NON-PRINCIPAL ULTRAFILTERS, PROGRAM EXTRACTION AND HIGHER ORDER REVERSE MATHEMATICS

ALEXANDER P. KREUZER

ABSTRACT. We investigate the strength of the existence of a non-principal ultrafilter over fragments of higher order arithmetic.

Let (\mathcal{U}) be the statement that a non-principal ultrafilter exists and let ACA_0^{ω} be the higher order extension of ACA_0 . We show that $ACA_0^{\omega} + (\mathcal{U})$ is Π_2^1 -conservative over ACA_0^{ω} and thus that $ACA_0^{\omega} + (\mathcal{U})$ is conservative over PA.

Moreover, we provide a program extraction method and show that from a proof of a strictly Π_2^1 statement $\forall f \exists g A_{qf}(f,g)$ in $ACA_0^{\omega} + (\mathcal{U})$ a realizing term in Gödel's system T can be extracted. This means that one can extract a term $t \in T$, such that $\forall f A_{qf}(f, t(f))$.

In this paper we will investigate the strength of the existence of a non-principal ultrafilter over fragments of higher order arithmetic. We will classify the consequences of this statement in the spirit of reverse mathematics. Furthermore, we will provide a program extraction method.

Let (\mathcal{U}) be the statement that a non-principal ultrafilter on \mathbb{N} exists. Let $\operatorname{RCA}_0^{\omega}$, ACA₀^{ω} be the extensions of RCA₀ resp. ACA₀ to higher order arithmetic as introduced by Kohlenbach in [12]. In RCA₀^{ω} or ACA₀^{ω} the statement (\mathcal{U}) can be formalized using an object of type $\mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}$.

Further, let Feferman's μ be a functional of type $\mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}$ satisfying

$$f(\mu(f)) = 0 \quad \text{if} \quad \exists x f(x) = 0$$

and let (μ) be the statement that such a functional exists. It is clear that (μ) implies arithmetical comprehension.

We will show that

- over $\operatorname{RCA}_0^{\omega}$ the statement (\mathcal{U}) implies (μ) and therefore also $\operatorname{ACA}_0^{\omega}$, and that
- ACA₀^{ω} + (μ) + (\mathcal{U}) is Π_2^1 -conservative over ACA₀^{ω} and therefore also conservative over PA. Moreover, we will show that from a proof of $\forall f \exists g A_{qf}(f,g)$ in ACA₀^{ω} + (μ) + (\mathcal{U}), where A_{qf} is quantifier free, one can extract a realizing term t in Gödel's system T, i.e. a term such that $\forall f A_{qf}(f, t(f))$.

The system $ACA_0^{\omega} + (\mu) + (\mathcal{U})$ is strong, one can carry out nearly all ultralimit and non-standard arguments. For instance one can carried out in this theory the construction of Banach limits and many Loeb measure constructions. Our results show that this system is weak with respect to Π_2^1 sentences. Moreover, our program extraction result show that one can still obtain constructive (even primitive recursive in the sense of Gödel) realizers and bounds from proofs using highly non-constructive objects like non-principal ultrafilter.

Date: October 8, 2018.

²⁰¹⁰ Mathematics Subject Classification. 03B15, 03B30, 03F35, 03F60.

Key words and phrases. ultrafilter, conservation, program extraction, functional interpretation.

The author is supported by the German Science Foundation (DFG Project KO 1737/5-1).

I am grateful to Ulrich Kohlenbach for useful discussions and suggestions for improving the presentation of the material in this article.

ALEXANDER P. KREUZER

Using this technique it is possible to extract bounds from proofs using ultralimits and non-standard technique. Such proofs do occur in mathematics, for instance in metric fixed point theory, see [1] and [9]. In [6] Gerhardy extracted a rate of proximity of such a proof by eliminating the ultrafilter by hand. Our result here show that this can be done with any such argument.

Comparison to other approaches. Solovay first used partial ultrafilter. He constructed a filter which acts on the hyperarithemtical sets like a non-principal ultrafilter. With this he show an effective version of the Galvin-Prikry theorem, see [17]. His construction of the partial ultrafilter is similar to ours. Avigad analyzed his result in terms of reverse mathematics and formalized this particular proof in ATR_0 , see [2]. However, this result does not follow from our meta-theorem, since it not only uses a non-principal ultrafilter but also substantial amounts of transfinite recursion.

Using our approach one also obtains upper bounds on the strength of nonstandard analysis and program extraction methods. This can be done by constructing a ultrapower model of non-standard analysis in $ACA_0^{\omega} + (\mu) + (\mathcal{U})$. If one is not interested in the ultrafilter but only in the axiomatic treatment of nonstandard analysis one can obtain refined results by interpreting it directly, see for instance [3], [8] and for program extraction [5].

Palmgren used in [15] an approach similar to ours to interpret non-standard arithmetic. He builds (partial) non-principal ultrafilters for the definable sets of a fixed level in the arithmetic hierarchy. He obtains conservations result very similar to ours. However he cannot treat ultrafilter nor obtains program extraction.

In reverse mathematics idempotent ultrafilters are considered in the context of Hindman's theorem, which can be proven using an idempotent ultrafilter (or at least a countable part of it), see Hirst [7] and Towsner [18]. We code ultrafilter over countable fields like Hirst does. However, our construction of ultrafilters is different since we are not aiming for idempotent ultrafilters. An idempotent ultrafilter is a very special ultrafilter and it seems that even the construction of countable parts of an idempotent ultrafilter requires a system that is proof theoretically stronger than $ACA_0^{\omega} + (\mu)$ and is therefore beyond our method.

Logical system. We will work in fragments of Peano arithmetic in all finite types. The set of all finite types \mathbf{T} is defined to be the smallest set that satisfies

$$0 \in \mathbf{T}, \quad \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

The type 0 denotes the type of natural numbers and the type $\tau(\rho)$ denotes the type of functions from ρ to τ . The type 0(0) is abbreviated by 1 the type 0(0(0)) by 2. The degree of a type is defined by

$$deg(0) := 0 \qquad deg(\tau(\rho)) := \max(deg(\tau), deg(\rho) + 1).$$

The type of a variable will sometimes be written as superscript of a term or as subscript of an equality sign.

The system RCA_{0}^{ω} is the extension of RCA_{0} to all finite types. The systems WKL_{0}^{ω} , ACA_{0}^{ω} are defined to be $\text{RCA}_{0}^{\omega} + \text{WKL}$ resp. $\text{RCA}_{0}^{\omega} + \Pi_{1}^{0}$ -CA. All of these system are conservative over their second order counterpart via the embedding of sets as characteristic functions. For details see [12].

Let $QF-AC^{1,0}$ be the schema

$$\forall f^1 \exists x^0 \operatorname{A}_{qf}(f, x) \to \exists F^2 \forall f^1 \operatorname{A}_{qf}(f, F(f)).$$

All of the above defined systems include QF-AC^{1,0}. This schema is the higher order equivalent to recursive comprehension (Δ_1^0 -CA).

The terms of RCA₀^{ω} consist of 0⁰, the successor function S^1 , lambda combinators II and Σ for all types, which provide lambda abstraction, and the recursor R_0 . The recursor R_0 satisfies the following equations

$$R_0 0yz =_0 0,$$
 $R_0(x+1)yz =_0 z(Rxyz)x.$

It provides primitive recursion (in the sense of Kleene). The closed terms of $\operatorname{RCA}_0^{\omega}$ are also called T_0 (for the restriction of Gödel's system T to recursion of type 0). If one adds (impredicative) recursors R_{ρ} for all types $\rho \in \mathbf{T}$ to T_0 one obtains the full system T of Gödel. The functions in T are called primitive recursive in the sense of Gödel. By $T_0[F]$ we will denote the system resulting from adding a function(al) F to T_0 .

The system RCA_{0}^{ω} has a functional interpretation (always combined with elimination of extensionality and a negative translation) in T_{0} . The system ACA_{0}^{ω} has a functional interpretation in $T_{0}[\mu]$ if one interprets comprehension using μ or in $T_{0}[B_{0,1}]$ if one interprets comprehension using the bar recursor of lowest type $B_{0,1}$. See [12] and [4] for the interpretation using μ and [13, Section 11] for the interpretation using $B_{0,1}$. For a general survey on the functional interpretation see [13] and [4].

Definition 1 (non-principal ultrafilter, (\mathcal{U})). Let (\mathcal{U}) be the statement that there exists a non-principal ultrafilter (on \mathbb{N}):

$$(\mathcal{U}): \begin{cases} \exists \mathcal{U}^2 \left(\ \forall X \ \left(X \in \mathcal{U} \lor \overline{X} \in \mathcal{U} \right) \\ \land \forall X^1, Y^1 \ \left(X \cap Y \in \mathcal{U} \to Y \in \mathcal{U} \right) \\ \land \forall X^1, Y^1 \ \left(X, Y \in \mathcal{U} \to (X \cap Y) \in \mathcal{U} \right) \\ \land \forall X^1 \ \left(X \in \mathcal{U} \to \forall n \ \exists k > n \ (k \in X) \right) \\ \land \forall X^1 \ \left(\mathcal{U}(X) =_0 \mathcal{U}(\lambda n. \min(X(n), 1)) \right) \end{cases}$$

Here $X \in \mathcal{U}$ is an abbreviation for $\mathcal{U}(X) = 0$. The type 1 variables X, Y are viewed as characteristic function of sets, where $n \in X$ is defined to be X(n) = 0. The operation \cap is defined as taking the pointwise maximum of the characteristic functions. With this the intersection of two sets can be expressed in a quantifier-free way. The last line of the definition states that \mathcal{U} yields the same value for different characteristic functions of the same set.

For notational ease we will usually add a Skolem constant \mathcal{U} and denote this also with (\mathcal{U}) .

The second line in the definition of (\mathcal{U}) is equivalent to the following axiom usually found in the axiomatization of (ultra)filters:

$$\forall X, Y \ (X \subseteq Y \land X \in \mathcal{U} \to Y \in \mathcal{U}) \,.$$

We avoided this statement in (\mathcal{U}) since \subseteq cannot be expressed in a quantifier free way.

Lemma 2 (finite partition property). The ultrafilter \mathcal{U} satisfies the finite partition property over $\operatorname{RCA}_0^{\omega}$.

This means that for each finite partition $(X_i)_{i < n}$ of \mathbb{N} the following holds

$$\operatorname{RCA}_{0}^{\omega} + (\mathcal{U}) \vdash \exists ! i < n X_{i} \in \mathcal{U}.$$

Proof. We prove by quantifier-free induction on m the statement

(1)
$$\exists ! i \le m \left(\left(i < m \to X_i \in \mathcal{U} \right) \land \left(i = m \to \bigcup_{j=m}^{n-1} X_j \in \mathcal{U} \right) \right).$$

In the cases $m \leq 2$ the statement follows directly from (\mathcal{U}) . For the induction step we assume that the statement for m holds. This means there exists an i as stated in (1). If i < m then this i also satisfies (1) with m replaced by m + 1 and we are done. Otherwise we have $\bigcup_{j=m}^{n-1} X_j \in \mathcal{U}$.

The axiom (\mathcal{U}) yields

$$\bigcup_{j=0}^{m} X_j \in \mathcal{U} \quad \lor \bigcup_{j=m+1}^{n-1} X_j \in \mathcal{U}.$$

If the left side of the disjunction holds then

$$X_m = \bigcup_{j=0}^m X_j \cap \bigcup_{j=m}^{n-1} X_j \in \mathcal{U}$$

and i := m satisfies the (1) with m replaced by m + 1. If the right side of the disjunction holds i := m + 1 satisfies (1).

The lemma follows from (1) by taking m := n.

Theorem 3.

$$\operatorname{RCA}_{0}^{\omega} + (\mathcal{U}) \vdash (\mu)$$

 $\operatorname{RCA}_{0}^{\omega} + (\mathcal{U}) \vdash (\mu)$ In particular $\operatorname{RCA}_{0}^{\omega} + (\mathcal{U}) \vdash \operatorname{ACA}_{0}^{\omega}.$

Proof. Let $f: \mathbb{N} \to \mathbb{N}$ be a function. The set $X_f := \{x \in \mathbb{N} \mid \exists x' < x f(x') = 0\}$ is cofinal if $\exists x f(x) = 0$, if not then the set X_f is empty. Hence

$$X_f \in \mathcal{U} \quad \text{iff} \quad \exists x f(x) = 0$$

From this it follows that

$$\forall f \exists x \ (X_f \in \mathcal{U} \to f(x) = 0)$$

An application of QF-AC^{1,0} now yields a functional satisfying (μ).

Theorem 4 (Program extraction). Let $A_{qf}(f,g)$ be a quantifier free formula of $\operatorname{RCA}_0^{\omega}$ containing only f, g free. In particular A_{qf} must not contain μ or \mathcal{U} . If

$$ACA_0^{\omega} + (\mu) + (\mathcal{U}) \vdash \forall f^1 \exists g^1 A_{af}(f,g)$$

then one can extract a closed term $t \in T$ such that

 $\forall f A_{qf}(f, tf).$

The proof of this theorem proceeds in five steps:

1. Using the functional interpretation and proof theoretic methods developed in [14] we show that a proof of the statement

$$ACA_0^{\omega} + (\mu) + (\mathcal{U}) \vdash \forall f \exists g A_{qf}(f,g)$$

can be normalized in such a way that each application of the functional \mathcal{U} that occurs in the proof has the form $\mathcal{U}(t[n^0])$, where t is a term that contains only n free and with $\lambda n.t \in T_0[\mathcal{U}]$. (We do not have to consider μ here, since it can be defined from \mathcal{U} by Theorem 3.) In particular this shows the ultrafilter \mathcal{U} is used only on countable many sets.

- 2. We show that we can construct in $RCA_0^{\omega} + (\mu)$ a partial ultrafilter, that is an object that behaves like an ultrafilter on the sets that occur in the proof. We then replace \mathcal{U} by this partial ultrafilter and obtain a proof of $\forall f \exists g \operatorname{A}_{qf}(f,g) \text{ in } \operatorname{RCA}_0^{\omega} + (\mu).$
- 3. The theory $RCA_0^{\omega} + (\mu)$ is conservative over ACA_0^{ω} , see [4], hence we obtain a proof in this theory.
- 4. Applying the functional interpretation to this statement and interpreting the comprehension using $B_{0,1}$ yields a term $t^2 \in T_0[B_{0,1}]$, such that

$$\forall f A_{qf}(f, tf).$$

5. Since this term t is only of type 2, one can use an ordinal analysis of the bar recursor to eliminated it and obtain a new term $t' \in T$, such that $t' =_2 t$ and hence that

$$\forall f A_{qf}(f, t'f).$$

Before we prove this theorem we show how to construct a partial ultrafilter and provide some proof theoretic lemmata.

Partial ultrafilter.

Definition 5 (partial ultrafilter).

- Call a set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ of subsets of natural numbers, that is closed under complement, finite unions and finite intersections, an *algebra*.
- Let \mathcal{A} be an algebra. Call a set $\mathcal{F} \subseteq \mathcal{A}$ a partial non-principal ultrafilter for \mathcal{A} iff \mathcal{F} satisfies the non-principal ultrafilter axioms in Definition 1 relativized to \mathcal{A} , i.e.

$$\begin{cases} \forall X \in \mathcal{A} \ \left(X \in \mathcal{F} \lor \overline{X} \in \mathcal{F} \right) \\ \land \forall X, Y \in \mathcal{A} \ \left(X \cap Y \in \mathcal{F} \to Y \in \mathcal{F} \right) \\ \land \forall X, Y \in \mathcal{A} \ \left(X, Y \in \mathcal{F} \to (X \cap Y) \in \mathcal{F} \right) \\ \land \forall X \in \mathcal{A} \ \left(X \in \mathcal{F} \to \forall n \, \exists k > n \, k \in X \right) \\ \land \forall X^{1} \ \left(\mathcal{F}(X) =_{0} \mathcal{F}(\lambda n. \min(X(n), 1)) \right). \end{cases}$$

It is easy to see that one can extend in RCA_0^{ω} every sequence of sets to a countable algebra. One should also note that partial non-principal ultrafilters for countable algebras are also countable. A partial ultrafilter \mathcal{F} can be viewed as the closed subset $\{\mathcal{U} \in \beta \mathbb{N} \mid \mathcal{U} \supseteq \mathcal{F}\}$ of the Stone-Čech compactification $\beta \mathbb{N}$.

Proposition 6. Let \mathcal{A} be a countable algebra and let $\mathcal{F} = (F_i)_{i \in \mathbb{N}}$ be a countable partial non-principal ultrafilter for \mathcal{A} . Then $\operatorname{RCA}_0^{\omega} + (\mu)$ proves that for each countable extension $\tilde{\mathcal{A}} = (\tilde{A}_i)_{i \in \mathbb{N}} \supseteq \mathcal{A}$ there exists a partial non-principal ultrafilter $\tilde{\mathcal{F}} \supseteq \mathcal{F}$.

Proof. In the following let x be the code for a tuple $\langle x_0, \ldots, x_{\mathrm{lth}(x)-1} \rangle$ in $2^{<\mathbb{N}}$. Let

$$\tilde{A}^x := \bigcap_{i < \mathrm{lth}\,x} \begin{cases} \tilde{A}_i & \mathrm{if}\ x_i = 0, \\ \overline{\tilde{A}_i} & \mathrm{if}\ x_i = 1. \end{cases}$$

Using quantifier free induction one easily sees that for every n the set $\{\tilde{A}^x \mid x \in 2^n\}$ defines a partition of \mathbb{N} , i.e.

 $\forall n \exists ! x \in 2^n \ \left(z \in \tilde{A}^x \right) \quad \text{for all } z.$

Define a Π_2^0 -0/1-tree T by

(2)

T(x) iff $\forall j (\tilde{A}^x \cap F_j \text{ is infinite}).$

The tree T is infinite because otherwise we would have

$$\exists n \,\forall x \in 2^n \,\exists j \,\exists y \,\forall z > y \ z \notin \tilde{A}^x \cap F_j.$$

The bounded collection principle Π_1^0 -CP yields

(3)
$$\exists n \, \exists j^*, y^* \, \forall x \in 2^n \, \forall z > y^* \, z \notin \tilde{A}^x \cap \bigcap_{j \le j^*} F_j.$$

The set $\bigcap_{j \leq j^*} F_j$ is in \mathcal{F} and is therefore infinite. In particular it contains an element z which is bigger than y^* . Because \tilde{A}^x with $x \in 2^n$ defines a partition of \mathbb{N} there is an x such that $z \in \tilde{A}^x$. This contradicts (3) and therefore the tree T is infinite.

Hence we obtain using Π_2^0 -WKL (which is provable in ACA₀^{ω} and hence using μ) an infinite branch b of T. The set

$$\tilde{\mathcal{F}} := \mathcal{F} \cup \left\{ \tilde{A}_i \mid b(i) = 0 \right\}$$

defines then a partial non-principal ultrafilter for $\tilde{\mathcal{A}}$. The characteristic function of $\tilde{\mathcal{F}}$ is given by

$$\chi_{\tilde{\mathcal{F}}}(B) := \begin{cases} 0 & \text{if } (B \in \mathcal{F}) \lor \exists i \ (b(i) =_0 0 \land A_i = B), \\ 1 & \text{otherwise.} \end{cases}$$

The set equality $(A_i = B)$ can be defined using μ , therefore $\tilde{\mathcal{F}}$ is definable. \Box

Proof theory. The system $\operatorname{RCA}_0^{\omega}$ contains full extensionality. This means roughly that for a functional Φ and functions f, g one has $\Phi(f) =_0 \Phi(g)$ if f and g are extensionally equal (i.e. $\forall x f(x) =_0 g(x)$). Extensionality cannot be expressed in a purely universal statement and therefore contains some constructive content. For this reason the functional interpretation cannot handle this general form of extensionality directly and it has to be eliminated beforehand. The system $\operatorname{RCA}_0^{\omega}$ is formulated in a way that this can be done using standard methods, i.e. the elimination of extensionality, see for instance [13, Section 10.4]. Since we added a new higher order constant \mathcal{U} we have to check manually that this constant is extensional. This will be done in the following lemma. To formulate it we will need a *weakly extensional system*, i.e. a system in which extensionality is restricted to a rule of extensionality that only allows quantifier free premises. We will use $\widehat{\operatorname{WE-PA}}^{\omega} \upharpoonright + \operatorname{QF-AC}^{1,0}$. This system is the weakly extensional counterpart to $\operatorname{RCA}_0^{\omega}$ in the sense that $\operatorname{RCA}_0^{\omega}$ results from $\widehat{\operatorname{WE-PA}}^{\omega} \upharpoonright + \operatorname{QF-AC}^{1,0}$ by adding the extensionality axioms. (In other words $\operatorname{RCA}_0^{\omega} \equiv \widehat{\operatorname{E-PA}}^{\omega} \upharpoonright + \operatorname{QF-AC}^{1,0}$.)

Lemma 7 (Elimination of extensionality). The system $\widehat{WE}-\widehat{PA}^{\omega} \upharpoonright + (\mathcal{U})$ proves that \mathcal{U} is extensional, i.e.

$$\forall X, Y (\forall k \ (k \in X \leftrightarrow k \in Y) \to (X \in \mathcal{U} \leftrightarrow Y \in \mathcal{U})).$$

In particular, the elimination of extensionality is applicable to $\operatorname{RCA}_0^{\omega} + (\mathcal{U})$. This means the following rule holds: If A is a statement that contains only quantification over variables of degree ≤ 1 and

$$\operatorname{RCA}_0^\omega \vdash (\mathcal{U}) \to \mathcal{A}$$

then

$$\widehat{\mathrm{WE}}\operatorname{-PA}^{\omega} \upharpoonright + \mathrm{QF}\operatorname{-AC}^{1,0} \vdash (\mathcal{U}) \to \mathrm{A}.$$

Proof. Suppose that \mathcal{U} is not extensional. Then there exist two sets X, Y, such that

 $\forall k \ (k \in X \leftrightarrow k \in Y)) \quad \text{and} \quad X \in \mathcal{U} \land Y \notin \mathcal{U}.$

By the axiom (\mathcal{U}) we obtain that $\overline{Y} \in \mathcal{U}$ and with this

$$X \cap \overline{Y} \in \mathcal{U}.$$

By the last line of (\mathcal{U}) there exists an $n \in X \cap \overline{Y}$. This contradicts the assumption and we conclude that \mathcal{U} is extensional.

For the elimination of extensionality we use the techniques presented in Section 10.4 of [13]. We will also use the notation introduced in this section for the rest of this proof.

The extensionality of \mathcal{U} translates into $\mathcal{U} =^{e} \mathcal{U}$. Since (\mathcal{U}) is (after the Skolemization) analytic and the constant \mathcal{U} is extensional, we obtain $(\mathcal{U})_{e} \leftrightarrow (\mathcal{U})$. Because A does not contain quantification of degree > 1 we also obtain that A_{e} is equivalent to A. Hence $(\mathcal{U}) \rightarrow A$ does not change under the $(\cdot)_{e}$ relativization.

6

The lemma follows now from Proposition 10.45 in [13] relativized according to [13, Section 10.5] to RCA_0^{ω} .

The next theorem will provide the term normalization that is need for the proof of Theorem 4.

Theorem 8 (term-normalization for degree 2). Let F_1, \ldots, F_n be constants of degree ≤ 2 .

For every term $t^1 \in T_0[F_1, \ldots, F_n]$ there is a term $\tilde{t} \in T_0[F_0, \ldots, F_{n-1}]$ with

$$\widehat{\text{WE-PA}^{\omega}} \vdash t =_1 \hat{t}$$

and such that every occurrence of an F_i in \tilde{t} is of the form

$$F_i(\tilde{t}_0[y^0],\ldots,\tilde{t}_{k-1}[y^0])$$

Here k is the arity of F_i , and $\tilde{t}_j[y^0]$ are fixed terms whose only free variable is y^0 .

Proof. See Theorem 20 in [14]. For a reference see also [11, proof of proposition 4.2]. This normalization is similar to the normalization described in Section 8.3 of [4]. \Box

The axiom (\mathcal{U}) can be prenext to a statement of the form

$$\exists \mathcal{U}^2 \,\forall X^1, Y^1 \,\forall n \,\exists k \left(\begin{array}{c} \left(X \in \mathcal{U} \lor \overline{X} \in \mathcal{U} \right) \\ \land \left(X \cap Y \in \mathcal{U} \to Y \in \mathcal{U} \right) \\ \land \left(X, Y \in \mathcal{U} \to (X \cap Y) \in \mathcal{U} \right) \\ \land \left(X \in \mathcal{U} \to (k > n \land k \in X) \right) \\ \land \left(\mathcal{U}(X) =_0 \mathcal{U}(\lambda n. \min(Xn, 1)) \right) \right).$$

By coding the sets X, Y together into one set Z and calling the quantifier free matrix of the above statement $(\mathcal{U})_{qf}$ we arrive at

$$\exists \mathcal{U}^2 \,\forall Z^1 \,\forall n \,\exists k \,(\mathcal{U})_{qf}(\mathcal{U}, Z, n, k).$$

Applying QF-AC^{1,0} yields

(4)
$$\exists \mathcal{U}^2 \ \exists K^2 \ \forall Z^1 \ \forall n \ (\mathcal{U})_{qf}(\mathcal{U}, Z, n, KnZ)$$

Note that \mathcal{U} and K are only of degree 2. This will be crucial for the following proof. For K one may always choose

(5)
$$K'(n,X) := \begin{cases} \min\{k \in X \mid k > n\} & \text{if exists,} \\ 0 & \text{otherwise.} \end{cases}$$

The functional K' is definable using μ . Therefore the real difficulty lies in finding a solution for \mathcal{U} .

We are now in the position to give a proof of Theorem 4.

Proof of Theorem 4. In the light of Theorem 3 it is sufficient to prove only that $\operatorname{RCA}_0^{\omega} + (\mathcal{U})$ is conservative.

Let $A_{qf}(f,g)$ be a quantifier-free statement not containing \mathcal{U} , such that

$$\operatorname{RCA}_{0}^{\omega} + (\mathcal{U}) \vdash \forall f^{1} \exists g^{1} \operatorname{A}_{qf}(f,g)$$

By the deduction theorem we obtain

$$\operatorname{RCA}_{0}^{\omega} \vdash (\mathcal{U}) \to \forall f \exists g \operatorname{A}_{qf}(f, g).$$

Using Lemma 7 we obtain

$$\widehat{\mathrm{WE-PA}}^{\omega} \upharpoonright + \mathrm{QF-AC}^{1,0} \vdash (\mathcal{U}) \to \forall f \exists g \operatorname{A}_{qf}(f,g).$$

Reintroducing a variable \mathcal{U} for the ultrafilter together with (4) gives

 $\left(\exists \mathcal{U}^2 \exists K^2 \forall Z^1 \forall n \left(\mathcal{U}\right)_{qf}(\mathcal{U}, Z, n, KnZ)\right) \to \forall f \exists g \operatorname{A}_{qf}(f, g)$

which is equivalent to

 $\forall f \forall \mathcal{U}^2 \forall K^2 \exists Z^1, n \exists g ((\mathcal{U})_{af}(\mathcal{U}, Z, n, KnZ) \to A_{af}(f, g)).$

A functional interpretation yields terms $t_Z, t_n, t_q \in T_0[\mathcal{U}, K, f]$ such that

(6) $\widehat{\text{WE-PA}}^{\omega} \models \forall f \forall \mathcal{U}^2 \forall K^2 \left((\mathcal{U})_{qf} (\mathcal{U}, t_Z, t_n, Kt_n t_Z) \to A_{qf} (f, t_g) \right),$

see for instance Theorem 10.53 in [13]. Now by Theorem 8 applied to t_Z, t_n, t_g we obtain normalized term t'_Z, t'_n, t'_g which are provably (relative to $\widehat{WE-PA}^{\omega}$) equal and such that every occurrence of \mathcal{U} and K is of the form

$$\mathcal{U}(t[j^0])$$
 resp. $K(n^0, t[j^0]),$

where t is a term in $T_0[\mathcal{U}, K, f]$.

Let $(t_i)_{i < n}$ be the list of all of these terms t to which \mathcal{U} and K are applied. Assume that this list is partially sorted according to the subterm ordering, i.e. if t_i is a subterm of t_j then i < j.

We now build for each f a partial non-principal ultrafilter \mathcal{F} which acts on these occurrences like a real non-principal ultrafilter. For this fix an arbitrary f.

The filter \mathcal{F} is build by iterated applications of Proposition 6:

To start the iteration let \mathcal{A}_{-1} be the trivial algebra $\{\emptyset, \mathbb{N}\}$ and \mathcal{F}_{-1} be the partial non-principal ultrafilter for \mathcal{A}_{-1} .

Let \mathcal{A}_i be the algebra spanned by \mathcal{A}_{i-1} and the sets described by t_i where \mathcal{U}, K are replaced by \mathcal{F}_{i-1} and K' from (5), i.e. $(t_i[\mathcal{U}/\mathcal{F}_{i-1}, K/K'](j))_{j \in \mathbb{N}}$. Let \mathcal{F}_i be an extension of \mathcal{F}_{i-1} to the new algebra \mathcal{A}_i as constructed in Proposition 6.

Obviously in a term t_i the functional \mathcal{F} is only applied to subterms of t_i . Since the (t_i) is sorted according to the subterm ordering the partial non-principal ultrafilter is already fixed for this applications.

For the resulting partial non-principal ultrafilter $\mathcal{F} := \mathcal{F}_n$ we then get

$$(\mathcal{U})_{qf}(\mathcal{F}, t_Z[\mathcal{F}_n, K', f], t_n[\mathcal{F}, K', f], K't_n[\mathcal{F}, K', f]t_Z[\mathcal{F}, K', f]).$$

and in total

 $\operatorname{RCA}_{0}^{\omega} + (\mu) \vdash \forall f \exists \mathcal{F}(\mathcal{U})_{qf}(\mathcal{F}, t_{Z}[\mathcal{F}, K', f], t_{n}[\mathcal{F}, K', f], K't_{n}[\mathcal{F}, K', f]t_{Z}[\mathcal{F}, K', f]).$

Combining this with (6) yields

$$\operatorname{RCA}_{0}^{\omega} + (\mu) \vdash \forall f \exists \mathcal{F} \operatorname{A}_{qf}(f, t_{g}[\mathcal{F}, K', f])$$

and hence

$$\operatorname{RCA}_{0}^{\omega} + (\mu) \vdash \forall f \exists g \operatorname{A}_{qf}(f, g).$$

With this we have eliminated the use of (\mathcal{U}) in the proof.

By Theorem 8.3.4 of [4] the theory $\text{RCA}_0^{\omega} + (\mu)$ is conservative over ACA_0^{ω} and therefore

$$\operatorname{ACA}_{0}^{\omega} \vdash \forall f \exists g \operatorname{A}_{qf}(f, g).$$

To obtain a realizer for g use again the functional interpretation on the last statement. This extracts a realizer $t \in T_0[B_{0,1}]$ where $B_{0,1}$ is the bar recursor of lowest type, see Section 11.3 in [13]. Since t_g is only a term of type 2 one can find a term $t' \in T$ which is equal to t, see [11, Corollary 4.4.(1)]. This t' solves the theorem.

If one is not interested in the extracted program then one can obtain a stronger conservation result:

Theorem 9 (Conservation). The system $ACA_0^{\omega} + (\mu) + (\mathcal{U})$ is Π_2^1 -conservative over ACA_0^{ω} and therefore also conservative over PA.

Proof. Let $\forall f \exists g \operatorname{A}(f,g)$ be an arbitrary Π_2^1 statement which is provable in $\operatorname{ACA}_0^{\omega} + (\mu) + (\mathcal{U})$ and does not contain μ or \mathcal{U} . We will show that this statement is provable in $\operatorname{ACA}_0^{\omega}$ and if it is arithmetical also in PA.

Relative to (μ) each arithmetical formula is equivalent to a quantifier free formula. Hence there exists a quantifier free formula A'_{af} such that

$$\operatorname{RCA}_{0}^{\omega} + (\mu) \vdash \operatorname{A}(f,g) \leftrightarrow \operatorname{A}'_{qf}(f,g).$$

This gives

$$\operatorname{RCA}_{0}^{\omega} + (\mu) + (\mathcal{U}) \vdash \forall f \exists g \operatorname{A}_{af}'(f, g).$$

Since the system $\text{RCA}_0^{\omega} + (\mu)$ has a functional interpretation in $T_0[\mu]$, see [4, 8.3.1], one can now apply the same argument as in the proof of Theorem 4 with T_0 is replaced by $T_0[\mu]$, and obtains that

$$\operatorname{RCA}_{0}^{\omega} + (\mu) \vdash \forall f \exists g \operatorname{A}_{af}'(f,g)$$

and therefore also

$$\operatorname{RCA}_{0}^{\omega} + (\mu) \vdash \forall f \exists g \operatorname{A}(f,g)$$

The result follows now also from Theorem 8.3.4 of [4].

Appendix A. Elimination of Skolem functions for monotone formulas

We will show in this appendix that uses of a partial non-principal ultrafilter for an algebra given by a fixed term over a weak basis theory does not lead to more than primitive recursive growth. For this we will make use of Kohlenbach's elimination of Skolem functions for monotone formulas, see [10], [13, Chapter 13].

Let WKL₀^{*} be the system WKL where Σ_1^0 -IA is replaced by QF-IA and the exponential function and let WKL₀^{ω *} be the corresponding finite type extension. For a formal definition of see [16, X.4.1] and [12] for the finite type system.

Let Π_1^0 -CA(f) be the restriction of Π_1^0 -comprehension to the Π_1^0 formula given by f, i.e. the statement

$$\exists g \,\forall n \, (g(n) = 0 \leftrightarrow \forall x \, f(n, x) = 0)$$

Further, let $\mathcal{U}(\mathcal{A})$ be the principle that states that for the algebra $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$ given by $(f(n))_{n \in \mathbb{N}}$ there exists a set $F \subseteq \mathbb{N}$, such that

$$\mathcal{F} = \{A \mid \exists n \in F (A = A_i)\}$$

satisfies (\mathcal{U}) relativized to \mathcal{A} . This means that

$$\begin{cases} \forall i, j \ \left(A_i = \overline{A_j} \to (i \in F \lor j \in F)\right) \\ \land \forall i, j \ \left((A_i \subseteq A_j \land i \in F) \to j \in F\right) \\ \land \forall i, j, k \ \left((i, j \in F \land A_k = A_i \cap A_j\right) \to k \in F) \\ \land \forall i \ (i \in F \to \forall n \exists k > n \ (k \in A_i)) \,. \end{cases}$$

We obtain the following theorem:

Theorem 10. Let $A_{qf}(f, x)$ be a quantifier free formula that contains only f, x free and let t_1, t_2 be terms in WKL₀^{ω^*}. If

WKL₀<sup>$$\omega$$
*</sup> $\vdash \forall f (\Pi_1^0 \text{-CA}(t_1 f) \land \mathcal{U}(t_2 f) \rightarrow \exists x \operatorname{A}_{qf}(f, x))$

then one can extract a primitive recursive (in the sense of Kleene) functional Φ such that

$$\operatorname{RCA}_{0}^{\omega} \vdash \forall f \operatorname{A}_{qf}(f, \Phi(f)).$$

In particular if f is only of type 0 one obtains that there exists a primitive recursive function g such that

$$PRA \vdash \forall x A_{qf}(x, g(x)).$$

Proof. We will show, by formalizing the construction of b in the proof of Proposition 6, that there exists a term t' such that

$$\forall h (\Pi_1^0 \text{-} CA(t'h) \rightarrow \mathcal{U}(h)).$$

The theorem follows then from the elimination of Skolem functions for monotone formulas and the fact that one can code the two instances of Π_1^0 -CA given by t_1 and $t't_2$ into one. For the elimination of Skolem functions see for instance Proposition 13.20 in [13] — the statement of this proposition is essentially the same as of this theorem without \mathcal{U} . For the conservativity over PRA, see [4].

In the construction of b in the proof of Proposition 6 only two steps cannot be formalized in WKL₀^{ω *}. The first step is the application of Π_1^0 -CP and the second is the use of Π_2^0 -WKL. The use of Π_1^0 -CP can be reduced to a suitable instance of Π_1^0 -CA (with the parameters $\mathcal{F}, \tilde{\mathcal{A}}$) and QF-AC^{1,0}. The use of Π_2^0 -WKL follows from Π_1^0 -WKL and another instance of Π_1^0 -CA (also with the parameters $\mathcal{F}, \tilde{\mathcal{A}}$). Since Π_1^0 -WKL is equivalent to WKL and one can code the two instances of comprehension together one obtains in total that the index function b can be constructed in WKL₀^{ω *} + Π_1^0 -CA($t\mathcal{F}\tilde{\mathcal{A}}$) for a suitable t. (Note that the set \mathcal{F} cannot be defined since it involves μ .)

Using this one can extend the partial ultrafilter $\mathcal{F} = \{\mathbb{N}\}$ on the trivial algebra $\mathcal{A} = \{\emptyset, \mathbb{N}\}$ to an (index set of an) ultrafilter satisfying $\mathcal{U}(h)$. From this one can easily construct a term t'. This provides the theorem.

Remark 11. Although the restriction of \mathcal{U} to an algebra given by a term seems to be weak, it is strong enough to prove instances of ultralimit, i.e. that the ultralimit exists for (a sequence of) sequences given by one fixed term.

To see this let $(x_n)_{n \in \mathbb{N}}$ be a sequence in the interval [0, 1]. We will prove that the ultralimit of this sequence exists using $(\mathcal{U})(t[(x_n)])$ for a term t. For this let

$$A_{i,k} := \left\{ n \in \mathbb{N} \mid x_n \in \left[\frac{i}{2^k}, \frac{i+1}{2^k} \right] \right\}.$$

Let \mathcal{A} be the algebra created by this sets. It is clear that \mathcal{A} can be described by a term $t[(x_n)]$.

Observed that the proof of Lemma 2 can also be carried out in RCA₀^{*}. Since $(A_{i,k})_{i\leq 2^k}$ defines a finite partition of \mathbb{N} , Lemma 2 provides

$$\forall k \exists ! i \le 2^k \ (A_{i,k} \in \mathcal{U}),$$

(strictly speaking we obtain that the index of $A_{i,k}$ is in an index set of \mathcal{U}) and QF-AC^{1,0} yields a choice function f(k) for *i*. Note that the ultrafilter properties provide that each $A_{f(k),k}$ is infinite and that

$$\forall k \,\forall k' > k \,\left(A_{f(k'),k'} \subseteq A_{f(k),k}\right).$$

Let g(k) be the k-th element of $A_{f(k),k}$ then the sequence $(x_{g(k)})_k$ defines a Cauchy-sequence with Cauchy-rate 2^{-k} which converges to $\lim_{n\to\mathcal{U}} x_n$.

References

1. Asuman G. Aksoy and Mohamed A. Khamsi, *Nonstandard methods in fixed point theory*, Universitext, Springer-Verlag, New York, 1990, With an introduction by W. A. Kirk. MR 1066202

Jeremy Avigad, An effective proof that open sets are Ramsey, Arch. Math. Logic 37 (1998), no. 4, 235–240. MR 1635557

_____, Weak theories of nonstandard arithmetic and analysis, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 19–46. MR 2185426

Jeremy Avigad and Solomon Feferman, Gödel's functional ("Dialectica") interpretation, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, North-Holland, Amsterdam, 1998, pp. 337–405. MR 1640329

^{5.} Benno van den Berg, Eyvind Briseid, and Pavol Safarik, NN, in preparation.

11

- Philipp Gerhardy, A quantitative version of Kirk's fixed point theorem for asymptotic contractions, J. Math. Anal. Appl. 316 (2006), no. 1, 339–345. MR 2201765
- Jeffry L. Hirst, Hindman's theorem, ultrafilters, and reverse mathematics, J. Symbolic Logic 69 (2004), no. 1, 65–72. MR 2039345
- H. Jerome Keisler, Nonstandard arithmetic and reverse mathematics, Bull. Symbolic Logic 12 (2006), no. 1, 100–125. MR 2209331
- Mohamed A. Khamsi and Brailey Sims, Ultra-methods in metric fixed point theory, Handbook of metric fixed point theory, Kluwer Acad. Publ., Dordrecht, 2001, pp. 177–199. MR 1904277
- Ulrich Kohlenbach, Elimination of Skolem functions for monotone formulas in analysis, Arch. Math. Logic 37 (1998), 363–390. MR 1634279
- On the no-counterexample interpretation, J. Symbolic Logic 64 (1999), no. 4, 1491– 1511. MR 1780065
- <u>Higher order reverse mathematics</u>, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295. MR 2185441
- 13. _____, Applied proof theory: Proof interpretations and their use in mathematics, Springer Monographs in Mathematics, Springer Verlag, 2008. MR 2445721
- 14. Alexander P. Kreuzer and Ulrich Kohlenbach. Termextraction andpreprint Ramsey's theoremfor pairs, submitted, available at http://www.mathematik.tu-darmstadt.de/~akreuzer/files/TermExtractionAndRT22.rev.pdf.
- Erik Palmgren, An effective conservation result for nonstandard arithmetic, Math. Log. Q. 46 (2000), no. 1, 17–23. MR 1736646
- Stephen G. Simpson, Subsystems of second order arithmetic, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999. MR 1723993
- Robert M. Solovay, Hyperarithmetically encodable sets, Trans. Amer. Math. Soc. 239 (1978), 99–122. MR 0491103
- Henry Towsner, Hindman's theorem: an ultrafilter argument in second order arithmetic, J. Symbolic Logic 76 (2011), no. 1, 353–360.

FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT DARMSTADT, SCHLOSSGARTENSTRASSE 7, 64289 DARMSTADT, GERMANY

 $E\text{-}mail\ address:\ akreuzer@mathematik.tu-darmstadt.de\ URL:\ http://www.mathematik.tu-darmstadt.de/~akreuzer$