

A POLYNOMIAL TIME ALGORITHM FOR LOCAL TESTABILITY AND ITS LEVEL

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Abstract

A locally testable semigroup S is a semigroup with the property that for some nonnegative integer k , called the order or level of local testability, two words u and v in some set of generators for S are equal in the semigroup if (1) the prefix and suffix of the words of length k coincide, and (2) the set of intermediate substrings of length k of the words coincide. The local testability problem for semigroups is, given a finite semigroup, to decide, if the semigroup is locally testable or not.

Recently, we introduced a polynomial time algorithm for the local testability problem and to find the level of local testability for semigroups based on our previous description of identities of k -testable semigroups and the structure of locally testable semigroups.

The first part of the algorithm we introduce solves the local testability problem.

The second part of the algorithm finds the order of local testability of a semigroup. The algorithm is of order n^2 , where n is the order of the semigroup.

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1 INTRODUCTION

The concept of local testability was first introduced by McNaughton and Papert [5] and since then has been extensively investigated from different points of view (see [1]-[6], [8]). This concept is connected with languages, finite automata and semigroups. The purely algebraic approach proved to be fruitful (see [6], [8]) and in this paper we use this technique.

The algorithms for the problem of local testability can be found in [1], [5]. They are polynomial in terms of the size of the semigroup. In [3] a polynomial time algorithm for the local testability problem for a given deterministic finite automaton was given. The order of this algorithm is sn^2 , where n is the number of states of the automaton, and s is the size of the alphabet.

We introduce in this paper a new polynomial time algorithm for finding the level of local testability for a given semigroup. Necessary and sufficient conditions for a semigroup to be locally testable from [6] are used here.

The solution is connected with the problem from [2] (see [1], [3] too): “Is there a practical algorithm which, given a locally testable deterministic automaton, finds k such that the automaton is properly k -testable (i.e. k -testable but not $(k-1)$ -testable)?” We give the answer to this question in the case that the automaton is the right regular representation of a semigroup S . In the general case computing the order of local testability of a locally testable automaton is NP-hard [4].

The order of our algorithm is n^2 , where n is the order of the semigroup. The first part of the algorithm solves the local testability problem. The second part finds the order of local testability.

There are two different definitions of k -testability (see [2]). Our algorithm gives an answer in the sense of the definition used in [1] and [2].

2 NOTATION AND DEFINITIONS

Let Σ be an alphabet and let Σ^+ denote the free semigroup on Σ . If $w \in \Sigma^+$, let $|w|$ denote the length of w . Let k be a positive integer. Let $i_k(w)$ ($t_k(w)$) denote the prefix (suffix) of w of length k or w if $|w| < k$. Let $F_k(w)$ denote the set of factors of w of length k . That is, $F_k(w) = \{x \in \Sigma^+ \mid |x| = k \text{ and } w = uxv \text{ for some } u, v \in \Sigma^+\}$.

A semigroup S is called *k-testable* if there is an alphabet Σ and a surjective morphism $\phi : \Sigma^+ \rightarrow S$ such that for all $u, v \in \Sigma^+$, if $i_k(u) = i_k(v)$, $t_k(u) = t_k(v)$ and $F_k(u) = F_k(v)$, then $u\phi = v\phi$. This definition follows [4], [5], [6], [8], but [2] and [1]. In [1] the definition differs by considering prefixes and suffixes of length $k-1$.

A semigroup S is *locally testable* if it is k -testable for some k . For local testability the two definitions mentioned above are equivalent [2].

It is known that the set of k -testable semigroups forms a variety of semigroups [8]. Let T_k denote the variety of k -testable semigroups.

We need the following notation and definitions.

$|S|$ is the number of elements of the set S .

S^m denotes the ideal of the semigroup S containing products of elements of S of length m and greater.

$[u = v]$ denotes the variety of semigroups defined by identity $u = v$. By definition, $S \in [u = v]$ if and only if the identity $u = v$ holds in S .

A semigroup $S \in [xy = x]$ [$S \in [xy = y]$] is called *semigroup of left [right] zeroes*.

We will need to consider the following semigroup $A_2 = \langle a, b \mid aba = a, bab = b, aa = a, bb = 0 \rangle$. S is a 5-element 0-simple semigroup, $A_2 = \{a, b, ab, ba, 0\}$, in which only b is not an idempotent. The basis of identities A_2 is the following [7].

$$x^2 = x^3, xyx = xyxyx, xyxzx = xzxyx. \quad (1)$$

According to [6] T_n has the following basis of identities:

$$\alpha_r : (x_1 \dots x_r)^{m+1} x_1 \dots x_p = (x_1 \dots x_r)^{m+2} x_1 \dots x_p, \quad (2)$$

where $r \in \{1, \dots, n\}$, $p = n - 1 \pmod{r}$, $m = (n - p - 1)/r$,

$$\beta : t_1 x_1 \dots x_{n-1} y x_1 \dots x_{n-1} z x_1 \dots x_{n-1} t_2 = t_1 x_1 \dots x_{n-1} z x_1 \dots x_{n-1} y x_1 \dots x_{n-1} t_2. \quad (3)$$

For instance, α_1 is the identity: $x^n = x^{n+1}$. A locally testable semigroup S has only trivial subgroups [5] and so a locally testable semigroup S with n elements satisfies identity α_1 .

3 SOME AUXILIARY RESULTS

Lemma 1. *Let a semigroup S be locally testable and assume that $S^k = S^{k+1}$. Then the ideal S^k is 2-testable and belongs to $\text{var} A_2$.*

Proof. A locally testable semigroup S satisfies identities (2), (3) for some n and for all numbers greater than n . So we may suppose that $n \geq k$. All words from S^k may be presented as words of length k and greater and so $S^n = S^k$. The identity α_{k-1} : $(x_1 \dots x_k)^2 = (x_1 \dots x_k)^3$ of S implies the identity $x^2 = x^3$ in S^k . The identity α_k of S implies the identity $xyx = xyxyx$ in S^k . Now consider a word a from S^k . The word a may be presented in the form $a = t_1 b t_2$, where b is of length k . Then

$$ayaza = t_1 b t_2 y t_1 b t_2 z t_1 b t_2$$

. Using identity (3) for the words b , $t_2 y t_1$, $t_2 z t_1$ we see that

$$ayaza = t_1 b t_2 y t_1 b t_2 z t_1 b t_2 = t_1 b t_2 z t_1 b t_2 y t_1 b t_2 = azaya.$$

So all the identities (1) are true in S^k . Thus S^k belongs to $\text{var} A_2$ and is 2-testable because A_2 is 2-testable.

The lemma is proved.

Now from the necessary and sufficient conditions of local testability [6] we have the following.

Corollary. *Let $S^n = S^{n+1}$ for some n in some semigroup S . Then S is locally testable iff S^n belongs to $\text{var} A_2$.*

The following statement is well known.

Lemma 2. *Let S be a finite semigroup, $S^m = S^{m+1}$ for some m . Let E be the set of idempotents of S . Then any element of S^m is divided by an idempotent, that is $S^m = SES$.*

Proof. Let a belong to S^m . Then a may be represented by a word of length greater than $|S|$. So we can construct a chain of left subwords of a such that each element of the chain is a left divisor of the following element and the number of elements is greater than $|S|$. This implies that there are two different left subwords b and bc such that $b = bc$. Then $b = bc = bcc$ and $b = bc^n$. For some n , c^n is an idempotent and a right unit of b . The element b divides a and the right unit of b divides a too. Thus S^m is contained in SES . The opposite inclusion is obvious. So $S^m = SES$.

Lemma 3. Let S be a finite semigroup such that $S^2 = S$ and $S \in [x^2 = x^3, xyx = xyxyx]$.

The following two conditions are equivalent in S :

a) S satisfies the identity $xyxzx = xzxxyx$.

b) No two distinct idempotents e, i from S such that $eie = e, iei = i$ have a common unit in S . That is, there is no idempotent $f \in S$ such that $ef = e = fe$ and $if = i = fi$.

Proof Suppose S belongs to $[xzxxyx = xyxzx]$ and for some idempotents e, i in S $eie = e, iei = i$. Suppose f is a common unit of e, i . The identity $xyxzx = xzxxyx$ implies that $ei = fefif = fifef = ie$. Now $e = eie = eei = ei = eii = iei = i$. Thus the idempotents e, i are not distinct.

Suppose now that S does not belong to $[xyxzx = xzxxyx]$. So for some a, b, c of S $abaca \neq acaba$. Since $S^2 = S$, Lemma 2 implies that a is divided by some idempotent e , $a = peq$. Then $peqbpeqcpeq \neq peqcpeqbpeq$. This implies that $eqbpeqcpe$ and $eqcpeqbpe$ are distinct. From the identity $xyx = xyxyx$ it follows that $i = eqbpe$ and $j = eqcpe$ are idempotents. They have the common unit e and are distinct, because ij and ji are distinct.

Consider the elements iji and jij . In view of the identity $xyx = xyxyx$ they are idempotents too. It is routine to prove that they belong to an idempotent subsemigroup of S . The element e is then a common unit for iji and jij .

Now suppose that $iji = jij$. We have $ij = eije = eijeije = eijije = ejijje = jij$. Analogously $ji = jji$. So $ij = ji$ in contradiction to the above result. We conclude that $iji \neq jij$. So the distinct idempotents iji, jij belong to a band and have common unit e .

The lemma is proved.

From the Corollary to lemma 1 and lemmas 2 and 3 we obtain the following result.

Theorem 3.1 *Let S be a finite semigroup, and let E be the subset of idempotents of S . Suppose that SES satisfies the identities $x^2 = x^3$, $xyx = xyxyx$ and every two idempotents i, j in S having a common unit and such that $ij = i$, $ji = j$ or $ij = j$, $ji = i$ coincide. Then S is locally testable.*

This theorem will be the basis for the first part of the algorithm to verify the local testability of a finite semigroup.

Recall that a semigroup S is called *locally idempotent* iff eSe is an idempotent subsemigroup for any idempotent $e \in S$. Obviously, the set SES for a locally idempotent semigroup S with set of idempotents E satisfies identities $x^2 = x^3$, $xyx = xyxyx$. A locally testable semigroup is locally idempotent [1], [5]. Then from the result of the last theorem follows.

Theorem 3.2 (1) *A finite semigroup S is locally testable iff it is locally idempotent and S does not contain the three-element monoid with two left [right] zeroes. That is, S is locally testable iff eSe is a semilattice for all $e = e^2 \in S$.*

Now we consider the definition of n -testability from [1] and [2]. The results of [8] and [6] may be repeated in this case too. We present this fact without proof. Consider the following identity.

$$x_1 \dots x_{n-1} y x_1 \dots x_{n-1} z x_1 \dots x_{n-1} = x_1 \dots x_{n-1} z x_1 \dots x_{n-1} y x_1 \dots x_{n-1}. \quad (4)$$

Let B_n be the set of n -testable semigroups in the sense of [1].

Theorem 3.3 a) B_n is a variety,

b) A basis of identities for B_n for $n \geq 2$ consists of identities (2) and the identity (4),

c) $B_1 = [x^2 = x, xy = yx]$.

The only difference between the identities (3) and (4) is the omission in (4) of the first and last letters t from (3). Therefore, we have the following.

Corollary. T_n contains B_n . B_n contains T_{n-1} . $B_2 = \text{var } A_2$.

The following lemma enables us to find the level of local testability of a semigroup S .

Lemma 4. Let S be a finite semigroup satisfying the identities (2) for some n . Then the following two conditions are equivalent in S :

a) The semigroup S is n -testable.

b) Every two distinct idempotents e, i in S such that $ei = e, ie = i$ [$ei = i, ie = e$] have no common left [right] divisor in S^{n-1} .

Proof: Let us denote $X = x_1 \dots x_{n-1}$.

Suppose first that S is n -testable. Then according to theorem 3.3, S satisfies the identity

$$XyXzX = XzXyX. \quad (5)$$

Consider idempotents e, i such that $ie = i, ei = e$ having common left divisor a in S^{n-1} . We will prove that $e = i$. Let $i = ab, e = ac$. We have $i = ie = iei = abacab$. The identity (5) implies that $abacab = acabab = eii = ei = e$. So $e = i$, and a) implies b).

Suppose now that S is not n -testable. Then S does not satisfy identity (5). So for some elements a, b, c , where $a \in S^{n-1}$, we have $abaca \neq acaba$. One of the equalities $abaca = abacaba$, $acaba = abacaba$ does not hold in S . Without loss of generality suppose that $acaba$ and $abacaba$ are distinct. In view of the identity

$$XyX = XyXyX, \quad (6)$$

we have $abacaba \neq acabacaba$. Then $abacab$ and $acabacab$ are distinct. Let us denote $e = abacab, i = acabacab$. e, i are distinct. Using (6) we have

$e = abacab = abacabacab = abacababacab = e^2$. Analogously i is idempotent too. Now $ei = abacabacabacab = abacab = e$. Analogously $ie = i$. So we find two distinct left zeroes having a common left divisor in S^{n-1} .

Let S be a finite locally testable semigroup and let $\phi : \Sigma^+ \rightarrow S$ be a surjective morphism for some alphabet Σ . Let a be an element from $S \setminus SES$. Let m be the maximal number such that $a^{m+1} \neq a^{m+2}$. Suppose $a = bc$. Since a belongs to $S \setminus SES$, it follows that a, b, c have only a finite number of preimages in Σ^+ . Denote $|a|, |b|, |c|$ the maximal length of the preimages of the elements a, b, c in the alphabet Σ , correspondingly. Suppose $n = \max((|b| + |c|)m + |b| + 1$ if $a^{m+1}b \neq a^{m+2}b, |a|m + 1$ otherwise) for all $a \in S \setminus SES$ and for all b, c such that $bc = a$.

Lemma 5. The minimal number for which S satisfies identities (2) is equal to $n+1$.

Proof: Consider some identity α_r from (2) and the corresponding words $(a_1 \dots a_r)^{m+1} a_1 \dots a_p, (a_1 \dots a_r)^{m+2} a_1 \dots a_p$. Denote $a = a_1 \dots a_r, b = a_1 \dots a_p, c = a_{p+1} \dots a_r$. Because $a^{m+1} \neq a^{m+2}$ the

semigroup S is not n -testable for $n = m|a|+1$. If the words $a^{m+1}b$ and $a^{m+2}b$ are not equal then the semigroup S is not n -testable for $n = m(|b| + |c|) + |b| + 1$. So the maximum of all such numbers n gives us the precise bound for which the identities (2) are not valid.

From Lemmas 4 and 5 we have the following theorem.

Theorem 3.4 . *Let S be a finite locally testable semigroup. Let a be an element of $S \setminus SES$. Let b be a proper left divisor of a and $a = bc$. Let $|a|$, $|b|$, $|c|$ be the maximal length of the words a, b, c in some alphabet Σ . Let m be the maximal number such that $a^{m+1} \neq a^{m+2}$. Let $n = \max((|b| + |c|)m + |b| + 1$ if $a^{m+1}b \neq a^{m+2}b$, $|a|m + 1$ otherwise) for all $a \in S \setminus SES$ and for all b, c such that $bc = a$.*

Let e, i be idempotents of a left [right] zero subsemigroup and let a be a left [right] common divisor of maximal length. Let $l(e, i) = |a| + 1$ [$r(e, i) = |a| + 1$] and let $l = \max(l(e, i))$ [$r = \max(r(e, i))$] for all pairs of left [right] zeroes.

Then $\max(n, r, l) + 1$ is equal to the exact level of local testability of the semigroup S .

Recall that a semigroup S is a left (right) zero semigroup if S satisfies the identity $xy = x(xy = y)$. The following proposition is useful for the next algorithm.

Proposition Let E be the set of idempotents of a semigroup and let $|E| = n$. We represent E as an ordered list $[e_1, \dots, e_n]$. Then there exists an algorithm of order n^2 that reorders the list so that the maximal left (right) zero subsemigroups of S appear consecutively in the list.

Finding the maximal semigroup of left zeroes containing a given idempotent needs n steps. So to reorder E we need at most n^2 steps.

4 ALGORITHMS

1. Testing whether a finite semigroup S is locally testable.

Suppose $|S| = k$. We begin by finding the set of idempotents E . This is a linear time algorithm. After this we find SE and then SES using two times $O(k^2)$ steps.

In view of Theorem 3.1 we begin by verifying the first two identities from (1) in SES . Verifying the identity $x^2 = x^3$ needs $O(k)$ steps, verifying the identity $xyx = xyxyx$ needs $O(k^2)$ steps.

Now consider the last identity from (1). In view of Lemma 3 consider the set E . We can reorder E according to the proposition above in a chain such that the subsemigroups of left zeroes form intervals in this chain. We note the bounds of these intervals. We find for each element e of E the first element i in the chain such that e is a unit for i . Then we find in the chain the next element j with the same unit e . If i and j belong to the same subsemigroup of left zeroes we conclude that S is not testable (Lemma 3) and end the process. If they are in different left zero semigroups, we replace i by j and continue the process of finding a new j . This takes $O(k^2)$ steps.

Then we repeat the same process for right zeroes. According to Theorem 3.1 we can give a positive answer to the question in the case we do not find two different left [right] idempotents with the same unit.

2. Finding the level of local testability.

The idea is based on theorem 3.4.

Suppose the semigroup S is locally testable, $|S| = k$, E is the set of idempotents of S , as above.

We use the sets E , SE and SES found above. According to lemma 1, $SES = S^l$, where $S^l = S^{l+1}$. We find $G = S \setminus SES$ as well.

In the case that G is empty the semigroup S belongs to $varA_2$ and is 2-testable in both senses. Verifying of 1-testability reduces to testing the identities $x = x^2$, $xy = yx$. So we are done in this case. Now suppose that G is not empty.

We start with the assignment $n := 2$ to a variable n that will change during the algorithm below.

For each element a we find a maximal number m that $a^{m+1} \neq a^{m+2}$. Denote this number by $m(a)$. This is an algorithm of order at most k . The maximum of all such $m(a)$ gives us a first lower bound of the order of local testability.

Then we find generators for G . It is easy to see that the unique minimal generating set for G is $G \setminus G^2$. This takes k^2 steps. We denote the set of these elements by G_1 . The maximal length of elements from G_1 is 1. We now define a sequence of sets $G_i, i \geq 1$. We want G_i to be equal to the set of elements in G that can be written as a product of i elements from G_1 , but that cannot be written as a product of more than i elements of G_1 . Assume that we have correctly defined G_i for some $i \geq 1$. Then we let $G_{i+1} = (G_1 G_i) \setminus ((G \setminus G_1) G_i)$. It is easy to see by induction that this correctly defines G_i for $1 \leq i \leq l-1$. Elements of G_i are said to have level i . Each element of G has a well defined level. This process need k^2 steps, because $|G| = |G_1| + \dots + |G_{l-1}|$. We know that the level of an element $g \in G$ is equal to its maximal length in any set of generators for G and will be denoted by $|g|$.

Consider now all possible products bc for b, c from G . This takes $O(k^2)$ and on each step for $a \in G$ we do the following:

Suppose that $a = bc$. Let $n = \max(n, m(a)|a|+1)$. Consider the element $a^{m+2}b$. If this is not equal to the element $a^{m+1}b$, we make the following assignment $n := \max(n, (|b|+|c|)m+|b|+1)$.

After considering all pairs we get a value for n . In view of Lemma 5 the semigroup S does not satisfy identities (2) for n and satisfies (2) for $n+1$.

Now consider the identity (4). What follows will be based on Lemma 4. We first reorder the set of idempotents E as in the proposition for left zero subsemigroups. We note the bounds between the subsemigroups. In view of the Proposition this takes at most $O(k^2)$ steps.

Let us assign $L := n$.

For each g of G we form the intersection $gSES \cap E$. Then we verify: are all elements of the intersection within the bounds or not. If there are two idempotents not within the bounds the semigroup S is not $(|g|+1)$ -testable and may be only $|g|+2$ -testable. Now assign $L := \max(L, |g|+1)$. This needs at most $2k$ steps. Repeating this process for all g from G , we find the maximum L for all such g , using at most $2k^2$ steps. The semigroup S is not L -testable and may be only $L+1$ -testable.

Then we repeat the procedure for the right order of E and right divisors of idempotents. As a result the upper bound R may be obtained.

Theorem 3.3 gives us the level of local testability as $1 + \text{maximum of the three above-mentioned numbers } n, R \text{ and } L$.

The algorithm is a polynomial time algorithm of order $O(k^2)$, where k is the order of the semigroup.

Both parts of the algorithm give us a way to verify testability and to find its level.

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