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A FUZZY DATA ENVELOPMENT ANALYSIS FOR CLUSTERING OPERATING UNITS WITH IMPRECISE DATA

SABER SAATI
Department of Mathematics, Tehran-North Branch, Islamic Azad University
P.O. Box 19585-936, Tehran, Iran
s_saatim@iau-tnb.ac.ir

ADEL HATAMI-MARBINI*
Louvain School of Management, Center of Operations Research and Econometrics (CORE),
Université Catholique de Louvain,
L1.03.01, B-1348 Louvain-la-Neuve, Belgium
adel.hatamimarbini@uclouvain.be

MADJID TAVANA
Business Systems and Analytics Department, La Salle University,
Philadelphia, PA 19141, USA
tavana@lasalle.edu

PER J. AGRELL
Louvain School of Management, Center of Operations Research and Econometrics (CORE),
Université Catholique de Louvain,
L1.03.01, B-1348 Louvain-la-Neuve, Belgium
per.agrell@uclouvain.be

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Data envelopment analysis (DEA) is a non-parametric method for measuring the efficiency of peer operating units that employ multiple inputs to produce multiple outputs. Several DEA methods have been proposed for clustering operating units. However, to the best of our knowledge, the existing methods in the literature do not simultaneously consider the priority between the clusters (classes) and the priority between the operating units in each cluster. Moreover, while crisp input and output data are indispensable in traditional DEA, real-world production processes may involve imprecise or ambiguous input and output data. Fuzzy set theory has been widely used to formalize and represent the impreciseness and ambiguity inherent in human decision-making. In this paper, we propose a new fuzzy DEA method for clustering operating units in a fuzzy environment by considering the priority between the clusters and the priority between the operating units in each cluster simultaneously. A numerical example and a case study for the Jet Ski purchasing decision by the Florida Border Patrol are presented to illustrate the efficacy and the applicability of the proposed method.

Keywords: Data envelopment analysis; clustering; priority, ranking; fuzzy input and output data; Florida border patrol.

*Corresponding author.

1. Introduction

Data envelopment analysis (DEA) is a methodology for measuring the relative efficiency of a set of operating units that use multiple inputs to produce multiple outputs. Originally proposed by Farrell,[1] Charnes *et al.*[2] popularized the non-parametric frontier analysis when they proposed the first DEA model for constant returns to scale. Numerous developments to both theory and application have been proposed over the past three decades. Some of the pioneering work in DEA include: the Russell measure originated by Färe and Lovell[3] and later enhanced by Pastor *et al.*,[4] the free disposal hull model originated by Deprins *et al.*[5] and Tulkens,[6] the cross efficiency originated by Sexton *et al.*,[7] the window analysis introduced by Charnes *et al.*,[8] the absolute multiplier restrictions proposed by Roll *et al.*,[9] the application of chance constrained programming in DEA by Thore[10] and Land *et al.*,[11,12] the Malmquist productivity index by Färe *et al.*,[13] the network DEA by Färe and Grosskopf,[14] the range adjusted measure as a non-radial model by Cooper *et al.*,[15] and the slacks-based measure by Tone.[16]

The most widely used extensions of DEA models include those of the variable returns to scale model (Banker *et al.*[17]), the additive model (Charnes *et al.*[18]), the fuzzy DEA (Sengupta[19]), the imprecise model (Cooper *et al.*[20]), the robust DEA (Shokouhi *et al.*[21]), the assurance region model (Thompson *et al.*[22]), the cone ratio model (Charnes *et al.*[23]), the super-efficiency model (Li *et al.*[24]), and the chance-constrained and stochastic models (Cooper *et al.*[25]). A detailed review and taxonomy of various DEA models can be found in Cook and Seiford[26] and Emrouznejad and De Witte.[27]

One limitation of the conventional DEA methods is the need for accurate measurement of both the input and the output data. While crisp input and output data are fundamentally indispensable in the conventional DEA models, input and output data in real-world problems are often imprecise or ambiguous. Imprecise evaluations may be the result of unquantifiable, incomplete and non-obtainable information. Numerous fuzzy methods have been proposed to deal with this impreciseness and ambiguity in DEA since the original study by Sengupta.[19] In general, fuzzy DEA methods can be classified into four primary categories, namely, the tolerance approach (Sengupta[19]), the α-level based approach (Kao and Liu,[28] Saati *et al*.,[29] Hatami-Marbini *et al.*[30]), the fuzzy ranking approach (Guo and Tanaka[31]) and the possibility approach (Lertworasirikul *et al.*[32]). An exhaustive review and taxonomy of various fuzzy DEA models can be found in Hatami-Marbini *et al.*[33]

Clustering is the process of organizing a set of objects (operating units) into a useful set of mutually exclusive clusters such that the similarity of the objects *within* a cluster is maximized while the similarity of the objects *between* different clusters is minimized (e.g., Jain *et al.*,[34] Okazaki,[35] Rai *et al.*,[36] Samoilenko and Osei-Bryson,[37] Wallace *et al.*[38]). Generally, clustering methods are grouped into hierarchical, learning network, and distance-based clustering.

Hierarchical clustering groups the objects by creating a cluster tree called dendrogram. Clusters are then formed by either the agglomerative approach or the divisive approach (Johnson,[39] Kaufman and Rousseeuw[40]). Agglomerative methods assume that each object is its own cluster and then these clusters are combined to form larger clusters with each step of the process. Eventually, these clusters are combined to form a single cluster. Divisive methods assume a single cluster encompassing all the objects within the sample and then proceeds to divide this cluster into smaller dissimilar clusters.

Learning network clustering is a neural network based unsupervised clustering where high dimensional data is mapped into a discrete one or two-dimensional space. Learning network clustering performs clustering through a competitive learning mechanism (Bu *et al.*,[41] Choi and Yoo,[42] Harb and Chen,[43] Kohenen[44]).

Distance-based clustering is a partitioning method which creates an initial cluster and then uses an iterative relocation technique to maximize total similarity or minimize total dissimilarity by moving objects from one cluster to another. *K*-means (McQueen[45]), fuzzy *c*-means (Yang,[46] Wu and Yang[47]) and possibilistic c-means (Krishnapuram and Keller[48]) are various forms of distance-based clustering.

The integration of clustering with DEA is not novel (Lemos *et al.*,[49] Marroquin *et al.*,[50] Meimand *et al.*,[51] Po *et al.*,[52] Samoilenko and Osei-Bryson,[53,37] Schreyögg and von Reitzenstein,[54] Sharma and Yu,[55] Shin and Sohn[56]). In general, clustering is integrated with DEA in two different ways. In the first approach the clustering results are applied to the results of DEA to construct multiple reference subsets from the original set of DMUs (Meimand *et al.*[51]). In the second approach, the efficiency score of a DMU is defined not by its peer group (an efficient subset of all DMUs) but by an efficient subset of its peer subgroup. Consequently, this approach will results in isolation of the multiple homogeneous subsets in the presence of scale heterogeneity of the sample and then each DMU is compared only with the appropriate subset consisting of its peers within the subset.

Samoilenko and Osei-Bryson[53] proposed a solution for performing DEA of a scale heterogeneous data set and their method did not require (1) explicit partitioning of the sample of DMUs into multiple peer groups; (2) a large data set; or (3) any data external to DEA as suggested by Dyson *et al.*[57] and used by Sarrico and Dyson.[58] Instead, their method took into consideration the presence of heterogeneous subsets without actually dividing the sample. As a result, their approach was not incongruent with one suggested in Dyson *et al.*,[57] where grouping of DMUs into homogenous subsets was based on management information.

In this paper, we propose a new DEA method for clustering operating units in a fuzzy environment by considering the priority between the clusters and the priority between the operating units in each cluster simultaneously. The proposed clustering-based DEA model defines the group of operating units that are similar to the operating unit under evaluation. This clustering process results in clusters with homogenous members. In addition, we present a numerical example and a case study for the Jet Ski purchasing decision by the Florida Border Patrol to illustrate the efficacy and the applicability of the proposed method.

This paper is organized into eight sections. In Sec. 2, we provide some basic definitions of fuzzy sets. In Section 3, we present an overview of DEA and the fuzzy DEA framework followed by the DEA-based clustering method proposed in this study in Sec. 4. In Sec. 5 we present a numerical example to illustrate the efficacy of the proposed method and in Sec. 6 we present the Florida Border Patrol case study. We compare the proposed clustering algorithm with other methods in Sec. 7. In Sec. 8 we summarize with our conclusions and future research directions.

2. Fuzzy Background

Fuzzy sets were introduced by Zadeh[59] as a means of representing and manipulating imprecise and inexact data associated with human cognitive processes (such as thinking and reasoning) with fuzzy numbers. The conventional approaches to knowledge representation lack the means to represent fuzzy numbers. As a consequence, the approaches grounded in first order logic and classical probability theory cannot provide an appropriate conceptual framework for dealing with commonsense knowledge representation since such knowledge is both lexically imprecise and non-categorical. In the following section we review several basic definitions of fuzzy sets (Zimmermann,[60] Dubois and Prade,[61] Kauffman and Gupta[62]).

Definition 2.1. (*Fuzzy set*): Let X be a nonempty set. A fuzzy set $\tilde{A}$ in X is characterized by its membership function

$$\mu_{\tilde{A}}(x) \to [0,1]$$

and $\mu_{\tilde{A}}(x)$ is interpreted as the degree of membership of element x in fuzzy set $\tilde{A}$ for each $x \in X$.

It is clear that $\tilde{A}$ is completely determined by the set of tuples

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x) \right) \mid x \in X \right\}.$$

Definition 2.2. (α*-cut*): An α-level set of a fuzzy set $\tilde{A}$ of X is a non-fuzzy set denoted by $\tilde{A}_\alpha$ and is defined by

$$\tilde{A}_\alpha = \begin{cases} \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\} \\ cl(\sup A) \end{cases},$$

where cl(sup A) denotes the closure of the support of $\tilde{A}$.

Definition 2.3. (*Fuzzy number*): A fuzzy number $\tilde{A}$ is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support whereby its normality and convexity can be defined as follows:

Convexity: $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)), \forall x, y \in X, \forall \lambda \in [0,1],$

Normality: $\exists x \in X,\ \mu_{\tilde{A}}(x) = 1$.

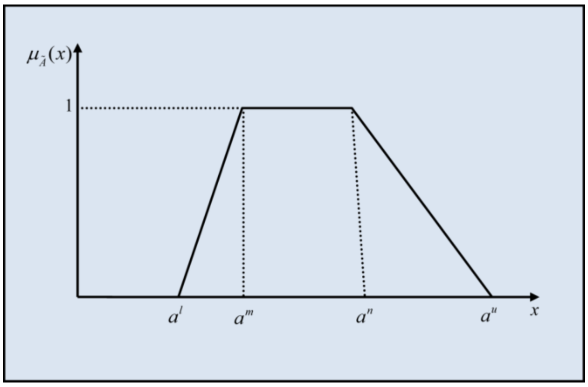

Fig. 1. A trapezoidal fuzzy number.

Definition 2.4. (*Generalized trapezoidal fuzzy number*): A fuzzy number $\tilde{A}$ = (a^l, a^m, a^n, a^u), is called a generalized trapezoidal fuzzy number with membership function $\mu_{\tilde{A}}(x)$ and has the following properties:

- $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to the closed interval [0, 1],
- $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, a^l]$,
- $\mu_{\tilde{A}}(x)$ is strictly increasing on [a^l , a^m],
- $\mu_{\tilde{A}}(x) = 1$ for all $x \in [a^m, a^n]$,
- $\mu_{\tilde{A}}(x)$ is strictly decreasing on [a^n , a^u], and
- $\mu_{\tilde{A}}(x) = 0$ for all $x \in [a^u, +\infty)$.

The membership function $\mu_{\tilde{A}}(x)$ of $\tilde{A}$ can be defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} f_{\tilde{A}}(x), & a^l \le x \le a^m, \\ 1, & a^m \le x \le a^n, \\ g_{\tilde{A}}(x), & a^n \le x \le a^u, \\ 0, & \textit{Otherwise.} \end{cases} \tag{1}$$

where $f_{\tilde{A}} : [a^l, a^n] \to [0,1]$ and $g_{\tilde{A}} : [a^m, a^u] \to [0,1]$.

Particularly, a special type of trapezoidal fuzzy number, plotted in Fig. 1, with a membership function $\mu_{\tilde{A}}(x)$ can be expressed as:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a^l}{a^m - a^l}, & a^l \le x \le a^m, \\ 1, & a^m \le x \le a^n, \\ \dfrac{a^u - x}{a^u - a^n}, & a^n \le x \le a^u, \\ 0, & \textit{Otherwise.} \end{cases} \tag{2}$$

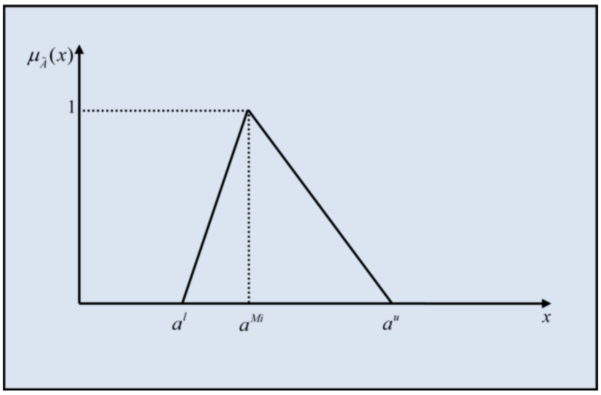

Fig. 2. A triangular fuzzy number.

If $a^l = a^m$ and $a^n = a^u$, then $\tilde{A}$ is called a crisp or simple interval. The trapezoidal fuzzy number $\tilde{A}=(a^l, a^m, a^n, a^u)$ is reduced to a real number *A* if $a^l = a^m = a^n = a^u$. In an opposite way, a real number *A* can be written as a trapezoidal fuzzy number $\tilde{A}=$ (*a*, *a*, *a*, *a*). If $a^{Mi} = a^m = a^n$, then, $\tilde{A}=(a^l, a^{Mi}, a^u)$ is called a triangular fuzzy number shown in Fig. 2. A triangular fuzzy number has the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a^l}{a^{Mi} - a^l}, & a^l \le x \le a^{Mi}, \\ \dfrac{a^u - x}{a^u - a^{Mi}}, & a^{Mi} \le x \le a^u, \\ 0, & \textit{Otherwise.} \end{cases} \tag{3}$$

For the sake of simplicity and without loss of generality, we assume that all fuzzy numbers used throughout the paper are triangular fuzzy numbers.

Definition 2.5. (*Linguistic variables*): Linguistic variables are represented in words or sentences or artificial languages, where each linguistic value can be modeled by a fuzzy set. For example, "very low", "low", "medium", "high", or "very high" are linguistic variables because their values are represented by verbal phrases rather than numerical values. It should be noted that there are several methods for representing linguistic variables. The representing method used in practice depends on the application and the domain experts' preferences. The concept of a linguistic variable is useful in dealing with settings that are too complex or too ill-defined to be reasonably described with quantitative values. Linguistic values can also be represented by fuzzy numbers.

Definition 2.6. (*Fuzzy arithmetic operation*): In fuzzy linear programming, the min T-norm is usually applied to assess a linear combination of fuzzy quantities. Therefore, for a given set of trapezoidal fuzzy numbers $\tilde{u}_j = (u_j^l, u_j^m, u_j^n, u_j^u),\ j = 1,...,n$ and $\lambda_j \geq 0$, $\sum_{j=1}^{n} \lambda_j \tilde{u}_j$ can be expressed as follows:

$$\sum_{j=1}^{n} \lambda_j \tilde{u}_j = \left(\sum_{j=1}^{n} \lambda_j u_j^l, \sum_{j=1}^{n} \lambda_j u_j^m, \sum_{j=1}^{n} \lambda_j u_j^n, \sum_{j=1}^{n} \lambda_j u_j^u \right) \tag{4}$$

where $\sum_{j=1}^{n} \lambda_j \tilde{u}_j$ denotes the combination $\lambda_1 \tilde{u}_1 \oplus \lambda_2 \tilde{u}_2 \oplus \ldots \oplus \lambda_n \tilde{u}_n$.

3. DEA and Fuzzy DEA Framework

DEA was initially developed as a fractional linear program to assess the comparative efficiencies of operating units that use multiple inputs to produce multiple outputs. Based on the economic notion of Pareto optimality, the DEA methodology states that a decision making unit (DMU) is considered to be inefficient if another DMUs can produce at least the same amount of output with less of the same resource input and not more of any other resource. Otherwise, a DMU is considered to be Pareto efficient. Assume that there are n DMUs to be evaluated where every DMU$_j$ ($j = 1,...,n$) produces the same s outputs in various amounts, y_{rj} ($r = 1,...,s$), using the same m inputs, x_{ij} ($i = 1,...,m$), also in various amounts. The relative efficiency of the DMU$_p$ can be obtained by using the following CCR model proposed by Charnes *et al.*[2]:

$$\begin{aligned} \max \quad & \theta_p = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}} \\ s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad \forall j, \\ & u_r, v_i \geq 0, \qquad \forall r, i. \end{aligned} \tag{5}$$

where u_r and v_i are the weights assigned to the rth output and ith input, respectively. The interpretation of the DEA model (5) is a ratio of a weighted sum of outputs to a weighted sum of inputs where the weights for both inputs and outputs are to be selected in a manner that calculates the efficiency of the evaluated unit. Model (5) can be solved using a linear form as shown below by performing the Charnes–Cooper[63] transformation:

$$\begin{aligned}
\max \quad & \theta_p = \sum_{r=1}^{s} u_r y_{rp} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ip} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad \forall j, \\
& u_r, v_i \ge 0, \qquad \forall r, i.
\end{aligned} \tag{6}$$

Note that the DMUs with $\theta_p^* = 1$, are called [technically] efficient units, and those units with $\theta_p^* \neq 1$ are called [technically] inefficient units. It is generally helpful for decision makers only to focus on the efficient DMUs. However, decision makers always face the problem of how to carry out additional comparisons among the efficient DMUs. The following duality form of model (6) also provides information about the contraction of resources or expansion of outputs for the DMUs to move from inefficiency to efficiency.

$$\begin{aligned}
\min \quad & \theta_p \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{ip}, \quad \forall i, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp}, \quad \forall r, \\
& \lambda_j \ge 0, \qquad \forall j.
\end{aligned} \tag{7}$$

Model (7) is referred to as the envelopment or primal problem, and (6) the multiplier or dual problem. In the original DEA and its extensions, all the inputs and outputs assume the form of specific numerical values. In many real-world problems, however, the data can be imprecise or vague or described by qualitative terms. How to deal with the imprecise and ambiguous data has been widely discussed in the DEA literature. Fuzzy logic and fuzzy sets can represent imprecise or ambiguous data in DEA by formalizing inaccuracy in decision making (Hatami-Marbini *et al.*[33]). A generic fuzzy CCR model and its dual are given as:

$$\begin{aligned}
\max \quad & \theta_p = \sum_{r=1}^{s} u_r \tilde{y}_{rp} \\
s.t. \quad & \sum_{i=1}^{m} v_i \tilde{x}_{ip} = 1 \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0, \quad \forall j, \\
& u_r, v_i \ge 0, \qquad \forall r, i.
\end{aligned} \tag{8}$$

$$\begin{aligned}
\min \quad & \theta_p \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \theta \tilde{x}_{ip}, && \forall i, \\
& \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \ge \tilde{y}_{rp}, && \forall r, \\
& \lambda_j \ge 0, && \forall j.
\end{aligned}$$

Note that in the multiplier form of the fuzzy CCR model (8) the right hand sides of the constraints are assumed to be crisp values because they are similar to the original CCR model used for normalization of the value of the efficiency in the objective function. The fuzzy DEA model can be used to cope with all kinds of fuzzy number shapes. In this study we use triangular fuzzy numbers to develop our model. However, our model is adaptable to other types of fuzzy numbers. The following model, therefore, can be obtained when fuzzy coefficients in model (8) are assumed to be triangular fuzzy numbers denoted as $\tilde{x}_{ij} = (x_{ij}^l, x_{ij}^m, x_{ij}^u)$ and $\tilde{y}_{rj} = (y_{rj}^l, y_{rj}^m, y_{rj}^u)$:

$$\begin{aligned}
\max \quad & \theta_p = (\sum_{r=1}^{s} u_r y_{rp}^l, \sum_{r=1}^{s} u_r y_{rp}^m, \sum_{r=1}^{s} u_r y_{rp}^u) \\
s.t.\ & (\sum_{i=1}^{m} v_i x_{ip}^l, \sum_{i=1}^{m} v_i x_{ip}^m, \sum_{i=1}^{m} v_i x_{ip}^u) = 1, \\
& (\sum_{r=1}^{s} u_r y_{rj}^l, \sum_{r=1}^{s} u_r y_{rj}^m, \sum_{r=1}^{s} u_r y_{rj}^u) - (\sum_{i=1}^{m} v_i x_{ij}^l, \sum_{i=1}^{m} v_i x_{ij}^m, \sum_{i=1}^{m} v_i x_{ij}^u) \le 0, \quad \forall j, \\
& u_r, v_i \ge 0, \quad \forall r, i.
\end{aligned}$$

$$\begin{aligned}
\min \quad & \theta_p \\
s.t. \quad & (\sum_{j=1}^{n} \lambda_j x_{ij}^m, \sum_{j=1}^{n} \lambda_j x_{ij}^l, \sum_{j=1}^{n} \lambda_j x_{ij}^u) \le (\theta x_{ip}^m, \theta x_{ip}^l, \theta x_{ip}^u), \quad \forall i, \\
& (\sum_{j=1}^{n} \lambda_j y_{rj}^m, \sum_{j=1}^{n} \lambda_j y_{rj}^l, \sum_{j=1}^{n} \lambda_j y_{rj}^u) \ge (y_{rp}^m, y_{rp}^l, y_{rp}^u), \quad \forall r, \\
& \lambda_j \ge 0, \quad \forall j.
\end{aligned} \tag{9}$$

The above models cannot be solved by a standard linear program solver program because of the fuzzy numbers. In the recent fuzzy DEA survey, Hatami-Marbini *et al.*[33] classified the existing approaches to solve models (9) into four general categories: (1) the tolerance approach, (2) the α-level based approach, (3) the fuzzy ranking approach, and (4) the possibility approach. The α-level-based approach is probably the most popular fuzzy DEA model among the aforementioned approaches (Hatami-Marbini *et al.*[33]). Therefore, the α-level-based approach is utilized here to consider the fuzzy data in performance assessment.

4. Fuzzy DEA-based Clustering Method

The purpose of the clustering methods is to identify partitions of data with respect to some form of similarity. The partitions of the set are called clusters. In other words, the predefined features such as color, quality, distance, number of observations and so on can be often utilized to categorize observations into various groups. In most conventional methods, the distance feature is used for classification of the observations that it is known as an absolute feature. Moreover, even though all data in the conventional DEA model are known precisely or given as crisp values, under many conditions, crisp data are inadequate or insufficient to model a real-life evaluation problem. In this section, we will propose an alternative DEA-based clustering algorithm to classify a set of evaluated DMUs when imprecise input-output data are characterized with fuzzy numbers. In order to cluster DMUs we will use the observations' ranking criterion in our method which is a rational feature. The proposed approach, in addition to the ranking of DMUs in the imprecise environment, considers the priority among classes and the priority among DMUs in each cluster. Suppose that we have *n* DMUs, DMU$_j$ $(j=1,\ldots,n)$, each using different amounts of *m* inputs to produce *s* outputs. Let $\tilde{x}_{ij}=(x_{ij}^l,x_{ij}^m,x_{ij}^u)$ $(i=1,2,...,m)$ and $\tilde{y}_{rj}=(y_{rj}^l,y_{rj}^m,y_{rj}^u)$ $(r=1,2,...,s)$ be the fuzzy triangular inputs and outputs, respectively, for DMU$_j$ $(j=1,\ldots,n)$, where $\tilde{x}_{ij}\geq 0$, $\tilde{y}_{ij}\geq 0$, $\tilde{x}_{ij}\neq 0$, and $\tilde{y}_{ij}\neq 0$. In the first step, we rank the DMUs based on the ranking method proposed by Saati *et al.*[29]:

$$\begin{aligned}
&\min \quad \theta_p \\
&\text{s.t:} \quad \theta_p(\alpha x_{ip}^m+(1-\alpha)x_{ip}^l)\geq \sum_{j\in J}\lambda_j(\alpha x_{ij}^m+(1-\alpha)x_{ij}^u), \qquad \forall i, \\
&\qquad \alpha y_{rp}^m+(1-\alpha)y_{rp}^u\leq \sum_{j\in J}\lambda_j(\alpha y_{rj}^m+(1-\alpha)y_{rj}^l), \qquad \forall r, \\
&\qquad \lambda_j\geq 0, \qquad \forall j.
\end{aligned} \tag{10}$$

Saati *et al.*[29] developed model (10) to rank the efficient DMUs in a fuzzy environment using the concept of α-cut. In their model, the best part of a DMU which is the lower and upper levels of inputs and outputs, respectively, are compared with the inner part of efficiency frontier. It is clear that in this case the efficient DMU increases its efficiency score to more than unity since the projection is made outside of the *possibility production set*. Consequently, after running model (10), the DMUs whose objective function values are greater than or equal to one are placed in the first cluster. Moreover, the DMUs placed in the first cluster can be ranked easily by their objective function values obtained from (10). In other words, the DMU with the greater θ has priority over the remaining DMUs. In the next step, we remove the DMUs that are assigned in the last step and solve model (10) for the remaining DMUs again. Accordingly, the DMUs whose objective function values are greater than or equal to one lie in the second cluster and, similarly, we can determine the priority among these selected DMUs. Likewise, we remove the assigned DMUs in the preceding step and the same method is applied until one DMU remains. It is important to note that in this DEA clustering method we cannot define the number of clusters before implementing the proposed algorithm and the number of clusters can be determined after applying the algorithm. It is also important to note that the importance of the first cluster is more than the second cluster, the importance of the second cluster is more important than the third cluster and so on. The following 10-step algorithm depicted in Fig. 3 summarizes the entire process:

Step 1. Assume a set of DMUs index ($J = \{1, 2, \ldots, n\}$),

Step 2. Set $k = 0$ as the cluster number,

Step 3. Set $M = \varnothing$ as an index of clustered DMUs,

Step 4. Consider $k = k + 1$,

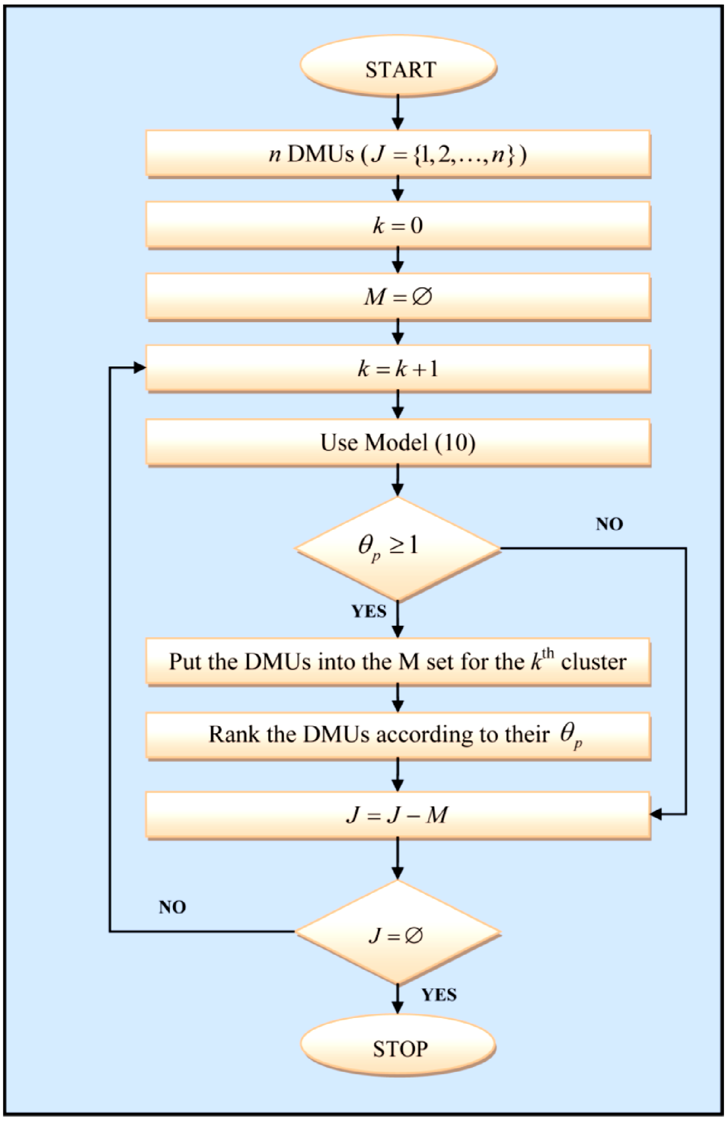

Fig. 3. The flowchart for the proposed algorithm.

Table 1. The numerical example of Saati *et al.* (2002).

DMU	Inputs		Outputs	
	1	2	1	2
A	(6, 7, 8)	(29, 30, 32)	(35.5, 38, 41)	(409, 411, 416)
B	(5.5, 6, 6.5)	(33, 35, 36.5)	(39.5, 40, 43)	(478, 480, 484)
C	(7.5, 9, 10.5)	(43, 45, 48)	(32.5, 35, 38)	(297, 299, 301)
D	(7, 8, 10)	(37.5, 39, 42)	(28, 31, 31)	(347.5, 352, 360)
E	(9, 11, 12)	(43, 44, 45)	(33, 35, 38)	(406, 411, 415)
F	(10, 10, 10)	(53, 55, 57.5)	(36, 38, 40)	(282, 286, 289)
G	(10, 12, 14)	(107, 110, 113)	(34.5, 36, 38)	(396, 400, 405)
H	(9, 13, 16)	(95, 100, 101)	(37, 41, 46)	(387, 393, 402)
I	(12, 14, 15)	(120, 125, 131)	(24, 27, 28)	(400, 404, 406)
J	(5, 8, 10)	(35, 38, 39)	(48, 50, 51)	(470, 470, 470)

Step 5. Utilize the proposed model (10) for the DMUs which consists of the J index,

Step 6. Assign the DMUs whose objective function values (obtained from step 5) are greater than or equal to one to the k^{th} cluster,

Step 7. Rank the DMUs obtained from step 6 according to their objective function values,

Step 8. Add the index of the DMUs in the k^{th} cluster to the set of M,

Step 9. Consider $J = J - M$,

Step 10. Stop the algorithm if $J = \varnothing$, otherwise, return to step 4.

5. Numerical Example

In this section, we illustrate the applicability and efficacy of the proposed algorithm with a numerical example taken from Saati *et al.*[29] This example includes two triangular fuzzy inputs and two trapezoidal fuzzy outputs with 10 DMUs as shown in Table 1.

A performance evaluation problem in the real world often consists of precise and imprecise data. Therefore, we can observe some crisp data as well as fuzzy data in Table 1. Note that the crisp data are described by triangular fuzzy numbers with equal medium, lower and upper values.

We first assume that $\alpha = 0.5$. The proposed algorithm is executed to cluster 10 DMUs and the procedure is briefly described here:

Step 1. We evaluate the following 10 DMUs in this clustering example:
$J = \{A, B, C, D, E, F, G, H, I, J\}$.
Steps 2 and 3. $k = 0$ (the first cluster) and $M = \varphi$.

<u>The first iterate/cluster</u>

Step 4. $k = 1$.
Step 5. Model (10) for ten DMUs is executed to obtain the objective function values presented in Table 2.

Table 2. The objective function values of model (10) for the ten DMUs in the first iteration.

DMU	A	B	C	D	E	F	G	H	I	J
Objective function value	1.10	1.14	0.73	0.71	0.72	0.64	0.53	0.63	0.41	1.24
Selected DMUs in the 1st cluster	*	*								*
Rank	3	2								1

Table 3. The objective function values of model (10) for the seven DMUs in the second iteration.

DMU	C	D	E	F	G	H	I
Objective function value	1.23	1.25	1.14	1.07	0.99	1.12	0.80
Selected DMUs in the 2nd cluster	*	*	*	*		*	
Rank	2	1	3	5		4	

Table 4. The objective function values of model (10) for the two DMUs in the third iteration.

DMU	G	I
Objective function value	1.24	1.09
Selected DMUs in the 3rd cluster	*	*
Rank	1	2

Step 6. As shown in Table 2, the objective function values for *A*, *B* and *J* are bigger than one. Therefore, these DMUs are placed in the first cluster.

Step 7. Rank the DMUs *A*, *B* and *J* presented in the last row of Table 2.

Step 8. $M = \{A, B, J\}$.

Step 9. $J = \{C, D, E, F, G, H, I\}$.

Step 10. $J \neq \varnothing$.

<u>The second iterate/cluster</u>

Step 4. $k = 2$.

Step 5. Model (10) for the remaining seven DMUs, $J = \{C, D, E, F, G, H, I\}$, is executed to obtain the objective function values presented in Table 3.

Step 6. C, D, E, F and H shown in Table 3 are placed in the second class.

Step 7. Rank the DMUs C, D, E, F and H presented in the last row of Table 3.

Step 8. $M = \{A, B, C, D, E, F, H, J\}$.

Step 9. $J = \{G, I\}$.

Step 10. $J \neq \varnothing$.

<u>The third iterate/cluster</u>

Step 4. $k = 3$.

Step 5. The objective function values of model (10) for the two remaining DMUs, $J = \{G, I\}$, are shown in Table 4.

Step 6. *G* and *I* shown in Table 4 are placed in the third cluster.

Step 7. Rank the DMUs *G* and *I* presented in the last row of Table 4.

Step 8. $M = \{A, B, C, D, E, F, G, H, I, J\}$.

Step 9. $J = \varnothing$.

Step 10. $J = \varnothing$.

Table 5. The clustering results.

DMU Ranking Within Clusters	Clusters		
	1st Cluster	2nd Cluster	3rd Cluster
1	DMU J	DMU D	DMU G
2	DMU B	DMU C	DMU I
3	DMU A	DMU E	
4		DMU H	
5		DMU F	

Table 5 presents the results for the DEA-based clustering algorithm proposed in this study. As shown in this table, ten DMUs are grouped into three clusters (1st, 2nd , and 3rd clusters) and the priority of the DMUs within each cluster is shown as: $J \succ B \succ A$ in the 1st cluster, $D \succ C \succ E \succ H \succ F$ in the 2nd cluster, and $G \succ I$ in the 3rd cluster (where " $\succ$ " means "is better than").

6. The Purchasing Decision at the Florida Border Patrol

The Florida Border Patrol plans to purchase 250 water jet skis for border protection and homeland security. They are evaluating 45 jet skis using a DEA model with four input variables and three output variables. The input variables include: Fuel Consumption (LPH), Weight (KG), Cost (USD) and Reliability Rating. The output variables include: Power (HP), Noise Level (dBA), and Emission Rating. The associated data are reported in Table 6.

The Reliability Rating among the inputs and Emission Rating among the outputs were measured with linguistic variables and the associated the trapezoidal fuzzy numbers are presented in Table 7.

Generally, the goal in efficiency theory is to maximize output while minimizing input. This goal is also pursued in DEA where multiple outputs are maximized while multiple inputs are minimized. We use the concept of "bad input-bad output" to maximize bad inputs and minimize good input. Lower input values are preferred to higher input values. However, higher bad input values are preferred to lower bad input variables and since the goal in efficiency theory is to minimize inputs, we must first invert the bad inputs. Similarly, higher output values are preferred to lower output values. However, lower bad output values are preferred to higher bad output variables and since the goal in efficiency theory is to maximize outputs, we must first invert the bad outputs. Bad inputs and outputs are the reciprocal of the inputs with higher desirable values and outputs with lower desirable values, respectively. In this case study, Noise Level (dBA) is a bad output and its inverse value is used in the model.

Using the above input and output data, we implemented the algorithm proposed in this study to evaluate 45 jet skis under consideration by the Florida Border Patrol with respect to $\alpha = \{0, 0.25, 0.5, 0.75, 1\}$ (see Fig. 3). The selected jet skis for each cluster at a given α-cut are shown in Tables 8-12 along with their respective ranking in each cluster.

Table 6. The input and output data for the jet skis.

	INPUTS				OUTPUTS		
Jet Ski	**Fuel Consumption (LPH)**	**Weight (KG)**	**Cost (USD)**	**Reliability Rating**	**Power (HP)**	**Noise Level (dBA) (BAD OUTPUT)**	**Emission Rating**
S01	48	262	4310	MH	110	80	H
S02	40	268	4250	L	88	85	L
S03	47	280	4020	MH	90	65	ML
S04	42	320	5810	VH	120	94	M
S05	55	350	5200	M	88	76	L
S06	54	366	5050	L	86	84	VL
S07	46	350	5500	VH	112	69	M
S08	42	280	5200	MH	95	94	ML
S09	38	310	4900	VH	110	96	H
S10	58	330	5620	MH	112	92	H
S11	54	288	4850	ML	98	78	MH
S12	56	292	4500	M	96	62	H
S13	58	320	4760	L	88	89	M
S14	44	312	4650	ML	94	93	M
S15	38	276	5320	MH	92	95	H
S16	42	267	5410	M	78	79	M
S17	46	296	5520	M	82	96	M
S18	40	294	5400	VH	84	89	H
S19	38	269	5380	L	92	84	VL
S20	40	330	4320	VH	106	76	M
S21	42	308	4820	M	110	84	M
S22	51	265	4530	L	95	97	MH
S23	43	286	4610	ML	98	94	VH
S24	38	302	5310	VH	86	68	M
S25	42	276	4530	VH	92	74	H
S26	58	320	4480	ML	96	92	MH
S27	60	322	4720	H	104	86	M
S28	54	306	4980	VH	106	90	H
S29	42	283	5320	M	92	82	VL
S30	38	276	5410	L	80	86	MH
S31	38	286	5670	L	88	88	H
S32	46	298	5430	VH	86	86	ML
S33	44	284	4980	ML	94	92	ML
S34	55	322	4680	M	94	92	ML
S35	54	269	5240	ML	88	82	VL
S36	48	296	5460	M	86	88	H
S37	46	274	5500	M	93	85	VL
S38	44	286	5380	M	95	81	L
S39	45	306	5760	MH	105	88	VH
S40	52	314	5310	MH	112	78	M
S41	49	305	4820	VH	97	68	VL
S42	39	285	4320	M	102	75	L
S43	44	295	4210	H	106	84	MH
S44	52	312	5220	M	110	92	M
S45	50	315	5680	MH	105	95	L

Table 7. The linguistic variables and their associated trapezoidal fuzzy numbers used in this study.

Linguistic variable	Trapezoidal fuzzy number
Very low (VL)	(0, 0, 10, 20)
Low (L)	(10, 20, 20, 30)
Medium low (ML)	(20, 30, 40, 50)
Medium (M)	(40, 50, 50, 60)
Medium high (MH)	(50, 60, 70, 80)
High (H)	(70, 80, 80, 90)
Very high (VH)	(80, 90, 100, 100)

Table 8. The clustering, objective function values (OFVs) and the ranking results (Alpha = 0).

1st Cluster			2nd Cluster		
Jet ski	**OFV**	**Rank**	**Jet ski**	**OFV**	**Rank**
S31	3.8571	1	**S17**	2.2500	1
S22	3.4286	2	**S34**	1.8750	2
S30	3.4286	2	**S05**	1.6948	3
S19	3.0388	3	**S38**	1.6402	4
S06	3.0279	4	**S27**	1.6215	5
S02	3.0149	5	**S29**	1.5396	6
S13	2.9686	6	**S37**	1.5256	7
S23	2.1429	7	**S08**	1.5000	8
S11	1.7143	8	**S45**	1.3925	9
S26	1.7143	8	**S32**	1.2640	10
S35	1.5505	9	**S41**	1.1820	11
S14	1.5333	10			
S33	1.5172	11			
S12	1.3982	12			
S09	1.2730	13			
S15	1.2730	13			
S01	1.2360	14			
S18	1.2094	15			
S25	1.1986	16			
S39	1.1961	17			
S36	1.1700	18			
S21	1.1547	19			
S42	1.1361	20			
S43	1.1199	21			
S03	1.1196	22			
S24	1.1144	23			
S20	1.0885	24			
S04	1.0534	25			
S28	1.0515	26			
S10	1.0306	27			
S44	1.0301	28			
S16	1.0149	29			
S40	1.0026	30			
S07	1.0005	31			

Table 9. The clustering, objective function values (OFVs) and the ranking results (Alpha = 0.25).

1st Cluster			2nd Cluster		
Jet ski	**OFV**	**Rank**	**Jet ski**	**OFV**	**Rank**
S31	2.6552	1	**S10**	1.7816	1
S22	2.3698	2	**S16**	1.6286	2
S30	2.3517	3	**S44**	1.5860	3
S19	2.2240	4	**S17**	1.5274	4
S06	2.2171	5	**S34**	1.3942	5
S02	2.2057	6	**S05**	1.3916	6
S13	2.1707	7	**S38**	1.3609	7
S23	1.6927	8	**S40**	1.3518	8
S11	1.3741	9	**S29**	1.3198	9
S26	1.3477	10	**S37**	1.3000	10
S12	1.2824	11	**S28**	1.2889	11
S35	1.2615	12	**S07**	1.1548	12
S14	1.2355	13	**S08**	1.1302	13
S33	1.2327	14	**S27**	1.1042	14
S09	1.2016	15	**S41**	1.0995	15
S15	1.2002	16	**S45**	1.0529	16
S01	1.1762	17	**3rd Cluster**		
S39	1.1582	18	**Jet ski**	**OFV**	**Rank**
S25	1.1458	19			
S18	1.1402	20	**S32**	2.5589	1
S21	1.1116	21			
S42	1.0997	22			
S03	1.0986	23			
S24	1.0851	24			
S20	1.0689	25			
S36	1.0664	26			
S43	1.0656	27			
S04	1.0409	28			

The results showed that the number of classes change with different α-cuts. In other words, we have two clusters for $\alpha = 0$ while we find three clusters for larger α-cuts. Table 13 presents the number of jet skis for each cluster with respect to various α-cuts.

As shown in Table 13, the number of jet skis in the 1st cluster has an indirect relationship with the α-cut. In other words, as we increase the α-cut from 0 to 1, the number of jet skis in the 1st cluster decreases from 34 to 23. On the other hand, the number of jet skis in the 2nd cluster slightly increases from 11 to 18 as we increase our α-cut from 0 to 1. Finally, the number of jet skis in the 3rd cluster increases from 1 to 4 as we increase our α-cut from 0 to 1. A close look at the results also reveals that the 1st cluster constitutes the largest (34) and the smallest (23) number of jet skis when $\alpha = 0$ and $\alpha = 1$, respectively. Furthermore, when $\alpha = 1$, we have 23 jet skis with declining α-cuts in the 1st cluster. Inversely, the 2nd and the 3rd clusters consist of the biggest and the smallest number of jet skis when $\alpha = 1$ and $\alpha = 0$, respectively. Therefore, the jet skis which are selected in the 2nd cluster for a given α-cut are also selected in the 2nd and/or 3rd cluster for larger α-cuts. For example, the 16 jet skis placed in the 2nd cluster for $\alpha = 0.75$, are placed in the 2nd and 3rd clusters for $\alpha = 1$. We should also point out that the first cluster formed is preferred to the 2nd cluster which in turn is preferred to the 3rd cluster. For example, in Table 8, when $\alpha = 0$, we have two clusters where the first one is preferred to the second one. It is also possible to have an identical objective function value for model (10) for some jet skis under consideration. In such cases, we assign an identical ranking order to all jet skis with equal objective function values. For instance, in Table 8, the objective function value for jet skis S11 and S26 is equal to 1.7143 and both jet skis share a common 8th place ranking.

A review of the DEA literature shows different views or interpretations of α-cuts (Kao and Liu,[28] Guo and Tanaka,[31] Lertworasirikul *et al.*[32]). In the decision sciences context, the α-cut concept is often used for incorporating the DMs' confidence level. Accordingly, the higher the α-value, the lower is the degree of uncertainty and in contrast, the lower the α-value, the higher is the degree of uncertainty. In this study, we provided the DMs with a compromise solution based on different α-values (i.e., 0, 0.25, 0.50, 0.75, and 1.00). Logically, the highest level of uncertainty $\alpha = 0$ implies the lowest discriminatory power with the highest number of fully efficiency units. As the uncertainty is decreased, i.e. an increase in α, the average efficiency score is decreased and the reference set can be decomposed in additional clusters. As a result a jet ski may be ranked differently with different α-cuts. For example when α is 0, 0.25 and 0.5, S31 is ranked first because of the highest objective function value while S22 is ranked second for $\alpha = 0.75$ and 1.00. This phenomenon is attributed to the intrinsic fuzzy character of the input and output data in combination with the DEA definition of the efficient frontier.

Table 10. The clustering, objective function values (OFVs) and the ranking results (Alpha = 0.5).

1st Cluster		
Jet ski	**OFV**	**Rank**
S31	1.8889	1
S22	1.7688	2
S30	1.7023	3
S19	1.6818	4
S06	1.6772	5
S02	1.6673	6
S13	1.6400	7
S23	1.5260	8
S11	1.2084	9
S26	1.2070	10
S12	1.1784	11
S09	1.1324	12
S15	1.1316	13
S39	1.1242	14
S01	1.1234	15
S25	1.0952	16
S03	1.0779	17
S18	1.0754	18
S21	1.0717	19
S42	1.0650	20
S24	1.0564	21
S14	1.0509	22
S20	1.0500	23
S35	1.0343	24
S43	1.0315	25
S04	1.0306	26
S33	1.0210	27

2nd Cluster		
Jet ski	**OFV**	**Rank**
S36	1.3852	1
S44	1.2892	2
S16	1.2583	3
S05	1.2441	4
S10	1.2329	5
S38	1.2181	6
S29	1.1935	7
S37	1.1744	8
S28	1.1641	9
S34	1.1598	10
S40	1.1261	11
S17	1.1044	12
S07	1.0819	13
S41	1.0804	14
S08	1.0653	15
S27	1.0571	16

3rd Cluster		
Jet ski	**OFV**	**Rank**
S32	2.1176	1
S45	1.9152	2

Table 11. The clustering, objective function values (OFVs) and the ranking results (Alpha = 0.75).

1st Cluster		
Jet ski	**OFV**	**Rank**
S23	1.3826	1
S31	1.3687	2
S22	1.3443	3
S30	1.3063	4
S19	1.2951	5
S06	1.2922	6
S02	1.2836	7
S13	1.2619	8
S11	1.1028	9
S39	1.0915	10
S12	1.0847	11
S26	1.0837	12
S01	1.0805	13
S15	1.0648	14
S09	1.0647	15
S03	1.0578	16
S25	1.0477	17
S24	1.0375	18
S21	1.0346	19
S20	1.0318	20
S42	1.0318	20
S04	1.0230	21
S18	1.0117	22
S43	1.0103	23
S14	1.0049	24

2nd Cluster		
Jet ski	**OFV**	**Rank**
S33	1.6794	1
S35	1.6211	2
S36	1.1766	3
S44	1.1351	4
S10	1.1314	5
S16	1.1182	6
S28	1.0874	7
S40	1.0839	8
S05	1.0657	9
S41	1.0634	10
S07	1.0556	11
S29	1.0550	12
S34	1.0525	13
S38	1.0484	14
S08	1.0317	15
S27	1.0295	16
S37	1.0163	17

3rd Cluster		
Jet ski	**OFV**	**Rank**
S17	1.2216	1
S45	1.1897	2
S32	1.0792	3

Table 12. The clustering, objective function values (OFVs) and the ranking results (Alpha = 1.00).

1st Cluster		
Jet ski	**OFV**	**Rank**
S23	1.2581	1
S22	1.1018	2
S39	1.0602	3
S01	1.0416	4
S03	1.0381	5
S24	1.0256	6
S30	1.0249	7
S09	1.0246	8
S04	1.0169	9
S25	1.0147	10
S20	1.0143	11
S19	1.0119	12
S11	1.0073	13
S26	1.0049	14
S06	1.0042	15
S43	1.0034	16
S15	1.0025	17
S02	1	18
S12	1	18
S13	1	18
S21	1	18
S31	1	18
S42	1	18

2nd Cluster		
Jet ski	**OFV**	**Rank**
S35	1.3649	1
S33	1.336	2
S14	1.3333	3
S10	1.0579	4
S41	1.0458	5
S40	1.0389	6
S18	1.0294	7
S07	1.0289	8
S28	1.0272	9
S29	1.0092	10
S08	1.0091	11
S37	1.0013	12
S05	1	13
S16	1	13
S27	1	13
S36	1	13
S38	1	13
S44	1	13

3rd Cluster		
Jet ski	**OFV**	**Rank**
S34	1.1289	1
S45	1.0821	2
S32	1.0504	3
S17	1	4

Table 13. The number of DMUs for each cluster with respect to various α-cuts.

	α-cut				
	0	**0.25**	**0.5**	**0.75**	**1**
No. (1st Cluster)	34	28	27	25	23
No. (2nd Cluster)	11	16	16	17	18
No. (3rd Cluster)		1	2	3	4

Table 14. The final ranking of the jet skis.

DMU	Sum of the ranking scores	Average ranking score	Final ranking
S22	11	2.2	1
S30	19	3.8	2
S31	23	4.6	3
S23	25	5	4
S06	35	7	5
S02	42	8.4	6
S19	44	8.8	7
S13	46	9.2	8
S11	48	9.6	9
S26	54	10.8	10
S01	63	12.6	11
S09	63	12.6	12
S12	63	12.6	13
S15	73	14.6	14
S25	78	15.6	15
S03	83	16.6	16
S24	92	18.4	17
S21	96	19.2	18
S42	100	20	19
S20	103	20.6	20
S04	108	21.6	21
S43	112	22.4	22

Table 15. Fuzzy input-output data for the comparison example 2 in Guo and Tanaka.[31]

DMU	Inputs		Outputs	
	1	**2**	**1**	**2**
A	(3.5, 4.0, 4.5)	(1.9, 2.1, 2.3)	(2.4, 2.6, 2.8)	(3.8, 4.1, 4.4)
B	(2.9, 2.9, 2.9)	(1.4, 1.5, 1.6)	(2.2, 2.2, 2.2)	(3.3, 3.5, 3.7)
C	(4.4, 4.9, 5.4)	(2.2, 2.6, 3.0)	(2.7, 3.2, 3.7)	(4.3, 5.1, 5.9)
D	(3.4, 4.1, 4.8)	(2.2, 2.3, 2.4)	(2.5, 2.9, 3.3)	(5.5, 5.7, 5.9)
E	(5.9, 6.5, 7.1)	(3.6, 4.1, 4.6)	(4.4, 5.1, 5.8)	(6.5, 7.4, 8.3)

In order to narrow down the list of 50 alternative jet skis, we focused on the DMUs placed in the 1st cluster for all five α-cuts. We then ranked the DMUs according to an average ranking score calculated by averaging the five α-cuts associated with each product. The results presented in Table 14 shows lower average ranking scores for jet skis with better performance. As a result, the Florida Border Patrol selected the top-eight jet skis: S22, S30, S31, S23, S06, S02, S19 and S13 for further consideration and invited the manufacturers of theses eight jet skis to make their product available for further testing.

7. Discussion

In this section, we use a commonly used numerical example, first introduced by Guo and Tanaka,[31] to compare our results with three popular methods in the fuzzy DEA literature. In this example, presented in Table 15, we consider five DMUs, two fuzzy inputs and two fuzzy outputs. Lertworasirikul *et al.*[32] and Saati *et al.*[29] have also used this example to illustrate the effectiveness of their methods.

Guo and Tanaka[31] developed a fuzzy CCR model in which fuzzy constraints were transformed into crisp forms by predefining a possibility level. According to Guo and Tanaka,[31] a DMU is α-possibilistic efficient if the maximum value of the fuzzy efficiency at that α level is greater than or equal to 1. The set of all possibilistic efficient DMUs is called the α-possibilistic non-dominated set. By means of the possibility approach in fuzzy set theory, Lertworasirikul *et al.*[32] proposed a fuzzy CCR model where a DMU becomes α-possibilistic efficient if its objective value is greater than or equal to 1 at the specified α level. Saati *et al.*[29] also suggested a fuzzy CCR model as a possibilistic programming problem and converted it into an interval programming problem using an α-cut based approach. In their method, Saati *et al.*[29] call a DMU efficient if its efficiency score is one. The solutions from Guo and Tanaka[31] (GT), Lertworasirikul *et al.*[32] (L), Saati *et al.*[29] (S) and the proposed method (PM) in this study for four different α values are summarized in Table 16. In this table, the cluster of each DMU for four different α levels is presented in the parentheses and the rank order of each DMU is present with *italic* numbers.

In the case of α = 0, DMU *A* is classified as dominated and the remaining DMU as non-dominated with the Guo and Tanaka's method whereas all units are 0-possibilistically efficient with the Lertworasirikul *et al.*'s method and efficient with the Saati *et al.*'s method. According to the method proposed in this study, all DMUs are included in the 1st group with the highest priorities, similar to the other three methods. In addition, our method provides information about the ranking of the DMUs in each cluster (i.e., $D \succ E \succ B \succ C \succ A$).

In the case of α = 0.5, *B* and *D* are in the non-dominated set according to Guo and Tanaka's method whereas *B*, *C*, *D* and *E* are 0.5-possibilistically efficient with the Lertworasirikul *et al.*'s method and efficient with the Saati *et al.*'s method. As it is shown in the Table 2, five DMUs are clustered into two clusters (1st and 2nd clusters) according to our method in which the first cluster is preferred to the second cluster. The priority of the DMUs within each cluster is shown as: $D \succ E \succ B \succ C$ in the 1st cluster and *A* in the 2nd cluster. It is interesting that the efficiency scores for the Lertworasirikul *et al.*'s method and the method proposed here are identical for the four DMUs *B*, *C*, *D* and *E*.

Table 16. Results for the comparison example for the proposed method (PM) and the methods in Guo and Tanaka[31] (GT), Lertworasirikul *et al.*[32] (L), and Saati *et al.*[29] (S).

α	Method	DMU				
		A	**B**	**C**	**D**	**E**
0	**GT**	(0.66,0.81,0.99)	(0.88,0.98,1.09)	(0.60,0.82,1.12)	(0.71,0.93,1.25)	(0.61,0.79,1.02)
	L	1.11 ***5***	1.24 ***4***	1.28 ***3***	1.52 ***1***	1.30 ***2***
	S	1 ***1***	1 ***1***	1 ***1***	1 ***1***	1 ***1***
	PM	1.03 (1st) ***5***	1.17 (1st) ***3***	1.15 (1st) ***4***	1.39 (1st) ***1***	1.23 (1st) ***2***
0.5	**GT**	(0.75,0.83,0.92)	(0.94,0.97,1.00)	(0.71,0.83,0.97)	(0.85,0.97,1.12)	(0.72,0.82,0.93)
	L	0.96 ***5***	1.11 ***3***	1.04 ***4***	1.26 ***1***	1.16 ***2***
	S	0.95 ***5***	1 ***1***	1 ***1***	1 ***1***	1 ***1***
	PM	1.22 (2nd) ***5***	1.11 (1st) ***3***	1.04 (1st) ***4***	1.26 (1st) ***1***	1.16 (1st) ***2***
0.75	**GT**	(0.80,0.84,0.88)	(0.96,0.99,1.02)	(0.77,0.83,0.90)	(0.92,0.98,1.05)	(0.78,0.83,0.89)
	L	0.90 ***5***	1.06 ***3***	0.93 ***4***	1.13 ***1***	1.10 ***2***
	S	0.90 ***5***	1 ***1***	0.93 ***4***	1 ***1***	1 ***1***
	PM	1.11 (2nd) ***5***	1.06 (1st) ***3***	1.14 (2nd) ***4***	1.13 (1st) ***1***	1.10 (1st) ***2***
1	**GT**	0.86	1	0.86	1	1
	L	0.86 ***4***	1 ***1***	0.86 ***4***	1 ***1***	1 ***1***
	S	0.86 ***4***	1 ***1***	0.86 ***4***	1 ***1***	1 ***1***
	PM	1 (2nd) ***4***	1 (1st) ***1***	1 (2nd) ***4***	1 (1st) ***1***	1 (1st) ***1***

In the case of α = 0.75, DMUs *B* and *D* are considered non-dominated with Guo and Tanaka's method while *B*, *D* and *E* are 0.75-possibilistically efficient with the Lertworasirikul *et al.*'s method and efficient with the Saati *et al.*'s method. Similar to the results from Lertworasirikul *et al.* and Saati *et al.*, DMUs *B*, *D* and *E* are placed in the highest priority cluster (1st class) according to our method. We here point out that the discriminatory power of the methods proposed by Lertworasirikul *et al.* and Saati *et al.* is weak since three of the five DMUs under consideration are efficient when α = 0.75**.** However**,** our approach first takes into account these units in the 1st cluster and then determines their rankings in each class to increase the discriminatory power. Finally, in the case of α = 1, the results from all four methods are qualitatively almost identical.

We also applied a Spearman's rank correlation coefficient for the four various α to measure the correlation between the rankings proposed in this study with the rankings of Lertworasirikul *et al.*[32] and Saati *et al.*[29] The Spearman's rank correlation coefficient between the rankings of the method proposed by Lertworasirikul *et al.*[32] and our method is 0.9 for α = 0, 1 for α = 0.5, 1 for α = 0.75, and 1 for α = 1. The Spearman's rank correlation coefficient between the rankings of the method proposed by Saati *et al.*[29] and our method is 0.5 for $\alpha = 0$, 0.75 for $\alpha = 0.5$, 0.9 for $\alpha = 0.75$, and 1 for $\alpha = 1$. The Spearman's rank correlation coefficients show the similarity between the rankings of the method proposed in this study with the rankings of the methods proposed by Lertworasirikul *et al.*[32] and Saati *et al.*[29] The rankings of the DMUs for different α (see the ***italic*** rankings in Table 17) confirms the rank-order convergence of the three methods.

In summary, the results for the methods proposed by Lertworasirikul *et al.*[32] and Saati *et al.*[29] are similar to those obtained by our algorithm. However, as shown here, the method proposed in this paper offers additional performance information compared with the other three competing methods, i.e. simultaneous inter-cluster and intra-cluster performance assessment. In certain settings where a complete ordering among the DMU is desired, this feature is preferred.

8. Conclusions and Future Research Directions

The field of DEA has grown exponentially since the pioneering work of Charnes *et al.*[2] DEA measures the relative efficiency of an operating unit by comparing it against a peer group. One limitation of the conventional DEA methods is the need for accurate measurement of the inputs and output data. However, input and output data in real-world problems are often imprecise or ambiguous. Numerous fuzzy methods have been proposed to deal with this impreciseness and ambiguity in DEA.

In this study, we proposed a new fuzzy DEA method for clustering operating units in a fuzzy environment by considering the priority between the clusters and the priority between the operating units in each cluster simultaneously. The proposed clustering-based DEA model defined the group of operating units that were similar to the operating unit under evaluation. This clustering process resulted in clusters with homogenous members. The contribution of this paper is threefold: (1) we consider ambiguous, uncertain or imprecise input and output data in DEA; (2) we propose a new fuzzy DEA method for clustering operating units in a fuzzy environment; and (3) we consider the priority between the clusters and the priority between the operating units in each cluster simultaneously.

The framework developed in this study can potentially lend itself to many practical applications. However, there are a number of challenges involved in the proposed research that provide a great deal of fruitful scope for future research. For example, there is no mechanism in the proposed algorithm to identify the number of clusters prior to the implementation of the algorithm. Another potential for future research is the integration of the proposed algorithm into other ranking methods such as the tolerance, fuzzy ranking, and possibility approaches. We also hope that the concepts introduced here provides the groundwork for comparing our clustering method with the other clustering methods commonly used in the literature such as hierarchical, *K*-means, possibilistic, and learning network clustering.

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