \Delta F_{0} = 0.$$

We have shown how to find functions F and G. To prove Lemma 5.1 it only remains to compare the elements of the same degree in y in the right-hand and the left-hand side, i.e., to verify the following three equalities:

$$(-1)^{p(F)+1}D_{\xi}^{3}F = 2(-1)^{p(f)+p(g)+1}D_{\xi}^{3}f \cdot D_{\xi}^{3}g$$

$$\operatorname{Le}_{F} + A_{G} = [\operatorname{Le}_{f}, A_{g}] + (-1)^{p(f)p(g)+p(f)+1}\Delta g \cdot A_{f}$$

$$A_{F} = [A_{f}, A_{g}] + 2 \cdot (-1)^{p(f)p(g)+p(f)+1}D_{\xi}^{3}g \cdot A_{f}$$

The verification is a direct one.

5.3. Lemma. The representation of i_1h in the form (5.1) is as follows:

$$i_1 h = \begin{cases} i_2(\Delta(h\xi_1\xi_2\xi_3)) & \text{if } \deg_\xi h = 0, \\ i_2 h + \alpha_{(\Delta h)\xi_1\xi_2\xi_3} & \text{if } \deg_\xi h = 1, \\ \alpha_h & \text{if } \deg_\xi h = 2, \\ \alpha_{\Delta^{-1}(D^3_\xi h)} & \text{if } \deg_\xi h = 3. \end{cases}$$

Proof. It suffices to compare the definition of α_g with the definitions of i_1 and i_2 . If $\deg_{\xi} h = 0$ use the equalities $\sum \frac{\partial f}{\partial u_i} \xi_j \xi_k = \Delta(f \xi_1 \xi_2 \xi_3)$ and $A_{\Delta(f \xi_1 \xi_2 \xi_3)} = \operatorname{Le}_f$. In the remaining cases the verification is not difficult. \square

Making use of the Lemmas 5.1, Lemma 5.2 and Lemma 4.2 we can compute the whole multiplication table of $[i_2f, i_1h]$:

• $\deg_{\mathcal{E}} h = 0$. Then

$$i_1h = i_2(\Delta(h\xi_1\xi_2\xi_3))$$
 and $[i_2f, i_1h] = i_2\{f, \Delta(h\xi_1\xi_2\xi_3)\}.$

We also have

$$\{f, \Delta(h\xi_1\xi_2\xi_3)\} = \begin{cases} 0 & \text{if } \deg_{\xi} f = 3\\ -\{\Delta f, h\}\xi_1\xi_2\xi_3 & \text{if } \deg_{\xi} f = 2. \end{cases}$$

• $\deg_{\xi} h = 1$. Then

$$\begin{split} [i_2f,i_1h] &= [i_2f,i_2h + \alpha_{(\Delta h)\xi_1\xi_2\xi_3}] = \\ i_2\{f,h\} - i_2(f\Delta h) + i_2(\Delta h \cdot \sum \xi_i \partial_{\xi_i} f) + \alpha_{-f \cdot \Delta((\Delta h)\xi_1\xi_2\xi_3)}. \end{split}$$

• $\deg_{\xi} h = 2$. Then

$$\begin{aligned} [i_2f,i_1h] &= [i_2f,\alpha_h] = (-1)^{p(f)}i_2(A_hf) - \alpha_{(f\Delta h)} = \\ \begin{cases} i_1(\{f,\Delta h\}\xi_1\xi_2\xi_3) & \text{if } \deg_\xi f = 0 \\ i_1(\Delta(fh) - f\Delta h) & \text{if } \deg_\xi f = 1 \\ i_2(A_hf) - i_2(\Delta^{-1}D_\xi^3(f\Delta h)) + i_1(\Delta^{-1}D_\xi^3(f\Delta h)) & \text{if } \deg_\xi f = 2 \\ -i_2(hD_\xi^3f) & \text{if } \deg_\xi f = 3. \end{cases}$$

• $\deg_{\varepsilon} h = 3$. Then

$$\begin{split} [i_2f,i_1h] &= [i_2f,\alpha_{\Delta^{-1}(D_\xi^3h)}] = -\alpha_{f\cdot D_\xi^3h} = \\ 0 & \text{if } \deg_\xi f = 0 \\ i_1(-\Delta(f\cdot D_\xi^3h)\xi_1\xi_2\xi_3) &= i_1(-f\Delta h - \Delta f\cdot h) & \text{if } \deg_\xi f = 1 \\ i_1(-f\cdot D_\xi^3g) & \text{if } \deg_\xi f = 2 \\ i_1(\Delta^{-1}(D_\xi^3f\cdot D_\xi^3g)) - i_2(\Delta^{-1}(D_\xi^3f\cdot D_\xi^3g)) & \text{if } \deg_\xi f = 3. \end{split}$$

The final result is represented in the following tables.

The brackets $[i_2f, i_1h]$

| $\deg_{\xi}(f)$ | $\deg_{\xi}(h) = 0$ | $\deg_{\xi}(h) = 1$ |
|-----------------|--|--|
| 0 | $i_2(\{f,\Delta(h\xi_1\xi_2\xi_3)\})$ | $-i_1(\{\Delta(f\xi_1\xi_2\xi_3),h\})$ |
| 1 | $i_2(\{f,\Delta(h\xi_1\xi_2\xi_3)\})$ | $i_1(\Delta^{-1}\{f,\Delta h\})+$ |
| | | $i_2(\{f,h\} - \Delta^{-1}\{f,\Delta h\})$ |
| 2 | $-i_2(\{\Delta f, h\}\xi_1\xi_2\xi_3)$ | $i_2(\Delta(fh) - \Delta(f)h)$ |
| 3 | 0 | $i_2(f\Delta(h) + \Delta(f)h)$ |

| $\deg_{\xi}(f)$ | $\deg_{\xi}(h) = 2$ | $\deg_{\xi}(h) = 3$ |
|-----------------|---|--|
| 0 | $i_1(\{f,\Delta h\}\xi_1\xi_2\xi_3)$ | 0 |
| 1 | $-i_1(\Delta(fh)+f\Delta h)$ | $i_1(-f\Delta(h)-\Delta(f)h)$ |
| 2 | $i_1(\Delta^{-1}D^3_{\xi}(f\Delta h))+$ | $i_1(-fD_{\xi}^3h)$ |
| | $i_2(A_hf-\Delta^{-1}D_{\xi}^3(f\Delta h))$ | • |
| 3 | $i_2(-hD_{\xi}^3f)$ | $i_1(\Delta^{-1}(D_{\xi}^3 f \cdot D_{\xi}^3 h)) -$ |
| | , | $ \begin{vmatrix} i_1(\Delta^{-1}(D_{\xi}^3 f \cdot D_{\xi}^3 h)) - \\ i_2(\Delta^{-1}(D_{\xi}^3 f \cdot D_{\xi}^3 h)) \end{vmatrix} $ |

References

- Alekseevsky D., Leites D., Shchepochkina I., New examples of simple Lie superalgebras of vector fields. C.r. Acad. Bulg. Sci. 34, N 9, 1980, p. 1187–1190 (in Russian).
- [2] Grozman P., Leites D., Shchepochkina I., Lie superalgebras of string theories, preprint hep-th/9702120, 1997.
- [3] Kac V.G. Lie superalgebras. Adv. Math. 26, 1977, p. 8-96.
- [4] Leites D., New Lie superalgebras and mechanics. Soviet Math. Doklady 18, N5, 1977, p. 1277–1280.
- [5] Leites D., Lie superalgebras. In: Modern Problems of Mathematics. Recent developments 25, VINITI, Moscow, 1984, p. 3–49 (English translation = JOSMAR 30(6), 1985, p. 2481–2512).
- [6] Leites D., Quantization. Supplement 3. In: F. Berezin, M. Shubin. Schrödinger equation, Kluwer, Dordrecht, 1991, p. 483–522.
- [7] Leites D., Shchepochkina I., Classification of simple Lie superalgebras (in preparation).
- [8] Shchepochkina I., New exceptional simple Lie superalgebras C.R. bulg. Sci. 36, 3, 1983, p. 313–314.
- [9] Shchepochkina I., Maximal subalgebras of simple Lie superalgebras In: Leites D.(ed.) Seminar on Supermanifolds vol. 32, Reports of Stockholm University, 1992, p. 1–43.
- [10] Shchepochkina I., Five exceptional simple Lie superalgebras, preprint hep-th/9702121, 1997.

I.Shch.: on leave of absence from the Independent University of Moscow. Correspondence: c/o D. Leites, Department of Mathematics, University of Stockholm, Roslagsv. 101, Kräftriket hus 6, S-106 91, Stockholm, Sweden.

G.P.: Department of Applied Mathematics, University Twente,, P.O. Box 217, 7500 AE Enschede, The Netherlands.