

# Strong Subgraph Connectivity of Digraphs: A Survey

Yuefang Sun<sup>1</sup> and Gregory Gutin<sup>2</sup>

<sup>1</sup> Department of Mathematics, Shaoxing University  
Zhejiang 312000, P. R. China, yuefangsun2013@163.com

<sup>2</sup> School of Computer Science and Mathematics  
Royal Holloway, University of London  
Egham, Surrey, TW20 0EX, UK, g.gutin@rhul.ac.uk

## Abstract

In this survey we overview known results on the strong subgraph  $k$ -connectivity and strong subgraph  $k$ -arc-connectivity of digraphs. After an introductory section, the paper is divided into four sections: basic results, algorithms and complexity, sharp bounds for strong subgraph  $k$ -(arc-)connectivity, minimally strong subgraph  $(k, \ell)$ -(arc-) connected digraphs. This survey contains several conjectures and open problems for further study.

**Keywords:** Strong subgraph  $k$ -connectivity; Strong subgraph  $k$ -arc-connectivity; Subdigraph packing; Directed  $q$ -linkage; Directed weak  $q$ -linkage; Semicomplete digraphs; Symmetric digraphs; Generalized  $k$ -connectivity; Generalized  $k$ -edge-connectivity.

**AMS subject classification (2010):** 05C20, 05C35, 05C40, 05C70, 05C75, 05C76, 05C85, 68Q25, 68R10.

## 1 Introduction

The generalized  $k$ -connectivity  $\kappa_k(G)$  of a graph  $G = (V, E)$  was introduced by Hager [14] in 1985 ( $2 \leq k \leq |V|$ ). For a graph  $G = (V, E)$  and a set  $S \subseteq V$  of at least two vertices, an  $S$ -Steiner tree or, simply, an  $S$ -tree is a subgraph  $T$  of  $G$  which is a tree with  $S \subseteq V(T)$ . Two  $S$ -trees  $T_1$  and  $T_2$  are said to be *edge-disjoint* if  $E(T_1) \cap E(T_2) = \emptyset$ . Two edge-disjoint  $S$ -trees  $T_1$  and  $T_2$  are said to be *internally disjoint* if  $V(T_1) \cap V(T_2) = S$ . The *generalized local connectivity*  $\kappa_S(G)$  is the maximum number of internally disjoint  $S$ -trees in  $G$ . For an integer  $k$  with  $2 \leq k \leq n$ , the *generalized  $k$ -connectivity* is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Observe that  $\kappa_2(G) = \kappa(G)$ . Li, Mao and Sun [18] introduced the following concept of generalized  $k$ -edge-connectivity. The *generalized local edge-connectivity*  $\lambda_S(G)$  is the maximum number of edge-disjoint  $S$ -trees in  $G$ .

For an integer  $k$  with  $2 \leq k \leq n$ , the *generalized  $k$ -edge-connectivity* is defined as

$$\lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Observe that  $\lambda_2(G) = \lambda(G)$ . Generalized connectivity of graphs has become an established area in graph theory, see a recent monograph [17] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized  $k$ -connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [23] observed that in the definition of  $\kappa_S(G)$ , one can replace “an  $S$ -tree” by “a connected subgraph of  $G$  containing  $S$ .” Therefore, Sun et al. [23] defined *strong subgraph  $k$ -connectivity* by replacing “connected” with “strongly connected” (or, simply, “strong”) as follows. Let  $D = (V, A)$  be a digraph of order  $n$ ,  $S$  a subset of  $V$  of size  $k$  and  $2 \leq k \leq n$ . A subdigraph  $H$  of  $D$  is called an  *$S$ -strong subgraph* if  $H$  is strong and  $S \subseteq V(H)$ . Two  $S$ -strong subgraphs  $D_1$  and  $D_2$  are said to be *arc-disjoint* if  $A(D_1) \cap A(D_2) = \emptyset$ . Two arc-disjoint  $S$ -strong subgraphs  $D_1$  and  $D_2$  are said to be *internally disjoint* if  $V(D_1) \cap V(D_2) = S$ . Let  $\kappa_S(D)$  be the maximum number of internally disjoint  $S$ -strong subgraphs in  $D$ . The *strong subgraph  $k$ -connectivity* of  $D$  is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}.$$

By definition,  $\kappa_k(D) = 0$  if  $D$  is not strong.

As a natural counterpart of the strong subgraph  $k$ -connectivity, Sun and Gutin [22] introduced the concept of strong subgraph  $k$ -arc-connectivity. Let  $D = (V(D), A(D))$  be a digraph of order  $n$ ,  $S \subseteq V$  a  $k$ -subset of  $V(D)$  and  $2 \leq k \leq n$ . Let  $\lambda_S(D)$  be the maximum number of arc-disjoint  $S$ -strong digraphs in  $D$ . The *strong subgraph  $k$ -arc-connectivity* of  $D$  is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$

By definition,  $\lambda_k(D) = 0$  if  $D$  is not strong.

The strong subgraph  $k$ -(arc-)connectivity is not only a natural extension of the concept of generalized  $k$ -(edge-)connectivity, but also relates to important problems in graph theory. For  $k = 2$ ,  $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$  [23] and  $\lambda_2(\overleftrightarrow{G}) = \lambda(G)$  [22]. Hence,  $\kappa_k(D)$  and  $\lambda_k(D)$  could be seen as generalizations of connectivity and edge-connectivity of undirected graphs, respectively. For  $k = n$ ,  $\kappa_n(D) = \lambda_n(D)$  is the maximum number of arc-disjoint spanning strong subgraphs of  $D$ . Moreover, since  $\kappa_S(G)$  and  $\lambda_S(G)$  are the number of internally disjoint and arc-disjoint strong subgraphs containing a given set  $S$ , respectively, these parameters are relevant to the subdigraph packing problem, see [4–7, 11].

Some basic results will be introduced in Section 2. In Section 3, we will sum up the results on algorithms and computational complexity for  $\kappa_S(D)$ ,  $\kappa_k(D)$ ,  $\lambda_S(D)$  and  $\lambda_k(D)$ . We will collect many upper and lower bounds for the parameters  $\kappa_k(D)$  and  $\lambda_k(D)$  in Section 4. Finally, in Section 5, results on minimally strong subgraph  $(k, \ell)$ -(arc-)connected digraphs will be surveyed.

**Additional Terminology and Notation.** For a digraph  $D$ , its *reverse*  $D^{\text{rev}}$  is a digraph with same vertex set and such that  $xy \in A(D^{\text{rev}})$  if and only if  $yx \in A(D)$ . A digraph  $D$  is *symmetric* if  $D^{\text{rev}} = D$ . In other words, a symmetric digraph  $D$  can be obtained from its underlying undirected graph  $G$  by replacing each edge of  $G$  with the corresponding arcs of both directions, that is,  $D = \overleftrightarrow{G}$ . A 2-cycle  $xyx$  of a strong digraph  $D$  is called a *bridge* if  $D - \{xy, yx\}$  is disconnected. Thus, a bridge corresponds to a bridge in the underlying undirected graph of  $D$ . An *orientation* of a digraph  $D$  is a digraph obtained from  $D$  by deleting an arc in each 2-cycle of  $D$ . A digraph  $D$  is *semicomplete* if for every distinct  $x, y \in V(D)$  at least one of the arcs  $xy, yx$  is in  $D$ . We refer the readers to [2, 3, 9] for graph theoretical notation and terminology not given here.

## 2 Basic Results

The following proposition can be easily verified using definitions of  $\lambda_k(D)$  and  $\kappa_k(D)$ .

**Proposition 2.1** [22, 23] *Let  $D$  be a digraph of order  $n$ , and let  $k \geq 2$  be an integer. Then*

$$\lambda_{k+1}(D) \leq \lambda_k(D) \text{ for every } k \leq n-1 \quad (1)$$

$$\kappa_k(D') \leq \kappa_k(D), \lambda_k(D') \leq \lambda_k(D) \text{ where } D' \text{ is a spanning subdigraph of } D \quad (2)$$

$$\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\} \quad (3)$$

By Tillson's decomposition theorem [26], we can determine the exact values for  $\kappa_k(\overleftrightarrow{K}_n)$  and  $\lambda_k(\overleftrightarrow{K}_n)$ .

**Proposition 2.2** [23] *For  $2 \leq k \leq n$ , we have*

$$\kappa_k(\overleftrightarrow{K}_n) = \begin{cases} n-1, & \text{if } k \notin \{4, 6\}; \\ n-2, & \text{otherwise.} \end{cases}$$

**Proposition 2.3** [22] *For  $2 \leq k \leq n$ , we have*

$$\lambda_k(\overleftrightarrow{K}_n) = \begin{cases} n-1, & \text{if } k \notin \{4, 6\}, \text{ or, } k \in \{4, 6\} \text{ and } k < n; \\ n-2, & \text{if } k = n \in \{4, 6\}. \end{cases}$$

**Proposition 2.4** [22] *For every fixed  $k \geq 2$ , a digraph  $D$  is strong if and only if  $\lambda_k(D) \geq 1$ .*

## 3 Algorithms and Complexity

### 3.1 Results for $\kappa_S(D)$ and $\kappa_k(D)$

For a fixed  $k \geq 2$ , it is easy to decide whether  $\kappa_k(D) \geq 1$  for a digraph  $D$ : it holds if and only if  $D$  is strong. Unfortunately, deciding whether

$\kappa_S(D) \geq 2$  is already NP-complete for  $S \subseteq V(D)$  with  $|S| = k$ , where  $k \geq 2$  is a fixed integer.

The well-known DIRECTED  $q$ -LINKAGE problem was proved to be NP-complete even for the case that  $q = 2$  [13]. The problem is formulated as follows: for a fixed integer  $q \geq 2$ , given a digraph  $D$  and a (terminal) sequence  $((s_1, t_1), \dots, (s_q, t_q))$  of distinct vertices of  $D$ , decide whether  $D$  has  $q$  vertex-disjoint paths  $P_1, \dots, P_q$ , where  $P_i$  starts at  $s_i$  and ends at  $t_i$  for all  $i \in [q]$ .

By using the reduction from the DIRECTED  $q$ -LINKAGE problem, we can prove the following intractability result.

**Theorem 3.1** [23] *Let  $k \geq 2$  and  $\ell \geq 2$  be fixed integers. Let  $D$  be a digraph and  $S \subseteq V(D)$  with  $|S| = k$ . The problem of deciding whether  $\kappa_S(D) \geq \ell$  is NP-complete.*

In the above theorem, Sun et al. obtained the complexity result of the parameter  $\kappa_S(D)$  for an arbitrary digraph  $D$ . For  $\kappa_k(D)$ , they made the following conjecture.

**Conjecture 1** [23] *It is NP-complete to decide for fixed integers  $k \geq 2$  and  $\ell \geq 2$  and a given digraph  $D$  whether  $\kappa_k(D) \geq \ell$ .*

Recently, Chudnovsky, Scott and Seymour [12] proved the following powerful result.

**Theorem 3.2** [12] *Let  $q$  and  $c$  be fixed positive integers. Then the DIRECTED  $q$ -LINKAGE problem on a digraph  $D$  whose vertex set can be partitioned into  $c$  sets each inducing a semicomplete digraph and a terminal sequence  $((s_1, t_1), \dots, (s_q, t_q))$  of distinct vertices of  $D$ , can be solved in polynomial time.*

The following nontrivial lemma can be deduced from Theorem 3.2.

**Lemma 3.3** [23] *Let  $k$  and  $\ell$  be fixed positive integers. Let  $D$  be a digraph and let  $X_1, X_2, \dots, X_\ell$  be  $\ell$  vertex disjoint subsets of  $V(D)$ , such that  $|X_i| \leq k$  for all  $i \in [\ell]$ . Let  $X = \cup_{i=1}^\ell X_i$  and assume that every vertex in  $V(D) \setminus X$  is adjacent to every other vertex in  $D$ . Then we can in polynomial time decide if there exists vertex disjoint subsets  $Z_1, Z_2, \dots, Z_\ell$  of  $V(D)$ , such that  $X_i \subseteq Z_i$  and  $D[Z_i]$  is strongly connected for each  $i \in [\ell]$ .*

Using Lemma 3.3, Sun, Gutin, Yeo and Zhang proved the following result for semicomplete digraphs.

**Theorem 3.4** [23] *For any fixed integers  $k, \ell \geq 2$ , we can decide whether  $\kappa_k(D) \geq \ell$  for a semicomplete digraph  $D$  in polynomial time.*

Now we turn our attention to symmetric graphs. We start with the following structural result.

**Theorem 3.5** [23] *For every undirected graph  $G$  we have  $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$ .*

Theorem 3.5 immediately implies the following positive result, which follows from the fact that  $\kappa(G)$  can be computed in polynomial time.

**Corollary 3.6** [23] *For a graph  $G$ ,  $\kappa_2(\overleftrightarrow{G})$  can be computed in polynomial time.*

Theorem 3.5 states that  $\kappa_k(\overleftrightarrow{G}) = \kappa_k(G)$  when  $k = 2$ . However when  $k \geq 3$ , then  $\kappa_k(\overleftrightarrow{G})$  is not always equal to  $\kappa_k(G)$ , as can be seen from  $\kappa_3(\overleftrightarrow{K_3}) = 2 \neq 1 = \kappa_3(K_3)$ . Chen, Li, Liu and Mao [10] introduced the following problem, which they proved to be NP-complete.

**CLLM PROBLEM:** Given a tripartite graph  $G = (V, E)$  with a 3-partition  $(\overline{U}, \overline{V}, \overline{W})$  such that  $|\overline{U}| = |\overline{V}| = |\overline{W}| = q$ , decide whether there is a partition of  $V$  into  $q$  disjoint 3-sets  $V_1, \dots, V_q$  such that for every  $V_i = \{v_{i_1}, v_{i_2}, v_{i_3}\}$   $v_{i_1} \in \overline{U}, v_{i_2} \in \overline{V}, v_{i_3} \in \overline{W}$  and  $G[V_i]$  is connected.

**Lemma 3.7** [10] *The CLLM Problem is NP-complete.*

Now restricted to symmetric digraphs  $D$ , for any fixed integer  $k \geq 3$ , by Lemma 3.7, the problem of deciding whether  $\kappa_S(D) \geq \ell$  ( $\ell \geq 1$ ) is NP-complete for  $S \subseteq V(D)$  with  $|S| = k$ .

**Theorem 3.8** [23] *For any fixed integer  $k \geq 3$ , given a symmetric digraph  $D$ , a  $k$ -subset  $S$  of  $V(D)$  and an integer  $\ell$  ( $\ell \geq 1$ ), deciding whether  $\kappa_S(D) \geq \ell$ , is NP-complete.*

The last theorem assumes that  $k$  is fixed but  $\ell$  is a part of input. When both  $k$  and  $\ell$  are fixed, the problem of deciding whether  $\kappa_S(D) \geq \ell$  for a symmetric digraph  $D$ , is polynomial-time solvable. We will start with the following technical lemma.

**Lemma 3.9** [23] *Let  $k, \ell \geq 2$  be fixed. Let  $G$  be a graph and let  $S \subseteq V(G)$  be an independent set in  $G$  with  $|S| = k$ . For  $i \in [\ell]$ , let  $D_i$  be any set of arcs with both end-vertices in  $S$ . Let a forest  $F_i$  in  $G$  be called  $(S, D_i)$ -acceptable if the digraph  $\overleftrightarrow{F_i} + D_i$  is strong and contains  $S$ . In polynomial time, we can decide whether there exists edge-disjoint forests  $F_1, F_2, \dots, F_\ell$  such that  $F_i$  is  $(S, D_i)$ -acceptable for all  $i \in [\ell]$  and  $V(F_i) \cap V(F_j) \subseteq S$  for all  $1 \leq i < j \leq \ell$ .*

Now we can prove the following result by Lemma 3.9:

**Theorem 3.10** [23] *Let  $k, \ell \geq 2$  be fixed. For any symmetric digraph  $D$  and  $S \subseteq V(D)$  with  $|S| = k$  we can in polynomial time decide whether  $\kappa_S(D) \geq \ell$ .*

The DIRECTED  $q$ -LINKAGE problem is polynomial-time solvable for planar digraphs [19] and digraphs of bounded directed treewidth [16]. However, it seems that we cannot use the approach in proving Theorem 3.4 directly as the structure of minimum-size strong subgraphs in these two classes of digraphs is more complicated than in semicomplete digraphs. Certainly, we cannot exclude the possibility that computing strong subgraph  $k$ -connectivity in planar digraphs and/or in digraphs of bounded directed treewidth is NP-complete.

**Problem 3.11** [23] *What is the complexity of deciding whether  $\kappa_k(D) \geq \ell$  for fixed integers  $k \geq 2$ , and  $\ell \geq 2$  and a given planar digraph  $D$ ?*

**Problem 3.12** [23] *What is the complexity of deciding whether  $\kappa_k(D) \geq \ell$  for fixed integers  $k \geq 2$ , and  $\ell \geq 2$  and a digraph  $D$  of bounded directed treewidth?*

It would be interesting to identify large classes of digraphs for which the  $\kappa_k(D) \geq \ell$  problem can be decided in polynomial time.

### 3.2 Results for $\lambda_S(D)$ and $\lambda_k(D)$

Yeo proved that it is an NP-complete problem to decide whether a 2-regular digraph has two arc-disjoint hamiltonian cycles (see, e.g., Theorem 6.6 in [6]). (A digraph is 2-regular if the out-degree and in-degree of every vertex equals 2.) Thus, the problem of deciding whether  $\lambda_n(D) \geq 2$  is NP-complete, where  $n$  is the order of  $D$ . Sun and Gutin [22] extended this result in Theorem 3.13.

Let  $D$  be a digraph and let  $s_1, s_2, \dots, s_q, t_1, t_2, \dots, t_q$  be a collection of not necessarily distinct vertices of  $D$ . A *weak  $q$ -linkage* from  $(s_1, s_2, \dots, s_q)$  to  $(t_1, t_2, \dots, t_q)$  is a collection of  $q$  arc-disjoint paths  $P_1, \dots, P_q$  such that  $P_i$  is an  $(s_i, t_i)$ -path for each  $i \in [q]$ . A digraph  $D = (V, A)$  is *weakly  $q$ -linked* if it contains a weak  $q$ -linkage from  $(s_1, s_2, \dots, s_q)$  to  $(t_1, t_2, \dots, t_q)$  for every choice of (not necessarily distinct) vertices  $s_1, \dots, s_q, t_1, \dots, t_q$ . The DIRECTED WEAK  $q$ -LINKAGE problem is the following. Given a digraph  $D = (V, A)$  and distinct vertices  $x_1, x_2, \dots, x_q, y_1, y_2, \dots, y_q$ ; decide whether  $D$  contains  $q$  arc-disjoint paths  $P_1, \dots, P_q$  such that  $P_i$  is an  $(x_i, y_i)$ -path. The problem is well-known to be NP-complete already for  $q = 2$  [13]. By using the reduction from the DIRECTED WEAK  $q$ -LINKAGE problem, we can prove the following intractability result.

**Theorem 3.13** [22] *Let  $k \geq 2$  and  $\ell \geq 2$  be fixed integers. Let  $D$  be a digraph and  $S \subseteq V(D)$  with  $|S| = k$ . The problem of deciding whether  $\lambda_S(D) \geq \ell$  is NP-complete.*

Bang-Jensen and Yeo [6] conjectured the following:

**Conjecture 2** *For every  $\lambda \geq 2$  there is a finite set  $\mathcal{S}_\lambda$  of digraphs such that  $\lambda$ -arc-strong semicomplete digraph  $D$  contains  $\lambda$  arc-disjoint spanning strong subgraphs unless  $D \in \mathcal{S}_\lambda$ .*

Bang-Jensen and Yeo [6] proved the conjecture for  $\lambda = 2$  by showing that  $|\mathcal{S}_2| = 1$  and describing the unique digraph  $S_4$  of  $\mathcal{S}_2$  of order 4. This result and Theorem 4.4 imply the following:

**Theorem 3.14** [22] *For a semicomplete digraph  $D$ , of order  $n$  and an integer  $k$  such that  $2 \leq k \leq n$ ,  $\lambda_k(D) \geq 2$  if and only if  $D$  is 2-arc-strong and  $D \not\cong S_4$ .*

Now we turn our attention to symmetric graphs. We start from characterizing symmetric digraphs  $D$  with  $\lambda_k(D) \geq 2$ , an analog of Theorem 3.14. To prove it we need the following result of Boesch and Tindell [8] translated from the language of mixed graphs to that of digraphs.

**Theorem 3.15** *A strong digraph  $D$  has a strong orientation if and only if  $D$  has no bridge.*

Here is the characterization by Sun and Gutin.

**Theorem 3.16** [22] *For a strong symmetric digraph  $D$  of order  $n$  and an integer  $k$  such that  $2 \leq k \leq n$ ,  $\lambda_k(D) \geq 2$  if and only if  $D$  has no bridge.*

Theorems 3.14 and 3.16 imply the following complexity result, which we believe to be extendable from  $\ell = 2$  to any natural  $\ell \geq 2$ .

**Corollary 3.17** [22] *The problem of deciding whether  $\lambda_k(D) \geq 2$  is polynomial-time solvable if  $D$  is either semicomplete or symmetric digraph of order  $n$  and  $2 \leq k \leq n$ .*

Sun and Gutin gave a lower bound on  $\lambda_k(D)$  for symmetric digraphs  $D$ .

**Theorem 3.18** [22] *For every graph  $G$ , we have*

$$\lambda_k(\overleftrightarrow{G}) \geq \lambda_k(G).$$

*Moreover, this bound is sharp. In particular, we have  $\lambda_2(\overleftrightarrow{G}) = \lambda_2(G)$ .*

Theorem 3.18 immediately implies the next result, which follows from the fact that  $\lambda(G)$  can be computed in polynomial time.

**Corollary 3.19** [22] *For a symmetric digraph  $D$ ,  $\lambda_2(D)$  can be computed in polynomial time.*

Corollaries 3.17 and 3.19 shed some light on the complexity of deciding, for fixed  $k, \ell \geq 2$ , whether  $\lambda_k(D) \geq \ell$  for semicomplete and symmetric digraphs  $D$ . However, it is unclear what is the complexity above for every fixed  $k, \ell \geq 2$ . If Conjecture 2 is correct, then the  $\lambda_k(D) \geq \ell$  problem can be solved in polynomial time for semicomplete digraphs. However, Conjecture 2 seems to be very difficult. It was proved in [23] that for fixed  $k, \ell \geq 2$  the problem of deciding whether  $\kappa_k(D) \geq \ell$  is polynomial-time solvable for both semicomplete and symmetric digraphs, but it appears that the approaches to prove the two results cannot be used for  $\lambda_k(D)$ . Some well-known results such as the fact that the hamiltonicity problem is NP-complete for undirected 3-regular graphs, indicate that the  $\lambda_k(D) \geq \ell$  problem for symmetric digraphs may be NP-complete, too.

**Problem 3.20** [22] *What is the complexity of deciding whether  $\lambda_k(D) \geq \ell$  for fixed integers  $k \geq 2$  and  $\ell \geq 2$ , and a semicomplete digraph  $D$ ?*

**Problem 3.21** [22] *What is the complexity of deciding whether  $\lambda_k(D) \geq \ell$  for fixed integers  $k \geq 2$  and  $\ell \geq 2$ , and a symmetric digraph  $D$ ?*

It would be interesting to identify large classes of digraphs for which the  $\lambda_k(D) \geq \ell$  problem can be decided in polynomial time.

## 4 Bounds for Strong Subgraph $k$ -(Arc-)Connectivity

### 4.1 Results for $\kappa_k(D)$

By Propositions 2.1 and 2.2, Sun, Gutin, Yeo and Zhang obtained a sharp lower bound and a sharp upper bound for  $\kappa_k(D)$ , where  $2 \leq k \leq n$ .

**Theorem 4.1** [23] *Let  $2 \leq k \leq n$ . For a strong digraph  $D$  of order  $n$ , we have*

$$1 \leq \kappa_k(D) \leq n - 1.$$

*Moreover, both bounds are sharp, and the upper bound holds if and only if  $D \cong \overleftrightarrow{K}_n$ ,  $2 \leq k \leq n$  and  $k \notin \{4, 6\}$ .*

Sun and Gutin gave the following sharp upper bound for  $\kappa_k(D)$  which improves (3) of Proposition 2.1.

**Theorem 4.2** [21] *For  $k \in \{2, \dots, n\}$  and  $n \geq \kappa(D) + k$ , we have*

$$\kappa_k(D) \leq \kappa(D).$$

*Moreover, the bound is sharp.*

### 4.2 Results for $\lambda_k(D)$

By Propositions 2.1 and 2.2, Sun and Gutin obtained a sharp lower bound and a sharp upper bound for  $\lambda_k(D)$ , where  $2 \leq k \leq n$ .

**Theorem 4.3** [22] *Let  $2 \leq k \leq n$ . For a strong digraph  $D$  of order  $n$ , we have*

$$1 \leq \lambda_k(D) \leq n - 1.$$

*Moreover, both bounds are sharp, and the upper bound holds if and only if  $D \cong \overleftrightarrow{K}_n$ , where  $k \notin \{4, 6\}$ , or,  $k \in \{4, 6\}$  and  $k < n$ .*

They also gave the following sharp upper bound for  $\lambda_k(D)$  which improves (3) of Proposition 2.1.

**Theorem 4.4** [22] *For  $2 \leq k \leq n$ , we have*

$$\lambda_k(D) \leq \lambda(D).$$

*Moreover, the bound is sharp.*

Shiloach [20] proved the following:

**Theorem 4.5** [20] *A digraph  $D$  is weakly  $k$ -linked if and only if  $D$  is  $k$ -arc-strong.*

Using Shiloach's Theorem, Sun and Gutin [22] proved the following lower bound for  $\lambda_k(D)$ . Such a bound does not hold for  $\kappa_k(D)$  since it was shown in [23] using Thomassen's result in [25] that for every  $\ell$  there are digraphs  $D$  with  $\kappa(D) = \ell$  and  $\kappa_2(D) = 1$ .



**Proposition 4.6** [22] *Let  $k \leq \ell = \lambda(D)$ . We have  $\lambda_k(D) \geq \lfloor \ell/k \rfloor$ .*

For a digraph  $D = (V(D), A(D))$ , the *complement digraph*, denoted by  $D^c$ , is a digraph with vertex set  $V(D^c) = V(D)$  such that  $xy \in A(D^c)$  if and only if  $xy \notin A(D)$ .

Given a graph parameter  $f(G)$ , the Nordhaus-Gaddum Problem is to determine sharp bounds for (1)  $f(G) + f(G^c)$  and (2)  $f(G)f(G^c)$ , and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide attention; see a recent survey paper [1] by Aouchiche and Hansen. By using Proposition 2.4, the following Theorem 4.7 concerning such type of a problem for the parameter  $\lambda_k$  can be obtained.

**Theorem 4.7** [22] *For a digraph  $D$  with order  $n$ , the following assertions holds:*

- (i)  $0 \leq \lambda_k(D) + \lambda_k(D^c) \leq n - 1$ . Moreover, both bounds are sharp. In particular, the lower bound holds if and only if  $\lambda(D) = \lambda(D^c) = 0$ .
- (ii)  $0 \leq \lambda_k(D)\lambda_k(D^c) \leq (\frac{n-1}{2})^2$ . Moreover, both bounds are sharp. In particular, the lower bound holds if and only if  $\lambda(D) = 0$  or  $\lambda(D^c) = 0$ .

We now discuss Cartesian products of digraphs. The *Cartesian product*  $G \square H$  of two digraphs  $G$  and  $H$  is a digraph with vertex set

$$V(G \square H) = V(G) \times V(H) = \{(x, x') \mid x \in V(G), x' \in V(H)\}$$

and arc set

$$A(G \square H) = \{(x, x')(y, y') \mid xy \in A(G), x' = y', \text{ or } x = y, x'y' \in A(H)\}.$$

By definition, we know the Cartesian product is associative and commutative, and  $G \square H$  is strongly connected if and only if both  $G$  and  $H$  are strongly connected [15].

**Theorem 4.8** [22] *Let  $G$  and  $H$  be two digraphs. We have*

$$\lambda_2(G \square H) \geq \lambda_2(G) + \lambda_2(H).$$

*Moreover, the bound is sharp.*

By Proposition 2.1 and Theorem 4.8, we can obtain precise values for the strong subgraph 2-arc-connectivity of the Cartesian product of some special digraphs, as shown in the Table. Note that  $\overleftrightarrow{T}_m$  is the symmetric digraph whose underlying undirected graph is a tree of order  $m$ .

## 5 Minimally Strong Subgraph $(k, \ell)$ -(Arc-)Connected Digraphs

### 5.1 Results for Minimally Strong Subgraph $(k, \ell)$ -Connected Digraphs

A digraph  $D = (V(D), A(D))$  is called *minimally strong subgraph  $(k, \ell)$ -connected* if  $\kappa_k(D) \geq \ell$  but for any arc  $e \in A(D)$ ,  $\kappa_k(D - e) \leq \ell - 1$  [21]. By

	$\vec{C}_m$	$\overleftarrow{C}_m$	$\overleftrightarrow{T}_m$	$\overleftrightarrow{K}_m$
$\vec{C}_n$	2	3	2	$m$
$\overleftarrow{C}_n$	3	4	3	$m + 1$
$\overleftrightarrow{T}_n$	2	3	2	$m$
$\overleftrightarrow{K}_n$	$n$	$n + 1$	$n$	$n + m - 2$

Table 1. Precise values for the strong subgraph 2-arc-connectivity of some special cases.

the definition of  $\kappa_k(D)$  and Theorem 4.1, we know  $2 \leq k \leq n, 1 \leq \ell \leq n - 1$ . Let  $\mathfrak{F}(n, k, \ell)$  be the set of all minimally strong subgraph  $(k, \ell)$ -connected digraphs with order  $n$ . We define

$$F(n, k, \ell) = \max\{|A(D)| \mid D \in \mathfrak{F}(n, k, \ell)\}$$

and

$$f(n, k, \ell) = \min\{|A(D)| \mid D \in \mathfrak{F}(n, k, \ell)\}.$$

We further define

$$Ex(n, k, \ell) = \{D \mid D \in \mathfrak{F}(n, k, \ell), |A(D)| = F(n, k, \ell)\}$$

and

$$ex(n, k, \ell) = \{D \mid D \in \mathfrak{F}(n, k, \ell), |A(D)| = f(n, k, \ell)\}.$$

By the definition of a minimally strong subgraph  $(k, \ell)$ -connected digraph, we can get the following observation.

**Proposition 5.1** [21] *A digraph  $D$  is minimally strong subgraph  $(k, \ell)$ -connected if and only if  $\kappa_k(D) = \ell$  and  $\kappa_k(D - e) = \ell - 1$  for any arc  $e \in A(D)$ .*

A digraph  $D$  is *minimally strong* if  $D$  is strong but  $D - e$  is not for every arc  $e$  of  $D$ .

**Proposition 5.2** [21] *The following assertions hold:*

- (i) *A digraph  $D$  is minimally strong subgraph  $(k, 1)$ -connected if and only if  $D$  is minimally strong digraph;*
- (ii) *For  $k \neq 4, 6$ , a digraph  $D$  is minimally strong subgraph  $(k, n - 1)$ -connected if and only if  $D \cong \overleftrightarrow{K}_n$ .*

The following result characterizes minimally strong subgraph  $(2, n - 2)$ -connected digraphs.

**Theorem 5.3** [21] *A digraph  $D$  is minimally strong subgraph  $(2, n-2)$ -connected if and only if  $D$  is a digraph obtained from the complete digraph  $\overleftrightarrow{K}_n$  by deleting an arc set  $M$  such that  $\overleftrightarrow{K}_n[M]$  is a 3-cycle or a union of  $\lfloor n/2 \rfloor$  vertex-disjoint 2-cycles. In particular, we have  $f(n, 2, n-2) = n(n-1) - 2\lfloor n/2 \rfloor$ ,  $F(n, 2, n-2) = n(n-1) - 3$ .*

Note that Theorem 5.3 implies that  $Ex(n, 2, n-2) = \{\overleftrightarrow{K}_n - M\}$  where  $M$  is an arc set such that  $\overleftrightarrow{K}_n[M]$  is a directed 3-cycle, and  $ex(n, 2, n-1) = \{\overleftrightarrow{K}_n - M\}$  where  $M$  is an arc set such that  $\overleftrightarrow{K}_n[M]$  is a union of  $\lfloor n/2 \rfloor$  vertex-disjoint directed 2-cycles.

The following result concerns a sharp lower bound for the parameter  $f(n, k, \ell)$ .

**Theorem 5.4** [21] *For  $2 \leq k \leq n$ , we have*

$$f(n, k, \ell) \geq n\ell.$$

Moreover, the following assertions hold:

(i) If  $\ell = 1$ , then  $f(n, k, \ell) = n$ ; (ii) If  $2 \leq \ell \leq n-1$ , then  $f(n, n, \ell) = n\ell$  for  $k = n \notin \{4, 6\}$ ; (iii) If  $n$  is even and  $\ell = n-2$ , then  $f(n, 2, \ell) = n\ell$ .

To prove two upper bounds on the number of arcs in a minimally strong subgraph  $(k, \ell)$ -connected digraph, Sun and Gutin used the following result, see e.g. [2].

**Theorem 5.5** *Every strong digraph  $D$  on  $n$  vertices has a strong spanning subgraph  $H$  with at most  $2n-2$  arcs and equality holds only if  $H$  is a symmetric digraph whose underlying undirected graph is a tree.*

**Proposition 5.6** [21] *We have (i)  $F(n, n, \ell) \leq 2\ell(n-1)$ ; (ii) For every  $k$  ( $2 \leq k \leq n$ ),  $F(n, k, 1) = 2(n-1)$  and  $Ex(n, k, 1)$  consists of symmetric digraphs whose underlying undirected graphs are trees.*

The minimally strong subgraph  $(2, n-2)$ -connected digraphs was characterized in Theorem 5.3. As a simple consequence of the characterization, we can determine the values of  $f(n, 2, n-2)$  and  $F(n, 2, n-2)$ . It would be interesting to determine  $f(n, k, n-2)$  and  $F(n, k, n-2)$  for every value of  $k \geq 3$  since obtaining characterizations of all  $(k, n-2)$ -connected digraphs for  $k \geq 3$  seems a very difficult problem.

**Problem 5.7** [21] *Determine  $f(n, k, n-2)$  and  $F(n, k, n-2)$  for every value of  $k \geq 3$ .*

It would also be interesting to find a sharp upper bound for  $F(n, k, \ell)$  for all  $k \geq 2$  and  $\ell \geq 2$ .

**Problem 5.8** [21] *Find a sharp upper bound for  $F(n, k, \ell)$  for all  $k \geq 2$  and  $\ell \geq 2$ .*

## 5.2 Results for Minimally Strong Subgraph $(k, \ell)$ -Arc-Connected Digraphs

A digraph  $D = (V(D), A(D))$  is called *minimally strong subgraph  $(k, \ell)$ -arc-connected* if  $\lambda_k(D) \geq \ell$  but for any arc  $e \in A(D)$ ,  $\lambda_k(D - e) \leq \ell - 1$ . By the definition of  $\lambda_k(D)$  and Theorem 4.3, we know  $2 \leq k \leq n$ ,  $1 \leq \ell \leq n - 1$ . Let  $\mathfrak{G}(n, k, \ell)$  be the set of all minimally strong subgraph  $(k, \ell)$ -arc-connected digraphs with order  $n$ . We define

$$G(n, k, \ell) = \max\{|A(D)| \mid D \in \mathfrak{G}(n, k, \ell)\}$$

and

$$g(n, k, \ell) = \min\{|A(D)| \mid D \in \mathfrak{G}(n, k, \ell)\}.$$

We further define

$$Ex'(n, k, \ell) = \{D \mid D \in \mathfrak{G}(n, k, \ell), |A(D)| = G(n, k, \ell)\}$$

and

$$ex'(n, k, \ell) = \{D \mid D \in \mathfrak{G}(n, k, \ell), |A(D)| = g(n, k, \ell)\}.$$

Sun and Gutin [22] gave the following characterizations.

**Proposition 5.9** [22] *The following assertions hold:*

- (i) *A digraph  $D$  is minimally strong subgraph  $(k, 1)$ -arc-connected if and only if  $D$  is minimally strong digraph;*
- (ii) *Let  $2 \leq k \leq n$ . If  $k \notin \{4, 6\}$ , or,  $k \in \{4, 6\}$  and  $k < n$ , then a digraph  $D$  is minimally strong subgraph  $(k, n-1)$ -arc-connected if and only if  $D \cong \overleftrightarrow{K}_n$ .*

**Theorem 5.10** [22] *A digraph  $D$  is minimally strong subgraph  $(2, n-2)$ -arc-connected if and only if  $D$  is a digraph obtained from the complete digraph  $\overleftrightarrow{K}_n$  by deleting an arc set  $M$  such that  $\overleftrightarrow{K}_n[M]$  is a union of vertex-disjoint cycles which cover all but at most one vertex of  $\overleftrightarrow{K}_n$ .*

Sun and Jin characterized the minimally strong subgraph  $(3, n-2)$ -arc-connected digraphs.

**Theorem 5.11** [24] *A digraph  $D$  is minimally strong subgraph  $(3, n-2)$ -arc-connected if and only if  $D$  is a digraph obtained from the complete digraph  $\overleftrightarrow{K}_n$  by deleting an arc set  $M$  such that  $\overleftrightarrow{K}_n[M]$  is a union of vertex-disjoint cycles which cover all but at most one vertex of  $\overleftrightarrow{K}_n$ .*

Theorems 5.10 and 5.11 imply that the following assertions hold: (i) For  $k \in \{2, 3\}$ ,  $Ex'(n, k, n-2) = \{\overleftrightarrow{K}_n - M\}$  where  $M$  is an arc set such that  $\overleftrightarrow{K}_n[M]$  is a union of vertex-disjoint cycles which cover all but exactly one vertex of  $\overleftrightarrow{K}_n$ . (ii) For  $k \in \{2, 3\}$ ,  $ex'(n, k, n-2) = \{\overleftrightarrow{K}_n - M\}$  where  $M$  is an arc set such that  $\overleftrightarrow{K}_n[M]$  is a union of vertex-disjoint cycles which cover all vertices of  $\overleftrightarrow{K}_n$ .

Sun and Jin completely determined the precise value for  $g(n, k, \ell)$ . Note that  $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$  by Theorem 4.3 and the definition of  $g(n, k, \ell)$ .

**Theorem 5.12** [24] *For any triple  $(n, k, \ell)$  with  $2 \leq k \leq n, 1 \leq \ell \leq n - 1$  such that  $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$ , we have*

$$g(n, k, \ell) = n\ell.$$

Some results for  $G(n, k, \ell)$  were obtained as well.

**Proposition 5.13** [24] *We have (i)  $G(n, n, \ell) \leq 2\ell(n - 1)$ ; (ii) For every  $k$  ( $2 \leq k \leq n$ ),  $G(n, k, 1) = 2(n - 1)$  and  $Ex'(n, k, 1)$  consists of symmetric digraphs whose underlying undirected graphs are trees; (iii)  $G(n, k, n - 2) = (n - 1)^2$  for  $k \in \{2, 3\}$ .*

Note that the precise values of  $g(n, k, \ell)$  for each pair of  $k$  and  $\ell$  and the precise values of  $G(n, k, n - 2)$  for  $k \in \{2, 3\}$  were determined. Hence, similar to problems 5.7 and 5.8, the following problems are also interesting.

**Problem 5.14** [24] *Determine  $G(n, k, n - 2)$  for every value of  $k \geq 2$ .*

**Problem 5.15** [24] *Find a sharp upper bound for  $G(n, k, \ell)$  for all  $k \geq 2$  and  $\ell \geq 2$ .*

**Acknowledgements.** Yuefang Sun was supported by National Natural Science Foundation of China (No.11401389) and China Scholarship Council (No.201608330111). Gregory Gutin was partially supported by Royal Society Wolfson Research Merit Award.

## References

- [1] M. Aouchiche, P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete Appl. Math.* 161(4/5), 2013, 466–546.
- [2] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, 2nd Edition, Springer, London, 2009.
- [3] J. Bang-Jensen and G. Gutin, Basic Terminology, Notation and Results, in *Classes of Directed Graphs* (J. Bang-Jensen and G. Gutin, eds.), Springer, 2018.
- [4] J. Bang-Jensen and J. Huang, Decomposing locally semicomplete digraphs into strong spanning subdigraphs, *J. Combin. Theory Ser. B* 102, 2012, 701–714.
- [5] J. Bang-Jensen and M. Kriesell, Disjoint sub(di)graphs in digraphs, *Electron. Notes Discrete Math.* 34, 2009, 179–183.
- [6] J. Bang-Jensen and A. Yeo, Decomposing  $k$ -arc-strong tournaments into strong spanning subdigraphs, *Combinatorica* 24(3), 2004, 331–349.
- [7] J. Bang-Jensen and A. Yeo, Arc-disjoint spanning sub(di)graphs in Digraphs, *Theoret. Comput. Sci.* 438, 2012, 48–54.

- [8] F. Boesch and R. Tindell, Robbins' theorem for mixed multigraphs, Amer. Math. Monthly 87, 1980, 716–719.
- [9] J.A. Bondy and U.S.R. Murty, Graph Theory, Springer, Berlin, 2008.
- [10] L. Chen, X. Li, M. Liu and Y. Mao, A solution to a conjecture on the generalized connectivity of graphs, J. Combin. Optim. 33(1), 2017, 275–282.
- [11] J. Cheriyan and M.R. Salavatipour, Hardness and approximation results for packing Steiner trees, Algorithmica 45, 2006, 21–43.
- [12] M. Chudnovsky, A. Scott and P.D. Seymour. Disjoint paths in unions of tournaments. arXiv:1604.02317, April 2016.
- [13] S. Fortune, J. Hopcroft and J. Wyllie, The directed subgraphs homeomorphism problem, Theoret. Comput. Sci. 10, 1980, 111–121.
- [14] M. Hager, Pendant tree-connectivity, J. Combin. Theory Ser. B 38, 1985, 179–189.
- [15] R.H. Hammack, Digraphs Products, in J. Bang-Jensen and G. Gutin (eds.), *Classes of Directed Graphs*, Springer, 2018.
- [16] T. Johnson, N. Robertson, P.D. Seymour and R. Thomas, Directed Tree-Width, J. Combin. Th. Ser. B 82(1), 2001, 138–154.
- [17] X. Li and Y. Mao, Generalized Connectivity of Graphs, Springer, Switzerland, 2016.
- [18] X. Li, Y. Mao and Y. Sun, On the generalized (edge-)connectivity of graphs, Australas. J. Combin. 58(2), 2014, 304–319.
- [19] A. Schrijver, Finding  $k$  partially disjoint paths in a directed planar graph. SIAM J. Comput. 23(4), 1994, 780–788.
- [20] Y. Shiloach, Edge-disjoint branching in directed multigraphs, Inf. Process. Lett. 8(1), 1979, 24–27.
- [21] Y. Sun, G. Gutin, Strong subgraph  $k$ -connectivity bounds, arXiv:1803.00281v1 [cs.DM] 1 Mar 2018.
- [22] Y. Sun, G. Gutin, Strong subgraph  $k$ -arc-connectivity, arXiv:1805.01687v1 [cs.DM] 4 May 2018.
- [23] Y. Sun, G. Gutin, A. Yeo, X. Zhang, Strong subgraph  $k$ -connectivity, arXiv:1803.00284v1 [cs.DM] 1 Mar 2018.
- [24] Y. Sun, Z. Jin, Minimally strong subgraph  $(k, \ell)$ -arc-connected digraphs, in preparation.
- [25] C. Thomassen, Highly connected non-2-linked digraphs, Combinatorica 11(4), 1991, 393–395.
- [26] T.W. Tillson, A Hamiltonian decomposition of  $K_{2m}^*$ ,  $2m \geq 8$ , J. Combin. Theory Ser. B 29(1), 1980, 68–74.