The two halves of disjunctive correctness

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Abstract

Ali Enayat had asked whether two halves of Disjunctive Correctness (DC) for the compositional truth predicate are conservative over Peano Arithmetic. In this article, we show that the principle "every true disjunction has a true disjunct" is equivalent to bounded induction for the compositional truth predicate and thus it is not conservative. On the other hand, the converse implication "any disjunction with a true disjunct is true" can be conservatively added to PA. The methods introduced here allow us to give a direct nonconservativeness proof for DC.

1 Introduction

The area of axiomatic truth theories analyses the concept of truth by studying first-order theories which try to capture various properties of this notion. These theories are formulated as follows: We choose a base theory strong enough to represent syntax (this is typically Peano Arithmetic, PA). To this theory, we add a fresh predicate T(x) whose intended reading is "x is (a code of) a true sentence" together with axioms governing the behaviour of that predicate. A notable example of such a theory is CT^- which stipulates that the truth predicate satisfies Tarski's compositional conditions for arithmetical sentences. For instance, a conjunction $\phi \wedge \psi$ is true iff both ϕ and ψ are true.

If we add to CT⁻ full induction for the formulae containing the truth predicate, the resulting theory, called CT, is not conservative over PA, that is,

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it proves arithmetical theorems which cannot be demonstrated in PA itself. More specifically, we can show by induction on the lengths of proofs that every sentence provable in PA is true (a principle called global reflection over PA) and, consequently, that PA is consistent.¹ On the other hand, by a theorem of Kotlarski, Krajewski, and Lachlan (see (Kotlarski et al., 1981)) CT⁻ itself does not prove any new arithmetical theorems.

In (Cieśliński, 2010), it was shown that already CT⁻ extended with a principle of propositional reflection "sentences derived in propositional logic from true premises are true" suffices to prove global reflection over PA. In other words, an overtly nonconservative principle can be derived without explicitly assuming induction on a ground of a rather innocuous principle of overtly truth-theoretic nature. Subsequently, it turned out that a number of other truth-theoretic principles are not conservative over PA and are all equivalent to Δ_0 -induction for the compositional truth predicate (CT⁻ with Δ_0 -induction is called CT₀).² The picture that emerged was that there seems to be a "minimal natural" nonconservative extension of CT⁻.

The "dividing line" between conservative and nonconservative extensions of PA has been named "Tarski boundary" by Ali Enayat.³ One of the most striking results on Tarski boundary was obtained by Enayat and Pakhomov (2019). It was shown that among theories equivalent to CT_0 is CT^- together with the principle of disjunctive correctness, DC, which states that any finite (but possibly nostandard) disjunction is true iff one of the disjuncts is true. This axiom appears to be a mild, natural, extension of the compositional clauses and yet turns out to carry the full strength of Δ_0 -induction.

It was not clear whether this result can be pushed further in the following manner: Disjunctive correctness can be naturally split into two halves. The first half is a principle DC-in saying "a disjunction with true disjunct is true" and the second is DC-out which says "a true disjunction has a true disjunct". It has been asked in (Enayat and Pakhomov, 2019) whether the second of these principles can be added conservatively to PA.⁴

In this article, we analyse both of the above principles. We show that over CT⁻, DC-in gives rise to a conservative extension of PA while DC-out

¹This is one of the basic results of truth theory. A comprehensive introduction to the area, including this result, can be found in (Halbach, 2011).

²For the proof of nonconservativity of CT_0 , see (Lelyk and Wcisło, 2017).

³One can find more information on the Tarski boundary in (Cieśliński, 2017). A more concise discussion is also contained in (Łełyk and Wcisło, 2016) and (Łełyk, 2019).

⁴See Question 5.3 in (Enayat and Pakhomov, 2019). The conservativity of DC-in was settled in December 2018 and stated in the formulation of that question, but the proof was not published.

yields Δ_0 -induction. The methods used for the nonconservativeness result yield a new, direct proof that DC is yet another incarnation of Δ_0 -induction for the formulae containing the truth predicate.

2 Preliminaries

We consider truth theories over Peano arithmetic, PA, as our base theory.⁵ It is an axiomatic theory in a language $\mathscr{L}_{PA} = \{S, +, \times, 0\}$ whose axioms consist of inductive definitions of addition and multiplication in terms of the successor function *S* together with the full induction scheme. Although this theory overtly speaks of natural numbers, in fact it is strong enough to capture objects such as finite sets, finite sequences, or finite graphs. Crucially, PA is capable of expressing syntactic notions such as "term", "formula" or "proof" and proving basic facts about these notions such as "a conjunction of two formulae is a formula." We assume that the reader is familiar with the coding of syntax and the basic metamathematics. This is discussed in many sources such as (Kaye, 1991), Chapter 9 or (Hájek and Pudlák, 1993), Chapter I, Section 1(d), pp.50–61. Throughout the article, we will explain those bits of notation that seem not to explain themselves. We provide a glossary of all formalised notions we are using throughout the paper in the appendix.

Convention 1.

- We will often conflate Gödel numbers of syntactic objects with those objects. We will also use formulae denoting syntactic expressions as if they were sets. For instance, we will write φ ∈ Sent_{ℒPA} rather than Sent_{ℒPA}(φ), where Sent_{ℒPA} expresses that φ is (a Gödel code of) an arithmetical sentence.
- Provably functional formulae will be used as if denoting actual functions. For instance, we will write $FV(\phi)$ for the set of free variables of ϕ .
- In particular, we will often write the results of syntactic operations without explicitly mentioning the operations themselves. For instance, if $\phi, \psi \in \text{Form}_{\mathscr{L}_{PA}}$, we will freely speak of the conjunction $\phi \wedge \psi$ rather than "the only *z* such that *z* is a conjunction of ϕ, ψ ."

⁵The choice is motivated mostly by a certain tradition in the field. However, the results make sense and still hold true over much weaker theories, say $I\Delta_0 + \exp$.

The research on Tarski boundary concerns extensions of the compositional truth theory. Let us define it. By writing \underline{n} , we mean (the Gödel code of) a **numeral** denoting the number n, that is,

$$\begin{bmatrix} S(S \dots S(0) \dots) \\ "S" \text{ repeated } n \text{ times} \end{bmatrix}$$

If *t* is (a Gödel code of) a closed arithmetical term, then by t° we mean the value of that term. Thus we have, for instance,

$$\mathbb{N} \models \lceil S0 + S(S(0)) \rceil^{\circ} = \underline{3}^{\circ} = 3.$$

Note that a value of an arithmetical term can be computed in a primitive recursive way and the natural function computing it may be formalised in PA. Now we can proceed to the actual definition.

Definition 2. By CT^- (Compositional Truth) we mean a theory in the arithmetical language with a fresh unary predicate *T* extending PA with the following axioms:

- $\forall x \left(T(x) \to \operatorname{Sent}_{\mathscr{L}_{\operatorname{PA}}}(x) \right).$
- $\forall s, t \in \operatorname{ClTerm}_{\mathscr{L}_{\operatorname{PA}}}\Big(T(s=t) \equiv s^{\circ} = t^{\circ}\Big).$
- $\forall \phi \in \operatorname{Sent}_{\mathscr{L}_{\operatorname{PA}}}\Big(T(\neg \phi) \equiv \neg T\phi\Big).$
- $\forall \phi, \psi \in \operatorname{Sent}_{\mathscr{L}_{PA}} \Big(T(\phi \lor \psi) \equiv T\phi \lor T\psi \Big).$
- $\forall \phi, \psi \in \operatorname{Sent}_{\mathscr{L}_{\operatorname{PA}}}\Big(T(\phi \wedge \psi) \equiv T\phi \wedge T\psi\Big).$
- $\forall \phi \in \operatorname{Form}_{\mathscr{L}_{PA}} \forall v \in \operatorname{Var}\left(\operatorname{Sent}_{\mathscr{L}_{PA}}(\exists v\phi) \to T(\exists v\phi) \equiv \exists x \ T\phi(\underline{x})\right).$
- $\forall \phi \in \operatorname{Form}_{\mathscr{L}_{PA}} \forall v \in \operatorname{Var}\left(\operatorname{Sent}_{\mathscr{L}_{PA}}(\forall v\phi) \to T(\forall v\phi) \equiv \forall x \ T\phi(\underline{x})\right).$
- $\forall \phi \in \operatorname{Form}_{\mathscr{L}_{\operatorname{PA}}} \forall \bar{s}, \bar{t} \in \operatorname{ClTermSeq}_{\mathscr{L}_{\operatorname{PA}}} \Big(\bar{s}^{\circ} = \bar{t}^{\circ} \to T\phi(\bar{s}) = T\phi(\bar{t}) \Big).$

Above, ClTermSeq $_{\mathscr{L}_{PA}}(x)$ is a formula expressing "x is a sequence of closed arithmetical terms." The rest of the notation should be self-explanatory and, as we remarked earlier, it is discussed in the appendix.

By CT, we mean CT⁻ with full induction for the extended language. By CT₀, we mean CT⁻ with induction for Δ_0 formulae containing the truth predicate. (Note that already CT⁻ contains full arithmetical induction.)

The last clause in the above axioms for CT^- is called **Regularity Axiom**, REG. It is not included among the basic axioms in the standard presentations of this theory, like (Halbach, 2011) or (Cieśliński, 2017). The version of CT^- with REG appears, e.g., in (Enayat et al., 2020) or (Lełyk and Wcisło, 2021). Admittedly, the regularity axiom has a clearly different status than the rest of the axioms of CT^- . We add it mostly for two (admittedly technical) reasons which we will explain in Section 6.

Although CT^- seems to express the crucial properties of the truth predicate, it is not arithmetically stronger than PA. The result was essentially proved by Kotlarski et al. (1981).⁶

Theorem 3 (Kotlarski–Krajewski–Lachlan). CT^- *is conservative over* PA, *i.e.*, *for any sentence* $\phi \in \mathscr{L}_{PA}$ *if* $CT^- \vdash \phi$, *then* PA $\vdash \phi$.

The above result contrasts with the situation for CT. By induction on the length of proofs, we can show that every proof in PA has true conclusion. In other words, we can prove in CT the following principle of **Global Reflection**, GRef:

$$\forall \phi \in \operatorname{Sent}_{\mathscr{L}_{\operatorname{PA}}}(\operatorname{Pr}_{\operatorname{PA}}(\phi) \to T\phi).$$

Notice that $CT^- + GRef$ is clearly not conservative over PA, since in particular $\neg T(0 \neq 0)$ is provable in the compositional theory CT^- and, by contraposition $\neg Pr_{PA}(0 \neq 0)$. The latter is not provable in PA by Gödel's second theorem.

As we have already mentioned, a number of seemingly unrelated principles turned out to be equivalent to this canonical nonconservative axiom GRef. One of them is Δ_0 -induction for the compositional truth predicate. Another principle which will play an important role in this article is **Propositional Reflection**, PropRef, defined as follows:

$$\forall \phi \in \operatorname{Sent}_{\mathscr{L}_{\operatorname{PA}}}\left(\operatorname{Pr}_{\operatorname{Prop}}^{T}(\phi) \to T\phi\right),$$

⁶The original result concerns satisfaction classes over a purely relational language. This work has been extended in (Kaye, 1991) to languages with terms and in (Engström, 2002) to truth classes over languages with terms. Subsequently, Enayat and Visser (2015) introduced a new, elegant and flexible method of constructing satisfaction classes which allowed them to strengthen the previous results in a number of interesting ways. They worked again in purely relational languages. A version for functional languages can be found in (Cieśliński, 2017). A version of Enayat–Visser construction covering languages with functional symbols with the regularity axioms included is discussed in (Lełyk and Wcisło, 2021). A reader may find a proof of a stronger result also in this article in Section 4.

where $\Pr_{\text{Prop}}^{T}(x)$ means that x is derivable in propositional logic from the set of premises Γ , such that T(y) holds for each $y \in \Gamma$. As we have already noted, Cieśliński (2010) showed that over CT^- , PropRef is equivalent to Δ_0 induction for the truth predicate. Subsequently, CT_0 was shown by Lełyk and Wcisło (2017) to be arithmetically equivalent to GRef and then, in Lełyk (2017), to be exactly the same theory as $\operatorname{CT}^- + \operatorname{GRef}$. Another presentation of the last result can be also found in a recent preprint (Lełyk, 2021).

Related to propositional reflection is the following principle of **Propositional Soundness**, PropSnd:

$$\forall \phi \in \operatorname{Sent}_{\mathscr{L}_{\operatorname{PA}}}\left(\operatorname{Pr}_{\operatorname{Prop}}(\phi) \to T\phi\right).$$

In effect, this axiom expresses that any arithmetical sentence which is a propositional tautology is true. It is still unknown whether $CT^- + PropSnd$ is conservative over PA. In the next section, we shall present a partial result towards this problem.

Now, let us turn to the main subject of our article. If $M \models PA$ and $\bar{\phi} \in SentSeq_{\mathscr{L}_{PA}}(M)$ is a coded sequence of sentences, we can form their disjunction $\bigvee \bar{\phi}$, which we will also denote by $\bigvee_{i \leq c} \phi_i$ if the length of $\bar{\phi}$ is c. We always assume that in "big disjunctions" parentheses are grouped to the left, so that the following equality holds in M:

$$\bigvee_{i \le c+1} \phi_i = \bigvee_{i \le c} \phi_i \lor \phi_{c+1}.$$

In effect, $\bigvee_{i \le c} \phi$ denotes the following formula:

$$(((\phi_0 \lor \phi_1) \lor \ldots) \lor \phi_{c-1}) \lor \phi_c.$$

The precise definition of how disjunctions over multiple disjuncts are parenthesised can actually matter in some cases. In the presence of Δ_0 -induction for the extended language, one can show that any two disjunctions with the same disjuncts are equivalent, no matter how the disjuncts are ordered and grouped together. However, this is not the case in pure CT⁻. The precise parenthesising can become relevant, since it dictates the relations between disjunctions over arbitrarily many disjuncts and the usual binary disjunctions. We will return to this issue in Section 5.

By **Disjunctive Correctness**, DC, we mean the following axiom:

$$\forall \bar{\phi} \in \operatorname{SentSeq}_{\mathscr{L}_{\mathrm{PA}}}\left(T\left(\bigvee \bar{\phi}\right) \equiv \exists i \leq \operatorname{lh}(\bar{\phi}) T\phi_i\right).$$

Therefore, DC states that a finite (but possibly nonstandard) disjunction is true iff it has a true disjunct. Very surprisingly, DC is yet another incarnation of CT_0 as shown by Enayat and Pakhomov (2019).

Theorem 4 (Enayat–Pakhomov). $CT^- + DC$ and CT_0 are equivalent.

As we mentioned, CT_0 is not conservative over PA. Hence the following fact easily follows:

Corollary 5 (Enayat–Pakhomov). $CT^- + DC$ is not conservative over PA.

Disjunctive correctness can be naturally split into two principles. The first is DC-out ("a true disjunction has a true disjunct"):

$$\forall \bar{\phi} \in \mathsf{SentSeq}_{\mathscr{L}_{\mathsf{PA}}}\Big(T \bigvee \bar{\phi} \to \exists i \leq \mathsf{lh}(\bar{\phi}) \ T\phi_i\Big).$$

The second is DC-in ("a disjunction with a true disjunct is true"):

$$\forall \bar{\phi} \in \operatorname{SentSeq}_{\mathscr{L}_{\mathrm{PA}}} \left(\exists i \leq \operatorname{lh}(\bar{\phi}) \ T \phi_i \to T \bigvee \bar{\phi} \right).$$

As we have mentioned, the status of DC-out was not settled in (Enayat and Pakhomov, 2019) and the claim that DC-in is conservative was stated without a proof. The subsequent parts of this article will be devoted to the analysis of both principles.

Let us finish this section by summing up the positive results on equivalences of theories of truth relevant for this work:

Theorem 6. *The following theories are equivalent:*

- 1. CT₀.
- 2. $CT^- + GRef.$
- 3. $CT^- + PropRef.$
- 4. $CT^{-} + DC$.

3 Yablo sequences and disjunctive correctness

In this section, we prove that DC-out is equivalent to CT₀. The argument is indirectly inspired by the classical Visser–Yablo paradox: ⁷ Consider the

⁷See (Yablo, 1993).

sequence of sentences Y_n , $n \in \mathbb{N}$, such that Y_n says: "some sentence Y_k for k > n is false." If for some k, Y_k is false, then every sentence Y_l for l > k is true. However, if Y_l is true, then there is some m > l such that Y_m is false contradicting the assumption. Hence, every sentence in the Yablo sequence is true. However, if some sequence in the Yablo sequence is true, then some has to be false, so they cannot be all true. We reach a contradiction.

Using a construction inspired by the Yablo sequence, we will show that DC-out implies CT_0 . We will actually use two intermediate principles which are overtly related to Δ_0 -induction. By **sequential induction**, SInd, we mean the following axiom:⁸

$$\forall s \in \operatorname{FinSeq} \left(Ts_0 \land \forall i < \operatorname{lh}(s) - 1 (Ts_i \to Ts_{i+1}) \to \forall j < \operatorname{lh}(s) Ts_j \right)$$

The **sequential order induction**, SOInd, is a natural variant of the above principle:

$$\forall s \in \operatorname{FinSeq} \left(\forall j < \operatorname{lh}(s) \left((\forall i < jTs_i) \to Ts_j \right) \to \forall l < \operatorname{lh}(s) Ts_l \right).$$

As we already mentioned, SInd and SOInd are clearly related to Δ_0 -induction:

Proposition 7. $CT^- + SOInd and CT_0$ are equivalent.

Proof. CT_0 clearly entails SOInd. On the other hand, observe that SOInd implies PropRef which by Theorem 6 is equivalent to CT_0 . Working in $CT^- +$ SOInd, fix a proof (ϕ_0, \ldots, ϕ_c) in propositional logic from true premises. We can assume that the proof system is chosen so that for all *i*, either ϕ_i is an assumption of the proof or ϕ_i is obtained by modus ponens from two formulas appearing earlier in the proof. In particular, for every *j*, if every formula ϕ_j for j < i is true, then ϕ_i is true which, by SOInd, implies that the conclusion of the proof is true. Thus PropRef holds.

Now, we get to the core argument of the article:

Theorem 8. Over CT⁻, DC-out *implies* SInd.

⁸A related principle of **Modus Ponens correctness** was introduced earlier by Ali Enayat in an unpublished note (Enayat, 2014). The principle states that if a conjunction of the implications $\phi_i \rightarrow \phi_{i+1}$ is true for i = 0, 1, ..., c and ϕ_0 is true, then ϕ_{c+1} is true. However, as stated, this principle is conservative over PA. Namely, it holds in any model in which all conjunctions of nonstandard length are false and by a construction very similar to that presented in Section 4, for any completion U of PA, we can find models with this property satisfying U.

Proof. Working in CT⁻+DC-out, fix any sequence $(\phi_0, \ldots, \phi_c) \in \text{SentSeq}_{\mathscr{L}_{PA}}$ such that $T\phi_0$ holds and for each i, $T\phi_i$ entails $T\phi_{i+1}$. Let us define a sequence $\psi_i, i \leq c$ as follows:

$$\psi_0 := \phi_0$$

$$\psi_{j+1} := \neg \phi_{j+1} \to \bigvee_{i < j+1} \neg \psi_i.$$

We claim that for all $j \leq c$, the sentence ψ_j is true. Suppose that $\neg T\psi_j$ holds for some j. Then j > 0, so we have:

$$\neg T\phi_j \land \neg T\bigvee_{i< j} \neg\psi_i.$$

By compositional conditions, this is equivalent to:

$$\neg T\phi_j \wedge \neg T\left(\bigvee_{i< j-1} \neg \psi_i\right) \wedge T\psi_{j-1}.$$

Expanding the definition of ψ_{j-1} , we obtain:

$$\neg T\phi_j \wedge \neg T\left(\bigvee_{i< j-1} \neg \psi_i\right) \wedge \left(T\phi_{j-1} \vee T\left(\bigvee_{i< j-1} \neg \psi_i\right)\right).$$

However, $\neg T \phi_j$ implies $\neg T \phi_{j-1}$. Therefore, we have the following:

$$\neg T\phi_j \wedge \neg T\left(\bigvee_{i< j-1} \neg \psi_i\right) \wedge T\left(\bigvee_{i< j-1} \neg \psi_i\right).$$

This contradiction concludes the proof of the claim. Notice that the proof of the claim only uses the fact that provably in CT⁻,

$$T\left(\bigvee_{i\leq j+1}\eta_i\right)\equiv T\left(\bigvee_{i\leq j}\eta_i\right)\vee T\eta_{j+1}.$$

Now, we show that for all j < c, ϕ_j is true. Suppose otherwise and fix j such that $\neg T\phi_j$. Since $T\psi_j$ holds, we have:

$$T\bigvee_{i < j} \neg \psi_i$$

By DC-out, we can fix i < j such that $\neg T \psi_i$ holds. However, this contradicts the previous claim.

As an application of the above result, we show that DC-out is the same theory as DC. In particular, $CT^- + DC$ -out is not conservative over PA.

Theorem 9. CT^- + SInd *implies* DC-in. *Consequently,* DC-out *and* DC *are equivalent over* CT^- .

Proof. Working in CT^- + SInd fix any sequence $(\alpha_0, \ldots, \alpha_c)$ such that $\alpha_i \in$ Sent \mathcal{L}_{PA} for all $i \leq c$. Suppose that α_j is true for some $j \leq c$ and notice that

$$\bigvee_{i \le j} \alpha_i = \bigvee_{i \le j-1} \alpha_i \lor \alpha_j,$$

hence the disjunction of α_i up to and including j has to be true as well. Moreover, for any k, if $\bigvee_{i \leq k} \alpha_i$ is true, then $\bigvee_{i \leq k+1} \alpha_i$ is true. Hence by sequential induction (starting with $\bigvee_{i < j} \alpha_i$), $T \bigvee_{i < c} \alpha_i$ holds.

Another application is a more perspicuous proof of nonconservativity of $CT^- + DC$.

Theorem 10. $CT^- + DC$ implies SOInd. Hence, $CT^- + DC$ is equivalent to CT_0 .

Proof. Working in $CT^- + DC$, fix any sequence (ϕ_0, \ldots, ϕ_c) and suppose that for any j, if $T\phi_i$ holds for all i < j, then $T\phi_j$ holds. By DC, the following implication holds for all $j \leq c$:

$$T \neg \bigvee_{i \le j} \neg \phi_i \to T \neg \bigvee_{i \le j+1} \neg \phi_i.$$

(Of course, these are essentially big conjunctions, but *prima facie*, disjunctive correctness does not imply conjunctive correctness.) Then, by SInd, we conclude that

$$T \neg \bigvee_{i \le j} \neg \phi_i$$

holds for each *i* which, again using DC implies that $T\phi_j$ holds for each *j*.

By Proposition 7 it follows that $CT_0 \subseteq CT^- + DC$. Since DC is easily provable in CT_0 , we conclude that $CT^- + DC$ is equivalent to CT_0 .

Remark 11. The main nonconservativeness proof for DC in (Enayat and Pakhomov, 2019) consists of two parts: it is first shown that over CT^- , DC implies the axiom of internal induction INT, to be defined in the next section, and then a much more direct proof that DC + INT implies CT_0 follows. It is easy to verify that CT^- + SInd entails internal induction and since we know that CT^- + DC implies SInd, we obtain a still different proof that CT^- + DC is equivalent to CT_0 .

We can present the above results in a slightly more abstract manner. This will allow us to obtain a significantly simpler proof of the result from (Wcisło, 2020) on the strength of certain extensions of Propositional Soundness.

Definition 12. Let U be a theory extending CT^- . We say that U has **outer disjunctions** if there exists a provably functional formula D(x, y) such that for any $\bar{\phi} = \langle \phi_1, \dots, \phi_c \rangle \in \text{SentSeq}_{\mathscr{L}_{PA}}$, we have $D(\bar{\phi}) \in \text{Sent}_{\mathscr{L}_{PA}}$ and the following two properties hold provably in U:

• $\forall \bar{\phi} \in \text{SentSeq}_{\mathscr{L}_{PA}} \forall \psi \in \text{Sent}_{\mathscr{L}_{PA}} \left(TD(\bar{\phi} \frown \langle \psi \rangle) \equiv TD(\bar{\phi}) \lor T\psi \right).$

•
$$\forall \bar{\phi} \in \text{SentSeq}_{\mathscr{L}_{\text{PA}}} \left(TD(\bar{\phi}) \to \exists i \leq \ln(\bar{\phi}) T\phi_i \right).$$

We call *D* as above an outer disjunction of ϕ_1, \ldots, ϕ_c .

In other words, a theory of truth has outer disjunctions if it has some uniform construction that behaves like disjunctions in $CT^- + DC$ -out.

Proposition 13. Suppose that U has outer disjunctions. Then it satisfies SOInd and in particular, it contains CT_0 .

Sketch of the proof. This is literally the same argument as in Theorems 8, 9, and 10. We only used the fact that $CT^- + DC$ -out has outer disjunctions.

By **quantifier-free correctness**, QFC, we mean the following axiom:

$$\forall \phi \in \operatorname{qfSent}_{\mathscr{L}_{\operatorname{PA}}}\Big(\operatorname{Tr}_0(\phi) \to T\phi\Big).$$

Let us pause for a moment and explain the notation. The formula $qfSent_{\mathscr{L}_{PA}}(x)$ expresses that x is a quantifier-free sentence of \mathscr{L}_{PA} , i.e., a Boolean combination of closed term equations. Peano arithmetic has a canonical way of deciding whether such (possibly nonstandard) sentences should be true or false by applying partial arithmetical truth predicates. The formula Tr_0 is such a predicate for Δ_0 formulae.⁹ It can be checked that QFC does not bring any arithmetical strength to CT⁻.

Proposition 14. $CT^- + QFC$ is conservative over PA.

⁹For a more detailed explanation of what partial arithmetical truth predicates are, consult (Kaye, 1991), Section 9 or (Hájek and Pudlák, 1993), Chapter I, Section 1(d).

The above proposition follows by a routine application of the methods introduced by Enayat and Visser, see e.g. (Enayat and Visser, 2015). A proof of this result in the exact same setting in which we work (with regularity axioms included in the definition of CT^-) can be found in (Wcisło, 2020). In the same article, it was shown with a different argument using a propositional construction called disjunctions with stopping conditions that QFC becomes significantly stronger when combined with PropSnd. Now we can prove that result with a simpler argument:

Proposition 15. $CT^- + QFC + PropSnd$ has outer disjunctions. In particular, *it is equivalent to* CT_0 .

Proof. The "in particular" part follows by Proposition 13 and the fact that QFC and PropSnd are clearly implied by Δ_0 -induction. So it is enough to show that $CT^- + QFC + PropSnd$ has outer disjunctions.

For $\overline{\phi} = (\phi_1, \dots, \phi_c)$, let

$$D(\bar{\phi}) = \exists x \le c \bigvee_{i \le c} \underline{i} = x \land \phi_i.$$

Let us check that *D* is an outer disjunction. First, consider the formula $D(\bar{\phi} \frown \langle \phi_{c+1} \rangle)$, i.e.

$$\exists x \le c+1 \bigvee_{i \le c+1} \underline{i} = x \land \phi_i.$$

We want to check that it is true iff either $D(\bar{\phi})$ is true or ϕ_{c+1} is true. Suppose that

$$T \exists x \le c+1 \ \bigvee_{i \le c+1} \underline{i} = x \land \phi_i.$$

By the compositional clauses and the definition of big disjunctions, this implies:

$$\exists x \le c+1 \ T\left(\bigvee_{i \le c} \underline{i} = \underline{x} \land \phi_i\right) \lor \exists x \le c+1 \ T\left(\underline{c+1} = \underline{x} \land \phi_{c+1}\right).$$

It can be easily checked by compositional clauses that if the second clause holds, then $T\phi_{c+1}$ holds. So it is enough to check that the first clause in fact implies

$$\exists x \le c \ T\left(\bigvee_{i \le c} \underline{i} = \underline{x} \land \phi_i\right).$$

To this end, we have to check that

$$\neg T\left(\bigvee_{i\leq c}\underline{i}=\underline{c+1}\wedge\phi_i\right).$$

However, notice that the following implication is an instance of a propositional tautology:

$$\left(\bigwedge_{i\leq c} \underline{i}\neq \underline{c+1}\right) \to \neg \left(\bigvee_{i\leq c} \underline{i}=\underline{c+1} \land \phi_i\right).$$

Hence, by PropRef, the whole implication is true and by QFC the antecedent is true as well and the conclusion follows by compositionality. This ends the proof of the implication.

The verification that $TD(\bar{\phi}) \vee T\phi_{c+1}$ implies $TD(\bar{\phi} \frown \langle \phi_{c+1} \rangle)$ is similar, but simpler, as it only uses compositional clauses of CT⁻.

Now, suppose that the following holds:

$$T\exists x \le c \bigvee_{i \le c} \underline{i} = x \land \phi_i.$$

We want to check that for some $i \le c$, $T\phi_i$ holds. By compositional clauses, we know that there exists $a \le c$ such that:

$$T\bigvee_{i\leq c}\underline{i}=\underline{a}\wedge\phi_i.$$

Again, using QFC and PropSnd, we check that the following holds:

$$T\left(\neg\phi_a \to \neg\bigvee_{i\leq c} \underline{i} = \underline{a} \land \phi_i\right).$$

Thus we can conclude that $T\phi_a$ holds which ends the proof.

Remark 16. In the above proof, we could actually show that the formula $D(\bar{\phi})$ satisfies both directions of DC.

The principle SInd clearly looks very related to Δ_0 -induction for the truth predicate. Indeed, it turns out that over CT⁻ the two principles are equivalent.

Theorem 17. CT^- + SInd has outer disjunctions.

Proof. For a coded sequence (ϕ_1, \ldots, ϕ_k) of \mathscr{L}_{PA} -sentences, let $\bigwedge_{i \leq c} \phi_i$ be their conjunction with parentheses grouped to the left so that we have:

$$\bigwedge_{i \le c+1} \phi_i = \bigwedge_{i \le c} \phi_i \wedge \phi_{c+1}.$$

Using SInd, we can show that if $T\phi_i$ holds for every $i \leq c$, then $T \bigwedge_{i \leq c} \phi_i$. We show this by considering an auxiliary sequence

$$\bigwedge_{i\leq 1}\phi_i, \bigwedge_{i\leq 2}\phi_i, \ldots, \bigwedge_{i\leq c}\phi_i.$$

Now, we claim that the formula

$$D(\bar{\phi}) := \neg \bigvee_{i \le c} \neg \phi_i$$

is an outer disjunction. It is easy to verify that over CT⁻:

$$TD(\bar{\phi} \frown \langle \phi_{c+1} \rangle) \equiv TD(\bar{\phi}) \lor T\phi_{c+1}.$$

We show that *D* satisfies the second condition of outer disjunctions (DC-out) by contraposition. If there is no $i \le c$ such that $T\phi_i$, then $T\neg\phi_i$ holds for every *i*. Consequently, the following conjunction is true:

$$\bigwedge_{i \le c} \neg \phi_i$$

which implies that $D\bar{\phi}$ cannot be true. This shows that D is indeed outer disjunction provably in $CT^- + SInd$.

Let us summarise the above results:

Corollary 18. *The following theories are equivalent:*

- CT₀
- $CT^- + SInd.$
- $CT^- + SOInd$.
- $CT^- + DC$ -out.

We conclude this section with some results that clarify the status of SInd and SOInd. First of all, let us note that SInd and SOInd are really just some forms of induction axioms and as such they do not require us to assume CT^- to make sense. They both clearly follow from $I\Delta_0(T)$, the induction scheme for Δ_0 -formulae containing the predicate T (which is now treated just as some arbitrary predicate). It turns out that these principles form a strict hierarchy.

Theorem 19. Over PA, $I\Delta_0(T) \rightarrow SOInd \rightarrow SInd$. Moreover, none of the implication reverses.

Proof. Both implications are straightforward, so let us show that neither reverses.

(SOInd $\nvDash I\Delta_0(T)$). Let M be a countable nonstandard model of PA. Let $s_i, i < \omega$ be an enumeration of all coded sequences in M. We will inductively construct a sequence of finite subsets of the standard cut ω^M , $A_i, B_i, i < \omega$ such that for all $i, A_i \cap B_i = \emptyset$. The sets A are approximations to T, the sets B are approximations to the complement.

Let $A_0 = B_0 = \emptyset$. For an arbitrary $i < \omega$, we consider two cases: if all the values of s_i are elements of A_i , i.e.:

$$\{a \in M \mid \exists j < \mathrm{lh}(s_i) \ a = s_i(j)\} \subseteq A_i,$$

then we set $A_{i+1} = A_i$, $B_{i+1} = B_i$. Otherwise, let j be the least element such that $s_i(j) \notin A_i$ (this element exists, since $|A_i|$ is standard, and therefore, A_i is arithmetically definable). Let $b = s_i(j)$, let $a \neq b$ be an arbitrary element in $\omega \setminus (A_i \cup B_i)$ and set:

$$A_{i+1} = A_i \cup \{a\} B_{i+1} = B_i \cup \{b\}.$$

Finally, we set $T = \bigcup_{i \in \omega} A_i$. We claim that $(M, T) \models$ SOInd, but (M, T) does not satisfy $I\Delta_0(T)$. The latter claim holds, since by construction T is an infinite (hence cofinal) subset of the standard cut ω . To check that the first claim holds, fix any coded sequence s. Pick $i < \omega$ such that $s = s_i$. By construction, either all values of s_i are in A_i , hence in T or there exists an l such that for all $j < l, s_i(j) \in A_{i+1}$, but $s_i(l) \in B_{i+1}$ (which means that it is not in T). This shows that $(M, T) \models$ SOInd.

(SInd \nvDash SOInd) As before, fix an arbitrary countable nonstandard model $M \models$ PA and let $s_i, i < \omega$ be an (external) enumeration of coded sequences from M. We inductively construct two sequences of sets A_i, B_i such that for

all i, $A_i \cap B_i = \emptyset$, the sets B_i are finite, and the sets A_i contain only finitely many nonstandard elements of M.

We construct A_i, B_i as follows: let $A_0 = \omega^M$ (the standard initial cut of M) and $B_0 = \emptyset$.

For an arbitrary *i*, if the set of values of s_i is (standardly) finite, we set $A_{i+1} = A_i, B_{i+1} = B_i$. Otherwise, we consider two further subcases. Suppose that the set of *j* such that $s_i(j) \in B_i$ is an initial segment in *M* (possibly empty). Let j_0 be its supremum (it exists by arithmetical induction, since B_i is finite; if the considered set is empty, then by definition, its supremum is 0). On the other hand, the set of values of s_i is by assumption infinite, so by overspill there exists $j > j_0$ such that $s_i(j) \notin A_i$. Let $a := s_i(j-1)$ and let $b := s_i(j)$.

If the set $\{j < \ln(s) \mid s_i(j) \in B_i\}$ is not an initial segment of M, then (again by arithmetical induction and finiteness of B_i) there exists some jsuch that $s_i(j) \notin B_i$ and $s_i(j+1) \in B$. Let $a := s_i(j), b := s_i(j+1)$.

In both cases, set:

$$A_{i+1} = A_i \cup \{a\}$$

 $B_{i+1} = B_i \cup \{b\}.$

Finally, let $T = \bigcup_{i \in \omega} A_i$. We claim that $(M, T) \models$ SInd, but not SOInd.

To see that $(M,T) \models$ SInd, fix any coded sequence $s \in M$. If the set of values of s_i has (standard) finite number of elements, then T cannot violate the sequential induction axiom for s. So suppose that the number of values of s is nonstandard. Fix i such that $s = s_i$ in our enumeration. By construction, there exists $j \in M$ such that $s(j) \in A_{i+1} \subset T$ and $s(j+1) \in B_{i+1} \subset M \setminus T$. So it is not the case that for all $x, T(s(x)) \to T(s(x+1))$ and thus the sequential induction axiom for s is satisfied.

On the other hand, (M, T) does not satisfy SOInd. Indeed, fix any $c \in M \setminus \omega$ and consider the identity sequence: s(i) = i for i = 0, 1, ..., c. We claim that the sequential order induction fails for this sequence. Let us check that *s* is progressive, i.e., if for any $j < i \ s(j) \in T$, then $s(i) \in T$.

Fix any $i \leq c$. If $i \in \omega$, then T(i) holds. On the other hand, if i is nonstandard, then consider the sequence $t := s \upharpoonright i$. It has nonstandardly many values, all of which are strictly below i. By construction, there exists an element b occurring in this sequence which is not in T. This b witnesses that not all j < i are in T. Hence s is progressive. On the other hand, not all terms of s are in T which means that SOInd fails.

4 Conservativeness of DC-in

In the previous section, we have shown that $CT^- + DC$ -out is not conservative over PA and that, in fact, it is another incarnation of CT_0 . Now we will use methods introduced by Enayat and Visser (2015) to show that the related principle DC-in is actually conservative over PA. The Enayat–Visser technique typically allows us to combine various results, so we can show joint conservativity of several distinct principles. Here we will illustrate this point by requiring that the constructed truth predicate additionally satisfies internal induction. Let us recall that principle. If $(M, T) \models CT^-$, then each formula $\phi \in Form_{\mathscr{L}_{PA}}^{\leq 1}(M)$ "defines" a set $\{x \in M \mid \phi(\underline{x}) \in T\}$. Internal induction (INT) expresses that each such set satisfies the induction principle:

$$\forall \phi \in \operatorname{Form}_{\mathscr{L}_{\mathsf{PA}}}^{\leq 1} \left(T\phi(\underline{0}) \land \forall x \left(T\phi(\underline{x}) \to T\phi(\underline{x+1}) \right) \to \forall x \ T\phi(\underline{x}) \right)$$

It was essentially observed already by Kotlarski et al. (1981) that $CT^- + INT$ is conservative over PA. This result can be also proved using cut elimination as in (Leigh, 2015) or the methods invented by Enayat and Visser which we will use in this section in order to show the following theorem:

Theorem 20. $CT^- + DC$ -in + INT is conservative over PA.

The theorem is a direct corollary to the following model-theoretic result:

Theorem 21. Let $M \models PA$. Then there exists $M' \succeq M$ and a $T' \subseteq M'$ such that $(M', T') \models CT^- + INT$ and for all sequences $\bar{\phi} \in SentSeq_{\mathscr{L}_{PA}}(M)$ of nonstandard length, the disjunction $\bigvee \bar{\phi}$ is in T'.

In other words, we will construct a model in which every disjunction with infinitely many disjuncts is true. Such a model clearly satisfies DC-in: fix any sequence of sentences $\bar{\phi}$ and suppose that there is $i \leq \ln(\bar{\phi})$ such that $T'\phi_i$. If $\bar{\phi}$ has nonstandard length, then $\bigvee \bar{\phi} \in T'$ by assumption. On the other hand, if the length of $\bar{\phi}$ is standard, then $\bigvee \bar{\phi} \in T'$ directly by the compositional axioms. So it is enough to prove Theorem 21.

In the proof, we will construct an elementary sequence of models $M_i \models$ PA and a sequence of satisfaction classes $S_i \subset M_i^2$, $i < \omega$ such that S_{i+1} will satisfy compositional clauses for formulae in the model M_i . Let us first define what a satisfaction class actually is. Below, if *s* is (a code of) a term and α is a function whose domain contains the free variables of *s* (an *s*assignment), by s^{α} we mean the formally computed value of *s* under the valuation α . For instance, if α ascribes the value 2 to *x* and 5 to *y*, then

$$\mathbb{N}\models \lceil SSx\times Sy\rceil^{\alpha}=24$$

If α, β are assignments and v is a variable, then by $\beta \sim_v \alpha$ we mean that $\operatorname{dom}(\beta) \supseteq \operatorname{dom}(\alpha) \cup \{v\}$ and $\beta(w) = \alpha(w)$ for every $w \in \operatorname{dom}(\alpha) \setminus \{v\}$. In other words, β is just like α , possibly except for the value $\beta(v)$ which is not even required to be defined for α .

Definition 22. Let $M \models PA$ and let $\phi \in Form_{\mathscr{L}_{PA}}(M)$. By the **compositional** clauses for ϕ , Comp (ϕ) , we mean the disjunction of the following sentences:

1.
$$\exists s, t \in \operatorname{Term}_{\mathscr{L}_{PA}} \left(\phi = (s = t) \land \forall \alpha \in \operatorname{Asn}(\phi) \left(S(\phi, \alpha) \equiv s^{\alpha} = t^{\alpha} \right) \right).$$

2.
$$\exists \psi \in \operatorname{Form}_{\mathscr{L}_{PA}} \left(\phi = (\neg \psi) \land \forall \alpha \in \operatorname{Asn}(\phi) \left(S(\phi, \alpha) \equiv \neg S(\psi, \alpha) \right) \right).$$

- 3. $\exists \psi, \eta \in \operatorname{Form}_{\mathscr{L}_{\operatorname{PA}}}(M) \left(\phi = (\psi \lor \eta) \land \forall \alpha \in \operatorname{Asn}(\phi) \left(S(\phi, \alpha) \equiv S(\psi, \alpha) \lor S(\eta, \alpha) \right) \right).$
- 4. $\exists \psi, \eta \in \operatorname{Form}_{\mathscr{L}_{\operatorname{PA}}}(M) \left(\phi = (\psi \land \eta) \land \forall \alpha \in \operatorname{Asn}(\phi) \left(S(\phi, \alpha) \equiv S(\psi, \alpha) \land S(\eta, \alpha) \right) \right).$
- 5. $\exists \psi \in \operatorname{Form}_{\mathscr{L}_{PA}}(M) \exists v \in \operatorname{Var} \left(\phi = (\exists v \psi) \land \forall \alpha \in \operatorname{Asn}(\phi) \left(S(\phi, \alpha) \equiv \exists \beta \sim_v \alpha S(\psi, \beta) \right) \right).$
- 6. $\exists \psi \in \operatorname{Form}_{\mathscr{L}_{PA}}(M) \exists v \in \operatorname{Var} \left(\phi = (\forall v\psi) \land \forall \alpha \in \operatorname{Asn}(\phi) \left(S(\phi, \alpha) \equiv \forall \beta \sim_v \alpha S(\psi, \beta) \right) \right).$

We say that $S \subset M^2$ is a **satisfaction class** if there exists a subset $D \subseteq$ Form_{$\mathcal{L}_{PA}(M)$} such that the following holds:

- If $(\phi, \alpha) \in S$, then $\phi \in \operatorname{Form}_{\mathscr{L}_{PA}}(M)$ and $\alpha \in \operatorname{Asn}(\phi)$.
- If $(\phi, \alpha) \in S$ for some $\alpha \in Asn(\phi)$, then $Comp(\phi)$ holds.
- If $(\phi, \alpha) \in S$, then $\phi \in D$ or $\phi = \neg \psi$ for some $\psi \in D$.
- If $\phi \in D$, then Comp (ϕ) holds.
- If $\phi \in D$ and ψ is a direct subformula of ϕ , then $\psi \in D$.
- If $\phi \in D$, then for all $\alpha \in Asn(\phi)$, $(\phi, \alpha) \in S$ or $(\neg \phi, \alpha) \in S$.

By the **domain** of *S*, dom(*S*), we mean the maximal set *D* satisfying the above conditions (the maximality requirement is needed, as there is a slight ambiguity in how to count negations of formulae which are not satisfied by any assignment). A satisfaction class is **full** iff its domain is the whole set Form $\mathscr{L}_{PA}(M)$.

The definition of a satisfaction class presented above involves some technical requirements which might seem to be slightly too restrictive, so let us briefly explain our motivations. The definition of a satisfaction class completely agrees with the usual one in the case of full satisfaction classes. However, if S is not a full satisfaction class, i.e., if it does not satisfy compositional conditions for all formulae, it becomes ambiguous whether we should interpret the fact that $(\phi, \alpha) \notin S$ as saying that ϕ is not satisfied under the valuation α or that S simply does not decide ϕ which might be a technical nuisance in some proofs or statements of results. This includes results and arguments in this article, though the specific difficulties we overcome with this definition will not be really visible. Therefore, we essentially require that for every formula ϕ we can unambiguously tell whether it is decided by S or not. This requirement is harmless: Define a pre-satisfaction class on M as a set $S \subset M^2$ such that it satisfies compositional axioms on some set D of formulae closed under direct subformulae and does not contain any pair (ϕ, α) for $\phi \notin D$. Then for any pre-satisfaction class, we can canonically define a satisfaction class extending it. Simply take the maximal set D with the two mentioned properties and extend S with all pairs $(\neg \phi, \alpha)$ such that $\phi \in D$ and $(\phi, \alpha) \notin S$. One can check that after such a one-step extension, we obtain a satisfaction class in our sense.

Satisfaction classes and truth predicates satisfying CT^- are very closely related objects. However, the link between them is not as direct as one could hope (see (Wcisło, 2021) for a discussion of this connection). We have to introduce a certain technical condition in order to switch between them in a completely unproblematic manner.

Let $M \models PA$, $\phi, \psi \in Form_{\mathscr{L}PA}(M)$, $\alpha \in Asn(\phi)$, $\beta \in Asn(\psi)$. We say that the pairs $(\phi, \alpha), (\psi, \beta)$ are **extensionally equivalent**, $(\phi, \alpha) \simeq (\psi, \beta)$ if there exists a formula η and two sequences of closed terms $\bar{s}, \bar{t} \in ClTermSeq_{\mathscr{L}PA}$ of the same length such that the sequences of their values are equal and

$$\phi[\alpha] = \eta(\bar{s}), \psi[\beta] = \eta(\bar{t}),$$

where $\phi[\alpha]$ is a sentence obtained by substituting in ϕ the numeral $\alpha(v)$ for each variable v. For instance, let $\phi = \exists x(x + y = SS0), \psi = \exists x(x + u \times v = w + S0)$, and let $\alpha \in \operatorname{Asn}(\phi), \beta \in \operatorname{Asn}(\psi)$ be such that $\alpha(y) = 2, \beta(u) = 0$

 $2, \beta(v) = \beta(w) = 1$. Then $(\phi, \alpha) \simeq (\psi, \beta)$ as witnessed by the formula

$$\eta = \exists x(x + v_0 = v_1)$$

and the terms $(SS0, SS0), (SS0 \times S0, S0 + S0)$. Finally, we say that a satisfaction class *S* is **regular** iff for all pairs $(\phi, \alpha) \simeq (\psi, \beta), (\phi, \alpha) \in S$ iff $(\psi, \beta) \in S$.

Before describing the relation between the interpretations of the truth predicate and regular satisfaction classes, let us introduce one more definition.

Definition 23.

• If $M \models \text{PA}$ and $S \subset M^2$ is a satisfaction class, we say that **internal induction** holds for $\phi \in \text{Form}_{\mathscr{L}_{\text{PA}}}^{\leq 1}(M)$ if for every $v \in \text{FV}(\varphi)$ and every $\alpha \in \text{Asn}(\phi)$, the following holds:

$$(\phi, \alpha[0/v]) \in S \land \forall x \Big((\phi, \alpha[x/v]) \in S \to (\phi, \alpha[x+1/v]) \in S \Big) \to \forall x \, (\phi, \alpha[x/v]) \in S.$$

Above, $\alpha[y/v]$ denotes the assignment α' which is identical to α , except for the fact that $\alpha'(v) = y$.

• We say that internal induction holds for *S* if it holds for every $\phi \in \operatorname{Form}_{\mathscr{L}_{\mathbf{PA}}}^{\leq 1}(M)$.

Notice that the formula ϕ need not be in the domain of *S*

The following proposition establishes the link between truth predicates satisfying CT^- (possibly with INT) and regular satisfaction classes (possibly with internal induction). It can be proved via a direct verification.

Proposition 24.

- (a) Let $(M,T) \models CT^-$ and let $S = \{(\phi, \alpha) \in M^2 \mid \phi \in Form_{\mathscr{L}PA}(M), \alpha \in Asn(\phi), \phi[\alpha] \in T\}$. Then S is a full regular satisfaction class.
- (b) Conversely, let $M \models PA$, let $S \subset M^2$ be a full regular satisfaction class in M and let $T = \{\phi \in Sent_{\mathscr{L}PA}(M) \mid (\phi, \emptyset) \in S\}$. Then $(M, T) \models CT^-$.
- (c) Let $(M,T) \models CT^- + INT$ and let S be defined as in (a). Then S is a full regular satisfaction class and internal induction holds for S.
- (d) Conversely, let S be a full regular satisfaction class in M such that internal induction holds for S. Let T be defined as in (b). Then $(M,T) \models CT^- + INT$.

Now we are ready to prove Theorem 21.

Proof of Theorem 21, general idea. Let $M \models PA$. We will find an elementary extension $M \preceq M'$ and a full regular satisfaction class S on M' with internal induction such that for every disjunction $\phi \in \text{Sent}_{\mathscr{L}PA}(M)$ with non-standardly many disjuncts, $(\phi, \emptyset) \in S$. Then by Proposition 24, there exists $T \subset M'$ such that $(M', T) \models CT^- + INT$ and all disjunctions with nonstandardly many disjuncts are in T.

We will construct (M', S) in stages. We will produce a sequence (M_n, S_n) of models such that:

- $M_0 := M, S_0 = \emptyset.$
- The models $M_i \models$ PA form an elementary chain.
- $S_{n+1} \subset M_{n+1}^2$ is a regular satisfaction class whose domain contains Form $\mathcal{L}_{PA}(M_n)$.
- For every $n, S_n \subset S_{n+1}$.
- For every *n*, the model M_n expanded with all the predicates $S_{\phi}, \phi \in$ Form $\mathscr{L}_{PA}(M_n)$ defined by $S_{\phi}(x) \equiv S_n(\phi, x)$ satisfies the full induction scheme.
- If $\phi := \bigvee \phi_i$ is a disjunction with nonstandardly many disjuncts and ϕ is in the domain of S_{n+1} , then $(\phi, \alpha) \in S_{n+1}$ for every $\alpha \in Asn(\phi)$.

Finally, we set $M' = \bigcup M_n$ and $S = \bigcup S_n$. By a straightforward verification, we check that *S* is indeed a full regular satisfaction class and it clearly makes all disjunctions with infinitely many disjuncts true and satisfies internal induction.

To complete the proof, it is enough to check that a sequence (M_n, S_n) as above can be produced. This will be demonstrated in Lemma 26 which takes care of the induction step for n > 1 (the existence of (M_1, S_1) can be proved with a very similar, slightly simpler argument).

Some care is needed in order to make sure that the satisfaction classes which we will construct are indeed regular. Before we state and prove the induction step lemma, we will introduce one more technical notion. (It appeared in the same context earlier, e.g. in (Lełyk and Wcisło, 2021).)

Definition 25. Let $\phi \in \text{Form}_{\mathscr{L}_{PA}}$. By a **syntactic template** of ϕ , we mean the smallest formula $\widehat{\phi}$ such that the following conditions are satisfied:

- 1. There exists a sequence of arithmetical terms \bar{s} (not necessarily closed) such that $\phi = \hat{\phi}(\bar{s})$.
- 2. No variable occurs in $\hat{\phi}$ both free and bound.
- 3. No free variable occurs in $\hat{\phi}$ more than once.
- 4. No closed term occurs in $\hat{\phi}$.
- 5. No complex term containing only free variables occurs in $\hat{\phi}$.

For instance, if $\phi = \exists x (SSx + Sy = (z \times (y + S0)) \times x)$, then

 $\widehat{\phi} = \exists x (SSx + v_0 = v_1 \times x),$

where v_0, v_1 are chosen so as to minimise the formula.

We say that ϕ and ψ are **syntactically similar** if they have the same syntactic template. We denote it with $\phi \sim \psi$. Notice that if $(\phi, \alpha) \simeq (\psi, \beta)$ for some α, β , then $\phi \sim \psi$.

Lemma 26. Let $M \models PA$, let $S \subset M^2$ be a regular satisfaction class such that:

- The model $(M, S_{\phi})_{\phi \in M}$ satisfies full induction, where $S_{\phi}(x) \equiv S(\phi, x)$.
- If φ is a disjunction with nonstandardly many disjuncts in the domain of S, then (φ, α) ∈ S for all α ∈ Asn(φ).

Then there exists an elementary extension $M \preceq M'$ and a regular satisfaction class $S' \supseteq S$ such that (M', S') satisfies the above conditions and dom $(S') \supseteq$ Form $\mathcal{L}_{PA}(M)$.

Proof. Let M, S be as in the assumptions of the lemma. Let us consider a theory Θ in a language \mathscr{L}_{PA} extended with an additional predicate S' and a family of auxiliary predicates $S'_{\phi}, \phi \in M$ which comprises the following axioms:

- ElDiag(*M*), the elementary diagram of *M*.
- $\forall x \ S'_{\phi}(x) \equiv S'(\phi, x)$ where $\phi \in \operatorname{Form}_{\mathscr{L}_{PA}}(M)$. (The definition of S'_{ϕ}).
- Comp(ϕ) for S', where $\phi \in \operatorname{Form}_{\mathscr{L}_{PA}}(M)$. (Compositionality Scheme).
- $S'(\phi, \alpha)$, where $(\phi, \alpha) \in S$. (Preservation Scheme).
- $\forall \phi, \psi \in \operatorname{Form}_{\mathscr{L}_{PA}} \forall \alpha \in \operatorname{Asn}(\phi), \beta \in \operatorname{Asn}(\psi) \left((\phi, \alpha) \simeq (\psi, \beta) \to S'(\phi, \alpha) \equiv S'(\psi, \beta) \right).$ (Regularity Axiom)

- The full induction scheme for formulae in the language L_{PA} + S'_φ, φ ∈ Form_{L_{PA}}(M). (Internal Induction).
- $\forall \alpha \in \operatorname{Asn}(\phi) S'(\phi, \alpha)$, where $\phi = \bigvee_{i \leq c} \phi_i$ for some $(\phi_i)_{i \leq c} \in \operatorname{FormSeq}_{\mathscr{L}_{PA}}(M)$ and a nonstandard *c*. (Disjunction Scheme).

The predicates S'_{ϕ} are introduced to concisely express what form of internal induction we accept. Note that in the Internal induction scheme, we do not allow formulae containing the predicate S'. Crucially, we are not allowed to treat ϕ like a variable. On the other hand, the form of induction we accept is *prima facie* stronger than the internal induction condition introduced in Definition 23.

If $(M', S', S'_{\phi})_{\phi \in M} \models \Theta$, then by restricting S' to (ϕ, α) such that $\phi \sim \phi'$ for some $\phi' \in M$ and ignoring the predicates S'_{ϕ} , we obtain a model satisfying the conclusion of the lemma (note that S' itself need not be a satisfaction class, as the compositional conditions may possibly fail badly for formulae in $M' \setminus M$). Therefore, it is enough to check that Θ is consistent. We will prove it by compactness. Let $\Theta_0 \subset \Theta$ be a finite subtheory. It is enough to check the consistency of Θ_0 .

Let $\phi_1, \ldots, \phi_n \in \text{Form}_{\mathscr{L}_{PA}}(M)$ be all formulae such that the instances of their compositionality, preservation, and disjunction schemes are in Θ_0 . We will define $S' \subset M^2$ which satisfies these instances of schemes, as well as all the other axioms in Θ . Note that ElDiag(M) will be automatically satisfied, since the interpretation of the relation S' will be defined in M.

Let us consider the classes $[\phi_i]_{\sim}$, where \sim is the syntactic similarity relation (see Definition 25 and the remark below the definition). Let \triangleleft be a relation defined on the classes as follows: $[\phi] \triangleleft [\psi]$ iff there exist $\phi' \in [\phi], \psi' \in [\psi]$ such that ϕ' is a direct subformula of ψ' . We define a sequence of relations $S_0, S_1, \ldots, S_k \subset M^2$ by induction on the rank of classes in the relation \triangleleft . We let $(\phi, \alpha) \in S_0$ if $[\phi]$ is minimal in the relation \triangleleft and one of the following holds:

- There exist $s, t \in \text{Term}_{\mathscr{L}_{PA}}(M)$ such that $\phi = (s = t)$ and $s^{\alpha} = t^{\alpha}$.
- $(\phi, \alpha) \in S$.
- ϕ is a disjunction with nonstandardly many disjuncts.

Above, we want to define S_0 on all classes minimal with respect to the relation \triangleleft . If *S* already holds of ϕ and α , then we preserve it in S_0 . If ϕ happens to be a disjunction with infinitely many disjuncts, then we make it true under all valuations. Otherwise, we set $(\phi, \alpha) \notin S_0$ for all α , but we do not have to mention it explicitly.

We define S_{i+1} as the union of S_i and the pairs (ϕ, α) such that $[\phi]$ has rank i + 1 and one of the following holds:

- There exists $\psi \in \operatorname{Form}_{\mathscr{L}_{PA}}(M)$ such that $\phi = \neg \psi$ and $(\psi, \alpha) \notin S_i$.
- There exist $\psi, \eta \in \operatorname{Form}_{\mathscr{L}_{PA}}(M)$ such that $\phi = \psi \lor \eta$ and $(\psi, \alpha) \in S_i$ or $(\eta, \alpha) \in S_i$.
- There exist $\psi, \eta \in \operatorname{Form}_{\mathscr{L}_{PA}}(M)$ such that $\phi = \psi \wedge \eta$ and $(\psi, \alpha) \in S_i$ and $(\eta, \alpha) \in S_i$.
- There exists $\psi \in \operatorname{Form}_{\mathscr{L}_{PA}}(M), v \in \operatorname{Var}(M)$ such that $\phi = \exists v \psi$ and $(\psi, \beta) \in S_i$ for some $\beta \sim_v \alpha$.
- There exists $\psi \in \operatorname{Form}_{\mathscr{L}_{PA}}(M), v \in \operatorname{Var}(M)$ such that $\phi = \forall v\psi$ and $(\psi, \beta) \in S_i$ for all $\beta \sim_v \alpha$.

In other words, we extend S_i to S_{i+1} so as to satisfy the compositional clauses. Finally, since we consider only finitely many classes, they can only attain some finite rank k. Let $S' = S_k$. Let $S'_{\phi}, \phi \in M$ be defined so that the definition-axiom of S'_{ϕ} is satisfied. We claim that $(M, S', S'_{\phi})_{\phi \in M} \models \Theta$.

It is obvious that (M, S') satisfies ElDiag(M). It follows directly by construction that it satisfies the compositional clauses for the formulae ϕ_1, \ldots, ϕ_n . It satisfies the instances of the preservation scheme for these formulae: for the formulae of minimal rank it follows by the definition of S_0 , since if a disjunction with infinitely many disjuncts is in the domain of S, then it is satisfied under all assignments. For formulae of higher rank, this follows by construction, since S is compositional on its domain and the compositional clauses uniquely determine the extension of a satisfaction predicate on a formula, given its extension on the direct subformulae.

Let us verify that S' satisfies the regularity axiom. For formulae in the \triangleleft -minimal classes, this follows by construction and the assumption that S is regular. For formulae in the classes of rank > 0, we can directly check that compositional clauses preserve regularity and thus show by induction that all the predicates S_i satisfy the regularity axiom.

The model $(M, S'_{\phi})_{\phi \in M}$ satisfies the full induction scheme, since we can verify by an easy induction on *i* that S_i is arithmetically definable in terms of S_{ϕ_i} , $i \leq n$, such that the class $[\phi_i]$ is \triangleleft -minimal and $\phi_i \in \text{dom}S$. By assumption, $(M, S_{\phi})_{\phi \in M}$ satisfies the full induction scheme and since S' is definable in that structure, it satisfies induction. The conclusion follows.

The predicate S' satisfies the disjunction scheme for formulae ϕ_1, \ldots, ϕ_n . Indeed, if ϕ is a disjunction with a nonstandard number of disjuncts and $[\phi]$ is minimal in the relation \triangleleft , then $(\phi, \alpha) \in S'$ by construction. Let us check by induction that the disjunction scheme is satisfied for formulae whose classes have higher rank. If the rank of $[\phi]$ is i + 1, then there exist formulae ψ, η such that ψ is a disjunction with a nonstandard number of disjuncts and $\phi = \psi \lor \eta$, where the rank of $[\psi]$ is $\leq i$. By induction hypothesis, $(\psi, \alpha) \in S'$ for all $\alpha \in \operatorname{Asn}(\psi)$, hence by the compositional clauses $(\psi \lor \eta, \beta) \in S'$ for all $\beta \in \operatorname{Asn}(\psi \lor \eta)$. This concludes the proof.

Remark 27. As we have already pointed out, Enayat–Visser methods of building satisfaction classes typically allow us to combine various conservativeness results. For instance, in Theorem 21, we could additionally require that the constructed predicate is correct with respect to blocks of existential or universal quantifiers.

It might appear that since a class we build in the proof of this theorem is somewhat pathological, there are certain obvious limits to what truththeoretic principles can be additionally satisfied. For instance, if the constructed truth predicate makes true all infinite disjunctions, then it cannot agree with Σ_n arithmetical truth predicates on sentences from the respective syntactic classes. However, as pointed out by Ali Enayat, we could still construct a model in which the truth predicate T agrees with all the usual partial arithmetical Σ_n -truth predicates and DC-in is satisfied. It is enough that in the proof of Lemma 26 we change the definition of S_0 so that for a minimal class $[\phi]$, where ϕ is a Σ_n -formula for some standard n which is not in the domain of S, S_0 agrees with the partial arithmetical satisfaction predicates. An alternative approach, mixing an Enayat–Visser approach and resplendence (by building a satisfaction class from recursively saturated partial classes) was proposed in an unpublished note by Enayat (2018).

5 Alternative versions of DC

In our paper, we defined $\bigvee_{i \leq c} \phi_i$ as $((\dots (\phi_0 \lor \phi_1) \lor \dots) \lor \phi_{c-1}) \lor \phi_c$. This is not the only possible definition of disjunction (and admittedly not the most natural one) and one could wonder whether modifying that definition has some impact on the presented results.

This point has been partially addressed in the definition of outer disjunctions: our proof of nonconservativity of $CT^- + DC$ -out used the clause

$$TD(\phi_1, \dots, \phi_c, \phi_{c+1}) \equiv TD(\phi_1, \dots, \phi_c) \lor T\phi_{c+1}.$$
(*)

It turns out that this assumption cannot be weakened much further. Let us define **balanced disjunctions** of a sequence of formulae $\phi_1, \ldots, \phi_c, B(\phi_1, \ldots, \phi_c)$

by recursion in the following way:

$$B(\emptyset) := 0 \neq 0$$

$$B(\phi_1) := \phi_1$$

$$B(\phi_1, \dots, \phi_c) := B(\phi_1, \dots, \phi_{\lfloor c/2 \rfloor}) \vee B(\phi_{\lfloor c/2 \rfloor + 1}, \dots, \phi_c).$$

In other words, we are grouping disjunctions of ϕ_1, \ldots, ϕ_k so that the parentheses form a binary tree. Over CT^- , balanced disjunctions do not satisfy the condition * and in fact, if we add an analogue of DC-out for that kind of disjunctions, we do not gain any arithmetical strength.

Theorem 28. CT⁻ together with the axiom $\forall \bar{\phi} \in \text{SentSeq}_{\mathscr{L}_{PA}}(TB(\bar{\phi}) \to \exists i \leq \ln(\bar{\phi}) T\phi_i)$ is a conservative extension of PA.

Sketch of a proof. Analogously to the proof of Theorem 20, we fix a model $M \models$ PA and we construct an elementary extension $M' \succeq M$ and $T \subset M'$ such that any balanced disjunction of a sequence $(\phi_1, \ldots, \phi_c) \in$ SentSeq $_{\mathcal{L}_{PA}}(M')$ of nonstandard length is false.

It is enough to observe that if ϕ is a balanced disjunction of nonstandardly many sentences, then there exist ϕ_1, ϕ_2 such that $\phi = \phi_1 \lor \phi_2$ and ϕ_i are themselves such balanced disjunctions, so we can maintain the requirement that all sentences of this form are false throughout the whole construction.

Notice that in the model produced in the proof of Theorem 20, all balanced disjunctions with nonstandardly many disjuncts are in fact true. This shows that both the analogues of DC-in and DC-out for the balanced formulae are conservative over PA and that in general DC-out is sensitive to the specific kind of forming disjunctions for which they are formulated. This leads to a natural question whether some natural forms of disjunction, say, balanced disjunctions, can satisfy full DC while remaining conservative. It turns out that DC behaves stably under such varying implementations.

Theorem 29. Suppose that U is a theory extending CT^- . Suppose that there exists a provably functional formula D(x) such that provably in U for any $\overline{\phi} = (\phi_1, \ldots, \phi_c) \in \text{SentSeq}_{\mathscr{L}_{PA}}$, $D(\overline{\phi})$ is in $\text{Sent}_{\mathscr{L}_{PA}}$ and the following holds:

$$TD(\phi) \equiv \exists i \le c \, T\phi_i.$$

Then U extends CT_0 .

Proof. Under the assumptions of the theorem, D is an outer disjunction provably in U. The conclusion follows by Proposition 13.

Remark 30. As we have noted above, the results on DC-out are sensitive to how we exactly define disjunctions over finite sets of sentences. One way to express abstractly what properties of disjunctions are used is the notion of outer disjunction. Ali Enayat has proposed a further generalisation of this concept.

Let U be a theory extending CT^- . We say that a provably functional formula D is a **selective disjunction** if there exists a formula F which provably in U defines a choice function for finite sets and the following conditions hold provably in U:

- For any $\Phi \in \text{SentSet}_{\mathscr{L}_{PA}}$, $D(\Phi) \in \text{Sent}_{\mathscr{L}_{PA}}$.
- $TD(\Phi) \equiv TF(\Phi) \lor TD(\Phi \setminus \{F(\phi)\}).$

One can check that if a theory U has selective disjunctions satisfying the analogue of DC-out, namely:

$$TD(\Phi) \to \exists \phi \in \Phi \ T\phi,$$

then *U* is again equivalent to CT_0 .

6 The role of the regularity axiom

In Section 2, we briefly mentioned that we have certain technical reasons to include Regularity, REG among the axioms for the compositional truth. Let us now explain why we adopt this principle and how this choice affects our results.

First of all, without REG, we have to be careful about the exact formulation of the compositional axioms for quantifiers. Our basic choice is between two options. We can require that, say, an existential statement $\exists v \phi(v)$ is true iff some sentence obtained by substituting a numeral in $\phi(v)$ is true. This amounts to adopting quantifier axioms in the form which we have chosen in this article:

$$T \exists v \phi(v) \equiv \exists x \ T \phi(\underline{x}).$$

The second option is to require that an existential statement is true iff a sentence obtained by substituting some closed arithmetical term for the quantified variable is true. Then, the quantifier axioms would have a form:

$$T \exists v \phi(v) \equiv \exists t \in \operatorname{ClTerm}_{\mathscr{L}_{\operatorname{PA}}} T \phi(t).$$

Note that these two versions of the quantifier axiom are not immediately comparable with respect to their strength. The left-to-right implication is stronger in the numeral version:

$$T \exists v \phi(v) \to \exists x T \phi(\underline{x})$$

whereas the reverse implication is formally stronger for the term version:

$$\exists t \in \operatorname{ClTerm}_{\mathscr{L}_{\operatorname{PA}}} T\phi(t) \to T \exists v \phi(v).$$

The term version was chosen, for instance, by Halbach (2011). The numeral version appears, e.g., in (Friedman and Sheard, 1987) or (Horsten and Leigh, 2017). This choice, in turn, may have a significant bearing on our results. Most importantly, we do not know whether adding Δ_0 -induction to a version of CT⁻ whose compositional axioms for quantifiers involve terms results in a non-conservative extension of PA. So, when regularity is missing, some of the main results in this article will depend on a technical choice in the formulation of compositional axioms which may lead to some further confusions.

Another issue is that without regularity, there is a mismatch between the notion of a truth predicate as discussed in the philosophical literature and the notion of satisfaction classes, as discussed in the literature on models of PA (e.g., as in (Kaye, 1991)). There is a direct link between the compositional truth and satisfaction classes assuming that we include some form of extensionality conditions in both these cases. (A more comprehensive discussion of why these kind of assumptions are relevant can be found in (Wcisło, 2021).) This link is important, as some techniques used in our proofs are designed to work specifically in the context of satisfaction classes. Most importantly, the conservativity arguments using the Enayat–Visser technique which we employ in Section 4 do not really work when directly applied to truth classes. Therefore, we would most likely need to add regularity assumptions to statements of some technical lemmas which could make for a potentially awkward reading.

Finally, let us discuss the impact of REG on the results in this article. The main results in Section 4, namely Theorems 20 and 21 are also true for CT⁻ without the regularity axiom, as the latter versions are formally weaker. The same applies to Remark 27, Proposition 14, and Theorem 28.

The situation of the results in Section 3 is slightly more complicated. As we have already mentioned, Theorem 6 is only known to hold if we consider a version of CT^- in which the numeral variant of the compositional quantifier axioms is assumed. In this case, all results from Section 3, still

hold. Unfortunately, this is not necessarily true for the term variant. Let us discuss in some detail what can be salvaged in this scenario.

Theorem 8 does not depend at all on quantifier axioms and thus it holds without assuming regularity in either version of CT^- ; similarly for Theorem 9. The first part of Theorem 10 and Theorem 17 also hold true. It is clear that SOInd implies PropRef for CT^- in the term version as well. Moreover, PropRef clearly implies DC using just compositional axioms for boolean connectives. However, it is unclear whether PropRef and CT_0 are equivalent without assuming REG and with the quantifier axioms in the term version. Similarly, the first part of Proposition 13 does not depend of regularity or the quantifier axioms, but we do not know whether such theories contain CT_0 . Proposition 15 may be entirely false if we do not assume the numeral version of the quantifier axioms, as the proof relies significantly on that assumption.

To sum it up: by the results of Section 3, DC-out, DC, SInd, SOInd, and PropRef are pairwise equivalent over any version of CT^- without assuming regularity. They are equivalent to CT_0 and not conservative over PA if we consider CT^- with quantifier axioms for numerals. If we consider CT^- with the term variant of the quantifier axioms, we do not know either whether DC or PropRef are equivalent to CT_0 , or whether the latter theory is arithmetically stronger than PA.

Appendix: a glossary of formalised notions

Throughout the article, we referred to a number of formalised notions and used some rather technical notation. Let us now gather it in a glossary for the convenience of the reader.

- $\alpha \sim_v \beta$ means that dom $(\beta) \supseteq$ dom $(\alpha) \cup \{v\}$ and $\beta(w) = \alpha(w)$ for every $w \in \text{dom}(\alpha) \setminus \{v\}$.
- *t*°. If *t* is (a Gödel code of) a closed arithmetical term, then *t*° is the value of a term (whose Gödel code is) *t*. We use the same expression to denote the formalised version of this function.
- t^α. If t is (a Gödel code of) an arithmetical term, and α is a t-assignment, then t^α is the value of term t under the assignment α. We use the same expression to denote the formalised version of this function.
- \bar{t}° . If \bar{t} is a sequence of (Gödel codes of) closed arithmetical terms, then \bar{t}° is the sequence of their values. We also use this expression to

denote the formalised version of this function.

- <u>x</u> is (a Gödel code of) a canonical numeral denoting the number x. We also use this expression to denote the formalised version of this function.
- $\phi[\alpha]$. If ϕ is a formula and $\alpha \in Asn(\phi)$, then by $\phi[\alpha]$ we mean a sentence obtained by substituting in ϕ the numeral $\alpha(v)$ for each variable v. We also use this expression to denote the corresponding formalised notion.
- Asn(*x*) is a set of *x*-assignments, that is, functions whose domain contains the set of free variables of *x*, where *x* is a term, a formula, or a sequence thereof. We also use this expression to denote the corresponding formalised notion.
- ClTerm L_{PA}(x) is a formula expressing "x is (a Gödel code of) a closed arithmetical term." (That is, a term with no free variables.)
- ClTermSeq_{LZPA}(x) is a formula expressing "x is a sequence of (Gödel codes of) closed arithmetical terms."
- FinSeq(*x*) is a formula expressing "*x* is a finite sequence of numbers."
- Form_{LPA}(x) is a formula expressing "x is (a Gödel code of) an arithmetical formula."
- Form ≤ 1 is a formula expressing "x is (a Gödel code of) an arithmetical formula with at most one free variable."
- FormSeq_{\mathcal{L}_{PA}}(*x*) is a formula expressing "*x* is a sequence of (Gödel codes of) arithmetical formulae."
- lh(s) = x is a formula expressing "s is a sequence and its length is x."
- Pr_{PA}(*x*) is a formula expressing "*x* is (a Gödel code of) an arithmetical sentence provable in PA."
- Pr_{Prop}(*x*) is a formula expressing "*x* is (a Gödel code of) an arithmetical sentence which is provable in pure propositional logic."
- $\Pr_{\text{Prop}}^{T}(x)$ is a formula expressing "x is (a Gödel code of) an arithmetical sentence which is provable in propositional logic from the set of premises Γ such that T(y) holds for all $y \in \Gamma$."

- qfSent $_{\mathscr{L}_{PA}}(x)$ is a formula expressing "x is (a Gödel code of) a quantifier-free arithmetical sentence."
- Sent_{LPA}(x) is a formula expressing "x is (a Gödel code of) an arithmetical sentence."
- SentSeq $_{\mathscr{L}_{PA}}(x)$ is a formula expressing "x is a sequence of (Gödel codes of) arithmetical sentences."
- SentSet $_{\mathscr{L}_{PA}}(x)$ is a formula expressing "x is a finite set of (Gödel codes of) arithmetical sentences."
- Term $\mathcal{L}_{PA}(x)$ is a formula expressing "x is (a Gödel code of) an arithmetical term."
- Tr₀ is the arithmetical truth predicate for Δ_0 -formulae.
- Var(*x*) is a formula expressing "*x* is (a Gödel code of) a first-order variable."

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