# Traversal with Enumeration of Geometric Graphs in Bounded Space 

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July 12, 2021


#### Abstract

In this paper, we provide an algorithm for traversing geometric graphs which visits all vertices, and reports every vertex and edge exactly once. To achieve this, we combine a given geometric graph $G$ with the integer lattice, seen as a graph, in such a way that the resulting hypothetical graph can be traversed using the algorithm in [6]. To overcome the problem with hypothetical vertices and edges, we develop an algorithm for visiting any $k$ th neighborhood of a vertex in a graph straight-line drawn in the plane using $O(\log k)$ memory. The memory needed to complete the traversal of a geometric graph then turns out to depend on the maximum ratio of the graph distance and Euclidean distance for pairs of distinct vertices of $G$ at Euclidean distance greater than one and less than $2 \sqrt{2}$.


## 1 Introduction

The problem of graph traversal is one of the fundamental combinatorial problems. Given as an input an undirected graph $G$ and a vertex $v$, the graph traversal problem is to start on $v$ and continue visiting vertices of $G$ by a sequence of moves along edges of $G$ in such a way that at some point every vertex of $G$ reachable by a path from $v$ is visited at least once. The
moves are guided by an edge labelling; every vertex $v$ has the edges incident to it labelled with integers 1 to $d(v)$ so that labels are a permutation. So a traversal algorithm at each vertex will choose a label of an edge and then moves along the edge with that label to the neighbor of the vertex. A problem related to graph traversal is $s, t$-connectivity where we are given two vertices $s$ and $t$ of $G$, and need to decide whether or not there is a path from $s$ to $t$ in $G$. It is obvious that a solution to the graph traversal problem also gives a solution to the $s, t$-connectivity problem. Many basic graph algorithms involve making traversal or determining connectivity.

The time complexity of both problems is well understood and it is linear in the number of edges of the graph. This can be achieved by classical breadthfirst search or depth-first search algorithms. Note that these algorithms can solve the two problems also in their directed version, i.e., when the graph is a directed graph. The space required to run these algorithms is linear as well, and understanding the space complexity of these problems was a major problem for several decades.

It was long known [1] that a random walk (a walk that starts at a vertex and chooses subsequent vertices uniformly at random from current available neighbors) will visit all the vertices reachable from the original vertex in polynomial number of steps. The major problem was to derandomize this simple algorithm without substantially increasing the space.

The first deterministic super-polynomial time algorithm for the traversal problem is the algorithm by Savitch [15] that needs $O\left(\log ^{2} n\right)$ space. The first improvement on this classical result was done by Nisan et al. [10] by showing that the traversal can be performed in $O\left(\log ^{3 / 2} n\right)$ space (the time is still super-polynomial). Building on this work as well as on related works [11, 14, Armoni et al. [2] improved the space complexity to $O\left(\log ^{4 / 3} n\right)$. Note that the most space-efficient polynomial time algorithm for the traversal problem was Nisan's [12] which required $O\left(\log ^{2} n\right)$ space. A big improvement was achieved by Trifonov [16] whose polynomial algorithm required $O(\log n \log \log n)$ space only. Note at this point that $\Omega(\log n)$ space is needed for any algorithm to solve the traversal problem. In 2005, Reingold in his seminal work [13] solved the long standing problem by presenting an $O(\log n)$ space deterministic polynomial algorithm that solves the traversal problem on any $n$-vertex graph.

The traversal problem asks to visit every vertex at least once, and in fact in all the algorithms mentioned above a vertex may be visited many times. In many applications one needs to have a list of vertices, edges, or some other combinatorial objects defined on graphs. For example, in Kruskal's
minimum cost spanning tree algorithm, one needs an ordered list of edges by their weights. In the following we describe work on traversal algorithms which provide such lists. We will refer to such algorithms as traversal with enumeration algorithms. Note at this point that the edge labelings in the traversal problem considered above are arbitrary. Also it is easy to see that the breadth-first search and depth-first search algorithms can be adapted to traversals with enumeration. However, these algorithms are linear in space as we mentioned already. We are interested in most space-efficient traversal with enumeration algorithms. The lists produced by such algorithms are write-only for the algorithms and contain every vertex/edge exactly once. The information provided in graphs considered just as purely combinatorial objects is not enough to achieve this in the same $O(\log n)$ space as the original traversal algorithms. Instead, the research concentrated on graphs embedded in a space and, hence, considered them as geometric objects.

One of the first traversal algorithms with enumeration was described in [8], which traverses triangulations of the plane. The main idea is used in all subsequent works, so we outline it here. One uses the geometric information from the embedding as part of the input to the traversal algorithm and using this information orders edges of the triangulation. The trick is that the query to compare two given edges can be locally computed in $O(\log n)$ space. Then each face will have an "entering edge" (say, first in the order of the three edges) which will be used to traverse the triangulation as a tree. The embedding of the triangulation is given using the rotation system (a cyclic ordering of edges in the embedding of the graph given at every vertex). These orderings are then used to traverse along faces using $O(\log n)$ space and search for the minimum edges, move from face to face, etc. Therefore such traversal can be performed in $O(\log n)$ space. Moreover, the geometric information can be further used to determine exact time when to output a visited vertex/edge to the list. In [3], authors describe a traversal algorithm with enumeration that can traverse all arrangements of convex polytopes. In [4], the algorithm is extended to all planar subdivisions and its running time is improved in [5]. All of the works in [3, 5, 4] consider plane subdivisions. First result on traversal with enumeration that extends to nonplanar graphs has been presented in [6]. The authors define a, so called, quasi-planar subdivision as a graph (straight-line) embedded into the plane, whose vertex set can be partitioned into two sets $V_{p}$ and $V_{c}$, vertices in $V_{p}$ induce a plane graph $P$ (backbone), the outer-face of $P$ does not contain any vertex of $V_{c}$, and no edge of $P$ is crossed by any other edge. One can imagine a quasi-planar graph
as a plane graph in which every face may contain an arbitrary graph which joins only to the vertices of the face. A quasi-planar subdivision is said to satisfy the left-neighbor rule if every vertex of a subgraph inside of a face of its backbone has a neighbor on the face that is to the left of the vertex or has a smaller $x$-coordinate; see Figure 1 for an example. It was proved in [6] that every quasi-planar subdivision that satisfies the left-neighbor rule can be traversed with enumeration in $O(\log n)$ space and polytime.


Figure 1: 6] An example of a quasi-planar subdivision satisfying the leftneighbour rule. The bold edges and filled vertices are those of the underlying planar subgraph.

Note at this point that traversal algorithms are often modelled using so called walking (or jumping) automata on graphs [7]. Such an automaton has a set of states, and a number of pebbles that represent vertex names and are used to mark certain vertices temporarily. The pebbles can be moved from vertex to adjacent vertex ("walk") or directly to a vertex containing another pebble ("jump"). The position of the automaton on the graph is itself marked by a pebble. Thus, walking represents replacing a vertex name by some adjacent vertex found in the input, and jumping represents copying a previously recorded vertex name. It follows that the space required by such an automaton depends on the number of pebbles used, and obviously a walking automaton that uses $c$ pebbles and can traverse a graph corresponds to an $O(c \log n)$ space traversal algorithm. In this paper we will model our traversal algorithm using a walking automaton.

We adapt the traversal algorithm from [6] to a traversal algorithm for
geometric graphs which reports every vertex and every edge exactly once. To achieve this, we develop a walking automaton for local exploration of graphs straight-line drawn in the plane (Section 3). This automaton will visit all vertices within graph distance $k$ (for any fixed $k \in \mathbb{N}$ ) from a given vertex, and will use as few as $k+1$ pebbles, i.e., its space complexity is $O(k \log n)$.

Definition 1 (basic notions).

- Let $G=(V, E)$ be a graph. For every $v \in V$ we denoted the set of neighbors of $v$ in $G$ by $N_{G}(v)$ and define $N_{G}[v]=N_{G}(v) \cup\{v\}$. Moreover, for every $A \subseteq V$ we define $N_{G}[A]=\bigcup_{v \in A} N_{G}[v]$ and set $N_{G}(A)=N_{G}[A] \backslash A$.
- A geometric graph is a (drawing of a) graph having a finite subset $V$ of the plane as its vertex set, and whose edge set consists of line-segments between all pairs of distinct points in $V$ with Euclidean distance $\leq 1$.
- For every connected geometric graph $G$ of order $\geq 2$ we define

$$
r(G)=\min _{\forall x, y \in V(G)}\left\{M: 1<d_{E}(x, y)<2 \sqrt{2} \Rightarrow d_{G}(x, y) \leq M d_{E}(x, y)\right\}
$$

where $d_{G}$ and $d_{E}$ denote the graph distance in $G$ and the Euclidean distance in the plane, respectively.

We shall use the invariant $r(G)$ to bound the space complexity of a traversal with enumeration algorithm for $G$. In particular, we provide such an algorithm that uses $\lfloor 2 \sqrt{2} r(G)\rfloor$ pebbles. We call a geometric graph $G$ well-embedded if $r(G)$ is a constant. Hence as a consequence we obtain a traversal with enumeration algorithm with $O(\log n)$ space complexity for well-embedded geometric graphs.

The main idea behind such algorithm is to obtain from $G$, which in general can have many crossings, a quasi-planar graph satisfying the left-neighbor rule. We achieve this by fixing a Cartesian coordinate system and assuming, with no loss of generality (since $G$ is finite), that no vertex of $G$ has an integer coordinate and no edge of $G$ passes through a point in $\mathbb{Z} \times \mathbb{Z}$. Then we create a virtual graph which is an augmentation of $G$ and the grid graph given by the integer lattice. For traversing the resulting graph, we use an adaptation of the traversal algorithm in [6] combined with an algorithm (Algorithm 1) for exploring the $k$ th neighborhood of a vertex in a graph straight-line drawn in the plane. In the following section, we present the formal construction of the virtual graph.

## 2 The Virtual Graph

In this section we provide details how to augment a geometric graph $G$ with a grid graph defined on the points of a sub-lattice of the plane.

Definition 2. For every graph $G$ and $k \in \mathbb{N}$, we define the $k$ th neighborhood of a vertex $v$ in $G$ by

$$
\begin{equation*}
N_{G}^{k}[v]=\left\{u \in V(G): d_{G}(u, v) \leq k\right\} . \tag{1}
\end{equation*}
$$

(We shall drop the subscript $G$ in $N_{G}^{k}[v]$ when $G$ is understood from the context.)

Given a connected geometric graph $G$, we shall regard the infinite integral lattice in the plane as an infinite plane graph and denote it by $L$. For every $v \in V(G)$ we define the square of $v$, denoted $s_{v}$, to be the face of $L$ that encloses $v$, and $\operatorname{cor}\left(s_{v}\right)$ to be the lower left corner of $s_{v}$.

We combine $G$ and $L$ into a new graph $G \boxtimes L$ as follows. Let $L_{G}$ be the smallest subgraph of $L$ whose face set contains every $s_{v}, v \in V(G)$. Let $C(L, G)$ be the set of all edges of $G$ that cross an edge of $L$, and for every vertex $v \in V(G)$ let $\ell_{v}$ be the line-segment joining $v$ to $\boldsymbol{\operatorname { c o r }}\left(s_{v}\right)$. We now define the graph $G \boxtimes L$ by

$$
\begin{equation*}
V(G \boxminus L)=V(G) \cup V\left(L_{G}\right), \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
E(G \boxtimes L)=(E(G) \backslash C(L, G)) \cup E\left(L_{G}\right) \cup\left\{\ell_{v}: v \in V(G)\right\} . \tag{3}
\end{equation*}
$$

See Figure 2 for an example. Again, as $G$ is finite we may assume that no newly added edge $\ell_{v}$ passes through a vertex of $G$. Then, as the planar graph $L_{G}$ is an induced subgraph of $G \backsim L$ and no edge of $L$ is crossed by any edge of $G$, it follows that $G \boxtimes L$ is a quasi-planar graph with the underlying plane graph $L_{G}$. Moreover, as every vertex $v$ of $G$ is incident, via $\ell_{v}$, to a vertex of $L_{G}$ having a smaller $x$ coordinate than that of $v, G \boxtimes L$ also satisfies the left-neighbor rule. The only problem with applying the quasi-planar subdivision traversal algorithm from [6] is that $G \boxtimes L$ contains hypothetical vertices and edges which cannot be visited or used by a traversing agent. But whenever $r(G)$ (or an upper bound for it) is known, this problem will be resolved. The idea is that the traversing agent on $G$, denoted $R^{r}$, will simulate a virtual agent, denoted $R^{v}$, that will traverse $G \boxtimes L$. To be able


Figure 2: (a): A geometric graph $G$ in the graph $L_{G}$, (b): The quasi-planar graph $G \boxtimes L$.
to perform this simulation in space bounded by $r(G)$, the traversing agent and the virtual agent must maintain a constant graph distance from each other, which turns out to be $\lfloor 2 \sqrt{2} r(G)\rfloor$. Then, the local explorations by the virtual agent can be simulated by traversing agent using $\lfloor 2 \sqrt{2} r(G)\rfloor$ pebbles. The virtual agent needs to know only a local portion of $G \backsim L$. Before each computation step of $R^{v}$ (e.g., a test to determine the next edge in the rotation system) the traversing agent $R^{r}$ explores a local portion of $G$ to provide the next arc of $G \square L$ that $R^{v}$ has to test to determine the next step (e.g., moving along an edge). Determination of such an arc depends not only on the current position of $R^{v}$ but also on the last edge of $G \square L$ which was traversed by $R^{v}$. If $R^{v}$ is on a vertex of $G$, then $R^{r}$ starts exploration of the graph starting from that vertex. Moreover, in any step of the game if $R^{v}$ is moving on an edge of $G, R^{r}$ will make the same move. An algorithm for an exploration used by $R^{v}$ is described in Section 3 in a more general settings.

In the description of our algorithms each edge $e=u v$ is associated with two oppositely directed $\operatorname{arcs} \mathbf{e}=(u, v)$ and its reverse, $\operatorname{rev}(\mathbf{e}):=(v, u)$. As $G$ is assumed to be a graph straight-line drawn into the plane, we can define the successor of an $\operatorname{arc} \mathbf{e}=(u, v)$, denoted $\operatorname{succ}(\mathbf{e})$, as the first $\operatorname{arc}(v, w)$ succeeding $\operatorname{rev}(\mathbf{e})$ counterclockwise around $v$. The predecessor of $\mathbf{e}$, denoted $\operatorname{pred}(\mathbf{e})$, is the $\operatorname{arc} \mathbf{e}^{\prime}$ such that $\operatorname{succ}\left(\mathbf{e}^{\prime}\right)=\mathbf{e}$. Furthermore, for every vertex (point) $v$ in the plane, we denote its $x$ and $y$ coordinates by $\operatorname{xcor}(v)$ and $\operatorname{ycor}(v)$, respectively. We shall also use the notation of the cone defined by a triple of the points in the plane, as follows. Suppose $a, b$, and $c$ are points in the plane. We denote by $\angle(a, b, c)$ the counter-clockwise angle with apex $b$ from the ray $b a$ to the ray $b c$. Then, cone $(a, b, c)$ is the cone with apex $b$ and
interior angle $\angle(a, b, c)$, including the supporting ray passing through $a$ and not the one through $c$, see [6].

## 3 Local Exploration of Graphs Straight-Line Drawn in the Plane

Let $v$ be any vertex in a straight-line drawn graph $G$. Given $k \in \mathbb{N}$ we want to visit the $k$ th neighborhood $N_{G}^{k}[v]$ of $v$. We shall follow the depth-first search tree of $G$ to distribute a set of $k+1$ pebbles (initially located at $v$ ) among vertices of the paths of the DFS tree that start from the root $v$, only backtracking when we reach a vertex in $N_{G}^{k}(v)$ or when every candidate for a "new" vertex is adjacent to an already pebbled vertex. Here is a more detailed description of the algorithm.
Here is a more detailed description of the algorithm. Consider a graph $G$ with a given straight-line drawing in the plane. On input

- any $v \in V(G)$, and
- $k+1$ pebbles initially located at $v$,
our algorithm visits every vertex in $N_{G}^{k}[v]$ at least once and does not visit any vertex outside $N_{G}^{k}[v]$. Visiting a vertex is defined by moving at least one pebble from a neighboring pebbled vertex to this vertex (by which this vertex will be considered visited then). We move pebbles around in $N_{G}^{k}[v]$ using the operators succ and rev, in a depth-first-search fashion.

At any stage (or iteration) the algorithm will be either in forward (fw) or backward (bw) mode, and on an arc e of $G$. Thus, at any time we can define the state ( $P, \mathbf{e}$, mode), where $P$ is the set of currently pebbled vertices (i.e., vertices where the algorithm stores at least one pebble), $\mathbf{e}$ is the current arc, and mode $\in\{\mathrm{fw}, \mathrm{bw}\}$. The computation of the algorithm can be described as a sequence of such states.

Exploration in our algorithm can begin on any arc of $G$ with tail $v$. But for definiteness we set $\mathbf{e}_{\mathbf{0}}$ as the arc with $\operatorname{tail}\left(\mathbf{e}_{\mathbf{0}}\right)=v$ having the smallest counterclockwise angle with the $x$-axis and set the initial state to be $\left(\left\{\operatorname{tail}\left(\mathbf{e}_{0}\right), \operatorname{head}\left(\mathbf{e}_{0}\right)\right\}, \mathbf{e}_{\mathbf{0}}, \mathrm{fw}\right)$.

In our algorithm, changing from a state ( $P, \mathbf{e}$, mode) to ( $P^{\prime}, \mathbf{e}^{\prime}$, mode ${ }^{\prime}$ ) depends on mode, adjacency of vertices in $N($ head $(\mathbf{e}))$ to the other pebbled
vertices, and whether or not $\operatorname{card}(P)=k+1$. Moreover, in such a change of states we will always have $\operatorname{card}\left(P \triangle P^{\prime}\right)=1$ and head $(\mathbf{e})=\operatorname{tail}\left(\mathbf{e}^{\prime}\right)$.

To visit all vertices in the $k$ th neighborhood of $v$, we choose the arc $\mathbf{e}_{\mathbf{0}}$ starting at $v$ as we described above, and initially put all pebbles at $v$. The algorithm then starts moving pebbles in a DFS way, and in such a way that at any time, the pebbled vertices form the vertex set of an induced path in $G$. This condition is essentially maintained by Steps 10-14 of the algorithm in the forward mode, and by Steps 29-33 in the backward mode. By Steps $10-14$, when in a state $(P, \mathbf{e}, \mathrm{fw}), \operatorname{tail}(\mathbf{e})$ is the only pebbled vertex which is adjacent to head(e). Furthermore, the mode in the next iteration remains forward only if there is an arc succeeding $\mathbf{e}$ whose head is an unpebbled vertex with no pebbled neighboring vertex except head(e). If no such arc exist, the algorithms backtracks along $\operatorname{rev}(\mathbf{e})$ and the following state will be ( $P \backslash\{\operatorname{head}(\mathbf{e})\}, \operatorname{rev}(\mathbf{e}), \mathrm{bw})$ (Steps 15-18). Transition from a state with the backward mode is similar (Steps 29-33 and 36-38). The algorithm terminates when it returns to the $\operatorname{arc} \mathbf{e}_{\mathbf{0}}$, by which time all of the vertices in the $k$ th neighborhood of $v$ will have been visited. We establish this fact, among other things, in Theorem 1 .


Figure 3: An instance of Algorithm 1 on input vertex 1 with the task of exploring the second neighborhood of vertex 1 that produces a list of 12 states before it terminates.

| $1:(\{1,2\},(1,2), \mathrm{fw})$ | $2:(\{1\},(2,1), \mathrm{bw})$ | $3:(\{1,3\},(1,3), \mathrm{fw})$ |
| :--- | :--- | :--- |
| $4:(\{1,3,5\},(3,5), \mathrm{fw})$ | $5:(\{1,3\},(5,3), \mathrm{bw})$ | $6:(\{1,3,6\},(3,6), \mathrm{fw})$ |
| $7:(\{1,3\},(6,3), \mathrm{bw})$ | $8:(\{1\},(3,1), \mathrm{bw})$ | $9:(\{1,4\},(1,4), \mathrm{fw})$ |
| $10:(\{1,4,5\},(4,5), \mathrm{fw})$ | $11:(\{1,4\},(5,4), \mathrm{bw})$ | $12:(\{1\},(4,1), \mathrm{bw})$ |

Figure 3 shows an instance of Algorithm 1 for exploring the second neighborhood of vertex 1 in the given graph. The algorithm first recognizes $\mathbf{e}_{0}=(1,2)$ as the current arc in the initial state, and places all pebbles but one on $\operatorname{head}\left(\mathbf{e}_{0}\right)$. Starting from the initial state $(\{1,2\},(1,2)$, fw), it eventually reaches the state $(\{1\},(4,1)$, bw) where all the pebbles are back on vertex 1 . Since $\mathbf{e}_{0}=\boldsymbol{\operatorname { s u c c }}((4,1))$ that is the terminal state, according to Step 34 in Algorithm 1 .

```
Algorithm 1 Visiting all vertices in \(N^{k}[v]\)
Input: a straight-line drawing of a graph \(G\) in the plane, a vertex \(v \in V(G)\)
with \(k+1\) pebbles on \(v\)
Objective: Visit all vertices in \(N^{k}[v]\)
    denote the set of currently pebbled vertices by \(P\).
    let \(\mathbf{e}_{\mathbf{0}}\) be the arc with \(v=\boldsymbol{\operatorname { t a i l }}\left(\mathbf{e}_{\mathbf{0}}\right)\) and the smallest counterclockwise
    angle with \(x\)-axis
    mode \(\leftarrow \mathrm{fw}\)
    Keep one pebble at \(\operatorname{tail}\left(\mathbf{e}_{\mathbf{0}}\right)\) and move the rest to \(\operatorname{head}(\mathbf{e})\)
    \(P=\left\{\operatorname{head}\left(\mathbf{e}_{\mathbf{0}}\right), \operatorname{tail}\left(\mathbf{e}_{\mathbf{0}}\right)\right\}\)
    \(\mathbf{e} \leftarrow \mathbf{e}_{0}\)
    repeat
        if mode \(=\mathrm{fw}\) then
            if \(|P|<k+1\) then
            \(\mathbf{e}^{\prime} \leftarrow \operatorname{succ}(\mathbf{e})\)
            while head \(\left(\mathbf{e}^{\prime}\right) \in(N(P \backslash\{\operatorname{head}(\mathbf{e})\}) \backslash P)\) do
                \(e^{\prime} \leftarrow \operatorname{succ}\left(\operatorname{rev}\left(\mathbf{e}^{\prime}\right)\right)\)
            end while
            \(\mathrm{e} \leftarrow \mathrm{e}^{\prime}\)
            if head \((\mathbf{e}) \in P\) then
                    move all pebbles on \(\operatorname{tail}(\mathbf{e})\) to head \((\mathbf{e})\)
                    \(P=P \backslash\{\operatorname{tail}(\mathbf{e})\}\)
                mode \(\leftarrow \mathrm{bw}\)
            else
                move all but one of the pebbles on \(\operatorname{tail}(\mathbf{e})\) to \(\operatorname{head}(\mathbf{e})\)
                \(P=P \cup\{\operatorname{head}(\mathbf{e})\}\)
            end if
            else
```

```
Algorithm 2 Visit all vertices in \(N^{k}[v]\) (continued)
24: move all pebbles from head(e) to tail(e)
\(\mathrm{e} \leftarrow \operatorname{rev}(\mathrm{e})\)
mode \(\leftarrow \mathrm{bw}\)
    end if
    else \(\quad \triangleright\) When mode \(=\mathrm{bw}\)
    \(\mathrm{e}^{\prime} \leftarrow \operatorname{succ}(\mathrm{e})\)
    while head \(\left(\mathbf{e}^{\prime}\right) \in(N(P \backslash\{\operatorname{head}(\mathbf{e})\}) \backslash P)\) do
        \(\mathbf{e}^{\prime} \leftarrow \operatorname{succ}\left(\operatorname{rev}\left(\mathbf{e}^{\prime}\right)\right)\)
        end while
        \(\mathbf{e} \leftarrow \mathrm{e}^{\prime}\)
        if \(\mathbf{e}=\mathbf{e}_{\mathbf{0}}\) then move all pebbles on \(v\) and terminate.
        end if
        if head \((\mathbf{e}) \in P\) then
            move all pebbles on \(\operatorname{tail}(\mathbf{e})\) to \(\operatorname{head}(\mathbf{e})\)
                \(P=P \backslash\{\operatorname{tail}(\mathbf{e})\}\)
            else
                move all but one of the pebbles on \(\operatorname{tail}(\mathbf{e})\) to \(\operatorname{head}(\mathbf{e})\)
                \(P=P \cup\{\operatorname{head}(\mathbf{e})\}\)
                mode \(=\mathrm{fw}\)
            end if
    end if
    until \(e=e_{0}\)
```

Theorem 1. Let $G$ be graph straight-line drawn in the plane, $v \in V(G)$, and $\mathbf{e}_{\mathbf{0}}$ an arc of $G$ with $\boldsymbol{\operatorname { t a i l }}\left(\mathbf{e}_{\mathbf{0}}\right)=v$. Then the following holds:
(a) For every iteration $i$,
(b) The states at different times are all distinct.
(c) The algorithm terminates after a finite number of iterations.
(d) For every $u \in N^{k}[v]$ there is an iteration $i$ such that $u \in P_{i}$.

Proof. We denote the state in every $i$ th iteration $(i \in \mathbb{N} \cup\{0\})$ by state $(i)=$ ( $P_{i}, \mathbf{e}_{\mathbf{i}}$, mode $_{i}$ ), where $i=0$ corresponds to the initial state, so that state $(0)=$ ( $\left.\{v\}, \mathbf{e}_{\mathbf{0}}, \mathrm{fw}\right)$. (a) We use induction on $i$. The claim holds for $i=0$ and 1 ,
since $G\left[P_{0}\right]$ is a single vertex, and $G\left[P_{1}\right]$ is a 2-path between $v$ and head $\left(\mathbf{e}_{0}\right)$. Suppose $j \in \mathbb{N}$ and the claim holds for every iteration $i<j$. In the case $\operatorname{mode}_{j}=\mathrm{fw}, \operatorname{head}\left(\mathbf{e}_{\mathbf{j}-\mathbf{1}}\right)=\operatorname{tail}\left(\mathbf{e}_{\mathbf{j}}\right)$ is the only vertex in $P_{j-1}$ which is adjacent to head $\left(\mathbf{e}_{\mathbf{j}}\right)$. Hence, by induction hypothesis, $G\left[P_{j}\right]$ is an induced path between $v$ and head $\left(e_{j}\right)$. On the other hand, if mode $j_{j}=\mathrm{bw}$ then $P_{j}:=P_{j-1} \backslash \operatorname{head}\left(\mathbf{e}_{\mathbf{j}-1}\right)$ where $\operatorname{head}\left(e_{j-1}\right)=\operatorname{tail}\left(e_{j}\right)$. Hence, $G\left[P_{j}\right]$ is the sub-path of $G\left[P_{j-1}\right]$ between $v$ and head $\left(\mathbf{e}_{\mathbf{j}}\right)$ and the claim follows by the induction hypothesis.(b) Suppose by the way of contradiction, that there is a repeated state and let state $e_{*}:=\left(P_{*}, \mathbf{e}_{*}\right.$, mode $\left._{*}\right)$ be the first state that occurs at least twice during an application of the algorithm. Let iterations $i$ and $j$ be the first two iterations whose states are equal to state $_{*}$. Note that because of the termination criterion (Step 34), state (0) cannot be repeated; thereby, $i, j>0$. It is not difficult to observe that under our assumptions and due to how the algorithm extends paths, we have

$$
\operatorname{head}\left(\mathrm{e}_{\mathbf{i}-\mathbf{1}}\right)=\operatorname{head}\left(\mathbf{e}_{\mathbf{j}-\mathbf{1}}\right)=\operatorname{tail}\left(\mathbf{e}_{*}\right)
$$

Let

$$
\mathbf{e}_{*}=(a, b) \quad \& \quad \mathbf{e}_{\mathbf{k}}=\left(c_{k}, a\right) \quad(\text { for } k \in\{i-1, j-1\})
$$

Then, by the choice of iterations $i$ and $j$ we have

$$
\begin{equation*}
c_{i-1} \neq c_{j-1} \tag{4}
\end{equation*}
$$

Indeed, if $c_{i-1}=c_{j-1}$, then it follows that $P_{i-1}=P_{j-1}$ and mode $_{i-1}=$ mode $_{j-1}$.

Moreover, for $k \in\{i-1, j-1\}$ we have $c_{k} \in P_{*}$ iff mode $_{k}=\mathrm{fw}$. It follows from this that $c_{k} \notin N\left(P_{*} \backslash\{a, b\}\right)$ whenever mode $_{k}=\mathrm{bw}$. Observe that in light of part (a) we also have

$$
\mathrm{bw} \in\left\{\operatorname{mode}_{k}: k \in\{i-1, j-1\}\right\},
$$

for otherwise we would have $c_{i-1}=c_{j-1}$, contradicting (4). Furthermore, since either $c_{i-1} \in \operatorname{cone}\left(c_{j-1}, a, b\right)$ or $c_{j-1} \in \operatorname{cone}\left(c_{i-1}, a, b\right)$, we also have

$$
\mathrm{fw} \in\left\{\operatorname{mode}_{k}: k \in\{i-1, j-1\}\right\} .
$$

Hence, we may assume that

$$
\begin{equation*}
\operatorname{mode}_{i-1}=\mathrm{fw} \quad \& \quad \operatorname{mode}_{j-1}=\mathrm{bw} \tag{5}
\end{equation*}
$$

Note that we have

$$
\begin{equation*}
c_{j-1} \in \operatorname{cone}\left(c_{i-1}, a, b\right) \quad \text { or } \quad c_{i-1} \in \operatorname{cone}\left(c_{j-1}, a, b\right) . \tag{6}
\end{equation*}
$$

But $c_{j-1} \in \operatorname{cone}\left(c_{i-1}, a, b\right)$ (resp. $\left.c_{i-1} \in \operatorname{cone}\left(c_{j-1}, a, b\right)\right)$ implies in iteration $i$ (resp. $j$ ) the algorithm would choose $\mathbf{r e v}\left(\mathbf{e}_{\mathbf{j}-\mathbf{1}}\right)$ (resp. $\mathbf{r e v}\left(\mathbf{e}_{\mathbf{i}-\mathbf{1}}\right)$ ) ahead of $\mathbf{e}_{*}$, contradicting that state $(i)=\operatorname{state}_{*}=\operatorname{state}(j)$. Therefore, no two states can be equal. (c) Since the set of possible states is finite, it follows by part (b) that the termination condition $\mathbf{e}=\mathbf{e}_{0}$ will hold after a finite number of states. (d) For every $i \in \mathbb{N}$ let $\mathcal{P}_{i+1}$ be the union of all first components of the states occurred during an application of the algorithm with $k=i+1$ pebbles. As every pebbled vertex is joined to $v$ via a path of pebbled vertices, we have $\mathcal{P}_{i+1} \subseteq N^{i}[v]$ for each $i$. We use induction on $i$ to show that the reverse inclusion also holds. For $i=2$ it is easy to see that every vertex in $N(v)$ appears in $\mathcal{P}_{2}$; i.e. $\mathcal{P}_{2} \supseteq N(v)$. Furthermore, since for every $i$ every state occurred using $i+1$ pebbles will also occur using more than $i+1$ pebbles, $\left\{\mathcal{P}_{i}\right\}$ is an increasing sequence with respect to $\subseteq$. Now, suppose $\mathcal{P}_{i+1}=N^{i}[v]$ for some $i$ and let $u \in V(G)$ such that $d_{G}(v, u)=i+1$ and consider a vertex $w \in N(u) \cap N^{i}[v]$. By the induction hypothesis and that $\mathcal{P}_{i} \subseteq \mathcal{P}_{i+1}$, there is an iteration $j$ with state ( $P^{\prime}, e^{\prime}$, mode $e^{\prime}$ ) in the application of the algorithm with $i+1$ pebbles where $w$ is pebbled for the first time. As such, we will have mode ${ }^{\prime}=\mathrm{fw}$ and the vertex head $\left(\mathbf{e}^{\prime}\right)$ will have two pebbles in iteration $j$. But then all vertices in $N(w) \backslash\left(N\left(P^{\prime} \backslash\left\{\operatorname{head}\left(\mathbf{e}^{\prime}\right)\right\}\right)\right.$, including $u$, will be pebbled after iteration $j$ and before head $\left(\mathbf{e}^{\prime}\right)$ is unpebbled.

## 4 A Traversal Algorithm with Enumeration for Geometric Graphs

Throughout this section, $G$ is a fixed geometric graph. We shall provide an algorithm for traversal of $G$ using $G \backsim L$, as defined in Section 2 , using a real robot $R^{r}$ and a virtual robot $R^{v}$. The virtual robot $R^{v}$ essentially does the task of traversing $G \square L$, while $R^{r}$ moves along edges in $G$ with the objective of simulating $G \boxtimes L$ for $R^{v}$.

The following conditions will be maintained by the algorithm:

- $R^{r}$ will simulate $R^{v}$ which will traverse $G \square L$,
- $R^{r}$ moves along edges of $G$ and does the task of local exploration of $G$ to ,
- in the traversal phase, the Euclidean distance between $R^{r}$ and $R^{v}$ can be maintained within the constant value of $2 \sqrt{2}$,
- in the exploration phase $R^{r}$ can efficiently get back to its initial position (at the start of the exploration phase), and
- the algorithm reports every vertex or edge of $G$ exactly once.

The entire algorithm consists of two main phases: the preliminary phase with the goal of finding the minimum edge $e_{\min }$ of $G \boxtimes L$, as defined in [6] for quasi-planar graphs, and the traversal phase with the goal of traversing $G \square L$ and reporting every vertex and edge of $G$ exactly once. In both phases, $R^{v}$ runs traversal on $G \square L$ via executing succ. In order for $R^{v}$ to execute $\operatorname{succ}(\mathbf{e})$ for some arc $\mathbf{e}, R^{r}$ finds e, essentially by running Algorithm 1. Similarly, all other queries are handled by $R^{r}$.
Preliminary Phase: Since, as one can easily see, $\mathbf{e}_{\text {min }}$ is also the minimum arc of $L_{G}$, it suffices to perform the traversal algorithm on $L_{G}$ until $\mathbf{e}_{\text {min }}$ is found. To this end, in stage 0 of the preliminary phase we put $R^{r}$ at any vertex $v \in V(G)$, choose the minimum arc of $s_{v}$ as the starting arc, and place $R^{v}$ in its head. In every subsequent stage, $R^{v}$ executes $\operatorname{succ}(\mathbf{e})$ for some arc $\mathbf{e}$, which requires $R^{r}$ to find $\operatorname{succ}(\mathbf{e})$ and provide it to $R^{v}$ by running Algorithm 1. Note that traversing $L_{G}$ can be carried out in accordance with the traversal algorithm in either [6] or in [5]. In addition, since the vertices of $L_{G}$ are all virtual and $R^{r}$ has to remain in $V(G)$, in no stage of the preliminary phase will $R^{r}$ and $R^{v}$ be in the same vertex of $G \boxtimes L$. At the end of a stage of the traversal $R^{r}$ either stays in its current face of $L_{G}$ or leaves it to a neighboring face, as described below:
a. If $R^{v}$ has not left to a new face, $R^{r}$ does not move.
b. If $R^{v}$ has left to a new face on an $\operatorname{arc} \mathbf{e}$, then starting from its current vertex, $R^{r}$ follows Algorithm 1 with $k=\lfloor 2 \sqrt{2} r(G)\rfloor$ until a vertex of $G$ in one of the two faces of $L_{G}$ that contain $\mathbf{e}$ is found.

The final stage of the preliminary phase is the one in which $\mathbf{e}_{\text {min }}$ is reached by the traversing robot $R^{v}$.

```
Algorithm 3 Traversing a Geometric Graphs \(G\)
Input: A graph \(G \in \mathcal{G}\) where \(\mathcal{G}\) is a class of well-embedded geometric graphs,
and a vertex \(v \in V(G)\)
Output: The list of (coordinate of) vertices of \(G\)
    let \(e\) be the lower arc of \(s_{v}\) pointing away from \(\operatorname{cor}\left(s_{v}\right)\)
    \(\mathbf{e} \leftarrow \mathbf{e}_{\mathbf{0}}\)
    \(R^{r} \leftarrow v, R^{v} \leftarrow \operatorname{head}(\mathbf{e})\)
    repeat \(\triangleright\) finds \(e_{\text {min }}\)
        \(\mathrm{e} \leftarrow \operatorname{rev}(\mathrm{e})\)
        \(R^{v} \leftarrow \operatorname{head}(\mathbf{e})\)
        while \(\mathbf{e} \neq\) entry \(_{L_{G}}\left(\right.\) qface \(\left._{L_{G}}(e)\right)\) do
            \(\mathbf{e} \leftarrow \operatorname{succ}_{L_{G}}(e)\)
            \(R^{v} \leftarrow \operatorname{head}(\mathbf{e})\)
            reposition \(R^{r}\) if necessary
        end while
    until \(\mathrm{e}=\mathbf{e}_{\text {min }}\)
    \(p \leftarrow \operatorname{left}(\mathbf{e})-(0,1)\)
    repeat \(\triangleright\) start the traversal on \(G \boxtimes L\)
    \(\mathbf{e} \leftarrow \operatorname{succ}(\mathbf{e})\)
    \(R^{v} \leftarrow \operatorname{head}(\mathrm{e})\)
    reposition \(R^{r}\) if necessary
    if \(p \in \operatorname{cone}(\operatorname{tail}(\mathbf{e})\), head \((\mathbf{e}), \operatorname{head}(\operatorname{succ}(\mathrm{e})))\) then
        report tail(e)
        \(R^{v} \leftarrow \operatorname{head}(\mathbf{e})\)
        report every edge tail \((\mathbf{e}) w \in C\left(L_{G}\right)\) such that in the lexico-
    \(\operatorname{graphic} \operatorname{order}(\mathbf{x c o r}(\operatorname{tail}(\mathbf{e})), \mathbf{y c o r}(\operatorname{tail}(\mathbf{e}))<(\operatorname{xcor}(w), \mathbf{y c o r}(w))\)
        end if
    if \(e \in E(G)\) and \((|\operatorname{tail}(\mathbf{e}) p|<|\operatorname{head}(\mathbf{e}) p|\) or \((|\operatorname{tail}(\mathbf{e}) p|=|\boldsymbol{\operatorname { h e a d }}(\mathbf{e}) p|\)
    and \(\overrightarrow{\operatorname{tail}(\mathbf{e}) p}<\overrightarrow{\operatorname{head}(\mathbf{e}) p})\) then
        report \(e\)
    end if
    if \(\mathrm{e}=\operatorname{entry}(\operatorname{qface}(\mathrm{e}))\) then
        \(\mathrm{e} \leftarrow \operatorname{rev}(\mathrm{e})\)
    else
        if \(\operatorname{rev}(e)=\operatorname{entry}(q \operatorname{face}(\operatorname{rev}(e))\) then
            \(\mathrm{e} \leftarrow \operatorname{rev}(\mathrm{e})\)
            end if
        end if
    until \(e=e_{\text {min }}\)

\section*{Traversal Phase:}

The final stage of the preliminary phase defines stage 0 of the traversal phase. During this phase, as opposed to the preliminary one, \(R^{v}\) traverses the entire graph \(G \boxtimes L\), but \(R^{r}\) uses the same shadowing rules as in the preliminary phase to get within the Euclidean distance of \(\sqrt{2}\) from \(R^{v}\) at the end of each stage. Moreover, whenever \(R^{v}\) gets to a vertex of \(G, R^{r}\) runs Algorithm 1 until it gets to the same vertex where \(R^{v}\) is positioned. For reporting vertices and edges we use the same criterion as the one described in [6] with the exception that hypothetical vertices, i.e. vertices of \(L_{G}\), and edges, i.e. edges \(L_{G}\) and the ones in \(\left\{\ell_{v}: v \in V(G)\right\}\), will not be reported. Finally, an edge \(s t \in C_{L}\) with \((\operatorname{xcor}(s), \mathbf{y c o r}(s))<(\mathbf{x} \operatorname{cor}(t), \mathbf{y} \operatorname{cor}(t))\) gets reported whenever \(s\) is reported.

\section*{5 Minimum Spanning Tree in Bounded Space}

One of the problems that can be tackled using local traversal algorithms is the problem of finding minimum spanning trees in log space for certain classes of graphs. In this section we present Algorithm 4 for finding the minimum spanning tree for connected graphs straight-line embedded in the plane with an injective edge-weight assignment. Likewise Kruskal's algorithm [9], Algorithm 4 considers edges in the increasing order of their weights. Utilizing local traversal algorithms without mark bits, Algorithm 4 has the advantage of not requiring global information such as the order or size of the graph under consideration- all it needs is to keep track of the weight of the "current" candidate for being included in the minimum spanning tree, and a couple of pebbles for distinguishing between its end vertices.

Let \(G\) be a connected planar graph straight-line embedded in the plane with an injective edge-weight assignment \(w: E(G) \rightarrow(0, \infty)\). Algorithm 4 uses two auxiliary functions next.weight defined on \(\{w(e): e \in E(G)\} \cup\{0\}\) and discard.weight, defined on \(\{w(e): e \in E(G)\}\). For every \(w \in\{w(e)\) : \(e \in E(G)\}\) we define next.weight \((w)\) to be 0 if no edge of \(G\) has a weight \(>w\), and to be \(\min \{w(e): e \in E(G)\) and \(w(e)>w\}\) otherwise. Moreover, we define discard.weight \((w)\) to be 1 if there is a path in \(G\) between the end-points of the edge with weight \(w\) that has all of its edge-weights \(<w\); otherwise we put discard.weight \((w)=0\). Each of these functions can be implemented by an adaptation of the traversal algorithm in [5] starting from any current vertex. As such, next.weight \((w)\) require a traversal of \(G\)
with keeping track of the tentative lightest edge having weight \(>w\), while discard.weight \((w)\) requires a, possibly partial, traversal of the component of the current vertex in the graph obtained from \(G\) by removing every edge of a weight \(\geq w\).
```

Algorithm 4 Finding a Minimum-Weight Spanning Tree $T$ of $G$
$x \leftarrow$ next.weight(0)
$W_{T} \leftarrow\{x\}$
while $x \neq 0$ do
if next.weight $(x)=0$ then return $W_{T}$
else
$x \leftarrow$ next.weight $(x)$
end if
if discard.weight $(x)=0$ then $W_{T} \leftarrow W_{T} \cup\{x\}$
end if
end while

```

\subsection*{5.1 Validity of the Algorithm}

Lemma 2. Let \(G\) be a connected graph with an injective edge-weight assignment \(w\), and let \(T\) be the minimum-weight spanning tree of \(G\) resulting from Kruscal's algorithm. Then, for every edge e of \(G\) we have \(e \notin E(T)\) iff there is a cycle \(C \subseteq G\) containing e with every edge of a wight \(\leq w(e)\).

Proof. For every \(e \in E(G)\) let \(T_{e}\) be the spanning subgraph of \(G\) with \(E\left(T_{e}\right)=\left\{e^{\prime} \in E(T): w\left(e^{\prime}\right) \leq w(e)\right\}\). Moreover, for every \(e \in E(G) \backslash E(T)\) let \(C_{e}\) be the fundamental cycle of \(e\) with respect to \(T\). If \(e \in E(G) \backslash E(T)\), then \(e \notin E\left(T_{e}\right)\) and, in view of Kruskal's algorithm, we will also have \(C_{e}-e \subseteq T_{e}\). This establishes the "only-if" part. For the converse, let \(e \in E(C)\) where \(C\) is a cycle having every edge of a weight \(\leq w(e)\), and let
\[
W:=C+\left(\sum C_{e^{\prime}}: e^{\prime} \in(E(C) \backslash\{e\}) \backslash E(T)\right)
\]
where the summation is taken in the cycle space of \(G\). Note that \(E(W) \backslash\) \(\{e\} \subseteq E\left(T_{e}\right) \subseteq E(T)\). Moreover, since for every \(e^{\prime} \in E(C) \backslash\{e\}\) the edges of \(C_{e^{\prime}}\) are lighter than \(e, e\) is an edge in \(W\). Therefore, \(e \in E(G) \backslash E(T)\), as desired. (Indeed, \(W\) is the fundamental cycle \(C_{e}\) of \(e\) with respect to \(T\).)

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