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An Epistemic Rationale for Order-Independence

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Abstract

The issue of the order-dependence of iterative deletion processes is well-known in the game theory community, and meanwhile conditions on the dominance concept underlying these processes have been detected which ensure order-independence (see e.g. the criteria of Gilboa et al., 1990 and Apt, 2011). While this kind of research deals with the technical issue, whether certain iterative deletion processes are order-independent, or not, our focus is on the normative issue, whether there are good reasons for employing order-independent iterative deletion processes on strategic games. We tackle this question from an epistemic perspective and attempt to figure out, whether order-independence contains some specific epistemic meaning. It turns out that, under fairly general preconditions on the choice rules underlying the iterative deletion processes, the order-independence of these deletion processes coincides with the epistemic characterization of their solutions by the common belief of choice-rule following behavior. The presumably most challenging precondition of this coincidence is the property of the independence of irrelevant acts. We also examine the consequences of two weakenings of this property on our epistemic motivation for order-independence. Although the coincidence mentioned above breaks down for both weakenings, still there exist interesting links between the order-independence of iterative deletion processes and the common belief of following the choice rules, on which these processes are based.

JEL Classification Number: C72, D83.
Key words: Iterative deletion process, order-independence, choice rule, epistemic game theory

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1 Introduction

The issue of order-independence of iterative deletion processes on strategic games has attracted a great deal of attention in the game theory community and almost every textbook in game theory point to the order-dependence of the deletion processes generated by the iterated deletion of weakly dominated strategies. Meanwhile, scholars have established order-independence of different iterative deletion processes and identified, for rather order-dependent iterative deletion processes, subclasses of strategic games in which these deletion processes become essentially order-independent. For example, Gilboa et al. (1990) detected properties on the players' domination relations guaranteeing that the iterative process based on these relations is order-independent. Notably, a corollary of their result is that the deletion processes generated by the iterated deletion of strictly dominated strategies is order-independent in finite strategic games.[1] Furthermore, for the class of finite strategic games, Börgers (1993) proved that the iterative deletion processes based on his dominance concept (also known as inherent dominance) is order-independent, Osborne and Rubinstein (1994) proved that the deletion processes relying on the iterated deletion of strictly dominated strategies in mixtures is order-independent and Apt (2005) proved that the iterative deletion processes of rationalizability (a solution concept powerfully proposed by Bernheim, 1984 and Pearce, 1984) are order-independent. Recently, resorting to a theorem of Newman (1942) regarding abstract reduction systems, Apt (2011) provided a handy criterion for order-independent iterative deletion processes. He showed that, if the iterative deletion processes are based on some hereditary dominance operator, then these processes turn out to be order-independent in finite strategic games. While the works just sketched dealt with the order-independence of specific iterative deletion processes, Rochet (1980) considered the order-dependent deletion processes of iterated deletion of weakly dominated strategies and addressed himself to the issue, in which subclass of finite strategic games these processes become order-independent. He established that order-independence holds for all finite strategic games for which at least one of the iterative deletion processes has a unique solution and for which every player receives a different payoff at each strategy profile. Marx and Swinkels (1997, 2000) also looked at the deletion processes generated by the iterated deletion of weakly dominated strategies and established that, if the finite strategic game satisfies the transference of decision maker indifference (TDI) condition, then these deletion processes result in solutions which are unique up to the addition or removal of redundant strategies.

The previous research on the order-independence we just briefly sketched has been concerned with the technical question, whether specific iterative deletion processes satisfy this property, or not. The focus of our paper differs substantially from this research agenda. Instead of supplying further results on the order-independence of specific iterative deletion processes, we shall address ourselves to the normative issue, whether there are weighty reasons beyond that of convenience to prefer order-independent iterative deletion processes. Thereto, we will take up an epistemic perspective on strategic games, and we will attempt to provide an epistemic motivation for order-independence. In the following discussion, we will set up a hypothesis that attributes a pellucid epistemic meaning to order-independence. Unfortunately, we will figure out counterexamples establishing that our hypothesis is not true in general. Nevertheless, as will show in the succeeding

[1]However, as Dufwenberg and Stegeman (2002) demonstrate with simple examples of infinite strategic games (see e.g. their Example 1), the deletion processes generated by the iterated deletion of strictly undominated strategies are order-dependent in the class of general strategic games.

sections of this paper, under specific restrictions regarding the deletion operators (or choice rules, as we will call it from now on) on which the iterative deletion processes are based, our postulated epistemic motivation proves to be correct. But, how does our hypothesis about the intrinsic epistemic value of order-independence exactly run?

At first, consider the solution of the deletion processes generated by the iterated deletion of weakly dominated strategies in strategic game Γ_1 depicted in Figure 1.[2] Obviously, the iterated maximal deletion of weakly dominated strategies (i.e., in each round of deletion, all strategies of every player weakly dominated in the (remaining) game will be eliminated) result in the unique solution (d, r). An alternative iterative deletion process based on weak undominance would be to eliminate all weakly dominated strategies only of player R in the first round and afterwards to eliminate all strategies of player C weakly dominated in the remaining game. This process would result in the outcome $\{(m, r), (d, r)\}$. Indeed, any of the sets $\{(d, r)\}$, $\{(d, c\}$, $\{m, r\}$, $\{(d, r), (d, c)\}$ and $\{(d, r), (m, r)\}$ could be attained as a solution of some deletion process based on iterated deletion of weakly dominated strategies in strategic game Γ_1. This divergence in their solutions is known as the order-dependence of the iterated deletion of weakly dominated strategies.

		Player C		
		l	c	r
	u	$(1000, 1000)$	$(0, 1000)$	$(0, 0)$
Player R	m	$(1000, 0)$	$(1, 1)$	$(1, 2)$
	d	$(0, 0)$	$(2, 1)$	$(1, 1)$

Figure 1: Strategic game Γ_1

Next, let us analyze strategic game Γ_1 from an epistemic perspective. Being more precise, we would like to figure out the set of strategy profiles which are possible if there is common belief among the players that they never choose weakly dominated strategies given their conjectures about the possible opponent's choices of strategy. It turns out that any strategy profile of strategic game Γ_1 could occur under this epistemic assumption. Even the strategy profile (u, l) consisting of the two strategies that do not survive any conceivable deletion process of iterated deletion of weakly undominated strategies is consistent with this precondition. To see this, consider a state in which (i) the players R and C choose the strategies u and l, respectively, and (ii) both players believe that this state will occur. Obviously, according to our assumptions (i) and (ii), in this state, both players act as if they deem favorable only strategies which are weakly undominated given their conjectures about the other player's move. Applying again assumption (ii), we can infer that, in this state, both players believe that they apply the choice rule of weak undominance. Indeed, by repeated application of assumption (ii), it can be established that, in this state, there is common belief of applying the choice rule of weak undominance (i.e. every player believes that every player applies this choice rule, every player believes that every player believes that every player applies this choice rule, and so on ad infinitum). Summing up the discussion so far, we have observed that the deletion processes based on the iterated deletion of weakly dominated

[2] In this paper, whenever we speak of a *weakly dominated strategy*, we refer to a strategy that is weakly dominated by some pure strategy. A strategy that is weakly dominated by some mixed strategy is said to be *weakly dominated by mixtures*. Analogously, strictly dominated strategies and strictly dominated strategies by mixtures are defined.

strategies are order-dependent and that their solutions do not contain all strategy profiles being consistent with applying the weak undominance rule and the common belief of applying this rule.

Now, let us consider the deletion processes generated by the iterated deletion of strictly dominated strategies. Unlike the deletion processes based on the weak undominance rule, these processes are order-independent and, as it can be shown, their solution consists exactly of the strategy profiles which are possible if all players follow the strict undominance rule and this is also commonly believed among them.[3] Thus, while the order-dependent deletion processes based on the weak undominance rule fail to comprise all strategy profiles that could be realized if all players behave according to this choice rule and such behavior is commonly believed among them, the order-independent deletion process relying on the strict undominance rule satisfies this epistemic property. Therefore, in reviewing our two examples of iterative deletion processes, it becomes obvious to suggest the following hypothesis about the epistemic meaning of order-independence.

CLAIM 0: *The deletion processes based on the iterated deletion of unfavorable strategies are order-independent, if and only if the solution of the iterated maximal deletion of unfavorable strategies coincides with the set of strategy profiles being possible if all players follow the choice rules on which these deletion processes are based and this is commonly believed among the players.*

Unfortunately, this claim is incorrect. In what follows, we will discuss two counterexamples to this claim. It will be demonstrated that order-independence is neither a sufficient nor a necessary condition for the coincidence between the solutions obtained by the iterated maximal deletion of unfavorable strategies and solutions under the choice rule following behavior and the common belief of this.

In the first example, we will consider the choice rule which favors only the strategies that, yield at least at one combination of the opponents' strategies, at least the average payoff of all available strategies at this combination. Henceforth, we shall simply term this choice rule as the "average rule". It can be proved that the deletion processes based on this choice rule are order-independent.[4] Now consider, strategic game Γ_2 that is depicted in Figure 2 and that will be solved by the iterated maximal deletion of unfavorable strategies with respect to this choice rule. Obviously, in the first round strategy d is canceled, in the second round strategy m, and in the third round strategy r. Hence, the solution of this iterative deletion process is the strategy profile (u, l).

However, as we will demonstrate next, the solution of the iterated deletion of unfavorable strategies does not contain all strategy profiles which are consistent with the choice rule following behavior and the common belief of this. Consider a state in which player R chooses m and player C chooses r. Moreover, it is presupposed that, in this state, both players believe that this state will be realized. Given such beliefs, both players act as if they follow the so-called average rule. Since both players believe that this state will occur, both players believe that they act according to the average rule, both players believe that they believe that they act according to this rule, and so on ad infinitum. Thus, at this state, there is common belief that they follow the average

[3]In Section 4, we will reproduce a theorem of Trost (2010) which specifies properties on choice rules ensuring that the solution obtained by the iterated maximal deletion of unfavorable strategies with respect to these choice rules gives exactly the set of strategy profiles consistent with the common belief of following these choice rules. As it can be easily checked, the choice rule of strict undominance satisfies these properties.

[4]The criterion of order-independence proposed by Apt (2011) will be reproduced in Section 3. There, by means of this criterion, we will demonstrate that the average rule is order-independent.

		Player C	
		l	r
	u	$(3,1)$	$(3,0)$
Player R	m	$(2,0)$	$(2,1)$
	d	$(0,1)$	$(0,1)$

Figure 2: Strategic game Γ_2

rule. Consequently, we have discovered a strategy profile that is consistent with the common belief of applying the average rule, namely (m, r), but that does not survive the iterated deletion of unfavorable strategies. We conclude that order-independence is not a sufficient condition that the solutions of the iterated deletion of unfavorable strategies coincides with the set of strategy profiles being consistent with the choice rule following behavior and the common belief of this.

In the second example, we will consider a choice rule that proves to be a modification of the choice rule of strict dominance. This "modified strict dominance rule" is specified as follows. For strategy sets containing exactly two distinguishable strategies (i.e. which generate different payoff profiles on the opponents' strategy profiles), any strategy is considered favorable. Otherwise, only the strategies that strictly dominate any other distinguishable strategy are deemed favorable or, if such strategies do not exist, then any strategy is deemed favorable. It can be shown that, for any strategic game, the solution of the iterated maximal deletion of strategies unfavorable with respect to this choice rule is equal to the set of strategy profiles being consistent with the common belief of applying this choice rule.[5] Now, consider strategic game Γ_3 depicted below.

			Player C	
		l	c	r
	u	$(4,4)$	$(4,2)$	$(4,0)$
Player R	m	$(2,4)$	$(2,2)$	$(2,0)$
	d	$(0,4)$	$(0,2)$	$(0,0)$

Figure 3: Strategic game Γ_3

As it can be easily verified, the iterated maximal deletion of unfavorable strategies stops after the first round of deletion. In this round, the strategies m and d of player R and the strategies c and r of player C are eliminated. Thus, the solution of this iterative deletion process consists only of strategy profile (u, l). Remarkably, this solution is not reached by all deletion processes based on this choice rule. For example, if only strategies d and r are eliminated in the first round of deletion, then the reduced game consists of strategy sets that contain two distinguishable strategies. According to our specification, the deletion process would stop and the solution would be the product set $\{u, m\} \times \{l, c\}$. This example shows, that the equivalence between iterated maximal deletion of unfavorable strategies and the common belief of choice rule following behavior does not ensure order-independence.

Obviously, the two examples just discussed lead to the following interim finding.

[5] Again, this can be affirmed by checking the criteria on choice rules postulated in Trost (2010).

REMARK: *The* CLAIM 0 *is false. Moreover, neither order independence implies that the solution of the iterative maximal deletion process corresponds to the set of strategy profiles being possible if all players follow the choice rules underlying this deletion process and this is commonly believed among the players, nor the latter correspondence implies the order-independence of all iterative deletion processes based on these choice rules.*

Although our CLAIM 0 is not generally true, we admit that the choices rules specified in the two counterexamples seem to be quite unreasonable and artificial. An obvious question is, whether the epistemic motivation for order-independence as stated in CLAIM 0 is true, although not for all conceivable choice rules, but at least for a class of choice rules containing prominent and meaningful choice rules. Figuring out such class of choice rules is the main topic of this paper. As a conclusion of our previous discussion, we can summarize our research question as follows.

RESEARCH QUESTION: *For which class of choice rules can the order-independence of the iterative deletion processes, based on these choice rules, be identified with the coincidence between the solution obtained by the iterated maximal deletion and the solution under the choice rule following behavior and the common belief of this?*

To provide an answer to this question, we will proceed as follows. In the next section, we shall introduce the concept of a choice rule in situations of subjective uncertainty and we shall discuss properties of these rules that become relevant to our succeeding epistemic analysis. In Section 3, the strategic games shall be transformed to individual decision problems under subjective uncertainty so that we can apply the concept of the choice rule to strategic games. Furthermore, iterative deletion processes based on these choice rules will be defined and we will discuss criteria ensuring their order-independence. The main results of this paper are stated in Section 4. There, at first, we will specify our epistemic model and, afterwards, we will prove that the relationship suggested above between the order-independence of iterative deletion processes and the epistemic characterization of its solution with the common belief of choice rule following behavior holds for a wide class of choice rules. As we will see, one precondition for this relationship is that the choice rules satisfy the property of the independence of unfavorable acts. Since this property is quite demanding, we shall also consider two weakenings of this property. Although the relationship just mentioned breaks down, whenever in place of the independence of irrelevant alternatives one of these weakenings are presupposed, there still exist some interesting links between the order-independence of iterative deletion processes and the common belief of choice rule following behavior. In Section 5, we shall summarize our findings and point to their limitations. For reasons of clarity, we have postponed the lengthy proofs of Lemmata 3.5 and 4.4 to the Appendix.

2 Properties of Choice Rules

In order to decompose strategic games into individual decision problems under uncertainty, we will follow the state space framework proposed by Savage (1954) and describe the uncertainty of a decision maker by a set of possible states of the world. Each state represents a specific resolution of all relevant uncertain features. For example, given a strategic game, the factor of uncertainty with which a player is faced is the strategy choices of the other players.

Henceforth, Ω will denote the finite set of all possible states of the world and this set will be called *state space*. A choice of a decision maker induces a specific profile of outcomes on the state space. Here, we shall assume that the outcomes are real numbers and therefore they could be interpreted as monetary payoffs. Formally, a choice rule is a mapping assigning to each state of the world a real-valued payoff. The set of all these mappings will be denoted by $\mathbb{R}^\Omega$. According to Savage (1954) these mappings will be called *acts*. In literature, acts are also known as state-contingent claims (see e.g. Arrow, 1953). The ωth component of the act x shall be denoted by x_ω and shall indicate the payoff the decision maker receives, when she has chosen the act x and the state ω occurs. A subset E of Ω will be termed as *event* and the restriction of some act $x \in \mathbb{R}^\Omega$ on E will be denoted by $x|_E$. To save coherency to the following conventions, this notation differs slightly from the notation generally found in decision theory, where restriction on E is simply denoted by x_E. An act is said to be constant, whenever it yields the same payoff at each state. Constant acts are signed by an upper bar. Let α be a real number, then $\bar{\alpha}$ is the act yielding, in each state, the payoff α.

A non-empty finite subset B of $\mathbb{R}^\Omega$ shall be termed *constraint* and shall comprise all acts that are available for the decision maker. A *decision problem under uncertainty* will be described by a pair $\Phi := (\Omega, B)$, where Ω is the state space specifying the uncertainty the decision maker is faced with and B the constraint specifying the options the decision maker has at her disposal. Let $E \subseteq \Omega$ be a non-empty event and $\Phi := (\Omega, B)$ some decision problem under uncertainty, then $B|_E := \{x|_E : x \in B\}$ will denote the constraint reduced on E and $\Phi|_E := (E, B|_E)$ will denote the decision problem Φ reduced on event E. The latter can be understood as the new decision problem arising when the decision maker confide in the information that the actual state of the world belongs to E.

A *choice rule* $\mathcal{C}$ is a mapping that assigns to each decision problem $\Phi := (\Omega, B)$ a (possibly empty) set $\mathcal{C}(\Phi) \subseteq B$ of acts. The set $\mathcal{C}(\Phi)$ will be called *choice set* and the acts belonging to this set will be termed as *favorable acts* or *best acts* under choice rule $\mathcal{C}$. Available acts which do not belong to the choice set will be called *unfavorable acts*. In the following, we shall list properties of choice rules which will become relevant for our analysis in the succeeding sections.

Definition 2.1 *A choice rule $\mathcal{C}$ is said to be*

- *non-empty, if $\mathcal{C}(\Phi) \neq \emptyset$ holds for any decision problem Φ.*
- *non-trivial, if there are real numbers $\alpha, \beta \in \mathbb{R}$ so that $\bar{\beta} \notin \mathcal{C}(\Omega, \{\bar{\alpha}, \bar{\beta}\})$ holds for some state space Ω.*
- *independent of payoff-equivalent states, if*

$$\mathcal{C}(\tilde{\Omega}, \{x \circ \tau : x \in B\}) = \{x \circ \tau : x \in \mathcal{C}(\Omega, B)\}$$

  holds for any decision problem (Ω, B) and for any surjective mapping $\tau : \tilde{\Omega} \to \Omega$.
- *independent of unfavorable acts, if*

$$\mathcal{C}(\Phi) = \mathcal{C}(\Omega, \tilde{B})$$

  holds for any decision problem $\Phi := (\Omega, B)$ and for any constraint $\tilde{B}$ satisfying $\mathcal{C}(\Phi) \subseteq \tilde{B} \subseteq B$.

The property of non-emptiness says that any decision problem contains a favorable act and, thus, any decision problem is solvable. To avoid grappling with existence problems we presuppose this property throughout this paper. Nevertheless, we should be aware that prominent choice rules like e.g. the choice rule of strict dominance (i.e. the rule considering favorable only the available act that strictly dominates any other available act) violate this property.

The property of non-triviality requires the existence of a decision problem in which only two constant acts are available and in which at least one of these two acts is considered unfavorable.

Obviously, this property excludes the trivial choice rule that, for any decision problem, considers any available act as a favorable act. Because any reasonable choice rule fulfills this property, it seems unproblematic to presuppose this property in succeeding analysis.

The property of the independence of payoff-equivalent states is more disputable. It demands that neither relabeling of the acts nor the addition or removal of payoff-equivalent states (i.e. states which yield the same payoffs as some other state) do affect the choice behavior of the decision maker. Obviously, whenever both non-triviality and the independence of payoff-equivalent states are postulated, there exists real numbers $\alpha, \beta \in \mathbb{R}$ so that $\bar{\beta} \notin \mathcal{C}(\Omega, \{\bar{\alpha}, \bar{\beta}\})$ holds for any arbitrary state space Ω.

A prominent example of a choice rule that violates this property is the *Laplace rule* (also known as *the principle of insufficient reasoning*) according to which any act is favorable which maximizes the expected payoff, where any state of the world is supposed to be equally likely. To see its failure, consider the two decision problems Φ_1 and $\tilde{\Phi}_1$ depicted below in Figure 4.

Decision problem Φ_1

	States	
Acts	α	β
x	5	0
y	2	2

Decision problem $\tilde{\Phi}_1$

	States		
Acts	$(\alpha, 1)$	$(\beta, 1)$	$(\beta, 2)$
$\tilde{x}$	5	0	0
$\tilde{y}$	2	2	2

Figure 4: Decision problems Φ_1 and $\tilde{\Phi}_1$

As pictured, the decision problems $\Phi_1 := (\Omega, B)$ and $\tilde{\Phi}_1 := (\tilde{\Omega}, \tilde{B})$ consist of the state spaces $\Omega := \{\alpha, \beta\}$ and $\tilde{\Omega} := \{(\alpha, 1), (\beta, 1), (\beta, 2)\}$, respectively, and the constraints $B := \{x, y\}$ and $\tilde{B} := \{\tilde{x}, \tilde{y}\}$, respectively. The projection τ assigning, to each state from $\tilde{\Omega}$, the state from Ω having the same Greek lower case is surjective and has the property $\tilde{B} = \{z \circ \tau : z \in B\}$. In case, the decision maker always applies the Laplace rule, act x is chosen in decision problem Φ_1, whereas act $\tilde{y}$ is chosen in decision problem $\tilde{\Phi}_1$. Obviously, this contradicts the property of the independence of payoff-equivalent states.

The presumably most demanding property presented in Definition 2.1 is the property of the independence of unfavorable acts. It requires that the choice behavior is not affected by omitting unfavorable acts from the constraint. A prominent example of a choice rule violating the independence of unfavorable acts is the *minimax regret rule* proposed already by Niehans (1948) and Savage (1951). This rule considers favorable these available acts which minimize, over all states, the deviation from the highest possible payoff in each state. In order to recognize that the minimax regret rule is dependent of unfavorable acts, consider decision problem Φ_3 depicted in Figure 5.

	States	
Acts	α	β
x	5	0
y	2	2
z	0	4

Figure 5: Decision problem Φ_2

Obviously, act x as well as act z are unfavorable acts in this decision problem. However, if act z is omitted from the constraint, then act x becomes favorable, while act y becomes unfavorable. This simple example shows, that the choice behavior resulting from the minimax regret rule is not independent of the set of unfavorable acts.

Among the choice rules satisfying all properties listed in Definition 2.1 are prominent choice rules like the maximin rule, the leximin rule (see Sen, 1970), Börgers' dominance rule (see Börgers, 1993), the strict undominance rule in pure as well as in mixed acts, and the weak undominance rule in pure acts as well as in mixed acts. The property of the independence of unfavorable acts can be split up in the following two properties.[6]

Definition 2.2 *The choice rule $\mathcal{C}$ satisfies*

(a) *property α_0, if*
$$\mathcal{C}(\Phi) \subseteq \mathcal{C}(\Omega, \tilde{B})$$
holds for any decision problem $\Phi := (\Omega, B)$ and for any subset $\mathcal{C}(\Phi) \subseteq \tilde{B} \subseteq B$.

(b) *property β_0, if*
$$\mathcal{C}(\Phi) \supseteq \mathcal{C}(\Omega, \tilde{B})$$
holds for any decision problem $\Phi := (\Omega, B)$ and for any subset $\mathcal{C}(\Phi) \subseteq \tilde{B} \subseteq B$.

The property α_0 requires that favorable acts remain favorable, even if unfavorable acts are omitted from the constraint. Clearly, a choice rule $\mathcal{C}$ having this property satisfies the condition $\mathcal{C}(\Phi) = \mathcal{C}(\Omega, \mathcal{C}(\Phi))$. The property β_0 requires that unfavorable acts remain unfavorable, even if other unfavorable acts are omitted from the constraint. As it can be easily derived from the above decision problem Φ_2, the minimax regret rule satisfies neither property α_0 nor property β_0.

The modified strict dominance rule defined in the Introduction satisfies property α_0, but violates property β_0. In order to show that the modified strict dominance rule satisfies property α_0, we note that $x \in \mathcal{C}(\Phi)$ means either that (i) $\#B > 2$ (i.e. the cardinality of the constraint is larger than 2) and x strictly dominates every other available act, or (ii) $\#B > 2$ and there exists no act strictly dominating the other available acts, or (iii) $\#B \leq 2$ holds. In the first case, we observe $\{x\} = \mathcal{C}(\Phi)$. Obviously, for any subset $\tilde{B}$ of B containing act x, act x strictly dominates all other acts belonging to $\tilde{B}$. Thus, $x \in \mathcal{C}(\Omega, \tilde{B})$ holds for any $\mathcal{C}(\Phi) \subseteq \tilde{B} \subseteq B$. In the second and third case, we observe $B = \mathcal{C}(\Phi)$ and the desired property follows immediately. In order to show that property β_0 is violated, consider the decision problem of player R participating in strategic game Γ_3. Let $\Omega := \{l, c, r\}$ be her state space and, for simplicity, denote each available act by the letter pertaining to the strategy that induces this act, then $\{u\} = \mathcal{C}^R(\Omega, \{u, m, d\})$ and $\{u, m\} = \mathcal{C}^R(\Omega, \{u, m\})$ hold. Clearly, this contradicts property β_0.

The average rule discussed in the Introduction violates property α_0, but satisfies property β_0. The former claim can be established by the decision problem of player R participating in strategic game Γ_2. Let $\Omega := \{l, r\}$ be her state space and identify the available acts as just done. Obviously, $\{u, m\} = \mathcal{C}^R(\Omega, \{u, m, d\})$ and $\{u\} = \mathcal{C}^R(\Omega, \{u, m\})$ hold, but this choice behavior contradicts our property α_0. In order to show that the average rule satisfies property β_0, consider some decision problem $\Phi = (\Omega, B)$ with the constraint $B := \{x_1, \ldots, x_n\}$, where the distinct available acts x^{n-1}

[6]The naming of these properties follows the naming of Sen's (1969) requirements on choice rules. As it can be easily shown, our property α_0 is a weakening of Sen's property α, whereas our property β_0 is a weakening of a slightly modified version of Sen's property β demanding that, if $x, y \in \tilde{B} \subseteq B$, $x \in \mathcal{C}(\Omega, \tilde{B})$ and $y \in \mathcal{C}(\Omega, B)$ hold, then $x \in \mathcal{C}(\Omega, B)$ results.

and x^n are deemed unfavorable. Now, suppose that act x^n is removed form the constraint B. Since both acts are deemed unfavorable in the original decision problem, $x^{n-1}_\omega < \frac{1}{n}\sum_{k=1}^{n} x^k_w$ as well as $x^n_\omega < \frac{1}{n}\sum_{k=1}^{n} x^k_w$ hold for any state $\omega \in \Omega$. The latter strict inequality is equivalent to $x^n_\omega < \frac{1}{n-1}\sum_{k=1}^{n-1} x^k_w$. Because $x^{n-1}_\omega < \frac{1}{n}\sum_{k=1}^{n-1} x^k_w + \frac{1}{n}x^n_\omega$ is satisfied for any $\omega \in \Omega$, we obtain $x^{n-1}_\omega < \frac{1}{n-1}\sum_{k=1}^{n-1} x^k_w$ for any $\omega \in \Omega$. This strict inequality says that act x^{n-1} is deemed unfavorable even in the new decision problem without act x^n. By applying repeatedly this argument, it can be established that the average rule satisfies property β_0.

3 Order-Independence of Iterative Deletion Processes

Up to now it has been left open what is hidden behind a state of the world. Now, we concretize the environment with which the decision maker is faced. Henceforth, we shall consider situations in which a group of decision makers interact, that is, a situation in which the payoffs the decision makers receive are affected not only by their own choice, but also by the choice of the other decision makers. Such situations are usually called *games* and the rules how this interaction takes place are recorded in the *game form*. In this paper, we shall restrict ourselves to the most simple game form, the so-called *strategic games*. This class of games is characterized by the property that individuals decide only once and simultaneously (i.e. no individual has observed the decisions of the other individuals, when she decides to move).

Formally, a strategic game Γ will be described by a tuple $\Gamma := (S^i, z^i)_{i\in N}$, where N shall denote a non-empty, finite set of players, S^i a non-empty set of strategies for player i, and $z^i : \times_{j\in N} S^j \to \mathbb{R}$ player i's *payoff function*. Throughout this paper, we presuppose that the strategic game is finite, that is, the strategy set S^i of every player $i \in N$ is finite. Any combination $s := (s^i)_{i\in N}$ of the players' strategies will be referred to as a *strategy profile*. As usual, the set of all strategy profiles shall be denoted by $S := \times_{i\in N} S^i$ and the set of all profiles listing strategies of players different to i shall be denoted by $S^{-i} = \times_{j\in N\setminus\{i\}} S^j$. Note, the payoff function z^i assigns to each strategy profile s a real-valued number $z^i(s)$ which will be interpreted as the monetary payoff, which player i receives, if the strategy profile s is realized. The sequence $\left(z^i(s^i, s^{-i})\right)_{s^{-i}\in S^{-i}}$ gives the possible payoffs which player i can receive, if she chooses strategy s^i. Henceforth, this sequence will be called the payoff profile of the strategy s^i on S^{-i}. The two strategies $s^i, \tilde{s}^i \in S^i$ of player i shall be termed *distinguishable* in the strategic game Γ, whenever their payoff profiles are different (i.e. there exists some combination s^{-i} so that $z^i(s^i, s^{-i}) \neq z^i(\tilde{s}^i, s^{-i})$ holds). The Cartesian product $R := \times_{i\in N} R^i$ where $R^i \subseteq S^i$ holds for every player $i \in N$ will be called a *restriction* of the strategy space S, and the set of all restrictions of the strategic game Γ will be signed by $\mathfrak{S}_\Gamma$. A strategy $s^i \in S^i$ shall be said to belong to constraint R, whenever $s^i \in R^i$ holds. Consider some strategic game Γ and some restriction $R \in \mathfrak{S}_\Gamma$. The strategic game $\Gamma|_R := \left(R^i, z^i|_R\right)_{i\in N}$, where $z^i|_R$ denotes the restriction of the payoff function on the domain R, will be called the *reduction of game* Γ *on restriction* R.

In order to apply some family $(\mathcal{C}^i)_{i\in N}$ of choice rules to some strategic game $\Gamma := (S^i, z^i)_{i\in N}$ we have to decompose the game into individual decision problems. Thereto, we introduce, for each player $i \in N$, a mapping $\mathfrak{a}^i_\Gamma$ that associates with each strategy $s^i \in S^i$ the act $\mathfrak{a}^i_\Gamma(s^i)$ on S^{-i} which is specified by $\mathfrak{a}^i_\Gamma(s^i) := z^i(s^i, s^{-i})$ for each $s^{-i} \in S^{-i}$. Obviously, act $\mathfrak{a}^i_\Gamma(s^i)$ which will be also referred to as the *act induced by the strategy* s^i, gives the payoff profile of the strategy s^i. Let $R^i \subseteq S^i$. As usual, the set $\mathfrak{a}^i_\Gamma(R^i) := \{\mathfrak{a}^i_\Gamma(s^i) \in \mathbb{R}^{S^{-i}} : s^i \in R^i\}$ is the image of R^i under the mapping $\mathfrak{a}^i_\Gamma$. The set of acts that player i has to her disposal when she is participating in game Γ

is determined by $B^i_\Gamma := \mathfrak{a}^i_\Gamma(S^i)$. Following this convention, the decision problem of player i can be described by the pair $\Phi^i_\Gamma := (S^{-i}, B^i_\Gamma)$. Henceforth, this pair shall also be referred to as the *strategic decision problem of player i in strategic game* Γ. A strategy s^i will be called favorable in the strategic game Γ, if $\mathfrak{a}^i_\Gamma(s^i) \in \mathcal{C}^i(\Phi^i_\Gamma)$ applies, and unfavorable, if $\mathfrak{a}^i_\Gamma(s^i) \in B^i_\Gamma$ but $\mathfrak{a}^i_\Gamma(s^i) \notin \mathcal{C}^i(\Phi^i_\Gamma)$ holds. Let $(\mathfrak{a}^i_\Gamma)^{-1}$ denote the inverse of the mapping $\mathfrak{a}^i_\Gamma$. Obviously, the strategy s^i is favorable in the strategic game Γ, if and only if $s^i \in (\mathfrak{a}^i_\Gamma)^{-1}\left(\mathcal{C}^i(\Phi^i_\Gamma)\right)$ holds.

Consider some family $\mathcal{C} := (\mathcal{C}^i)_{i\in N}$ of choice rules. For each strategic game $\Gamma := (S^i, z^i)_{i\in N}$, we set up a binary relation $\xrightarrow{\Gamma}_{\mathcal{C}}$ on $\mathfrak{S}_\Gamma$ by

$$R \xrightarrow{\Gamma}_{\mathcal{C}} \tilde{R} \quad :\Leftrightarrow \quad \begin{array}{l} \tilde{R} \subseteq R, \tilde{R} \neq R, \text{ and } s^i \notin \left(\mathfrak{a}^i_{\Gamma|_R}\right)^{-1}\left(\mathcal{C}^i(\Phi^i_{\Gamma|_R})\right) \text{ for any player } i \in N \\ \text{and for any strategy } s^i \in R^i \setminus \tilde{R}^i \ . \end{array}$$

Henceforth, the binary relation $\xrightarrow{\Gamma}_{\mathcal{C}}$ will be called the *reduction relation* of the family $\mathcal{C}$ of choice rules on the restrictions $\mathfrak{S}_\Gamma$ of the strategic game Γ. In words, $R \xrightarrow{\Gamma}_{\mathcal{C}} \tilde{R}$ says, that the restriction $\tilde{R}$ is distinct from the restriction R and contains at least all strategies that are favorable with respect to the choice rules $\mathcal{C}$ in the reduced strategic game $\Gamma|_R$. In case some restriction $\tilde{R}$ exists, so that $R \xrightarrow{\Gamma}_{\mathcal{C}} \tilde{R}$ is satisfied, restriction R shall be termed *reducible*. Otherwise, it shall be called *irreducible*. Let $\xrightarrow{\Gamma}^*_{\mathcal{C}}$ be the reflexive and transitive closure of the reduction relation $\xrightarrow{\Gamma}_{\mathcal{C}}$.[7] A restriction R will be said to be *reachable*, whenever $S \xrightarrow{\Gamma}^*_{\mathcal{C}}$ applies.

A deletion process generated by some iterated deletion of unfavorable strategies with respect to the choice rules $\mathcal{C}$ in the strategic game Γ corresponds to a sequence $(R_t)^T_{t=0}$ of restrictions of $\mathfrak{S}_\Gamma$ satisfying the three conditions:

(i) $R_0 := S$,
(ii) restriction R_t is reachable for any $t \in \{1, \ldots, T\}$,
(iii) restriction R_T is irreducible.

Index t indicates the round of the iterative deletion process and set R^i_t (set R^{-i}_t) is the set of strategies of player i (of strategies of the opponents of i, respectively) surviving t rounds of deletion. The last component R_T of some iterative deletion process will be called the *solution* of this process.

Let us turn to the deletion process eliminating in each round all unfavorable acts. This process will be referred to as the *iterated maximal deletion of unfavorable strategies* and is determined by the sequence $(R_t)^T_{t=0}$ satisfying the condition

(iv) $R_t := \cup\{\tilde{R} : R_{t-1} \xrightarrow{\Gamma}_{\mathcal{C}} \tilde{R}\}$ holds for each $t \in \{1, \ldots, T\}$,

in addition to the above conditions (i)-(iii). Clearly, for any family of non-empty choice rules and for any finite strategic game, the solution produced by the iterated maximal deletion of unfavorable strategies is non-empty.

The iterative deletion processes based on the family $(\mathcal{C}^i)_{i\in N}$ of choice rules shall be said to be *order-independent* if, for all strategic games, all deletion processes generated by the iterated deletion of unfavorable strategies with respect to these choice rules have the same solution. In the remaining part of this section, we will discuss some sufficient and necessary conditions for order-independence. By referring to a result of Newman (1942) for abstract reduction systems,

[7]Let S be a set and $\rightarrow \subseteq S \times S$ a binary relation on S. The reflexive and transitive closure of the binary relation $\rightarrow$ is a reflexive and transitive binary relation $\rightarrow^*$ on S containing $\rightarrow$ and satisfying $\rightarrow^* \subseteq \rightsquigarrow$ for any reflexive and transitive binary relation $\rightsquigarrow$ on S that contains $\rightarrow$.

Apt (2011) demonstrated that the following property on choice rules ensures that the iterative deletion processes based on such rules are order-independent.[8]

Definition 3.1 *A family $(\mathcal{C}^i)_{i \in N}$ of non-empty choice rules is hereditary if for any strategic game $\Gamma := (S^i, z^i)_{i \in N}$ and for any restrictions R and $\tilde{R}$, where $R \xrightarrow{\Gamma}_{\mathcal{C}} \tilde{R}$ holds, the implication*

$$\text{if } \; s^i \notin \left(\mathfrak{a}^i_{\Gamma|_R}\right)^{-1}\left(\mathcal{C}^i(\Phi^i_{\Gamma|_R})\right) \; \text{ then } \; s^i \notin \left(\mathfrak{a}^i_{\Gamma|_{\tilde{R}}}\right)^{-1}\left(\mathcal{C}^i(\Phi^i_{\Gamma|_{\tilde{R}}})\right)$$

is satisfied for any player $i \in N$ and for any strategy $s^i \in S^i$.

The succeeding theorem reproduces the order-independence theorem of Apt (2011). We will resort to this theorem during our epistemic motivation for order-independence in the following section.

Theorem 3.2 (Apt, 2011) *If the family $(\mathcal{C}^i)_{i \in N}$ of choice rules is hereditary, then the iterated deletion processes based on these choice rules are order-independent.*

However, without any further preconditions, the property of heredity does not prove to be a sufficient condition for order-independence. In order to verify this claim, consider the following choice rule which we shall call the "modified strict undominance rule", henceforth. According to this choice rule, for decision problems, in which at least four distinguishable acts are available, the acts that are strictly undominated are considered unfavorable. For the remaining decision problems, this choice rule deems unfavorable only the strictly undominated acts that yield the lowest total payoff (i.e. with the lowest total sum of payoffs received over all strategy combinations of the opponents). Interestingly, although the modified strict undominance rule fails to be hereditary, the iterative deletion processes based on this choice rule are order-independent.

Definition 3.3 *A choice rule $\mathcal{C}$ is called monotone, if $x \notin \mathcal{C}(\Phi)$ implies $x|_E \notin \mathcal{C}(\Phi|_E)$ for any decision problem $\Phi := (\Omega, B)$ and for any non-empty event $E \subseteq \Omega$.*

The property of monotonicity demands that the restriction of an act, being unfavorable in the original decision problem, on some event E is also unfavorable in the decision problem reduced to E. This requirement rules out behavior in which acts being once considered unfavorable become favorable, when additional information about the true state of the world is disclosed. As it can be easily shown, while prominent choice rules like the maximin rule, the weak undominance in pure acts or the weak undominance in mixed acts violate monotonicity, other well-known choice rules like Börgers' concept of undominance, the strict undominance in pure acts or the strict undominance in mixed acts satisfy this property. Without any difficulty, it can also be verified that

[8] Indeed, the framework of which Apt (2011) makes use differs from ours but, as we will see, only in unessential aspects. Although not faithfully, we can say - without any substantial loss - that Apt (2011) starts with a dominance mapping $D : \mathfrak{S}_\Gamma \to \mathfrak{S}_\Gamma$, where $R \setminus D(R) \neq \emptyset$ and $D(R) \subseteq R$ hold for any restriction $R \in \mathfrak{S}_\Gamma$. The strategies of player i belonging to $D(R)^i$ are said to be $D-$*dominated in* R. By this dominance mapping, a binary relation $\xrightarrow{\Gamma}_D$ on $\mathfrak{S}_\Gamma$ is constructed, where $R \xrightarrow{\Gamma}_D \tilde{R}$ holds, whenever $\tilde{R} \subseteq R, \tilde{R} \neq R$ and $s^i \in R^i \setminus \tilde{R}^i$ is D-dominated in R for any player $i \in N$ and for any strategy $s^i \in R^i$. Apt (2011) terms a dominance relation D as hereditary if, for any strategic games Γ, for any restriction R and $\tilde{R}$, where $R \xrightarrow{\Gamma}_D \tilde{R}$ holds, for any player $i \in N$ and for any strategy $s^i \in \tilde{R}^i$, from s^i is D-dominated in R it follows that s^i is D-dominated in $\tilde{R}$. In order to bring into line these concepts to our framework, specify the dominance mapping D by $D(R) := \times_{i \in N}\{s^i \in R^i : s^i \notin \left(\mathfrak{a}^i_{\Gamma|_{\tilde{R}}}\right)^{-1}\left(\mathcal{C}^i(R^{-i}, \Phi^i_{\Gamma|_R})\right)\}$ for any restriction $R \in \mathfrak{S}_\Gamma$. Obviously, taking into account this specification, our definition of hereditary corresponds to that of Apt (2011).

the average rule as well as the modified strict dominance rule, both already specified in the Introduction, are monotone, too. The following remark establishes that property β_0 and monotonicity, together, imply hereditary.

Remark 3.4 *Consider a non-empty family $(\mathcal{C}^i)_{i\in N}$ of choice rules. If the choice rules of this family satisfy property β_0 and monotonicity, then this family is hereditary.*

Proof. Consider some strategic game $\Gamma := (S^i, z^i)_{i\in N}$ and let R and $\tilde{R}$ be restrictions of Γ, where $R \overset{\Gamma}{\to}_{\mathcal{C}} \tilde{R}$ is satisfied. Consider some player $k \in N$ and some strategy $s^k \in \tilde{R}^k$, where $s^k \notin \left(\mathfrak{a}^i_{\Gamma|_R}\right)^{-1}\left(\mathcal{C}^k(\Phi^i_{\Gamma|_R})\right)$ holds. By property β_0, we obtain $\mathcal{C}^k(R^{-k}, \mathfrak{a}^i_{\Gamma|_R}(R^k)) \supseteq \mathcal{C}^k(R^{-k}, \mathfrak{a}^i_{\Gamma|_R}(\tilde{R}^k))$. Hence, $s^k \notin \left(\mathfrak{a}^i_{\Gamma|_R}\right)^{-1}\left(\mathcal{C}^k(R^{-k}, \mathfrak{a}^i_{\Gamma|_R}(\tilde{R}^k))\right)$ holds. Finally, the property of monotonicity entails $s^k \notin \left(\mathfrak{a}^i_{\Gamma|_{\tilde{R}}}\right)^{-1}\left(\mathcal{C}^k(\Phi^k_{\Gamma|_{\tilde{R}}})\right)$. ■

Note, due to Theorem 3.2, the iterative deletion processes generated by choice rules satisfying property α_0 as well as monotonicity are order-independent. As we already know, our average rule satisfies both properties. Consequently, the iterative deletion processes based on this choice rule are order-independent, as already claimed in the Introduction.

The following lemma reverses the standard issue in game theory regarding the order-independence of iterative deletion processes as addressed e.g. in Gilboa et al. (1990) or in Apt (2011). Instead of figuring out catchy properties of choice rules ensuring order-independence of their iterative deletion processes, we intend to find out which properties are inherent to choice rules whose iterative deletion processes are order-independent. It turns out that, under certain presuppositions on the players' choice rules, order-independence entails that the players' choice rules satisfy property β_0 and monotonicity.

Lemma 3.5 *Suppose the non-empty and non-trivial choice rules of the family $(\mathcal{C}^i)_{i\in N}$ satisfy the independence of payoff-equivalent states and property α_0. If the deletion processes generated by the iterated deletion of unfavorable strategies with respect to the choice rules $(\mathcal{C}^i)_{i\in N}$ are order-independent, then*

(a) *any of these choice rules satisfy property β_0.*

(b) *any of these choice rules satisfy monotonicity.*

Our Lemma 3.5 together with Theorem 3.2 and Remark 3.4 entails that the heredity of a family of choice rules is equivalent to the order-independence of the iterative deletion processes based on this family of choice rules, whenever each of these choice rules satisfies non-emptiness, non-trivialness, the independence of payoff-equivalent states and property α_0. Remarkably, our modified strict undominance rule fulfills the first three preconditions of our equivalence result, however violates property α_0. As aforementioned, this choice rule is not hereditary, but order-independent. Thereby, it is demonstrated that the equivalence between order-independence and hereditary breaks down if property α is canceled from the preconditions stated in Lemma 3.5 .

4 An Epistemic Rationale for Order-Independence

In this section, we aim to provide an epistemic rationale for order-independent deletion processes in strategic games. We will demonstrate that, provided the choice rules $(\mathcal{C}^i)_{i\in N}$ fulfill the preconditions listed in Definition 2.1, the deletion processes of iterated deletion of unfavorable strategies with respect to these choice rules are order-independent if and only if, for any strategic game, the iterated maximal deletion of unfavorable strategies with respect to these choice rules gives exactly the strategy profiles that are consistent with the choice rule following behavior and the common belief of such behavior. This theorem relies on the order-independence theorem by Apt (2011), we reproduced in Section 3, an epistemic characterization theorem by Trost (2010), we will discuss after introducing our epistemic framework, and two results connecting these two theorems. Indeed, one of these two results is Lemma 3.5 saying that, under specific preconditions on the choice rules, the property of monotonicity follows from the order-independence of the iterative deletion processes based on these choice rules. The second result, which will be proved later in this section, is a complement to Lemma 3.5 and establishes that, under specific preconditions on the choice rules, the property of monotonicity is also a consequence from the coincidence of the set of strategy profiles surviving the iterated maximal deletion of unfavorable strategies with the set of strategy profiles being consistent with the choice rule following behavior and the common belief of it.

The objective of the epistemic approach on games is to detect the implications of the players' beliefs about the opponents' choices and beliefs on the outcomes of the games. To accomplish this objective, games are supplemented with epistemic models which enable to describe explicitly the beliefs of the players. The epistemic analysis carried out in this paper is based on models that are known as *extended Kripke frames* or *Aumann structures* (see Kripke, 1963 and Aumann, 1976).

Formally, our *epistemic model to a strategic game* $\Gamma := (S^i, z^i)_{i\in N}$ is a tuple $\mathfrak{M}_\Gamma := \big(\Omega, (P^i, \sigma^i)_{i\in N}\big)$ whose items are defined as follows. The finite set Ω denotes the *state space*. Its members will be called *states of the world* and will represent a certain resolution of all relevant issues for the players. The mapping $P^i : \Omega \to 2^\Omega$ is the serial, transitive and euclidean *possibility correspondence of player* i assigning to each state ω the possibility set $P^i(\omega)$ that comprises all states deemed possible by player i at state ω.[9] The mapping $\sigma^i : \Omega \to S^i$ will be referred to as *strategy function of player* i and reveals player i's choice of strategy for each state. It is taken for granted that $\sigma^i(\tilde\omega) = \sigma^i(\omega)$ holds for any $\tilde\omega \in P^i(\omega)$ and for any $\omega \in \Omega$. That is, at each state, player i does not err about (i.e. knows) her own choice of strategy.

An event $E \subseteq \Omega$ *is believed by player* i *at state* ω, if $P^i(\omega) \subseteq E$ applies. Let P_* the transitive closure of the possibility correspondences $(P^i)_{i\in N}$.[10] This correspondence will be referred to as the *common possibility correspondence*. Clearly, $P^i(\omega) \subseteq P_*(\omega)$ is satisfied for any $\omega \in \Omega$ and any $i \in N$. Moreover, it turns out (see e.g. Fagin et al., 1995, Lemma 2.2.1) that an *event* E *is commonly believed at state* ω (i.e. at state ω, every player believes E, every player believes that every player believes E, and so on), if and only if $P_*(\omega) \subseteq E$ applies. Note, if $P^i(\omega) \subseteq E$ holds for any $\omega \in E$ and for any player $i \in N$, then the event E is commonly believed at every state belonging to E.

Consider a strategic game $\Gamma := (S^i, z^i)_{i\in N}$, in which the beliefs of the players about the opponents' choices and beliefs are captured by the epistemic model $\mathfrak{M}_\Gamma := \big(\Omega, (P^i, \sigma^i)_{i\in N}\big)$. Let $\mathfrak{a}^i_{\mathfrak{M}_\Gamma}$ be the mapping that associates with each strategy $s^i \in S^i$ the act $\mathfrak{a}^i_{\mathfrak{M}_\Gamma}(s^i)$ determined by $\mathfrak{a}^i_{\mathfrak{M}_\Gamma}(s^i) := z^i(s^i, \sigma^{-i}(\omega))$ for any state $\omega \in \Omega$. That is, $\mathfrak{a}^i_{\mathfrak{M}_\Gamma}(s^i)$ is the payoff profile on the state space induced by the strategy s^i. A player $i \in N$ is said to *follow* (or to *apply*) *the choice rule*

[9] Recall, seriality means that, for any $\omega \in \Omega$, $P^i(\omega) \neq \emptyset$ is satisfied, transitivity means that, for any $\omega \in \Omega$, if $\tilde\omega \in P^i(\omega)$, then $P^i(\tilde\omega) \subseteq P^i(\omega)$, and euclideanness means that, for any $\omega \in \Omega$, if $\tilde\omega \in P^i(\omega)$, then $P^i(\omega) \subseteq P^i(\tilde\omega)$. Obviously, transitivity and euclideanness entail that, for any $\omega \in \Omega$, if $\tilde\omega \in P^i(\omega)$ holds, then $P^i(\tilde\omega) = P^i(\omega)$ is satisfied.

[10] The mapping $P_* : \Omega \to 2^\Omega$ is termed a transitive closure of the possibility correspondences $(P^i)_{i\in N}$, whenever $\tilde\omega \in P_*(\omega)$ holds, if and only if there is a finite sequence $(i_1, \ldots, i_m)$ in N and a finite sequence $(\omega_0, \omega_1, \ldots, \omega_m)$ in Ω such that $\omega_0 = \omega$, $\omega_m = \tilde\omega$ applies and, for every $k = 1, \ldots, m$, $\omega_k \in P^{i_k}(\omega_{k-1})$ holds.

$\mathcal{C}^i$, whenever $\mathfrak{a}^i_{\mathfrak{M}_\Gamma}(\sigma^i(\omega))|_{P^i(\omega)} \in \mathcal{C}^i\left(P^i(\omega), \mathfrak{a}^i_{\mathfrak{M}_\Gamma}(S^i)|_{P^i(\omega)}\right)$ holds, that is, whenever her choice of strategy at state ω is favorable with respect to the choice rule $\mathcal{C}^i$ and given her uncertainty $P^i(\omega)$. A player i is said to believe at state ω that player j follows choice rule $\mathcal{C}^j$, if $\mathfrak{a}^j_{\mathfrak{M}_\Gamma}(\sigma^j(\tilde{\omega}))|_{P^j(\tilde{\omega})} \in \mathcal{C}^j\left(P^j(\tilde{\omega}), \mathfrak{a}^j_{\mathfrak{M}_\Gamma}(S^j)|_{P^j(\tilde{\omega})}\right)$ applies to any state $\tilde{\omega} \in P^i(\omega)$. We remark that, if player i believes at state ω that she herself applies the choice rule $\mathcal{C}^i$, then she actually applies the choice rule $\mathcal{C}^i$ at this state. This results from our precondition that she knows her choice of strategy (formally, $\sigma^i(\omega) = \sigma^i(\tilde{\omega})$ holds for any $\tilde{\omega} \in P^i(\omega)$) and that she knows which states she considers possible (formally, $P^i(\omega) = P^i(\tilde{\omega})$ holds for any $\tilde{\omega} \in P^i(\omega)$). We say, that there is *common belief at state ω that the players follow the choice rules $(\mathcal{C}^i)_{i\in N}$*, if $\mathfrak{a}^i_{\mathfrak{M}_\Gamma}(\sigma^i(\tilde{\omega}))|_{P^i(\tilde{\omega})} \in \mathcal{C}^i\left(P^i(\tilde{\omega}), \mathfrak{a}^i_{\mathfrak{M}_\Gamma}(S^i)|_{P^i(\tilde{\omega})}\right)$ is satisfied for any $\tilde{\omega} \in P_*(\omega)$ and for any $i \in N$. Obviously, by the arguments stated above, the common belief of applying the choice rules $(\mathcal{C}^i)_{i\in N}$ at the state ω implies that the choice rules $(\mathcal{C}^i)_{i\in N}$ are actually applied by the players at the state ω. Therefore, whenever such common belief is presupposed, there is no need anymore for stating explicitly that the players will follow the choice rules, as we did it hitherto.

Consider some strategic game $\Gamma := (S^i, z^i)_{i\in N}$. An epistemic model $\mathfrak{M}_\Gamma$ to Γ is said to be *consistent with a statement about the world*, whenever the model contains a state, in which this statement is satisfied. A statement *characterizes a set of $T \subseteq S$ of strategy profiles*, if the following two conditions hold:

(i) (*Consistency*) If the epistemic model $\mathfrak{M}_\Gamma$ is consistent with the statement, then, at every state satisfying this statement, some strategy profile $s \in T$ is realized.

(ii) (*Existence*) For every strategy profile $s \in T$ there exists an epistemic model $\mathfrak{M}_\Gamma$ which contains a state in which this statement is satisfied and in which this strategy profile is realized.

An *epistemic statement* is a statement referring to the beliefs of the players. An *epistemic characterization for a solution concept* is given, whenever an epistemic statement is found which characterizes, for any strategic game, the set of strategy profiles resulting from this solution concept. For our purpose, it becomes crucial to characterize the iterated maximal deletion of unfavorable strategies with respect to a given family $(\mathcal{C}^i)_{i\in N}$ of choice rules. As shown in Trost (2010), whenever the choice rules satisfy specific preconditions, this solution concept is characterized by the common belief that the players follow these choice rules. One of these preconditions is the property of reflexivity, which is defined as follows.

Definition 4.1 *A choice rule $\mathcal{C}$ is called reflexive, if*

$$\mathcal{C}(E, \mathcal{C}(\Phi)|_E)) \subseteq \mathcal{C}(\Phi|_E)$$

holds for any decision problem $\Phi := (\Omega, B)$ and for any non-empty event $E \subseteq \Omega$.

To understand the property of reflexivity, recall that $\Phi|_E$ is the decision problem of a decision maker after receiving the reliable information that the actual state of the world belongs to the event E. Reflexivity demands that, if some act is considered unfavorable after the decision maker received information E, then this act is either unavailable or unfavorable in the decision problem whose constraint is limited to the acts which are favorable in the corresponding unrestricted decision problem and whose state space corresponds to the information E. Remarkably, the average rule as well as the modified strict dominance rule, which we defined in the Introduction, are reflexive. Furthermore, as shown next, reflexivity results from the properties β_0 and monotonicity.

Remark 4.2 *If some choice rule satisfies both property β_0 and monotonicity, then it is reflexive.*

Proof. Consider some decision problem $\Phi := (\Omega, B)$ and some non-empty event $E \subseteq \Omega$. Monotonicity leads to

$$\mathcal{C}(\Phi|_E) \subseteq \mathcal{C}(\Phi)|_E \subseteq B|_E \ .$$

Finally, by property β_0,

$$\mathcal{C}(E, \mathcal{C}(\Phi)|_E)) \subseteq \mathcal{C}(\Phi|_E)$$

results. ■

The following epistemic characterization of the iterated maximal deletion of unfavorable strategies with respect to some given family of choice rules has been proved in Trost (2010). It provides conditions on the choice rules ensuring that the solution of this iterative deletion process consists exactly of the strategy profiles that are consistent with the common belief, among the players, that they apply these choice rules.[11]

Theorem 4.3 (Trost, 2010) *If the non-empty choice rules of the family $(\mathcal{C}^i)_{i\in N}$ satisfy the independence of output-equivalent states, property α_0, reflexivity as well as monotonicity then, for every strategic game, the solution of the iterated maximal deletion of unfavorable strategies with respect to these choice rules is characterizable with the common belief of applying these choice rules.*

The objective we pursue in this paper is not to provide an epistemic characterization for some solution concept, rather we seek for an epistemic justification for applying order-independent deletion processes to strategic games. The next lemma proves to be a crucial step to accomplish this goal. It states, that under certain conditions on the players' choice rules, the coincidence between the solution generated by the iterated maximal deletion of unfavorable strategies and the solution generated by the common belief of choice-rule following behavior implies that the choice rules of all players satisfy property α_0 and monotonicity.

Lemma 4.4 *Suppose the non-empty and non-trivial choice rules of the family $(\mathcal{C}^i)_{i\in N}$ satisfy the independence of payoff-equivalent states and property β_0. If the iterated maximal deletion of unfavorable strategies is characterizable by the common belief of applying these choice rules, then*

(a) *any of the choice rules satisfies property α_0*

(b) *any of the choice rules satisfies monotonicity,*

[11] The following theorem can also be proved by resorting to two results of Apt (2010). Theorem 1(*i*) of Apt (2010) says that, if the non-empty choice rules of all players are monotone, then any strategy profile being consistent with the common belief of choice rule following behavior survives the *global iterated maximal deletion* of unfavorable strategies. Unlike our iterative deletion process introduced in Section 3, the global version requires that, in every round, each strategy being unfavorable in the *original strategy set* and given the remaining strategies of the opponents is eliminated. However, for the case that the choice rules satisfy additionally property α_0, it can be shown that the solution generated by our local version of iterated maximal deletion corresponds to the solution generated by the global version of iterated maximal deletion. Furthermore, according to Theorem 1(*iii*) of Apt (2010), every strategy profile surviving the global iterated maximal deletion of unfavorable strategies is consistent, within the subclass of *standard epistemic models*, with the common belief of choice rule following behavior. A standard epistemic model is defined as an epistemic model, in which the state space is equal to the strategy space (i.e. $\Omega := \times_{i\in N} S^i$) and the strategy functions are specified by $\sigma^i(s) := s^i$ for any state $s \in \Omega$ and for any player $i \in N$. It turns out that, if the choice rules satisfy additionally the independence of payoff-equivalent states, the previous restriction to the subclass of standard epistemic models can be omitted.

The following theorem is one central result of this paper. It states that, provided the requirements on the choice rules listed in Definition 2.1 are fulfilled, the order-independence of the iterative deletion processes based on these choice rules are satisfied if and only if, for each strategic game, the solution of the iterated maximal deletion of unfavorable strategies consists exactly of the strategy profiles being consistent with the common belief of applying these choice rules.

Theorem 4.5 *Suppose the non-empty and non-trivial choice rules of the family $(\mathcal{C}^i)_{i\in N}$ satisfy the properties of independence of payoff-equivalent states and of independence of unfavorable acts. Then the following statements are equivalent:*

(i) Iterated deletion of unfavorable strategies with respect to the choice rules $(\mathcal{C}^i)_{i\in N}$ is order-independent.

(ii) Each choice rule of the family $(\mathcal{C}^i)_{i\in N}$ satisfies monotonicity.

(iii) The iterated maximal deletion of unfavorable strategies is characterizable by the common belief of applying the choice rules $(\mathcal{C}^i)_{i\in N}$.

Proof. ("(i) $\Rightarrow$ (ii)") This implication results from Lemma 3.5(b). ("(ii) $\Rightarrow$ (i)") According to Remark 3.4, our presuppositions imply that the family $(\mathcal{C}^i)_{i\in N}$ of choice rules is hereditary. As Apt (2011) showed (see Theorem 3.2), this property entails the order-independence of iterative deletion processes which are based on this family of choice rules. ("(ii) $\Rightarrow$ (iii)") As Trost (2010) proved (see Theorem 4.3), our presuppositions entail that the iterated maximal deletion of unfavorable strategies with respect to the choice rules $(\mathcal{C}^i)_{i\in N}$ gives exactly these strategy profiles which are consistent with the common belief of following these choice rules. ("(iii) $\Rightarrow$ (ii)") This implication results from Lemma 4.4(b). ■

As postulated in Theorem 4.5, the equivalence between the order-independence of iterative deletion processes and the epistemic characterization of their solution by the common belief of choice rule following behavior hinges on several preconditions on the choice rules underlying these deletion processes. An obvious question is, whether these preconditions can be relaxed without loosing this equivalence. In what follows, we shall examine the consequences of weakening the assumption of the independence of unfavorable acts on our epistemic rationale for order-independence. Unfortunately, it turns out that these weakenings will result in a breakdown of this motivation. However, it is still possible to formulate either necessary or sufficient epistemic conditions for order-independence.

Theorem 4.6 *Suppose the non-empty and non-trivial choice rules of family $(\mathcal{C}^i)_{i\in N}$ satisfy the independence of output-equivalent states and property α_0. If the iterative deletion processes based on the choice rules $(\mathcal{C}^i)_{i\in N}$ are order-independent, then their solution is characterizable by the common belief of following the choice rules $(\mathcal{C}^i)_{i\in N}$.*

Proof. Suppose the iterative deletion processes based on the choice rules $(\mathcal{C}^i)_{i\in N}$ are order-independent. Due to Lemma 3.5, any of these choice rules satisfies property β_0 and monotonicity. As we already know, the properties α_0 and β_0 together correspond to the property of the independence of unfavorable acts. Hence, the presuppositions of Theorem 4.5 are fulfilled, and the solution of the iterated (maximal) deletion of unfavorable strategies with respect to the choice rules $(\mathcal{C}^i)_{i\in N}$ is characterizable by the common belief of applying these choice rules. ■

Remarkably, the antecedent of Theorem 4.6 does not imply the converse of its consequent. To establish this claim, consider the modified strict dominance rule we specified in the Introduction. As we already know, this choice rule is independent of output-equivalent states and satisfies property α_0 as well as reflexivity and monotonicity. According to the epistemic characterization result of Trost (2010), the solution of the iterated (maximal) deletion of unfavorable strategies with respect to this choice rule is characterizable by the common belief of applying this choice rule. However, as we argued in the Introduction, the iterative deletion processes based on this choice rule are order-dependent.

Theorem 4.7 *Suppose the non-empty and non-trivial choice rules of family $(\mathcal{C}^i)_{i\in N}$ of choice rules satisfy the independence of output-equivalent states and property β_0. If, for any strategic game, the solution of the iterated maximal deletion of unfavorable strategies is characterizable by the common belief of choice rule following behavior, then the iterative deletion processes based on these choice rules are order-independent.*

Proof. Suppose that, for any strategic game, the solution generated by the iterated maximal deletion of unfavorable strategies with respect to the choice rules $(\mathcal{C}^i)_{i\in N}$ coincides with the set of strategies consistent with the common belief of choice rule following behavior. According to Lemma 4.4, any of these choice rules satisfies both property α_0 and monotonicity. Hence, the choice rules are independent of unfavorable acts. Finally, Theorem 4.5 entails that the iterative deletion processes based on theses choice rules are order-independent. ■

However, it turns out that the preconditions on the choice rules stated in Theorem 4.7 are not sufficient for ensuring the converse of its conclusion. Take, as an example, the average rule presented in the Introduction. As we already know, this choice rule is independent of output-equivalent states and satisfies property β_0, but not property α_0. Furthermore, it is hereditary and according to Apt (2011) the iterative deletion processes relying on this choice rule are order-independent. However, applied to strategic game Γ_2, their solution does not contain all strategy profiles which are consistent with the common belief of following this choice rule.

5 Concluding Remarks

As we have seen, it is not generally true, that the order-independence of iterative deletion processes means that their solutions correspond to the set of strategies being consistent with the common belief of applying the choice rules underlying these processes. However, and this is the main result of this paper, if these choice rules are non-empty, non-trivial and independent of payoff-equivalent states as well as of unfavorable acts, then the order-independence is equivalent to this correspondence. Thereby, we have detected a comprehensible epistemic rationale for employing order-independent iterative deletion processes on strategic games.

We do not deny that the preconditions for this epistemic motivation of order-independence are quite demanding. Notably, the properties of the independence of payoff-equivalent states and of the independence of unfavorable acts are restrictive and narrow the applicability of our epistemic motivation for order-independence. As we will see, the following two prominent iterative deletion processes are not captured by our theorems, because they fail in one of these independence postulates.

The first well-known iterative deletion process not embraced by our analysis is the solution concept of rationalizability put forth by Bernheim (1984) and Pearce (1984). This deletion process is based on a choice rule, which we will call *uncorrelated best response rule*, henceforth, and which is a tightening of the choice rule of strict undominance in mixtures. While the latter choice rule picks out the acts which maximize the expected payoff for some uncorrelated probabilistic belief on the state space (see Pearce, 1984, Lemma 3 for this characterization), the choice rule of uncorrelated best response favors only the acts which maximize the expected payoff for some uncorrelated (i.e. independent) probabilistic belief on the state space. Without any difficulty, it can be shown that the uncorrelated best response rule satisfies the properties of non-emptiness, non-triviality and the independence of unfavorable acts. However, as we will demonstrate with the following two strategic games Γ_4 and $\tilde{\Gamma}_4$, this choice rule fails to be independent of payoff-equivalent states.

Let us presuppose that player M takes part in both games and applies always the uncorrelated best response rule. The strategic game Γ_4 depicted in Figure 6 has been taken from Osborne and Rubinstein (1994, Figure 58.1) and is a three-payer game, where player R's strategy space is $S^R := \{u, d\}$, player C's strategy space is $S^C := \{l, r\}$ and player M's strategy space is $S^M := \{m_1, m_2, m_3, m_4\}$. For simplicity, we have only recorded the payoffs player M receives. As aforementioned, player M sticks by the uncorrelated best response rule, where p and q measure the probability M could attach to the event that player R chooses strategy u and player C chooses strategy l, respectively. Obviously, in order for m_2 to be a favorable strategy to M the weak inequality $4pq + 4(1-p)(1-q) \geq max\{8pq, 8(1-p)(1-q), 3\}$ must be satisfied for some values of $p, q \in [0, 1]$. Since, the last weak inequality does not hold for any values of p and q, strategy m_2 turn outs to be unfavorable to player M.

Player C

Player R		l	r		l	r		l	r		l	r
	u	8	0	u	4	0	u	0	0	u	3	3
	d	0	0	d	0	4	d	0	8	d	3	3
		m_1			m_2			m_3			m_4	

Player M

Figure 6: Three-player strategic game Γ_4

The strategic game $\tilde{\Gamma}_4$ depicted in Figure 7 is a two-player game, where player D's strategy space is given by $\tilde{S}^D := \{\tilde{e}, \tilde{f}, \tilde{g}, \tilde{h}\}$ and player M's strategy space is given by $\tilde{S}^M := \{\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, \tilde{m}_4\}$. Again, we have only depicted the possible payoffs of player M. Since strategy $\tilde{m}_2$ maximizes the expected payoff to the probabilistic belief assigning the same probability of $\frac{1}{2}$ to player D's strategies $\tilde{e}$ and $\tilde{f}$, this strategy is deemed favorable by player M in his strategic decision problem $\Phi^M_{\tilde{\Gamma}}$.

Player M

Player D		$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$
	$\tilde{e}$	8	4	0	3
	$\tilde{f}$	0	0	0	3
	$\tilde{g}$	0	0	0	3
	$\tilde{h}$	0	4	8	3

Figure 7: Strategic Game $\tilde{\Gamma}_4$

Obviously, the mapping $\tau : \tilde{S}^D \to S^R \times S^C$, specified by $\tau(\tilde{e}) := (u, l)$, $\tau(\tilde{f}) := (d, l)$, $\tau(\tilde{g}) := (u, r)$ and $\tau(\tilde{h}) := (d, r)$, is bijective and $\mathfrak{a}^M_{\tilde{\Gamma}}(\tilde{m}_i) = \mathfrak{a}^M_{\Gamma}(m_i) \circ \tau$ applies to any $i = 1, \ldots, 4$. The latter equality says that the strategies $\tilde{m}_i$ and m_i induce the same payoff profile for player M. Thus, the condition $(\tilde{S}^D, B^M_{\tilde{\Gamma}}) = (\tilde{S}^D, \{x \circ \tau : x \in B^M_{\Gamma}\})$ is satisfied and the postulate of the independence of payoff-equivalent states would imply $\mathcal{C}^M(\tilde{S}^D, B^M_{\tilde{\Gamma}}) = \{x \circ \tau : x \in \mathcal{C}^M(S^R \times S^C, B^M_{\Gamma})\}$). However, as argued above, $\mathfrak{a}^M_{\Gamma}(m_2) \circ \tau \in \mathcal{C}^M(\tilde{S}^D, B^M_{\tilde{\Gamma}})$ and $\mathfrak{a}^M_{\Gamma}(m_2) \notin \mathcal{C}^M(S^R \times S^C, B^M_{\Gamma})\}$) hold, contradicting this independence postulate.

As Apt (2005) proved, the iterative deletion processes based on the uncorrelated best response rule are order-independent. Furthermore, in Apt (2010), it is shown that their solution can be characterized, within the subclass of standard epistemic models, by the common belief of following this choice rule. Standard epistemic models are defined as epistemic models, in which the state space corresponds exactly to the set of all strategy profiles (formally, $\Omega := \times_{i \in N} S^i$ must hold) and the strategy function of the players are projections on the players' strategy space (formally, $\sigma^i(s) := s^i$ must hold for all player $i \in N$).[12] Apart from this restriction on the epistemic models, the solution concept of rationalizability would fit in with the result of our main Theorem 4.5, although its underlying choice rule does not fulfill all preconditions of this theorem.

The other prominent iterative deletion process that is also not captured by our results is the deletion process of iterated regret minimization which has been forcefully advocated by Halpern and Pass (2012). Iterated regret minimization is defined as the iterated maximal deletion of unfavorable strategies with respect to the minimax regret rule of Niehans (1948) and Savage (1951), which, as we already know, fulfills all requirements listed in Definition 2.1 with the exception of the independence of unfavorable acts.

Applied to strategic game Γ_5 depicted in Figure 8 the iterative deletion processes based on this choice rule produce different solutions, namely $\{(u, l)\}$ or $\{(m, c)\}$, where the first solution results from the iterated regret minimization. Interestingly, this solution is not reconcilable with the assumption that the players have the common belief of minimax regret rule following behavior. To see this, consider player R. Obviously, her strategy u becomes only favorable according to the minimax regret rule, if she considers possible that player C chooses strategy r. The strategy r, in turn, is only favorable with respect to the minimax regret rule, if player C considers possible that player R chooses strategy d. However, this strategy is incompatible with minimax regret rule following behavior. Thus, strategy u will not be realized in strategic game Γ_5, if player R applies the minimax regret rule, believes that player C applies this choice rule and believes that player C believes that she applies this choice rule.

		Player C		
		l	c	r
	u	$(3,3)$	$(3,2)$	$(4,1)$
Player R	m	$(4,2)$	$(4,3)$	$(2,2)$
	d	$(0,1)$	$(0,1)$	$(3,3)$

Figure 8: Strategic Game Γ_5

Summing up, with this example, it has been demonstrated both that the iterative deletion processes based on the minimax regret rule are order-dependent and that there is no coincidence of their solutions with the set of strategies being characterizable by the common belief of following this choice rule. Unfortunately, although this observation conforms with the consequent of our Theorem 4.5, these iterative deletion processes are not covered by any of our results about order-

[12] See also the Footnote 11. There, we briefly sketched the epistemic framework used in Apt (2010).

independence, since their underlying choice rule does not satisfy all preconditions of these results (i.e. it does satisfy neither property α_0 nor property β_0).

Since the preconditions of our epistemic rationale for order-independence are quite restrictive, we have attempted to relax this precondition without abandoning above epistemic motivation of order-independence. For this purpose, we have split this independence postulate into two properties, called property α_0 and property β_0. It turns out that, whenever the individual choice rules are only required to satisfy property α_0 instead of the independence of unfavorable acts, order-independence of the iterative deletion processes continues to imply that, for any strategic game, their solution is characterizable by the common belief of applying the choice rules underlying these processes. In the other case - whenever the choice rules are only required to satisfy property β_0 instead of the independence of unfavorable acts - the fact that, for any strategic game, the set of strategies surviving the iterated maximal deletion of unfavorable strategies is characterizable by the common belief of choice rule following behavior implies the order-independence of all iterative deletion processes based on these choice rules. However, for both weakenings of the postulate of the independence of unfavorable acts, the converses of these implications are not valid, as we have demonstrated by the counterexamples of the so-called average rule and modified strict dominance rule. That means, in both cases, the equivalence between the order-independence of the iterative deletion processes and the epistemic characterization of the solution of the iterated maximal deletion of unfavorable acts by the common belief of applying the choice rules underlying these processes breaks down.

Even if the consequent of our main result Theorem 4.5 breaks down for both weakenings, this does not mean that the preconditions of our epistemic motivation for order-independence postulated in this theorem are the weakest possible one. In fact, they are not, as the example of iterated regret minimization reveals. Identifying a wider class of choice rules, for which our motivation for order-independence still holds, remains a question of further research.

Interestingly, even our specification of an iterative deletion process by individual choice rules proves to be restrictive. For example, the deletion processes of the iterated deletion of nicely weakly dominated strategies, as introduced by Marx and Swinkels (1997), can not be captured by our specification. According to Marx and Swinkels (1997), a strategy s_i nicely weakly dominates a strategy $\tilde{s}^i$ on restriction $R \subseteq S$ if (*i*) s^i weakly dominates $\tilde{s}^i$ on R^{-i} and (ii) if for any $s^{-i} \in R^{-i}$ the equality $z^i(s^i, s^{-i}) = z^i(\tilde{s}^i, s^{-i})$ implies the $z^j(s^i, s^{-i}) = z^j(\tilde{s}^i, s^{-i})$ for all player j different to i. Applying repeatedly their dominance concept to the trivial strategic game Γ_6 depicted in Figure 9, we obtain the sets $\emptyset, \{(u,m)\}, \{(d,m)\}, \{(u,m),(d,m)\}$ as possible solutions. However, if we define the iterative deletion process in terms of (possibly, empty) choice rules, as we did in Section 3, only the sets $\emptyset, \{(u,m),(d,m)\}$ are conceivable solutions. This divergence in the set of possible solutions shows that the deletion process introduced by Marx and Swinkels (1997) is incompatible with our specification of an iterative deletion process.

		Player C
		m
Player R	u	$(1,1)$
	d	$(1,1)$

Figure 9: Strategic Game Γ_6

A further limitation of our work is that we only consider finite strategic games (i.e. games in which the strategy sets of each player is finite) and finite epistemic models (i.e. epistemic models with finite state spaces). In literature, there have been worked out two ways of dealing with the issue of infinity. The first one is to add topological assumptions like e.g. in Dufwenberg and Stegeman (2002), who established that the solutions of the iterative deletion processes based on the strict undominance rule are unique, whenever the strategy sets are compact Hausdorff spaces and the payoff functions are continuous. The second one is to generalize the iterative deletion process to a transfinite recursion. This way has been pursued by Chen et al. (2007), who proved that transfinite deletion processes based on the choice rule of strict undominance are order-independent. Unquestionably, it might be a worthwhile project to extend our idea of an epistemic motivation of order-independence to general strategic games and epistemic models, but on the other side we also heavily presume that the properties listed in Definition 2.1 are still relevant for an epistemic rationale for order-independence in general strategic games.

Appendix

Proof of Lemma 3.5:

(*a*) Let $\mathcal{C} := (\mathcal{C}^i)_{i\in N}$ be some family of non-empty and non-trivial choice rules being independent of payoff-equivalent states and satisfying property α_0. Suppose there is some player k whose choice rule $\mathcal{C}^k$ violates property β_0. That is, there exists some decision problem $\Phi^k := (\Omega^k, B^k)$, some act $x \in B^k$ and some constraint $\tilde{B}^k$ satisfying $\mathcal{C}^k(\Phi^k) \subseteq \tilde{B}^k \subseteq B^k$ so that $x \notin \mathcal{C}^k(\Phi^k)$, but $x \in \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ hold. Consider the strategic game $\Gamma := (S^i, z^i)_{i\in N}$ consisting of the strategy sets

$$S^i := \begin{cases} B^k & \text{for player } i = k, \\ \Omega^k & \text{for some arbitrary player } i = l \text{ different to } k, \\ \{\emptyset\} & \text{for any player } i \text{ different to } k \text{ and } l, \end{cases}$$

and the payoff functions assigning to each strategy profile $s \in S$ the payoff

$$\begin{aligned} \tilde{z}^k(y, s^{-k}) &:= y_{s^l} && \text{for player } k \text{ and for any strategy } y \in B^k, \\ \tilde{z}^l(\omega, s^{-l}) &:= 0 && \text{for player } l \text{ and for any strategy } \omega \in \Omega^k, \\ \tilde{z}^j(\emptyset, s^{-j}) &:= 0 && \text{for any player } j \text{ different to } k \text{ and } l\,. \end{aligned}$$

In what follows, we shall show that, for the strategic game Γ, a reachable restriction exists which does not contain all strategies surviving some alternative iterative deletion process based on $\mathcal{C}$.

At first, we remark that the independence of payoff-equivalent states entails

$$\mathcal{C}^k(\Phi^k_\Gamma) = \{y \circ \tau : y \in \mathcal{C}^k(\Omega^k, B^k)\}\,, \tag{1}$$

where τ denotes the projection from the strategy space $S^{-k} = \Omega^k \times (\times_{j\neq l,k}\{\emptyset\})$ of player k's opponents on the state space Ω^k. Because $\mathfrak{a}^k_\Gamma(y) = y \circ \tau$ holds for any act $y \in B^k$, we obtain the identity

$$R^k_1 := (\mathfrak{a}^k_\Gamma)^{-1}\left(\mathcal{C}^k(\Phi^k_\Gamma)\right) = \mathcal{C}^k(\Omega^k, B^k)\,.$$

From our presupposition $x \notin \mathcal{C}^k(\Omega^k, B^k)$, it follows $R^k_1 \neq S^k$. Let $R_1 := \times_{i\in N} R^i_1$ be the restriction consisting of the sets $R^i_1 := S^i$ for any player i different to k. Obviously, $S \overset{\Gamma}{\to}_{\mathcal{C}} R_1$ holds. Remarkably, strategy x does not belong to the reachable restriction R_1.

Next, we shall construct an iterative deletion process on Γ based on the choice rules $\mathcal{C}$ whose solution contain strategy x. The restrictions of this deletion process will be denoted by the upper-case letter T, henceforth. Remember above identity (1). Due to this identity, our assumption $\mathcal{C}^k(\Omega^k, B^k) \subseteq \tilde{B}^k$ entails

$$\mathcal{C}^k(\Phi^k_\Gamma) \subseteq \{y \circ \tau : y \in \tilde{B}^k\} =: \mathfrak{a}^k_\Gamma(\tilde{B}^k) \subseteq \mathfrak{a}^k_\Gamma(B^k)\,.$$

The latter result says that any strategy of player k belonging to the set $B^k \setminus \tilde{B}^k$ is unfavorable in the strategic game Γ. Determine $T^k_1 := \tilde{B}^k$. Since the difference $B^k \setminus \tilde{B}^k$ is non-empty, $T^k_1 \neq S^k$ applies. Define $T_1 := \times_{i \in N} T^i_1$ where $T^i_1 := S^i$ is specified for any player i different to k. Obviously, $S \xrightarrow{\Gamma}_{\mathcal{C}} T_1$ holds. Since $x \in \tilde{B}^k$ is presupposed, strategy x belongs to the restriction T_1. Again, by the independence of payoff-equivalent states, we reach the identity

$$\mathcal{C}^k(\Phi^k_{\Gamma|_{T_1}}) = \{y \circ \tau : y \in \mathcal{C}^k(\Omega^k, \tilde{B}^k)\}\,. \tag{2}$$

Let us start with the case that $\tilde{B}^k = \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ is satisfied. Then, it follows

$$\mathcal{C}^k(\Phi^k_{\Gamma|_{T_1}}) = \{y \circ \tau : y \in \tilde{B}^k\} = \mathfrak{a}^k_{\Gamma|_{T_1}}(T^k_1)$$

from the identity (2). Consequently,

$$\left(\mathfrak{a}^k_{\Gamma|_{T_1}}\right)^{-1}\left(\mathcal{C}^k(\Phi^k_{\Gamma|_{T_1}})\right) = T^k_1 \tag{3}$$

holds. Furthermore, non-emptiness implies

$$\left(\mathfrak{a}^i_{\Gamma|_{T_1}}\right)^{-1}\left(\mathcal{C}^i(\Phi^i_{\Gamma|_{T_1}})\right) = T^i_1 \tag{4}$$

for any player i different to k. The equations (3) and (4) reveal that, whenever $\tilde{B}^k = \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ holds, the restriction T_1 is not further reducible and thus T_1 becomes the solution of an iterative deletion process on strategic game Γ.

Now, turn to the case, in which $\tilde{B}^k = \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ is violated. That is, there exists an act $\tilde{x} \in \tilde{B}^k$ which is unfavorable for player k in the decision problem $(\Omega^k, \tilde{B}^k)$. Due to the identity (2), we obtain

$$T^k_2 := \left(\mathfrak{a}^k_{\Gamma|_{T_1}}\right)^{-1}\left(\mathcal{C}^k(\Phi^k_{\Gamma|_{T_1}})\right) = \mathcal{C}^k(\Omega^k, \tilde{B}^k)\,.$$

Because $\tilde{x} \in T^k_1$ and $\tilde{x} \notin \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ hold, $T^k_2 \neq T^k_1$ results. Specify $T_2 := \times_{i \in N} T^i_2$ where $T^i_2 := T^i_1$ is defined for any player i different to k. Obviously, $T_1 \xrightarrow{\Gamma}_{\mathcal{C}} T_2$ holds. Remarkably, by our presupposition $x \in \mathcal{C}^k(\Omega^k, \tilde{B}^k)$, strategy x is contained in the restriction T_2. The property α_0 entails

$$T^k_2 = \mathcal{C}^k(\Omega^k, \tilde{B}^k) = \mathcal{C}^k\left(\Omega^k, \mathcal{C}^k(\Omega^k, \tilde{B}^k)\right) = \mathcal{C}^k(\Omega^k, T^k_2)\,.$$

By the independence of payoff-equivalent states, the identity

$$\left(\mathfrak{a}^k_{\Gamma|_{T_2}}\right)^{-1}\left(\mathcal{C}^k(\Phi^k_{\Gamma|_{T_2}})\right) = \left(\mathfrak{a}^k_{\Gamma|_{T_2}}\right)^{-1}\left(\{y \circ \tau : y \in \mathcal{C}^k(\Omega^k, T^k_2)\}\right) = T^k_2 \tag{5}$$

results, which says that each strategy of player k in her restriction T^k_2 is favorable in the reduced strategic game $\Gamma|_{T_2}$. Furthermore, by non-emptiness, we obtain the identity

$$\left(\mathfrak{a}^i_{\Gamma|_{T_2}}\right)^{-1}\left(\mathcal{C}^i(\Phi^i_{\Gamma|_{T_2}})\right) = T^i_2 \tag{6}$$

for any player i different to k. The two identities (5) and (6) establish that the restriction T_2 is not further reducible and, thus, it constitutes a solution of an iterative deletion process on strategic game Γ. As aforementioned, $x \in T_2^k$ holds and, hence, strategy x is contained in some solution of an iterative deletion process based on the choice rules $\mathcal{C}$.

Summing up, regardless whether our iterative deletion process consists of one or two rounds of deletion, strategy x always belongs to the solution of this process. Consequently, the solution of some iterative deletion process contains a strategy that does not belong to some reachable restriction (recall that $x \notin R_1$ applies). Thereby, it is established that the iterative deletion processes based on the choice rules $\mathcal{C}$ are order-dependent. Or putting it differently, order-independence requires that each of the choices rules underlying the iterative deletion processes must satisfy property β_0.

(*b*) Let $\mathcal{C} := (\mathcal{C}^i)_{i \in N}$ be some family of non-empty and non-trivial choice rules satisfying the independence of payoff-equivalent states and property α_0. Suppose there is some player k whose choice rule $\mathcal{C}^k$ fails to be monotone. That is, there exists some decision problem $\Phi^k := (\Omega^k, B^k)$, some act $x \in B^k$ and some event $E \subseteq \Omega^k$ so that $x \notin \mathcal{C}^k(\Phi^k)$, but $x|_E \in \mathcal{C}^k(E, B^k|_E)$ hold. Consider the strategic game $\tilde{\Gamma} := (\tilde{S}^i, \tilde{z}^i)_{i \in N}$ with the same strategy sets as specified in the previous part (*a*) (i.e. $\tilde{S} := S$), but with the payoff functions, assigning to any strategy profile $s \in S$, the payoff

$$\tilde{z}^k(y, s^{-k}) := y_{s^l} \qquad \text{for player } k \text{ and for any strategy } y \in B^k,$$

$$\tilde{z}^l(\omega, s^{-l}) := \begin{cases} \alpha\,, & \text{if } \omega \in E \\ \beta\,, & \text{if } \omega \notin E \end{cases} \qquad \text{for player } l \text{ and for any strategy } \omega \in \Omega^k,$$

$$\tilde{z}^j(\emptyset, s^{-j}) := 0 \qquad \text{for any player } j \text{ different to } k \text{ and } l,$$

where α, β are the real numbers for which $\bar{\beta} \notin \mathcal{C}^l(\Omega, \{\bar{\alpha}, \bar{\beta}\})$ holds. Note, because the choice rule $\mathcal{C}^l$ is supposed to be non-trivial, the existence of these numbers is guaranteed. Furthermore, by the independence of payoff-equivalent states, we can specify $\Omega := B^k$ without any loss of generality. Analogously to the part (a), we shall demonstrate that, for the strategic game $\tilde{\Gamma}$, a reachable restriction exists which does not contain all strategies surviving some alternative iterative deletion process. To differentiate the restrictions of $\tilde{\Gamma}$ from those of Γ we will sign the formers with the circumflex $\tilde{}$.

Let the restriction $\tilde{R}_1$ be defined as in part (*a*), but for which strategic game $\tilde{\Gamma}$ is considered instead of strategic game Γ. By the same arguments as in part (*a*), it turns out, that $S \xrightarrow{\tilde{\Gamma}}_{\mathcal{C}} \tilde{R}$ holds and that strategy x does not belong to the reachable restriction $\tilde{R}_1$.

Next, we will construct an iterative deletion process on $\tilde{\Gamma}$ based on the choice rules $\mathcal{C}$ whose solution contains strategy x. Its restrictions shall be denoted by upper-case letter T, henceforth. Due to the independence of output-equivalent states, the identity

$$\mathcal{C}^l(\Phi^l_{\tilde{\Gamma}}) = \left\{ y \circ \tilde{\tau}^l : y \in \mathcal{C}^l(B^k, \{\bar{\alpha}, \bar{\beta}\}) \right\} = \left\{ \bar{\alpha} \circ \tilde{\tau}^l \right\}$$

results, where $\tilde{\tau}^l$ is the projection from the strategy space $B^k \times (\times_{j \neq k,l}\{0\})$ of player l's opponents to the state space B^k. Because $\mathfrak{a}^l_{\Gamma}(\omega) = \bar{\alpha} \circ \tau^l$ holds, if and only if $\omega \in E$ holds, we obtain

$$\tilde{T}^l_1 := \left(\mathfrak{a}^l_{\tilde{\Gamma}}\right)^{-1} \left(\mathcal{C}^l(\Phi^l_{\tilde{\Gamma}})\right) = E\ .$$

By presupposition, $E \neq \Omega^k$ applies. Therefore, the restriction $\tilde{T}_1 := \times_{i \in N} \tilde{T}^i_1$, where $\tilde{T}^i_1 := S^i$ is specified for any player i different to l, satisfies $S \xrightarrow{\tilde{\Gamma}}_{\mathcal{C}} \tilde{T}_1$. Again, the independence of payoff-equivalent states guarantees the identity

$$\mathcal{C}^k(\Phi^k_{\Gamma|_{\tilde{T}_1}}) = \left\{ y \circ \tilde{\tau}^k : y|_E \in \mathcal{C}^k(E, B^k|_E) \right\}\ , \tag{7}$$

where $\tilde{\tau}^k$ is the projection from the restriction $E \times (\times_{j \neq k,l}\{0\})$ of player k's opponents to the event E.

At first, consider the case that $B^k|_E = \mathcal{C}^k(E, B^k|_E)$ is satisfied. Resorting to the arguments put forward in part (*a*), it follows that

$$\left(\mathfrak{a}^i_{\tilde{\Gamma}|_{\tilde{T}_1}}\right)^{-1}\left(\mathcal{C}^i(\Phi^i_{\tilde{\Gamma}|_{\tilde{T}_1}})\right) = \tilde{T}^i_1$$

holds for any player $i \in N$. The latter identity says that the restriction $\tilde{T}_1$ is no further reducible and thus $\tilde{T}_1$ is the solution of some iterative deletion process on the strategic game $\tilde{\Gamma}$.

Now, turn to the case that $B^k|_E = \mathcal{C}^k(E, B^k|_E)$ is violated. That is, there exists an act $\tilde{x} \in B^k$ which is unfavorable for player k in the reduced decision problem $(E, B^k|_E)$. Due to the identity (7), we attain

$$\tilde{T}^k_2 := \left(\mathfrak{a}^k_{\tilde{\Gamma}|_{\tilde{T}_1}}\right)^{-1}\left(\mathcal{C}^k(\Phi^k_{\tilde{\Gamma}|_{\tilde{T}_1}})\right) = \left\{y : y|_E \in \mathcal{C}^k(E, B^k|_E)\right\} .$$

Because $\tilde{x} \in \tilde{T}^k_1$, but also $\tilde{x}|_E \notin \mathcal{C}^k(E, B^k|_E)$ applies, $\tilde{T}^k_2 \neq \tilde{T}^k_1$ is satisfied. Specify $\tilde{T}^i_2 := \tilde{T}^i_1$ for each player i different to k and define $\tilde{T}_2 := \times_{i\in N}\tilde{T}^i_2$. Obviously, $\tilde{T}_1 \xrightarrow{\tilde{\Gamma}}_{\mathcal{C}} \tilde{T}_2$ holds. Remarkably, by our presupposition $x|_E \in \mathcal{C}^k(\Omega^k, B^k|_E)$, strategy x is contained in the restriction $\tilde{T}_2$. Similar to the arguments given in the part (*a*), property α_0, the independence of payoff-equivalent states and the non-emptiness lead to the identity

$$\left(\mathfrak{a}^i_{\tilde{\Gamma}|_{\tilde{T}_2}}\right)^{-1}\left(\mathcal{C}^i(\Phi^i_{\tilde{\Gamma}|_{\tilde{T}_2}})\right) = \tilde{T}^k_2$$

for any player $i \in N$. These identities establish that the restriction $\tilde{T}_2$ is not further reducible and, thus, $\tilde{T}_2$ constitutes a solution of some iterative deletion process on strategic game $\tilde{\Gamma}$. As aforementioned, $x \in \tilde{T}^k_2$ holds and, hence, strategy x is contained in the solution of some iterative deletion process which is based on the choice rules $\mathcal{C}$.

Summing up, regardless whether above iterative deletion process consists of one or two rounds of deletion, strategy x always belongs to the solution of this process. Hence, the solution of some iterative deletion process on $\tilde{\Gamma}$ contains a strategy that does not belong to some reachable restriction (recall that $x \notin \tilde{R}_1$ holds). Thereby, it is established that the iterative deletion processes based on the choice rules $\mathcal{C}$ are order-dependent. Or putting it differently, their order-independence requires that each of the choices rules underlying these deletion processes must be monotone. ■

Proof of Lemma 4.4:

(*a*) Let $\mathcal{C} := (\mathcal{C}^i)_{i\in N}$ be some family of non-empty and non-trivial choice rules satisfying the independence of payoff-equivalent states and property β_0. Suppose there is some player k whose choice rule $\mathcal{C}^k$ violates property α_0. That is, there exists some decision problem $\Phi^k := (\Omega^k, B^k)$, some act $x \in B^k$ and some constraint $\tilde{B}^k$ satisfying $\mathcal{C}^k(\Phi^k) \subseteq \tilde{B}^k \subseteq B^k$ so that $x \in \mathcal{C}^k(\Phi^k)$, but $x \notin \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ hold. Obviously, $B^k \neq \tilde{B}^k$ holds and, thus, $B^k \neq \mathcal{C}^k(\Phi^k)$ applies.

Consider the strategic game Γ specified in Lemma 3.5(a) and frame it with the epistemic model $\mathfrak{M}_\Gamma := \left(\Omega, (P^i, \sigma^i)_{i\in N}\right)$ consisting of state space $\Omega := \Omega^k$, the strategy functions assigning strategy

$$\sigma^i(\omega) := \begin{cases} x\,, & \text{if player } i = k, \\ \omega\,, & \text{if player } i = l, \\ \emptyset\,, & \text{otherwise,} \end{cases}$$

to any state $\omega \in \Omega$, and the possibility correspondences $P^i(\omega) := \Omega$ for any player $i \in N$ and any state $\omega \in \Omega$. Obviously, for every state, the players act as if they follow the choice rules $(\mathcal{C}^i)_{i \in N}$ and, hence, for every state, there is common belief, among the players, that they apply these choice rules. Since strategy x is realized in each state, this simple epistemic model confirms that strategy x is compatible with the common belief of choice rule following behavior.

Now, look at the deletion process generated by the iterated maximal deletion of unfavorable strategies with respect to $\mathcal{C}$. Specify

$$\hat{R}_1^i := \left(\mathfrak{a}_\Gamma^i\right)^{-1}\left(\mathcal{C}^i(\Phi_\Gamma^i)\right)$$

for each player $i \in N$. By the independence of payoff equivalent states, the identity

$$\hat{R}_1^k = \left(\mathfrak{a}_\Gamma^i\right)^{-1}\left(\{y \circ \tau : y \in \mathcal{C}^k(\Omega^k, B^k)\}\right) = \mathcal{C}^k(\Omega^k, B^k)$$

holds, where τ denotes the projection from the strategy space $S^{-k} = \Omega^k \times (\times_{j \neq l,k}\{\emptyset\})$ of player k's opponents on the state space Ω^k. Moreover, the non-emptiness ensures the identity $\hat{R}_1^i = S^i$ for any player i different to k. Because $B^k \neq \mathcal{C}^k(\Omega^k, B^k)$ applies, the restriction $\hat{R}_1 := \times_{i \in N}\hat{R}_1^i$ differs from the strategy space S and thus $S \xrightarrow{\Gamma}_{\mathcal{C}} \hat{R}_1$ holds. According to our construction, the restriction $\hat{R}_1$ is the first round of the process generated by the iterated maximal deletion of unfavorable strategies. Since we presupposed $x \in \mathcal{C}^k(\Phi^k)$, strategy x survives this round.

Proceed with specifying the sets

$$\hat{R}_2^i := \left(\mathfrak{a}_{\Gamma|_{\hat{R}_1}}^i\right)^{-1}\left(\mathcal{C}^i(\Phi_{\Gamma|_{\hat{R}_1}}^i)\right)$$

for each player $i \in N$. By the independence of payoff equivalent states, the identity

$$\hat{R}_2^k = \left(\mathfrak{a}_{\Gamma|_{\hat{R}_1}}^i\right)^{-1}\left(\{y \circ \tau : y \in \mathcal{C}^k(\Omega^k, \hat{R}_1^k)\}\right) = \mathcal{C}^k(\Omega^k, \hat{R}_1^k)$$

results. In what follows, we will demonstrate that strategy x does not belong to $\hat{R}_2^k$. Note, if $\tilde{B}^k = \mathcal{C}^k(\Phi^k)$ holds, then $x \notin \mathcal{C}^k(\Omega^k, \hat{R}_1^k)$ results and thus $x \notin \hat{R}_2^k$ follows immediately. Consider the remaining case that $\tilde{B}^k = \mathcal{C}^k(\Phi^k)$ is not satisfied. Define the set $\hat{B}^k := \{z \in \tilde{B}^k : z \notin \mathcal{C}^k(\Phi^k)\}$ consisting of all strategies of player k belonging to the constraint $\tilde{B}^k$ as well as being unfavorable in the original decision problem Φ^k. By property β_0, we obtain $z \notin \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ for any $z \in \hat{B}^k$. Remember, $x \in \tilde{B}^k$ as well as $x \notin \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ have been presupposed. Since $\hat{B}^k \subseteq \tilde{B}^k \setminus \mathcal{C}^k(\Omega^k, \tilde{B}^k)$ is satisfied, property β_0 implies $x \notin \mathcal{C}^k(\Omega^k, \tilde{B}^k \setminus \hat{B}^k)$. Due to the identity $\tilde{B}^k \setminus \hat{B}^k = \mathcal{C}^k(\Omega^k, B^k)$ we obtain $\hat{R}_2^k = \mathcal{C}^k(\Omega^k, \tilde{B}^k \setminus \hat{B}^k)$. Hence, $x \notin \hat{R}_2^k$ results.

Obviously, the restriction $\hat{R}_2 := \times_{i \in N}\hat{R}_2^i$ differs from restriction $\hat{R}_1$. Therefore, $\hat{R}_1 \xrightarrow{\Gamma}_{\mathcal{C}} \hat{R}_2$ applies. According to the above construction, the restriction $\hat{R}_2$ corresponds to the second round of the deletion process generated by the iterated maximal deletion of unfavorable strategies with respect to the choice rules $\mathcal{C}$. Remarkably, strategy x does not survive this deletion process, although it is consistent with the common belief of following these choice rules. Putting it differently, if for any strategic games the solution of the iterated maximal deletion of unfavorable strategies with respect to the choice rules $\mathcal{C}$ is characterizable by the common belief of applying these choice rules, then any of the choice rules must satisfy property α_0.

(*b*) Let $\mathcal{C} := (\mathcal{C}^i)_{i \in N}$ be some family of non-empty and non-trivial choice rules satisfying the independence of payoff-equivalent states and property β_0. Suppose there is some player k whose choice rule $\mathcal{C}^k$ violates the property of monotonicity. Hence, there exists some decision problem $\Phi^k := (\Omega^k, B^k)$, some act $x \in B^k$ and some event $E \subseteq \Omega^k$ so that $x \notin \mathcal{C}^k(\Phi^k)$, but $x|_E \in \mathcal{C}^k(E, B^k|_E)$ hold.

Consider the strategic game $\tilde{\Gamma}$ constructed in Lemma 3.5(b) and frame it with the epistemic model $\mathfrak{M}_{\tilde{\Gamma}} := \big(\Omega, (P^i, \sigma^i)_{i \in N}\big)$ consisting of the state space $\Omega := E$, the strategy functions defined by

$$\sigma^i(\omega) := \begin{cases} x\,, & \text{if player } i = k, \\ \omega\,, & \text{if player } i = l, \\ \emptyset\,, & \text{otherwise}\,. \end{cases}$$

for any state $\omega \in \Omega$, and the possibility correspondences $P^i(\omega) := \Omega$ for any state $\omega \in \Omega$ and for any player $i \in N$.

Obviously, in each state $\omega \in \Omega$, each player acts according to her choice rule and thus there is, in each state, common belief of choice rule following behavior. Furthermore, for each state $\omega \in \Omega$, strategy x is realized. Hence, choosing strategy x is consistent with the common belief of choice rule following behavior. However, as already shown in Lemma 3.5(b), strategy x is not contained in the restriction T_1. As it can be easily checked, restriction T_1 corresponds to the first round of the iterated maximal deletion of unfavorable strategies with respect to the choice rules $\mathcal{C}$. Hence, strategy x does not survive this iterative deletion process. We conclude that if, for any strategic game, the solution of the iterated maximal deletion of unfavorable strategies corresponds to the set of strategy profiles being consistent with the common belief of choice rule following behavior, then any of the choice rules must be monotone. ■

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