The g-extra edge-connectivity of balanced hypercubes

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Abstract The g-extra edge-connectivity is an important measure for the reliability of interconnection networks. Recently, Yang et al. [Appl. Math. Comput. 320 (2018) 464–473] determined the 3-extra edge-connectivity of balanced hypercubes BH_n and conjectured that the g-extra edge-connectivity of BH_n is $\lambda_g(BH_n) = 2(g+1)n - 4g + 4$ for $2 \leq g \leq 2n-1$. In this paper, we confirm their conjecture for $n \geq 6 - \frac{12}{g+1}$ and $2 \leq g \leq 8$, and disprove their conjecture for $n \geq \frac{3e_g(BH_n)}{g+1}$ and $9 \leq g \leq 2n-1$, where $e_g(BH_n) = \max\{|E(BH_n[U])| \mid U \subseteq V(BH_n), |U| = g+1\}$.

Keywords balanced hypercube, g-extra edge-connectivity, reliability evaluation

1 Introduction

The topology of interconnection networks can be modeled by a graph $G = (V, E)$ in which a vertex represents a processor and an edge represents a communication link between processors. We refer readers to [1, 10, 11] for terminology and notation unless stated otherwise. Once a network is running, some processors or links might be faulty. An interconnection network without faults is impossible. So the reliability evaluation of interconnection networks is significant.

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The traditional *edge-connectivity* $\lambda(G)$ is a measurement for the reliability of interconnection networks. However, in real situation, it is a small probability event that all links incident with the same processor fail simultaneously. To overcome this shortcoming, Esfahanian and Hakimi [2] proposed restricted edge-connectivity. Given a graph G, an edge-cut $S \subseteq E(G)$ is called a *restricted edge-cut* if there are no isolated vertices in $G-S$. The *restricted edge-connectivity* $\lambda'(G)$ is the minimum cardinality of all restricted edge-cuts. Inspired by the restricted edge-connectivity, Fàbrega and Foil [3] proposed the g-extra edge-connectivity of a graph. We restate this concept as follows.

- Given a graph G, an edge-cut F is called a *g-extra edge-cut* if every component of $G-F$ has at least $g+1$ vertices. The *g-extra edge-connectivity* of G, denoted by $\lambda_g(G)$, is the minimum cardinality of all g-extra edge-cuts, if exist.

A connected graph G is called *λ_g-connected* if $\lambda_g(G)$ exists.

In recent years, the g-extra edge-connectivity of a graph has received much attention [5–7, 12–16, 19]. For example, Montejano and Sau [7] proved that given a connected graph G and a positive integer g, determining $\lambda_g(G)$ or giving a correct report that G is not λ_g-connected is NP-hard. Yang [13] determined that the 1-extra edge-connectivity of balanced hypercubes BH_n is $\lambda_1(BH_n) = 4n-2$ for $n \geq 2$. Lü [6] showed that $\lambda_2(BH_n) = 6n-4$ for $n \geq 2$. Li et al. [5] and Yang et al. [12] independently proved that $\lambda_3(BH_n) = 8n-8$ for $n \geq 2$. In addition, Yang et al. [12] proposed a conjecture about the g-extra edge-connectivity of BH_n as follows.

Conjecture 1.1 *Let BH_n be an n-dimensional balanced hypercube. Then $\lambda_g(BH_n) = 2(g+1)n-4g+4$ for $2 \leq g \leq 2n-1$.*

Let $e_g(G) = \max\{|E(G[U])| \mid U \subseteq V(G), |U| = g+1\}$, where $G[U]$ is the subgraph of G induced by U. In this paper, we confirm their conjecture for $n \geq 6 - \dfrac{12}{g+1}$ and $2 \leq g \leq 8$, and disprove their conjecture for $n \geq \dfrac{3e_g(BH_n)}{g+1}$ and $9 \leq g \leq 2n-1$.

2 Balanced hypercubes

In 1997, Wu and Huang proposed balanced hypercubes BH_n.

Definition 2.1 ([9]) *An n-dimensional balanced hypercube $BH_n = (V(BH_n), E(BH_n))$ has vertex set $V(BH_n) = \{(a_0, a_1, \ldots, a_i, \ldots, a_{n-1}) \mid a_i \in \{0, 1, 2, 3\}, 0 \leq i \leq n-1\}$. Each vertex $(a_0, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$ of BH_n has $2n$ neighbors:*

(1) $((a_0 \pm 1) \bmod 4, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$,

(2) $((a_0 \pm 1) \bmod 4, a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0}) \bmod 4, a_{i+1}, \ldots, a_{n-1})$.

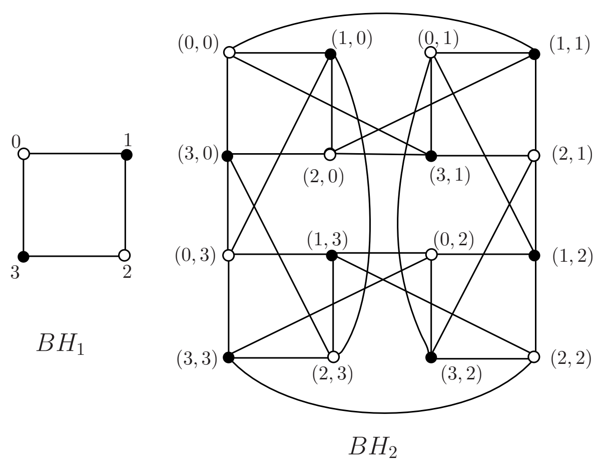

Figure 1: Illustration of BH_1 and BH_2.

Figure 1 depicts BH_1 and BH_2. Clearly, BH_n is a $2n$-regular graph. For a graph G and a vertex $v \in V(G)$, the set of neighbors of v in G is denoted by $N_G(v)$. Some useful properties of BH_n are listed below.

Lemma 2.2 ([9]) *The balanced hypercube BH_n is bipartite.*

Lemma 2.3 ([13]) *Let u be an arbitrary vertex of BH_n for $n \geq 1$. Then, for an arbitrary vertex v of BH_n, either $|N_{BH_n}(u) \cap N_{BH_n}(v)| = 0$, $|N_{BH_n}(u) \cap N_{BH_n}(v)| = 2$, or $|N_{BH_n}(u) \cap N_{BH_n}(v)| = 2n$. Furthermore, there is exactly one vertex w such that $|N_{BH_n}(u) \cap N_{BH_n}(w)| = 2n$.*

According to Lemma 2.3, we call the vertex w the *equivalent vertex* of u, denoted by u', if w satisfies that $|N_{BH_n}(u) \cap N_{BH_n}(w)| = 2n$ in BH_n, and u and u' are said to be a pair of equivalent vertices.

The following two lemmas are important observations about the structure of BH_n.

Lemma 2.4 *The balanced hypercube BH_n is $K_{3,3}$ free.*

Proof. Assume to the contrary that there exists a subgraph H_1 of BH_n which is isomorphic to $K_{3,3}$. By Lemma 2.2, suppose the bipartite graph $H_1 = (X_1, Y_1)$, where $X_1 = \{u_1, u_2, u_3\}$ and Y_1 are two parts of H_1. Since $N_{H_1}(u_1) \cap N_{H_1}(u_2) = Y_1$, $|N_{BH_n}(u_1) \cap N_{BH_n}(u_2)| \geq |N_{H_1}(u_1) \cap N_{H_1}(u_2)| = 3 > 2$. Thus, by Lemma 2.3, the vertex u_2 is the unique equivalent vertex of u_1. Similar to the above deduction, we see that the vertex u_3 is also the unique equivalent vertex of u_1, which contradicts $u_2 \neq u_3$.

This completes the proof of Lemma 2.4. $\square$

Let $\mathcal{F}_g$ be a collection of induced subgraphs of BH_n with $g+1$ vertices and $e_g(BH_n)$ edges for $g \geq 2$. By Lemma 2.2, H is bipartite for any graph $H \in \mathcal{F}_g$.

Lemma 2.5 *The vertex set X (or Y) must consist of several pairs of equivalent vertices besides at most one vertex for some $H = (X, Y) \in \mathcal{F}_g$.*

Proof. If $|X| = 1$, then this lemma holds obviously. Now we consider the case of $|X| \geq 2$. Assume to the contrary that there exist two vertices $u, v \in X$ such that their equivalent vertices are not in X for any graph $H = (X, Y) \in \mathcal{F}_g$. Without loss of generality, assume that $|N_H(u)| \geq |N_H(v)|$. We replace v with u' and obtain an induced subgraph $H' = (X', Y')$ of BH_n. If $|N_H(u)| > |N_H(v)|$, then $|V(H)| = |V(H')|$ and $|E(H)| < |E(H')|$, which contradicts the selection of H. If $|N_H(u)| = |N_H(v)|$, then $H' \in \mathcal{F}_g$. If X' contains a pair of vertices like $u, v \in X$, then this operation continues until we obtain a graph $H^* = (X^*, Y^*)$ satisfying that X^* consists of several pairs of equivalent vertices besides at most one vertex. Note that $H^* = (X^*, Y^*) \in \mathcal{F}_g$, a contradiction.

By the similar arguments as above, we see that the vertex set Y also consists of several pairs of equivalent vertices besides at most one vertex. $\square$

The *lexicographic product* $G \circ H$ of graphs G and H is defined as the graph with vertex set $V(G) \times V(H)$ and $(u_1, v_1)(u_2, v_2) \in E(G \circ H)$ if and only if $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. Zhou et al. [17] proved that BH_n is a lexicographic product of a Cayley graph X_n and an empty graph with two vertices. In addition, Zhou et al. [18] showed that BH_n is edge-transitive. Their results are presented as follows.

Lemma 2.6 **([17])** *For each $n \geq 1$, $BH_n \cong X_n \circ 2K_1$.*

Lemma 2.7 ([17, Page 151]) *For $n \geq 3$, the girth of X_n is 6.*

Lemma 2.8 **([18])** *The balanced hypercube is edge-transitive.*

3 Main Results

In this section, we will discuss the g-extra edge-connectivity of the balanced hypercube BH_n for $2 \leq g \leq 2n - 1$.

Let $G = (V, E)$ be a graph. For a nonempty proper subset $U \subseteq V$, the set of edges with one end in U and the other end in $\overline{U} = V \backslash U$ is denoted by $[U, \overline{U}]$ and $\partial(U) = |[U, \overline{U}]|$. The *$g$-th isoperimetric edge-connectivity* $\gamma_g(G)$ of a graph G was proposed by Hamidoune et al. [4]. We restate the definition of $\gamma_g(G)$, that is $\gamma_g(G) = \min\{\partial(U) \mid U \subseteq V, |U| \geq g + 1, |\overline{U}| \geq g + 1\}$. Wang and Li [8] gave a sufficient condition to ensure a regular edge-transitive graph such that $\lambda_g(G) = \gamma_g(G)$.

Lemma 3.1 **([8])** *Let G be a k-regular edge-transitive graph of order n with $k \geq 2$, and let $g+1$ be a positive integer. If $n \geq 3(g+1)$, then G is λ_g-connected, and $\lambda_g(G) = \gamma_g(G)$.*

A graph G satisfying that $\gamma_j(G) = \beta_j(G)$ $(j = 0, 1, \ldots, g)$ is called *γ_g-optimal*, where $\beta_g(G) = \min\{\partial(U) \mid U \subseteq V, |U| = g + 1\}$. Zhang [16] gave a sufficient condition for a regular edge-transitive graph to be γ_g-optimal.

Lemma 3.2 ([16]) *Let $g + 1$ be a positive integer, and G a connected k-regular edge-transitive graph with $k \geq \dfrac{6e_g(G)}{g+1}$. Then G is γ_g-optimal.*

The following lemma gives a lower bound of $e_g(BH_n)$ for $2 \leq g \leq 2n - 1$.

Lemma 3.3 *The balanced hypercube BH_n satisfies that $e_g(BH_n) \geq 2g - 2$ for $2 \leq g \leq 2n - 1$.*

Proof. Suppose that $u = (0, 0, \ldots, 0)$, $u' = (2, 0, \ldots, 0)$, $u_1 = (1, 0, \ldots, 0)$, $u_2 = (3, 0, \ldots, 0)$, $u_{2i-1} = (1, \overbrace{0, \ldots, 0}^{i-2}, 1, 0, \ldots, 0)$ and $u_{2i} = (3, \overbrace{0, \ldots, 0}^{i-2}, 1, 0, \ldots, 0)$ for $2 \leq i \leq g - 1$ are some vertices of BH_n. Let $A_g = \{u, u'\} \cup \{u_i \mid 1 \leq i \leq g - 1\}$. By Definition 2.1, we know that the induced subgraph $BH_n[A_g]$ is isomorphic to $K_{2,g-1}$ (see Figure 2). Therefore, $e_g(BH_n) \geq 2g - 2$ for $2 \leq g \leq 2n - 1$. $\square$

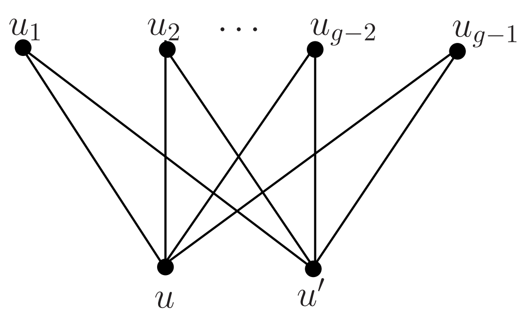

Figure 2: Illustration of $BH_n[A_g]$.

Now, we determine $e_g(BH_n)$ for $2 \leq g \leq 8$.

Lemma 3.4 *The balanced hypercube BH_n satisfies that $e_g(BH_n) = 2g - 2$ for $2 \leq g \leq 8$.*

Proof. By Lemma 3.3, we only need to prove that $e_g(BH_n) \leq 2g-2$ for $2 \leq g \leq 8$. Let H be an induced subgraph of BH_n with $|V(H)| = g+1$ and $|E(H)| = e_g(BH_n)$. By Lemma 2.2, suppose the bipartite graph $H = (X, Y)$, where X and Y are two parts of H. We divide our discussion into five cases.

Case 1. $2 \leq g \leq 4$.

In this case, $|X| + |Y| = |V(H)| = g+1$. Hence,

$$|E(H)| \leq |X| \cdot |Y| \leq \left\lfloor \frac{g+1}{2} \right\rfloor \cdot \left\lceil \frac{g+1}{2} \right\rceil = 2g-2.$$

Case 2. $g = 5$.

In this case, $|X| + |Y| = |V(H)| = 6$. Hence, $|E(H)| \leq |X| \cdot |Y| \leq 3 \times 3 = 9$. If $|E(H)| = 9$, then H is isomorphic to $K_{3,3}$, which contradicts Lemma 2.4. Thus, $|E(H)| \leq 8 = 2g-2$.

Case 3. $g = 6$.

In this case, $|X| + |Y| = |V(H)| = 7$. Hence, $|E(H)| \leq |X| \cdot |Y| \leq 3 \times 4 = 12$. If $|E(H)| = 12$, then H is isomorphic to $K_{3,4}$, which contradicts Lemma 2.4. If $|E(H)| = 11$, then H is isomorphic to $K_{3,4} - e$ for some $e \in E(K_{3,4})$, which also contradicts Lemma 2.4. Thus, $|E(H)| \leq 10 = 2g-2$.

Case 4. $g = 7$.

In this case, $|X| + |Y| = |V(H)| = 8$. Hence, $|E(H)| \leq |X| \cdot |Y| \leq 4 \times 4 = 16$. Note that $|E(K_{2,6})| = 12 = 2g-2$. Let $E_0 \subseteq E(K_{3,5})$ and $E_1 \subseteq E(K_{4,4})$. If $H = K_{3,5} - E_0$ and $|E_0| \leq 2$, then H contains a subgraph isomorphic to $K_{3,3}$, which contradicts Lemma 2.4. If $H = K_{4,4} - E_1$ and $|E_1| \leq 3$, then by Lemma 2.5, suppose that $X = \{u_1, u_1', u_2, u_2'\}$ and $Y = \{v_1, v_1', v_2, v_2'\}$. Since $H = (X, Y)$ is an induced subgraph of BH_n, $|E_1| = 0$. Then H is isomorphic to $K_{4,4}$, which contradicts Lemma 2.4. Thus, $|E(H)| \leq 12 = 2g-2$.

Case 5. $g = 8$.

In this case, $|X| + |Y| = |V(H)| = 9$. Hence, $|E(H)| \leq |X| \cdot |Y| \leq 4 \times 5 = 20$. Note that $|E(K_{2,7})| = 14 = 2g-2$. Let $E_0 \subseteq E(K_{3,6})$ and $E_1 \subseteq E(K_{4,5})$. Now, we only need to discuss the following two cases.

Case 5.1. $H = K_{3,6} - E_0$ and $|E_0| \leq 3$.

Note that $K_{3,6} - E_0$ with $|E_0| \leq 3$ contains a subgraph isomorphic to $K_{3,3}$, which contradicts Lemma 2.4.

Case 5.2. $H = K_{4,5} - E_1$ and $|E_1| \leq 5$.

By Lemma 2.5, without loss of generality, suppose that $X = \{u_1, u_1', u_2, u_2'\}$ and $Y = \{v_1, v_1', v_2, v_2', v\}$. Note that $H = (X, Y)$ is an induced subgraph of BH_n. Then $|E_1|$ can not be an odd integer. If $|E_1| \in \{0, 2\}$, then $K_{4,5} - E_1$ contains a subgraph isomorphic to $K_{3,3}$, which contradicts Lemmas 2.4. If $|E_1| = 4$ and $u_i v_j \in E_1$ for some $i, j \in \{1, 2\}$, then edges $u_i' v_j, u_i v_j', u_i' v_j' \in E_1$. Thus, $E_1 = \{u_i v_j, u_i' v_j, u_i v_j', u_i' v_j'\}$ and $H = K_{4,5} - E_1$ contains a subgraph isomorphic to $K_{3,3}$, which contradicts Lemmas 2.4. If $|E_1| = 4$ and $u_i v_j \notin E_1$ for all $i, j \in \{1, 2\}$, then $E_1 = \{u_1 v, u_2 v, u_1' v, u_2' v\}$. Therefore, $H = K_{4,5} - E_1$ contains a subgraph isomorphic to $K_{3,3}$, which contradicts Lemmas 2.4.

Thus, $|E(H)| \leq 14 = 2g - 2$.

So, $e_g(BH_n) = 2g - 2$ for $2 \leq g \leq 8$. □

The following lemma gives a lower bound of $e_g(BH_n)$ for $9 \leq g \leq 2n - 1$, which will be used to disprove Conjecture 1.1 for $9 \leq g \leq 2n - 1$.

Lemma 3.5 *The balanced hypercube BH_n satisfies that $e_g(BH_n) > 2g - 2$ for $9 \leq g \leq 2n - 1$.*

Proof. To prove this lemma, it suffices to construct a subgraph of BH_n with $g + 1$ vertices and at least $2g - 1$ edges.

By Lemma 2.7, the girth of X_n is 6 for $n \geq 3$. Suppose that $\overline{C_6}$ is a cycle of X_n with six vertices. Let $H_0 = \overline{C_6} \circ 2K_1$ be a subgraph of BH_n. Since BH_n is connected, by Lemma 2.6, X_n is a connected graph for $n \geq 1$. Let $\overline{U_t}$ be a connected subgraph of X_n with $|V(\overline{U_t})| = t \geq 6$ satisfying that $\overline{U_t}$ is a unicyclic graph which contains $\overline{C_6}$. Then

$|E(\overline{U_t})| = t \geq 6$. Now, we distinguish the following four cases.

Case 1. $g = 9$.

We consider the graph $H_0 - \{u, v\}$, where $u, v \in V(H_0)$, $u' \neq v$ and $uv \in E(H_0)$. Note that $|V(H_0 - \{u, v\})| = g + 1$ and $|E(H_0 - \{u, v\})| = 2g - 1$. Then $e_g(BH_n) \geq |E(H_0 - \{u, v\})| > 2g - 2$.

Case 2. $g = 10$.

We consider the graph $H_0 - v$, where $v \in V(H_0)$. Note that $|V(H_0 - v)| = g + 1$ and $|E(H_0 - v)| = 2g$. Then $e_g(BH_n) \geq |E(H_0 - v)| > 2g - 2$.

Case 3. g is an odd integer with $g \geq 11$.

Since g is an odd integer with $g \geq 11$, we have $\dfrac{g+1}{2} \geq 6$. We consider the graph $H_g = \overline{U_{\frac{g+1}{2}}} \circ 2K_1$ as a subgraph of BH_n. Note that $|V(H_g)| = g+1$ and $|E(H_g)| = 2g+2$. Then $e_g(BH_n) \geq |E(H_g)| > 2g - 2$.

Case 4. g is an even integer with $g \geq 12$.

Since g is an even integer with $g \geq 12$, $g - 1$ is an odd integer with $g - 1 \geq 11$. We consider the graph $H_{g-1} = \overline{U_{\frac{g}{2}}} \circ 2K_1$ as a subgraph of BH_n. Pick a vertex u from $V(BH_n) \setminus V(H_{g-1})$. Note that $|V(H_{g-1}) \cup \{u\}| = g+1$ and $|E(BH_n[V(H_{g-1}) \cup \{u\}])| \geq |E(H_{g-1})| = 2g$. Then $e_g(BH_n) \geq |E(BH_n[V(H_{g-1}) \cup \{u\}])| > 2g - 2$.

As mentioned above, we obtain the desired result. □

Now, we give the proof of our main theorem.

Theorem 3.6 *The g-extra edge-connectivity of balanced hypercubes BH_n is $\lambda_g(BH_n) = 2(g+1)n - 4g + 4$ for $n \geq 6 - \dfrac{12}{g+1}$ and $2 \leq g \leq 8$. In addition, $\lambda_g(BH_n) < 2(g+1)n - 4g + 4$ for $n \geq \dfrac{3e_g(BH_n)}{g+1}$ and $9 \leq g \leq 2n - 1$.*

Proof. By Lemma 2.8, BH_n is edge-transitive. Note that $|V(BH_n)| = 2^{2n} \geq 6n \geq 3(g+1)$ for $n \geq 2$. By Lemma 3.1, $\lambda_g(BH_n) = \gamma_g(BH_n)$ for $2 \leq g \leq 2n - 1$.

By Lemma 3.4, $e_g(BH_n) = 2g-2$ for $2 \le g \le 8$. Since $n \ge 6 - \dfrac{12}{g+1}$, we have $2n \ge \dfrac{6(2g-2)}{g+1} = \dfrac{6e_g(BH_n)}{g+1}$ for $2 \le g \le 8$. By Lemma 3.2, BH_n is γ_g-optimal. Thus, $\gamma_g(BH_n) = \beta_g(BH_n)$ for $n \ge 6 - \dfrac{12}{g+1}$ and $2 \le g \le 8$. Since $\beta_g(BH_n) = 2n(g+1) - 2e_g(BH_n) = 2n(g+1) - 2(2g-2) = 2(g+1)n - 4g + 4$, we have $\lambda_g(BH_n) = 2(g+1)n - 4g + 4$ for $n \ge 6 - \dfrac{12}{g+1}$ and $2 \le g \le 8$.

By Lemma 3.5, $e_g(BH_n) > 2g-2$ for $9 \le g \le 2n-1$. By Lemma 3.2, we have $\gamma_g(BH_n) = \beta_g(BH_n) = 2n(g+1) - 2e_g(BH_n) < 2n(g+1) - 2(2g-2) = 2(g+1)n - 4g + 4$ for $2n \ge \dfrac{6e_g(BH_n)}{g+1}$ and $9 \le g \le 2n-1$. Therefore, $\lambda_g(BH_n) < 2(g+1)n - 4g + 4$ for $n \ge \dfrac{3e_g(BH_n)}{g+1}$ and $9 \le g \le 2n-1$. □

4 Conclusions

The g-extra edge-connectivity is an important measure for the reliability of interconnection networks. We establish the g-extra edge-connectivity of balanced hypercubes BH_n, that is $\lambda_g(BH_n) = 2(g+1)n - 4g + 4$ for $n \ge 6 - \dfrac{12}{g+1}$ and $2 \le g \le 8$, which partially confirms Conjecture 1.1. This result can provide a more accurate measurement of edge fault tolerance of balanced hypercubes. Meanwhile, we prove that $\lambda_g(BH_n) < 2(g+1)n - 4g + 4$ for $n \ge \dfrac{3e_g(BH_n)}{g+1}$ and $9 \le g \le 2n-1$, which disproves Conjecture 1.1 for any n and g with $n \ge \dfrac{3e_g(BH_n)}{g+1}$ and $9 \le g \le 2n-1$.

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