# ELLIPTIC SCALING FUNCTIONS AS COMPACTLY SUPPORTED MULTIVARIATE ANALOGS OF THE B-SPLINES* 

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#### Abstract

In the paper, we present a family of multivariate compactly supported scaling functions, which we call as elliptic scaling functions. The elliptic scaling functions are the convolution of elliptic splines, which correspond to homogeneous elliptic differential operators, with distributions. The elliptic scaling functions satisfy refinement relations with real isotropic dilation matrices. The elliptic scaling functions satisfy most of the properties of the univariate cardinal B-splines: compact support, refinement relation, partition of unity, total positivity, order of approximation, convolution relation, Riesz basis formation (under a restriction on the mask), etc. The algebraic polynomials contained in the span of integer shifts of any elliptic scaling function belong to the null-space of a homogeneous elliptic differential operator. Similarly to the properties of the B-splines under differentiation, it is possible to define elliptic (not necessarily differential) operators such that the elliptic scaling functions satisfy relations with these operators. In particular, the elliptic scaling functions can be considered as a composition of segments, where the function inside a segment, like a polynomial in the case of the B-splines, vanishes under the action of the introduced operator.


Keywords: Elliptic scaling functions; isotropic dilation matrices; compact support; polyharmonic splines; cardinal B-splines; homogeneous elliptic differential operators

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## 1 Introduction

The $d$-variate polyharmonic splines (thin plate splines, etc.), see for example Refs. 9, 11, 16, 17, can be presented as some linear combinations of shifted versions of the Green functions of the polyharmonic operators $\Delta^{m}, m=1,2, \ldots,\left(\Delta:=\sum_{j=1}^{d} \frac{\partial^{2}}{\partial x_{j}^{2}}\right.$ is the Laplace operator). The Green function of the operator $\Delta^{m}$ is of the form

$$
\rho(\boldsymbol{x}):= \begin{cases}|\boldsymbol{x}|^{2 m-d} \log |\boldsymbol{x}| & \text { if } d \text { is even, } \\ |\boldsymbol{x}|^{2 m-d} & \text { if } d \text { is odd, } \quad \boldsymbol{x} \in \mathbb{R}^{d} ;\end{cases}
$$

[^0]and the Fourier transform of the Green function is
$$
\hat{\rho}(\boldsymbol{\xi}):=\frac{1}{|\boldsymbol{\xi}|^{2 m}}, \quad \boldsymbol{\xi} \in \mathbb{R}^{d}
$$

Here and in the sequel, we denote vectors by boldface symbols and shall not distinguish vectors as points of the Euclidean spaces and as column-matrices.

A particular case of the polyharmonic splines, the so-called elementary polyharmonic cardinal B-splines (the term of Ch. Rabut [20, see also Refs. [18, 21, 22]) are most similar to the univariate cardinal B-splines. In the Fourier domain, the elementary polyharmonic cardinal B-spline is of the form

$$
\begin{equation*}
\hat{B}_{m}(\boldsymbol{\xi}):=\frac{|2 \sin (\boldsymbol{\xi} / 2)|^{2 m}}{|\boldsymbol{\xi}|^{2 m}}=\left(\frac{4 \sum_{n=1}^{d} \sin ^{2}\left(\xi_{n} / 2\right)}{\sum_{n=1}^{d} \xi_{n}^{2}}\right)^{m} \tag{1.1}
\end{equation*}
$$

where $m \geq d / 2, \boldsymbol{\xi}:=\left(\xi_{1}, \ldots, \xi_{d}\right), \sin \boldsymbol{\xi}:=\left(\sin \xi_{1}, \ldots, \sin \xi_{d}\right)$. In the present paper, we shall say that functions (1.1) are called polyharmonic B-splines. The polyharmonic B-splines satisfy most of the properties of the univariate cardinal B-splines, see Refs. [20, 22. However an important property of the univariate B-splines being compactly supported is violated. (The polyharmonic B-splines decay like $O\left(1 /|\boldsymbol{x}|^{d+2}\right)$ as $|\boldsymbol{x}| \rightarrow \infty$ [20]. Note also that any (multivariate) polyharmonic spline (defined in the Cartesian coordinate system) is not compactly supported. See, for example, Ref. [13]. Furthermore, the refinement relation, another important property of the univariate cardinal B-splines, does not hold also, see Refs. [14, 15, 18, 22 .

In the paper, we present multivariate compactly-supported scaling functions. For any real isotropic dilation matrix, we can construct a trigonometric polynomial mask such that the Fourier transform of the corresponding scaling function is of the form

$$
\begin{equation*}
\hat{\phi}^{m}(\boldsymbol{\xi}):=\frac{G(\boldsymbol{\xi})}{(P(\boldsymbol{\xi}))^{m}} M(\boldsymbol{\xi}), \quad m \geq 1, \quad \boldsymbol{\xi} \in \mathbb{R}^{d} \tag{1.2}
\end{equation*}
$$

where $P(\boldsymbol{\xi})$ is a positive definite quadratic form (in particular, $\left.|\boldsymbol{\xi}|^{2}\right), G(\boldsymbol{\xi})$ is a trigonometric polynomial, and the function $M(\boldsymbol{\xi})$ is continuous on $\mathbb{R}^{d}$ and does not decay at the infinity. Thus scaling function (1.2) can be considered as the convolution of an elliptic spline, whose Fourier transform is $\frac{G(\boldsymbol{\xi})}{(P(\boldsymbol{\xi}))^{m}}$, with a distribution. In the paper, we shall say that the scaling functions of the form (1.2) are called elliptic scaling functions.

We state that any real isotropic (not necessarily dilation) matrix $A$ is similar, with a positive definite similarity transformation matrix, to an orthogonal matrix; and the similarity transformation matrix defines the positive definite quadratic form $P(\boldsymbol{\xi})$. Note that the quadratic form $P(\boldsymbol{\xi})$ (and the corresponding homogeneous elliptic operator) is invariant (up to the constant factor) under the coordinate transformation by the matrix $A$. The invariance of the quadratic form $P(\boldsymbol{\xi})$ is crucial for the properties of scaling function (1.2).

In the paper, we show that elliptic scaling functions (1.2) have many properties of the univariate B-splines and multivariate polyharmonic B-splines. Any elliptic scaling function $\phi^{m}$ partitions unity, is totally positive, its order of approximation is 2 m . We present a sufficient condition that the integer shifts of $\phi^{m}$ form a Riesz basis. Also note that the algebraic polynomials reproduced by the scaling function $\phi^{m}$ belong to the null-space of the elliptic operator whose symbol is $(P(\xi))^{m}$.

Unlike the polyharmonic B-splines and like the univariate B-splines, the elliptic scaling functions are compactly supported and, by definition, satisfy refinement relations.

Considering the function $\widehat{\Delta^{\sharp}}(\boldsymbol{\xi}):=\frac{P(\boldsymbol{\xi})}{\left.\left(M_{m}\right)\right)^{1 / m}}$, see (1.2), as the symbol of an operator, we see that the elliptic scaling function $\phi^{m}$, similarly to the B-splines, satisfies relations with this operator. In particular, we have: $\left(\Delta^{\sharp}\right)^{m} \phi^{m}(\boldsymbol{x})=0, \forall \boldsymbol{x} \in \mathbb{R}^{d} \backslash \mathbb{Z}^{d}$. Thus the elliptic scaling functions can be considered as a composition of segments, where the function inside a segment, like a polynomial in the case of the B-splines, vanishes under the action of the operator $\left(\Delta^{\sharp}\right)^{m}$. Note also that the elliptic scaling functions $\phi^{m}$, in the Fourier domain, can be written as $\hat{\phi}^{m}(\boldsymbol{\xi})=\left(\frac{G(\boldsymbol{\xi})}{\widehat{\Delta^{\sharp}}(\boldsymbol{\xi})}\right)^{m}$, where $G$ is a trigonometric polynomial and $\widehat{\Delta^{\sharp}}$ is the symbol of an elliptic operator. Hence we can say that the elliptic scaling functions can be considered as multivariate analogs of the univariate cardinal B-splines. On the other hand, we shall see that the univariate cardinal B-splines come under the presented approach. Namely the univariate cardinal B-splines of odd degree are a particular case of elliptic scaling functions (1.2).

Note that, in the paper, we consider only the scaling functions. The construction of the corresponding wavelets is not discussed. Moreover, the orthogonalization or construction of dual scaling functions (in the biorthogonal case) are not discussed also. This will be the object of another paper.

The paper is organized as follows. In Section 2, we present some properties of the isotropic matrices. In particular, the similarity to orthogonal matrices, the positive definite quadratic forms defined by the similarity transformation matrices, and the invariance of the quadratic forms are considered. Section 3 is devoted to the construction of compactly supported elliptic scaling functions of the first order. The explicit form of the Fourier transform of the scaling functions is derived. In this section, some properties of the first order scaling functions are presented. Section 4 is devoted to the definition and properties of the elliptic scaling functions of an arbitrary order. In Section 5, several bivariate isotropic dilation matrices are presented and some properties of the corresponding elliptic scaling functions are discussed.

### 1.1 Preliminaries and notations

Let us introduce here some general notation.
For any vector $\boldsymbol{v} \in \mathbb{R}^{d}$, by $|\boldsymbol{v}|:=\sqrt{v_{1}^{2}+\cdots+v_{d}^{2}}$ denote vector's length; in other words, $|\boldsymbol{v}|$ is the usual Euclidean norm of $\boldsymbol{v}$.

A multi-index $k$ is a $d$-tuple $\left(k_{1}, \ldots, k_{d}\right)$ with its components being nonnegative integers, i. e., $k \in \mathbb{Z}_{\geq 0}^{d}$. The length of the multi-index $k$ is $|k|:=k_{1}+\cdots+k_{d}$.

By $\boldsymbol{x}^{k}$ denote the monomial $x_{1}^{k_{1}} \cdots x_{d}^{k_{d}}$, where $x:=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{R}^{d}, k:=\left(k_{1}, \ldots, k_{d}\right) \in \mathbb{Z}_{\geq 0}^{d}$. Note that the total degree of $\boldsymbol{x}^{k}$ is $|k|$.

Let $\Pi$ denote the space of all polynomials on $\mathbb{R}^{d}$. Also by $\Pi_{N}, N \in \mathbb{Z}_{\geq 0}$, denote the space of polynomials with total degree less than or equal to $N$.

By $D^{k}, k \in \mathbb{Z}_{\geq 0}^{d}$, denote the differential operator $D_{1}^{k_{1}} \cdots D_{d}^{k_{d}}$, where $D_{n}, n=1, \ldots, d$, is the partial derivative with respect to the $n$th coordinate.

By $\boldsymbol{e}_{j}$ denote the $j$ th basis vector of the space $\mathbb{R}^{d}$, i.e., $\boldsymbol{e}_{j}:=\left(\delta_{j 1}, \ldots, \delta_{j d}\right)$, where $\delta_{j k}:=$ $\begin{cases}1, & j=k, \\ 0, & j \neq k .\end{cases}$

The Fourier transform of a function $f \in L^{1}\left(\mathbb{R}^{d}\right), d=1,2, \ldots$, is defined as

$$
\mathcal{F}(f)(\boldsymbol{\xi})=\hat{f}(\boldsymbol{\xi}):=\int_{\mathbb{R}^{d}} f(\boldsymbol{x}) e^{-i \boldsymbol{\xi} \cdot \boldsymbol{x}} d \boldsymbol{x}, \quad \boldsymbol{x}, \boldsymbol{\xi} \in \mathbb{R}^{d},
$$

where $\boldsymbol{x} \cdot \boldsymbol{\xi}:=x_{1} \xi_{1}+\cdots+x_{d} \xi_{d}$.

By $\mathcal{S}^{\prime}$ denote the space of tempered distributions and by $\delta$ denote the Dirac distribution.
In the paper, we shall suppose that the dilation matrices are real integer matrices whose eigenvalues are greater than 1 in absolute value. Then, for any dilation matrix $A$, we have the following property

$$
\begin{equation*}
\lim _{j \rightarrow \infty}\left|A^{j} \boldsymbol{x}\right| \rightarrow \infty, \quad \forall \boldsymbol{x} \in \mathbb{R}^{d}, \boldsymbol{x} \neq \mathbf{0} \tag{1.3}
\end{equation*}
$$

In the paper, we shall also suppose that the matrix-vector multiplication is (left-)distributive over a (finite or countable) set of vectors of an Euclidian space: $A\left\{s_{j}: j \in J\right\}:=\left\{A s_{j}: j \in J\right\}$, where $A$ is a $d \times d$ matrix and $s_{j}:=\left(s_{j, 1}, \ldots, s_{j, d}\right) \in \mathbb{R}^{d}$. Moreover, the distributive property can be extended to uncountable sets.

To simplify the notations, by $A^{-T}$ we shall denote the matrix $\left(A^{T}\right)^{-1} \equiv\left(A^{-1}\right)^{T}$.
A scaling function $\phi$ satisfies a refinement relation

$$
\begin{equation*}
\phi(\boldsymbol{x})=\sum_{\boldsymbol{k} \in \mathbb{Z}^{d}} h_{\boldsymbol{k}}|\operatorname{det} A|^{\frac{1}{2}} \phi(A \boldsymbol{x}-\boldsymbol{k}), \quad \boldsymbol{x} \in \mathbb{R}^{d} \tag{1.4}
\end{equation*}
$$

where $A$ is a $d \times d$ dilation matrix. Refinement relation (1.4) can be rewritten in the Fourier domain as

$$
\hat{\phi}(\boldsymbol{\xi})=m_{0}\left(A^{-T} \boldsymbol{\xi}\right) \hat{\phi}\left(A^{-T} \boldsymbol{\xi}\right), \quad \boldsymbol{\xi} \in \mathbb{R}^{d}
$$

where $m_{0}(\boldsymbol{\xi}), \boldsymbol{\xi} \in \mathbb{R}^{d}$, is a $2 \pi$-periodic function, which is called mask. The Fourier transform of the scaling function $\phi$ can be defined by the mask $m_{0}$ as follows

$$
\begin{equation*}
\hat{\phi}(\boldsymbol{\xi})=\prod_{j=1}^{\infty} m_{0}\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right) . \tag{1.5}
\end{equation*}
$$

### 1.2 Auxiliary propositions on dilation matrices

In this subsection, we present some well-known propositions and definitions concerning the dilation matrices.

Proposition 1.1. Let $A$ be a non-singular $d \times d$ matrix with integer entries. Then the number of the cosets of $\mathbb{Z}^{d}$ by modulo $A$ is equal to $|\operatorname{det} A|$ and the set $\mathbb{Z}^{d} \cap A[0,1)^{d}$ is a set of representatives of the quotient $\mathbb{Z}^{d} / A \mathbb{Z}^{d}$.

Definition 1.1. A full set of representatives of the quotient $\mathbb{Z}^{d} / A \mathbb{Z}^{d}$ is called the set of digits of the matrix $A$; and we shall denote the set of digits of the matrix $A$ by $\mathcal{W}(A)$. By $\mathcal{S}(A) \subset[0,1)^{d}$ we denote the set of points such that $A \mathcal{S}(A)=\mathcal{W}(A)$.

Remark 1.1. By definition, the set of digits of the matrix $A$ does not depend on any specific choice of the representatives of the quotient $\mathbb{Z}^{d} / A \mathbb{Z}^{d}$. Thus the sets $-\mathcal{W}(A)$ and $\boldsymbol{k}+\mathcal{W}(A)$, for some $\boldsymbol{k} \in \mathbb{Z}^{d}$, contain different from the initial set $\mathcal{W}(A)$ elements; nevertheless they can be considered (and denoted) as the set of digits $\mathcal{W}(A)$. On the other hand, the elements of the set $\mathcal{S}(A)$ are defined uniquely (up to a permutation).

Proposition 1.2. Let $A$ be a $d \times d$ nonsingular matrix with integer entries and $\mathcal{W}(A)$ be its set of digits, then we have

$$
\begin{aligned}
& \bigcup_{\boldsymbol{n} \in \mathcal{W}(A)}\left\{A \boldsymbol{k}+\boldsymbol{n}: \boldsymbol{k} \in \mathbb{Z}^{d}\right\}=\mathbb{Z}^{d} ; \\
& \left\{A \boldsymbol{k}+\boldsymbol{n}: \boldsymbol{k} \in \mathbb{Z}^{d}\right\} \cap\left\{A \boldsymbol{k}+\boldsymbol{n}^{\prime}: \boldsymbol{k} \in \mathbb{Z}^{d}\right\}=\emptyset, \quad \boldsymbol{n}, \boldsymbol{n}^{\prime} \in \mathcal{W}(A), \boldsymbol{n} \neq \boldsymbol{n}^{\prime} .
\end{aligned}
$$

The proof of Propositions 1.1, 1.2 can be found, for example, in the book 19 .
Proposition 1.3. Let $A$ be a $d \times d$ dilation matrix with integer entries, then we have

$$
\begin{aligned}
& \bigcup_{j=1}^{\infty} \bigcup_{\boldsymbol{s} \in \mathcal{S}(A) \backslash\{\mathbf{0}\}} A^{j}\left\{\boldsymbol{k}+\boldsymbol{s}: \boldsymbol{k} \in \mathbb{Z}^{d}\right\}=\mathbb{Z}^{d} \backslash\{\mathbf{0}\}, \\
& A^{j}\left\{\boldsymbol{k}+\boldsymbol{s}: \boldsymbol{k} \in \mathbb{Z}^{d}\right\} \cap A^{j^{\prime}}\left\{\boldsymbol{k}+\boldsymbol{s}^{\prime}: \boldsymbol{k} \in \mathbb{Z}^{d}\right\}=\emptyset \quad \text { if } j \neq j^{\prime} \text { or } \boldsymbol{s} \neq \boldsymbol{s}^{\prime},
\end{aligned}
$$

where the set $\mathcal{S}(A)$ is defined in Definition 1.1.
To prove Proposition 1.3, we can refer the reader to Ref. [24].

## 2 Similarity to Orthogonal Matrices

Let us recall the definition of isotropic (dilation) matrices.
Definition 2.1. A square matrix is called isotropic if the matrix is diagonalizable over $\mathbb{C}$ and all its eigenvalues are equal in absolute value.

Theorem 2.1. Let $\tilde{A}$ be a square non-singular real matrix. Suppose the matrix $\tilde{A}$ is diagonalizable over $\mathbb{C}$ and all its eigenvalues are equal to 1 in absolute value, then

$$
\begin{equation*}
\tilde{A}=Q U Q^{-1}, \tag{2.1}
\end{equation*}
$$

where $U$ is an orthogonal (real) matrix and $Q$ is a symmetric positive definite (real) matrix.
To prove the theorem, we can refer the reader to Refs. [24, 25. Here note only that the proof is based on the diagonalizability of the matrix $\tilde{A}$ and the polar decomposition of the similarity transformation matrix.

So we see that any real isotropic (dilation) matrix is similar (up to a constant factor) to an orthogonal matrix.

The next corollary directly follows from Theorem 2.1 and will be very useful hereinafter.
Corollary 2.2. From Theorem 2.1 it follows

$$
\begin{align*}
& \tilde{A} Q^{2} \tilde{A}^{T}=\tilde{A}^{-1} Q^{2} \tilde{A}^{-T}=Q^{2},  \tag{2.2}\\
& \tilde{A}^{T} Q^{-2} \tilde{A}=\tilde{A}^{-T} Q^{-2} \tilde{A}^{-1}=Q^{-2} \tag{2.3}
\end{align*}
$$

Using (2.1), the proof is straightforward.
Consider a real square matrix $A$ with integer entries. Suppose that $A$ is isotropic; then, using Theorem [2.1] $A^{-T}$ can be factored as follows

$$
\begin{equation*}
A^{-T}=\frac{1}{q^{1 / d}} Q^{-1} U Q \tag{2.4}
\end{equation*}
$$

where $q:=|\operatorname{det} A|, U$ is an orthogonal matrix, and $Q$ is a symmetric positive definite matrix. Now we can define a quadratic form

$$
\begin{equation*}
P(\boldsymbol{x}):=\boldsymbol{x}^{T} Q^{2} \boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^{d} . \tag{2.5}
\end{equation*}
$$

Since $Q^{2}$ is positive definite; therefore, quadratic form (2.5) is positive definite. By Corollary 2.2 we see that the quadratic form $P(\boldsymbol{x})$ is invariant (up to the constant value) under the variable transformation by the matrix $A^{-T}: \boldsymbol{x} \mapsto \boldsymbol{x}^{\prime}:=A^{-T} \boldsymbol{x}$. Indeed, using (2.2), we have

$$
\begin{equation*}
P\left(\boldsymbol{x}^{\prime}\right)=P\left(A^{-T} \boldsymbol{x}\right)=\boldsymbol{x}^{T} A^{-1} Q^{2} A^{-T} \boldsymbol{x}=\frac{1}{q^{2 / d}} \boldsymbol{x}^{T} Q^{2} \boldsymbol{x}=\frac{1}{q^{2 / d}} P(\boldsymbol{x}) \tag{2.6}
\end{equation*}
$$

(Similarly, the quadratic form $\boldsymbol{x}^{T} Q^{-2} \boldsymbol{x}$ will be invariant under the transformation by the matrix $A$, see (2.3).)

Remark 2.1. Note that the differential operator corresponding to quadratic form (2.5), i. e., $P(\boldsymbol{\xi})$ is the symbol of the operator, will be invariant under the coordinate transformation by the matrix $A$.

Remark 2.2. The matrix $Q$ in formulas (2.1), (2.4) (consequently, quadratic form (2.5)) is defined within a constant factor.

Remark 2.3. We conjecture that, if an isotropic matrix $A$ is a matrix with integer entries; then, multiplying by the appropriate real value, the matrix $Q^{2}$ that corresponds to the matrix $Q$ in formulas (2.1), (2.4) can be made a matrix with integer entries. This will be discussed elsewhere.

## 3 Elliptic Scaling Function of the First Order

### 3.1 Construction of the mask

Let $A$ be an isotropic dilation matrix and let $A^{-T}$ be factored by formula (2.4). Define a trigonometric function $G(\boldsymbol{\xi})$ such that its Taylor series about zero begins with quadratic form (2.5), i. e.,

$$
\begin{equation*}
G(\boldsymbol{\xi}):=P(\boldsymbol{\xi})+\text { higher order terms }, \quad \boldsymbol{\xi} \in \mathbb{R}^{d} \tag{3.1}
\end{equation*}
$$

Define the mask $m_{0}$ as follows

$$
\begin{equation*}
m_{0}(\boldsymbol{\xi}):=\frac{\prod_{\boldsymbol{s} \in \mathcal{S}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G(\boldsymbol{\xi}+2 \pi \boldsymbol{s})}{\prod_{\boldsymbol{s} \in \mathcal{S}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G(2 \pi \boldsymbol{s})} . \tag{3.2}
\end{equation*}
$$

(In formula (3.2), we suppose that $G(2 \pi s) \neq 0, \forall s \in \mathcal{S}\left(A^{T}\right) \backslash\{0\}$.)
Let the matrix $Q^{2}$ be presented in component-wise form as follows

$$
Q^{2}:=\left(\begin{array}{cccc}
q_{11} & q_{12} & \cdots & q_{1 d} \\
q_{12} & q_{22} & \cdots & q_{2 d} \\
\cdots & \ldots & \cdots & \cdots
\end{array}\right), \quad q_{i j} \in \mathbb{R}, \quad i, j=1, \ldots, d, i \leq j \text {; }
$$

and let $\boldsymbol{\xi}:=\left(\xi_{1}, \ldots, \xi_{d}\right)$. Then quadratic form (2.5) is

$$
\begin{equation*}
P(\boldsymbol{\xi}):=\sum_{1 \leq i \leq d} q_{i i} \xi_{i}^{2}+2 \sum_{\substack{1 \leq i, j \leq d \\ i<j}} q_{i j} \xi_{i} \xi_{j} \tag{3.3}
\end{equation*}
$$

It is easy to see that the following trigonometric polynomial has the required Taylor expansion about zero, see (3.1),

$$
\begin{equation*}
G\left(\xi_{1}, \ldots, \xi_{d}\right):=4 \sum_{1 \leq i \leq d} q_{i i} \sin ^{2} \frac{\xi_{i}}{2}+2 \sum_{\substack{1 \leq i, j \leq d \\ i<j}} q_{i j} \sin \xi_{i} \sin \xi_{j} \tag{3.4}
\end{equation*}
$$

Thus, using (3.4), the mask $m_{0}$ given by (3.2) is a trigonometric polynomial.

### 3.2 Explicit form of the Fourier transform of the scaling function

Let $m_{0}(\boldsymbol{\xi})$ be given by (3.2). Acting similarly to the classical univariate formula $\prod_{j=1}^{\infty} \cos \left(\frac{\xi}{2^{j}}\right)=$ $\frac{\sin \xi}{\xi}$, see the book [8], we can write

$$
\begin{align*}
\prod_{j=1}^{J} m_{0}\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right)=\prod_{j=1}^{J} \frac{m_{0}\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right) G\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right)}{G\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right)} \\
=\frac{G(\boldsymbol{\xi})}{G\left(\left(A^{-T}\right)^{J} \boldsymbol{\xi}\right)} \prod_{j=0}^{J-1} \frac{m_{0}\left(\left(A^{-T}\right)^{j+1} \boldsymbol{\xi}\right) G\left(\left(A^{-T}\right)^{j+1} \boldsymbol{\xi}\right)}{G\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right)} \tag{3.5}
\end{align*}
$$

where $G(\boldsymbol{\xi})$ is the same function as in formula (3.2). Unfortunately, unlike the univariate case, the fractions under the product sign in (3.5) are not canceled. Denoting

$$
\begin{equation*}
\mu(\boldsymbol{\xi}):=\frac{q^{2 / d} m_{0}\left(A^{-T} \boldsymbol{\xi}\right) G\left(A^{-T} \boldsymbol{\xi}\right)}{G(\boldsymbol{\xi})} \tag{3.6}
\end{equation*}
$$

where $q:=|\operatorname{det} A|$, we have

$$
\begin{equation*}
\prod_{j=1}^{J} m_{0}\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right)=\frac{G(\boldsymbol{\xi})}{q^{2 J / d} G\left(\left(A^{-T}\right)^{J} \boldsymbol{\xi}\right)} \prod_{j=0}^{J-1} \mu\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right) \tag{3.7}
\end{equation*}
$$

Using (2.6), (3.1), we get the limit of (3.7) as $J \rightarrow \infty$ :

$$
\begin{equation*}
\hat{\phi}(\boldsymbol{\xi})=\frac{G(\boldsymbol{\xi})}{P(\boldsymbol{\xi})} M(\boldsymbol{\xi}) \tag{3.8}
\end{equation*}
$$

where

$$
\begin{equation*}
M(\boldsymbol{\xi}):=\prod_{j=0}^{\infty} \mu\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right) \tag{3.9}
\end{equation*}
$$

Note that, in this subsection, we implicitly suppose that the mask $m_{0}$ is not necessarily a trigonometric polynomial, but we suppose that the infinite product in the right-hand side of (1.5) converges almost everywhere. However, in the sequel, we shall suppose that $m_{0}$ is a trigonometric polynomial, i. e., $G$ is given by (3.4).

### 3.3 Properties of the scaling function

### 3.3.1 Compact support

We recall that, for any $d \times d$ dilation matrix and trigonometric polynomial mask, there exists a unique up to a constant factor compactly supported solution $\phi \in S^{\prime}\left(\mathbb{R}^{d}\right)$ of refinement relation (1.4), see Ref. [4]. Thus we have the following proposition.

Proposition 3.1. For any real isotropic dilation matrix and mask (3.2), where $G$ is given by formula (3.4), the corresponding elliptic scaling function, whose Fourier transform is of the form (3.8), is compactly supported.

In formula (3.8), the Fourier transform of the elliptic spline is rather simple function. Thus, in the next subsection, we give our attention to the function $M(\boldsymbol{\xi})$. In particular, the behavior of $M(\boldsymbol{\xi})$ at the infinity determines the decay of $\hat{\phi}(\boldsymbol{\xi})$ as $|\boldsymbol{\xi}| \rightarrow \infty$.

### 3.3.2 Properties of $M(\boldsymbol{\xi})$

Below we present a lemma about the positive definiteness of the trigonometric polynomial $G$.
Lemma 3.2. For any quadratic positive definite form (3.3), trigonometric polynomial (3.4) is not negative on $\mathbb{R}^{d}$ and vanishes only at the points $2 \pi \boldsymbol{k}, \boldsymbol{k} \in \mathbb{Z}^{d}$.

Proof. Rewrite formula (3.4) as follows

$$
G\left(\xi_{1}, \ldots, \xi_{d}\right)=4 \sum_{1 \leq i \leq d} q_{i i} \sin ^{2} \frac{\xi_{i}}{2}+8 \sum_{\substack{1 \leq i, j \leq d \\ i<j}} q_{i j} \sin \frac{\xi_{i}}{2} \sin \frac{\xi_{j}}{2} \cos \frac{\xi_{i}}{2} \cos \frac{\xi_{j}}{2}
$$

Since the quadratic form $P(\boldsymbol{\xi})$ is positive definite; we have

$$
\sum_{1 \leq i \leq d} q_{i i} \sin ^{2} \frac{\xi_{i}}{2}+2 \sum_{\substack{1 \leq i, j \leq d \\ i<j}} q_{i j} \sin \frac{\xi_{i}}{2} \sin \frac{\xi_{j}}{2} \equiv P\left(\sin \frac{\xi_{1}}{2}, \ldots, \sin \frac{\xi_{d}}{2}\right) \geq 0, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{d}
$$

and the trigonometric polynomial $P\left(\sin \frac{\xi_{1}}{2}, \ldots, \sin \frac{\xi_{d}}{2}\right)$ vanishes iff $\sin \frac{\xi_{j}}{2}=0, j=1, \ldots, d$. Thus, since $0 \leq \cos \xi_{j} / 2 \leq 1$ for $\xi_{j} \in[-\pi, \pi], j=1, \ldots, d$, and $G(\boldsymbol{\xi})$ is $2 \pi$-periodic; it follows that $G(\boldsymbol{\xi}) \geq 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^{d}$ and $G(\boldsymbol{\xi})$ vanishes only at the points $2 \pi \boldsymbol{k}, \boldsymbol{k} \in \mathbb{Z}^{d}$.

It is convenient to rewrite formula (3.2) as follows

$$
\begin{equation*}
m_{0}(\boldsymbol{\xi}):=\frac{\prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G\left(\boldsymbol{\xi}+2 \pi A^{-T} \boldsymbol{n}\right)}{\prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G\left(2 \pi A^{-T} \boldsymbol{n}\right)} \tag{3.10}
\end{equation*}
$$

Now we can state and prove lemmas about some properties of the function $\mu(\boldsymbol{\xi})$.
Lemma 3.3. Let $\mu(\boldsymbol{\xi})$ be given by (3.6), where $G(\boldsymbol{\xi})$ is given by formula (3.4). Then $\mu(\boldsymbol{\xi})$ is $2 \pi$-periodic and

$$
\begin{equation*}
\mu(2 \pi \boldsymbol{k})=1, \quad \forall \boldsymbol{k} \in \mathbb{Z}^{d} \tag{3.11}
\end{equation*}
$$

Moreover, $\mu(\boldsymbol{\xi})$ is continuous on $\mathbb{R}^{d}$; and there exist constants $\mathcal{A}>0$ and $1 \leq \mathcal{B}<\infty$ such that $\mathcal{A} \leq \mu(\boldsymbol{\xi}) \leq \mathcal{B}, \forall \boldsymbol{\xi} \in \mathbb{R}^{d}$.

Proof. First, using (3.10), for any $\boldsymbol{k} \in \mathbb{Z}^{d}$, we have

$$
\begin{aligned}
& \mu(\boldsymbol{\xi}+2 \pi \boldsymbol{k})=\frac{q^{2 / d} \prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right)} G\left(A^{-T}(\boldsymbol{\xi}+2 \pi \boldsymbol{k})+2 \pi A^{-T} \boldsymbol{n}\right)}{G(\boldsymbol{\xi}+2 \pi \boldsymbol{k}) \prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G\left(2 \pi A^{-T} \boldsymbol{n}\right)} \\
&=\frac{q^{2 / d} \prod_{\boldsymbol{n}^{\prime} \in \mathcal{W}\left(A^{T}\right)} G\left(A^{-T} \boldsymbol{\xi}+2 \pi A^{-T} \boldsymbol{n}^{\prime}\right)}{G(\boldsymbol{\xi}) \prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G\left(2 \pi A^{-T} \boldsymbol{n}\right)}=\mu(\boldsymbol{\xi})
\end{aligned}
$$

(where $\left.\boldsymbol{n}^{\prime}=\boldsymbol{k}+\boldsymbol{n}\right)$.
Secondly consider the numerator of the fraction in the right-hand side of (3.6):

$$
\begin{equation*}
\prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right)} G\left(A^{-T} \boldsymbol{\xi}+2 \pi A^{-T} \boldsymbol{n}\right) \tag{3.12}
\end{equation*}
$$

Since $G(\boldsymbol{\xi})$ is $2 \pi$-periodic and vanishes only at the points $2 \pi \mathbb{Z}^{d}$, it follows that the previous expression vanishes only at the points: $2 \pi\left(A^{T} \boldsymbol{k}+\boldsymbol{n}\right), \forall \boldsymbol{k} \in \mathbb{Z}^{d}$ and $\forall \boldsymbol{n} \in \mathcal{W}\left(A^{T}\right)$. Using Proposition [1.2, we see that numerator (3.12) vanishes only at the points $2 \pi \boldsymbol{k}, \boldsymbol{k} \in \mathbb{Z}^{d}$; and the zeros of the numerator do not superimpose by different multipliers under the product sign.

Thirdly, using (2.6), (3.1), since $\mu(\boldsymbol{\xi})$ is $2 \pi$-periodic, $m_{0}(0)=1$, and the zeros of the numerator have the same multiplicity; we have, for all $\boldsymbol{k} \in \mathbb{Z}^{d}$,

$$
\mu(2 \pi \boldsymbol{k})=\lim _{\boldsymbol{\xi} \rightarrow \mathbf{0}} \frac{q^{2 / d} G\left(A^{-T} \boldsymbol{\xi}\right)}{G(\boldsymbol{\xi})}=\lim _{\boldsymbol{\xi} \rightarrow \mathbf{0}} \frac{q^{2 / d} P\left(A^{-T} \boldsymbol{\xi}\right)}{P(\boldsymbol{\xi})}=1
$$

Finally, since $\mu(\boldsymbol{\xi})$ is continuous, $2 \pi$-periodic, and vanishes nowhere; the condition $0<\mathcal{A} \leq$ $\mu(\boldsymbol{\xi}) \leq \mathcal{B}<\infty, \forall \boldsymbol{\xi} \in \mathbb{R}^{d}$, is obvious.

Lemma 3.4. The mask $m_{0}(\boldsymbol{\xi})$ given by (3.2) (or (3.10)), where $G(\boldsymbol{\xi})$ is given by (3.4), is even; i.e., $m_{0}\left(-\xi_{1}, \ldots,-\xi_{d}\right)=m_{0}\left(\xi_{1}, \ldots, \xi_{d}\right), \forall \boldsymbol{\xi}:=\left(\xi_{1}, \ldots, \xi_{d}\right) \in \mathbb{R}^{d}$.

Proof. Since polynomial (3.4) is even, we have

$$
\begin{aligned}
m_{0}(-\boldsymbol{\xi})=\frac{\prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G\left(-\boldsymbol{\xi}+2 \pi A^{-T} \boldsymbol{n}\right)}{\prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G\left(2 \pi A^{-T} \boldsymbol{n}\right)} & \\
& =\frac{\prod_{\boldsymbol{n}^{\prime} \in \mathcal{W}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G\left(\boldsymbol{\xi}+2 \pi A^{-T} \boldsymbol{n}^{\prime}\right)}{\prod_{\boldsymbol{n} \in \mathcal{W}\left(A^{T}\right) \backslash\{\mathbf{0}\}} G\left(2 \pi A^{-T} \boldsymbol{n}\right)}=m_{0}(\boldsymbol{\xi})
\end{aligned}
$$

(where $\left.\boldsymbol{n}^{\prime}=-\boldsymbol{n}\right)$.
Lemma 3.5. For function $\mu(\boldsymbol{\xi})$ given by (3.6), where $m_{0}(\boldsymbol{\xi})$ given by (3.2) and $G(\boldsymbol{\xi})$ by (3.4), the following estimations are valid

$$
\begin{equation*}
1-C^{\prime} P(\boldsymbol{\xi}) \leq \mu(\boldsymbol{\xi}) \leq 1+C^{\prime \prime} P(\boldsymbol{\xi}), \quad C^{\prime}, C^{\prime \prime}>0, \forall \boldsymbol{\xi} \in \mathbb{R}^{d} \tag{3.13}
\end{equation*}
$$

where $P(\boldsymbol{\xi})$ is given by (3.3).

Proof. The Taylor series expansion for trigonometric polynomial (3.4) about zero is of the form

$$
G(\boldsymbol{\xi})=P(\boldsymbol{\xi})+\sum_{\substack{k \in \mathbb{Z}_{\geq 0}^{d} \\|k|=4,6,8, \ldots}} a_{k} \boldsymbol{\xi}^{k}, \quad a_{k} \in \mathbb{R}, \boldsymbol{\xi} \in \mathbb{R}^{d}
$$

Since $m_{0}(\boldsymbol{\xi})$ is an even function, its Taylor series about zero includes only even powers:

$$
m_{0}(\boldsymbol{\xi})=1+\sum_{\substack{k \in \mathbb{Z}_{\geq 0}^{d} \\|k|=2,4,6, \ldots}} b_{k} \boldsymbol{\xi}^{k}, \quad b_{k} \in \mathbb{R}, \boldsymbol{\xi} \in \mathbb{R}^{d}
$$

Using property (2.6), we have

$$
\begin{align*}
& \mu(\boldsymbol{\xi})= q^{2 / d}\left(1+\sum_{\substack{k \in \mathbb{Z}_{\geq 0}^{d} \\
|k|=2,4,6, \ldots}} b_{k}\left(A^{-T} \boldsymbol{\xi}\right)^{k}\right)\left(P\left(A^{-T} \boldsymbol{\xi}\right)+\sum_{\substack{k \in \mathbb{Z}_{\geq 0}^{d} \\
|k|=4,6,8, \ldots}} a_{k}\left(A^{-T} \boldsymbol{\xi}\right)^{k}\right) \\
& P(\boldsymbol{\xi})+\sum_{\substack{k \in \mathbb{Z}_{\geq 0}^{d} \\
|k|=4,6,8, \ldots}} a_{k} \boldsymbol{\xi}^{k} \\
& P(\boldsymbol{\xi})+\sum_{\substack{k \in \mathbb{Z}_{\geq 0}^{d} \\
|k|=4,6,8, \ldots}} a_{k}^{\prime} \boldsymbol{\xi}^{k}  \tag{3.14}\\
& a_{k} \boldsymbol{\xi}^{k} \\
& P(\boldsymbol{\xi})+\sum_{\substack{k \in \mathbb{Z}_{\geq 0}^{d} \\
|k|=4,6,8, \ldots}} a^{\prime \prime}+\sum_{\substack{k \in \mathbb{Z}_{\geq 0}^{d} \\
|k|=2,4,6, \ldots}} a_{k}^{\prime \prime} \boldsymbol{\xi}^{k}, \quad a_{k}^{\prime}, a_{k}^{\prime \prime} \in \mathbb{R}, \quad \boldsymbol{\xi} \in \mathbb{R}^{d} .
\end{align*}
$$

By Lemma 3.3, $\mu(\boldsymbol{\xi})$ is bounded. Since expansion (3.14) includes only even powers, it follows that there exist constants $C^{\prime}, C^{\prime \prime}>0$ such that, for all $\boldsymbol{\xi} \in \mathbb{R}^{d}$, we have estimations (3.13).

In the following theorem, the convergence of infinite product (3.9) and continuity of the function $M(\boldsymbol{\xi})$ are considered.

Theorem 3.6. Let $\mu(\boldsymbol{\xi})$ be given by (3.6) and $M(\boldsymbol{\xi})$ be given by (3.9). Suppose $G(\boldsymbol{\xi})$ is of the form (3.4), then infinite product (3.9) converges absolutely and uniformly on any compact set. Moreover, the function $M(\boldsymbol{\xi})$ is continuous on $\mathbb{R}^{d}$.

Proof. From (3.13), we have

$$
|\mu(\boldsymbol{\xi})-1| \leq C P(\boldsymbol{\xi}), \quad C>0, \forall \boldsymbol{\xi} \in \mathbb{R}^{d}
$$

Using the previous inequality and property (2.6), we see that the series $\sum_{j=0}^{\infty}\left|\mu\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right)-1\right|$ uniformly converges on any compact set. Thus infinite product (3.9) also uniformly converges on any compact set.

Moreover, since $\mu(\boldsymbol{\xi})$ is continuous and the partial products converge uniformly to $M(\boldsymbol{\xi})$ on compact sets; the function $M(\boldsymbol{\xi})$ is continuous on $\mathbb{R}^{d}$.

Below we present an upper estimate of $M(\boldsymbol{\xi})$ at the infinity. In fact, this estimate is similar to the brute force estimates of the smoothness of compactly supported univariate wavelets [7, 8].

Theorem 3.7. Let the functions $\mu(\boldsymbol{\xi}), M(\boldsymbol{\xi})$ satisfy the conditions of Theorem 3.6. Suppose

$$
\begin{equation*}
\mathcal{B}:=\sup _{\boldsymbol{\xi} \in \mathbb{R}^{d}} \mu(\boldsymbol{\xi}), \tag{3.15}
\end{equation*}
$$

then we have

$$
\begin{equation*}
M(\boldsymbol{\xi}) \leq C(1+|\boldsymbol{\xi}|)^{d \log _{q} \mathcal{B}}, \quad C>0, \forall \boldsymbol{\xi} \in \mathbb{R}^{d} \tag{3.16}
\end{equation*}
$$

where $q=|\operatorname{det} A|$.
Proof. By $\mathcal{L}_{j}$ denote an ellipsoid

$$
\mathcal{L}_{j}:=\left\{\boldsymbol{\xi} \in \mathbb{R}^{d}: P(\boldsymbol{\xi}) \leq q^{2 j / d}\right\}
$$

Using (3.13), we can estimate the function $\mu(\boldsymbol{\xi})$ as follows

$$
\mu(\boldsymbol{\xi}) \leq 1+C P(\boldsymbol{\xi}) \leq \exp [C P(\boldsymbol{\xi})], \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{d} ;
$$

and, using property (2.6), we have

$$
\begin{equation*}
\sup _{\boldsymbol{\xi} \in \mathcal{L}_{0}} \prod_{j=0}^{\infty} \mu\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right) \leq \sup _{\boldsymbol{\xi} \in \mathcal{L}_{0}} \prod_{j=0}^{\infty} \exp \left[C q^{-2 j / d} P(\boldsymbol{\xi})\right]=\exp \left(C \frac{q^{2 / d}}{1-q^{2 / d}}\right) \tag{3.17}
\end{equation*}
$$

Suppose $\boldsymbol{\xi} \notin \mathcal{L}_{0}$, then there exists a number $J \geq 0$ such that $\boldsymbol{\xi} \in \mathcal{L}_{J+1} \backslash \mathcal{L}_{J}$. Thus,

$$
\begin{aligned}
& M(\boldsymbol{\xi})=\prod_{j=0}^{\infty} \mu\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right)=\prod_{j=0}^{J} \mu\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right) \prod_{j=J+1}^{\infty} \mu\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right) \\
&=\prod_{j=0}^{J} \mu\left(\left(A^{-T}\right)^{j} \boldsymbol{\xi}\right) \prod_{j=0}^{\infty} \mu\left(\left(A^{-T}\right)^{j}\left(A^{-T}\right)^{J+1} \boldsymbol{\xi}\right) .
\end{aligned}
$$

Since $\left(A^{-T}\right)^{J+1} \boldsymbol{\xi} \in \mathcal{L}_{0} ;$ by (3.17) and (3.15), we have $M(\boldsymbol{\xi}) \leq \mathcal{B}^{J+1} \exp \left(C \frac{q^{2 / d}}{1-q^{2 / d}}\right)$. If $\boldsymbol{\xi} \in$ $\mathcal{L}_{J+1} \backslash \mathcal{L}_{J}$; then $C_{1} q^{\frac{J}{d}} \leq|\boldsymbol{\xi}| \leq C_{2} q^{\frac{J+1}{d}}, C_{1}, C_{2}>0$. Consequently we have estimation (3.16).

Remark 3.1. Note that the brute force estimations are very non-optimal. But, in the present paper, we shall not consider better estimations.

On the other hand, if $\mathcal{B}$ is close to 1 (recall that $\mathcal{B} \geq 1$ ), then we can suppose that the function $M(\boldsymbol{\xi})$ decays as $|\boldsymbol{\xi}| \rightarrow \infty$. Nevertheless below we show that $M(\boldsymbol{\xi})$ does not decay at the infinity.
Theorem 3.8. Let the functions $\mu(\boldsymbol{\xi}), M(\boldsymbol{\xi})$ satisfy the conditions of Theorem 3.6, Let $\boldsymbol{k} \in \mathbb{Z}^{d}$ and $\boldsymbol{s} \in \mathcal{S}\left(A^{T}\right) \backslash\{\mathbf{0}\}$. Then, for an arbitrary large $r>|2 \pi(\boldsymbol{k}+\boldsymbol{s})|$, there exists a point $\boldsymbol{\xi} \in \mathbb{R}^{d}$, $|\boldsymbol{\xi}|>r$, such that $M(\boldsymbol{\xi})=M(2 \pi(\boldsymbol{k}+\boldsymbol{s}))>0$.
Proof. For some $J \in \mathbb{N}, \boldsymbol{k} \in \mathbb{Z}^{d}$, and $\boldsymbol{s} \in \mathcal{S}\left(A^{T}\right) \backslash\{\mathbf{0}\}$, consider the function $M(2 \pi \cdot)$ at the point $\left(A^{T}\right)^{J}(\boldsymbol{k}+\boldsymbol{s})$. Using (3.11), we have

$$
\begin{aligned}
& M\left(2 \pi\left(A^{T}\right)^{J}(\boldsymbol{k}+\boldsymbol{s})\right)=\prod_{j=0}^{\infty} \mu\left(2 \pi\left(A^{T}\right)^{J-j}(\boldsymbol{k}+\boldsymbol{s})\right) \\
& =\prod_{j=0}^{J-1} \mu\left(2 \pi\left(A^{T}\right)^{J-j}(\boldsymbol{k}+\boldsymbol{s})\right) \prod_{j=J}^{\infty} \mu\left(2 \pi\left(A^{T}\right)^{J-j}(\boldsymbol{k}+\boldsymbol{s})\right) \\
& =\prod_{j=0}^{\infty} \mu\left(2 \pi\left(A^{T}\right)^{-j}(\boldsymbol{k}+\boldsymbol{s})\right)=M(2 \pi(\boldsymbol{k}+\boldsymbol{s}))
\end{aligned}
$$

By (1.3), for an arbitrary large $r>|2 \pi(\boldsymbol{k}+\boldsymbol{s})|$, there exists a number $J \in \mathbb{N}$ such that $2 \pi\left|\left(A^{T}\right)^{J}(\boldsymbol{k}+\boldsymbol{s})\right|>r$ and $M\left(2 \pi\left(A^{T}\right)^{J}(\boldsymbol{k}+\boldsymbol{s})\right)=M(2 \pi(\boldsymbol{k}+\boldsymbol{s}))>0$.

Below we present a corollary of Proposition 1.3 ,
Corollary 3.9. Let $A$ be a $d \times d$ dilation matrix. Consider the set

$$
\Omega_{\boldsymbol{k}, \boldsymbol{s}}:=\left\{A^{j}(\boldsymbol{k}+\boldsymbol{s}): j \in \mathbb{N}\right\}, \quad \boldsymbol{k} \in \mathbb{Z}^{d}, \quad \boldsymbol{s} \in \mathcal{S}(A) \backslash\{\mathbf{0}\}
$$

then we have $\mathbb{Z}^{d} \backslash\{\mathbf{0}\}=\bigcup_{\boldsymbol{s} \in \mathcal{S}(A) \backslash\{\mathbf{0}\}}^{\boldsymbol{k} \in \mathbb{Z}^{d}} \Omega_{\boldsymbol{k}, \boldsymbol{s}}$ and $\Omega_{\boldsymbol{k}, \boldsymbol{s}} \cap \Omega_{\boldsymbol{k}^{\prime}, \boldsymbol{s}^{\prime}}=\emptyset$ if $\boldsymbol{k} \neq \boldsymbol{k}^{\prime}$ or $\boldsymbol{s} \neq \boldsymbol{s}^{\prime}$.
The proof is left to the reader.
Remark 3.2. By Corollary 3.9, Theorem 3.8 guarantees that the function $M(\boldsymbol{\xi})$ does not decay at all (including infinitely distant) points of the lattice $2 \pi \mathbb{Z}^{d}$. Whereas $G(\boldsymbol{\xi})$ vanishes at the same points $2 \pi \boldsymbol{k}, \boldsymbol{k} \in \mathbb{Z}^{d}$. Consequently we suppose that $\hat{\phi}(\boldsymbol{\xi})$ can decay faster than $1 /|\boldsymbol{\xi}|^{2}$ as $|\boldsymbol{\xi}| \rightarrow \infty$. In any case, Theorem 3.8 states that the function $\mathcal{F}^{-1} M$ can be interpreted only as a distribution.

In the next subsection, some corollaries of the properties of the function $M$ are presented.

### 3.4 Corollaries

Since any elliptic scaling function is the convolution of an elliptic spline with a distribution; we have obvious corollaries of the propositions of the previous subsection.

Corollary 3.10. If a mask $m_{0}$ is given by formula (3.2), where $G(\boldsymbol{\xi})$ is given by (3.4); then $\hat{\phi}(\boldsymbol{\xi})$ is continuous on $\mathbb{R}^{d}$.

Corollary 3.11. Under the conditions of Theorem 3.7, using formulas (3.8), (3.16), we have the following estimate

$$
\begin{equation*}
|\hat{\phi}(\boldsymbol{\xi})| \leq C(1+|\boldsymbol{\xi}|)^{d \log _{q} \mathcal{B}-2}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{d} \tag{3.18}
\end{equation*}
$$

where $q:=|\operatorname{det} A|$ and $\mathcal{B}$ is given by (3.15).

## 4 Elliptic Scaling Function of an Arbitrary Order

An elliptic scaling function of arbitrary order $m>1$ can be defined (in the Fourier domain) by the typical for the B-splines manner

$$
\begin{equation*}
\hat{\phi}^{m}(\boldsymbol{\xi}):=(\hat{\phi}(\boldsymbol{\xi}))^{m}=\frac{(G(\boldsymbol{\xi}))^{m}}{(P(\boldsymbol{\xi}))^{m}}(M(\boldsymbol{\xi}))^{m} \tag{4.1}
\end{equation*}
$$

where $G(\boldsymbol{\xi}), M(\boldsymbol{\xi})$, and $P(\boldsymbol{\xi})$ correspond to the function $\phi$, see (3.8). Moreover, for function $\phi^{m}$, the mask is $\left(m_{0}(\boldsymbol{\xi})\right)^{m}$, where $m_{0}(\boldsymbol{\xi})$ is the mask corresponding to $\phi$. In other words, the elliptic scaling function of the $m \mathrm{th}, m>1$, order is defined as follows

$$
\begin{equation*}
\phi^{m}:=\phi * \phi^{m-1}, \quad \text { where } \quad \phi^{1}:=\phi \tag{4.2}
\end{equation*}
$$

### 4.1 Properties of $\phi^{m}$

Now we consider some customary properties of the functions $\phi^{m}$. Note that these properties are similar to the properties of the univariate cardinal B-splines.

### 4.1.1 Compact support

By definition (4.2), and since any elliptic scaling function of the first order is compactly supported; the elliptic scaling function $\phi^{m}, m \geq 1$, has a compact support.

### 4.1.2 Riesz bases formation

Here we shall use ideas of the proof from the paper [22.
As is well known, to determine the Riesz bounds, we can consider the function

$$
\begin{equation*}
\sum_{\boldsymbol{k} \in \mathbb{Z}^{d}}\left|\hat{\phi}^{m}(\boldsymbol{\xi}+2 \pi \boldsymbol{k})\right|^{2} . \tag{4.3}
\end{equation*}
$$

Note that, using (3.18) and (4.1), we have $\left|\hat{\phi}^{m}(\boldsymbol{\xi})\right| \leq C(1+|\boldsymbol{\xi}|)^{m\left(d \log _{q} \mathcal{B}-2\right)}, \forall \boldsymbol{\xi} \in \mathbb{R}^{d}$, where $q:=|\operatorname{det} A|$ and $\mathcal{B}$ is given by (3.15). If $m\left(d \log _{|\operatorname{det} A|} \mathcal{B}-2\right)<-d / 2$, i. e., if

$$
\begin{equation*}
\mathcal{B}<|\operatorname{det} A|^{\frac{2}{d}-\frac{1}{2 m}}, \tag{4.4}
\end{equation*}
$$

then series (4.3) is convergent and an upper Riesz bound obviously exists. Note that, increasing the order $m$, estimation (4.4) cannot be made weaker than $\mathcal{B}<|\operatorname{det} A|^{\frac{2}{d}}$.

On the other hand, the existence of a lower bound follows from the fact that $\left|\hat{\phi}^{m}(\boldsymbol{\xi})\right|^{2}$ does not vanish in $[-\pi, \pi]^{d}$ (because $G(\boldsymbol{\xi}) / P(\boldsymbol{\xi})$ does not vanish in $[-\pi, \pi]^{d}$ ).

Remark 4.1. Note that condition (4.4) is also a sufficient condition on $\phi^{m}$ to belong to $L^{2}\left(\mathbb{R}^{d}\right)$.
In fact, $\mathcal{B}$ is determined completely by the dilation matrix (by the similarity transformation matrix $Q$ ). Note also that, since the brute force estimation is very non-optimal, we suppose that actually the restrictions on $\mathcal{B}$ can be essentially weakened. This will be discussed elsewhere.

### 4.1.3 Partition of unity

By Lemma 3.2, the partition of unity by integer shifts of the scaling function $\phi^{m}$ is obvious.

### 4.1.4 Polynomials representation

From Lemma 3.2 and formulas (3.8), (4.1), we see that $\phi^{m}$ satisfies the Strang-Fix conditions of $2 m-1$ order: $D^{n} \hat{\phi}^{m}(2 \pi \boldsymbol{k})=0, \forall \boldsymbol{k} \in \mathbb{Z}^{d} \backslash\{\mathbf{0}\}, \forall n \in \mathbb{Z}_{\geq 0}^{d}, 0 \leq|n| \leq 2 m-1$. Thus the span of integer shifts of any elliptic scaling function $\phi^{m}$ contains all the polynomials of total degree $\leq 2 m-1$.

Nevertheless the elliptic scaling functions supply an example of the functions that generalize the classical Strang-Fix conditions and can represent polynomials of higher degree.

In more detail, in the paper [6, W. Dahmen and Ch. Micchelli introduced a space of polynomials

$$
\begin{equation*}
\mathcal{V}:=\left\{p: p \in \Pi,(p(D) \hat{f})(2 \pi \boldsymbol{k})=0, \boldsymbol{k} \in \mathbb{Z}^{d} \backslash\{0\}\right\} \tag{4.5}
\end{equation*}
$$

where $\Pi$ is the space of all polynomials on $\mathbb{R}^{d}, p(D)$ is the differential operator induced by $p$, and $f$ is a compactly supported function (that, for example, belongs to $S^{\prime}\left(\mathbb{R}^{d}\right)$ ). In the paper [6], see Proposition 2.1., it has been proved that if $f$ satisfies conditions (4.5) and $\hat{f}(\mathbf{0}) \neq 0$, then the span of integer shifts of $f$ contains the affinely-invariant, i. e., scale- and shift-invariant, subspace $\mathcal{V}_{\text {aff }}$ of the space $\mathcal{V}$. Namely,

$$
\sum_{\boldsymbol{k} \in \mathbb{Z}^{d}} p(\boldsymbol{k}) f(\boldsymbol{x}-\boldsymbol{k})=\hat{f}(\mathbf{0}) p(\boldsymbol{x})+p^{*}(\boldsymbol{x}), \quad \text { where } p \in \mathcal{V}_{\mathrm{aff}} \subset \mathcal{V}, \quad \operatorname{deg} p^{*}<\operatorname{deg} p
$$

However, in the paper [2] of C. de Boor, a generalization on not scale-invariant (only shiftinvariant) polynomial spaces was considered.

The well-known functions that define nontrivial spaces $\mathcal{V}$ (i. e., $\mathcal{V} \varsubsetneqq \Pi_{N}$, where $N$ is the order of the Strang-Fix conditions) are box-splines. The elliptic scaling functions supply another example of the functions that define nontrivial spaces $\mathcal{V}$. Below we state the theorem.

Theorem 4.1. Let an elliptic scaling function $\phi^{m}$ be compactly supported and given by (4.1), where $\hat{\phi}$ is of the form (3.8). Then algebraic polynomials represented by $\phi^{m}$ belong to the nullspace of the elliptic differential operator whose symbol is $(P(\boldsymbol{\xi}))^{m}$.

The proof is omitted. And, to prove the theorem, we refer the reader to Ref. [24], where the proof is based essentially on the invariance of the quadratic form $P(\boldsymbol{\xi})$ under transformation by the dilation matrix $A^{T}$; and the proof can be performed even if the explicit form of the Fourier transform of the scaling function is not known.

Note that, in the case of the elliptic scaling functions, the shift-invariant subspace of the space $\mathcal{V}$ is also scale-invariant. Note also that $\mathcal{V}$ corresponding to $\phi^{m}$ contains polynomials of degree $\leq 2 m+1$.

### 4.1.5 Order of approximation

Let a function $f(\boldsymbol{x})$ and its derivatives up to order $2 m$ belong to $L^{2}\left(\mathbb{R}^{d}\right)$. Then the function $\phi^{m}$ supplies the $(2 m)$-order of approximation if

$$
\inf _{c(\boldsymbol{k})}\left|f(\boldsymbol{x})-\sum_{\boldsymbol{k} \in \mathbb{Z}^{d}} c(\boldsymbol{k}) \phi^{m}(\boldsymbol{x} / h-\boldsymbol{k})\right|_{L^{2}} \leq C h^{2 m}
$$

where the constant $C$ does not depend on $h$, see Refs. [1, 3, 22].
By Lemma 3.2 and formulas (3.8), (4.1), we have

$$
\forall \boldsymbol{k} \in \mathbb{Z}^{d} \backslash\{\mathbf{0}\}: \quad \hat{\phi}^{m}(\boldsymbol{\xi}+2 \pi \boldsymbol{k})=O\left(|\boldsymbol{\xi}|^{2 m}\right) \quad \text { as } \boldsymbol{\xi} \rightarrow \mathbf{0}
$$

that implies the $(2 m)$-order approximation of $\phi^{m}$.

### 4.1.6 Convolution relation

The convolution relation $\phi^{m_{1}+m_{2}}=\phi^{m_{1}} * \phi^{m_{2}}$ directly follows from definitions (4.1), (4.2).

### 4.1.7 Total positivity

Let us recall that if a function $f$ is totally positive, then the function satisfies the following conditions

$$
\forall \boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{k} \in \mathbb{R}^{d}, \quad \forall \xi_{1}, \ldots, \xi_{k} \in \mathbb{C}, \quad \sum_{i, j=1}^{k} \xi_{i} \bar{\xi}_{j} f\left(\boldsymbol{x}_{i}-\boldsymbol{x}_{j}\right) \geq 0
$$

It is known, see for example Ref. [10], that the previous conditions are equivalent to the condition: $\hat{f}(\boldsymbol{\xi}) \geq 0, \forall \boldsymbol{\xi} \in \mathbb{R}^{d}$.

From Lemmas 3.2, 3.3, it follows that $\hat{\phi}^{m}(\boldsymbol{\xi}) \geq 0, \forall \boldsymbol{\xi} \in \mathbb{R}^{d}$. So any elliptic scaling function $\phi^{m}$ is totally positive.

In the next subsection, we consider some differential properties of the elliptic scaling functions.

### 4.2 Differential properties

It is well known that the B-splines (univariate and multivariate polyharmonic) satisfy some differential (and integral) relations. Moreover, the B-splines are built by some simple (differentiable) functions; in the univariate B-splines case, it is algebraic polynomials. For the elliptic scaling functions, the situation is more complicated. Nevertheless, for any elliptic scaling function, we can introduce a (not necessarily differential) operator such that the scaling function will satisfy some (similar to differential) relations with this operator.

In detail, consider an elliptic scaling function $\phi^{m}$, see (4.1), and define the symbol of an operator as

$$
\begin{equation*}
\widehat{\Delta^{\sharp}}(\boldsymbol{\xi}):=\frac{P(\boldsymbol{\xi})}{M(\boldsymbol{\xi})} . \tag{4.6}
\end{equation*}
$$

Now we can present some relations with this operator. But before we must introduce notation.
Let $\mathcal{D}_{j}^{2}, j=1, \ldots, d$, be a finite-difference operator such that, for any function $f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^{d}$, we have $\mathcal{D}_{j}^{2} f:=f\left(\cdot-\boldsymbol{e}_{j}\right)-2 f+f\left(\cdot+\boldsymbol{e}_{j}\right)$; and let the operator $\mathcal{D}_{i j}:=\mathcal{D}_{i} \mathcal{D}_{j}, i \neq j$, be defined as $\mathcal{D}_{i j} f:=\frac{1}{4}\left(f\left(\cdot-\boldsymbol{e}_{i}-\boldsymbol{e}_{j}\right)+f\left(\cdot+\boldsymbol{e}_{i}+\boldsymbol{e}_{j}\right)-f\left(\cdot-\boldsymbol{e}_{i}+\boldsymbol{e}_{j}\right)-f\left(\cdot+\boldsymbol{e}_{i}-\boldsymbol{e}_{j}\right)\right)$.
Remark 4.2. Note that, since $\mathcal{D}_{j}^{2}$ and $\mathcal{D}_{i j}$ are the iterations of the different operators: $f\left(\cdot-\frac{1}{2} \boldsymbol{e}_{j}\right)-$ $f\left(\cdot+\frac{1}{2} \boldsymbol{e}_{j}\right)$ and $\frac{1}{2}\left(f\left(\cdot-\boldsymbol{e}_{j}\right)-f\left(\cdot+\boldsymbol{e}_{j}\right)\right)$, respectively; it follows that, if $i=j$, then the operator $\mathcal{D}_{i j}$ does not change into the operator $\mathcal{D}_{j}^{2}$.
Theorem 4.2. Let an elliptic scaling function $\phi^{m}$ be given by (4.2), see also (4.1); and let the symbol of an operator $\Delta^{\sharp}$ be given by (4.6). Then we have

$$
\begin{equation*}
\left(\Delta^{\sharp}\right)^{k} \phi^{m}=\mathcal{G}^{k} \phi^{m-k}, \quad k<m, \tag{4.7}
\end{equation*}
$$

where $\mathcal{G}$ is a finite-difference operator corresponding to trigonometric polynomial (3.4). The operator $\mathcal{G}$ is defined as: $\mathcal{G}:=-P\left(\mathcal{D}_{1}, \ldots, \mathcal{D}_{d}\right)$, where $P$ is quadratic form (3.3).
Proof. Using (4.1) and (4.6), the Fourier transform of the left-hand side of relation (4.7) can be written as

$$
\left(\frac{P(\boldsymbol{\xi})}{M(\boldsymbol{\xi})}\right)^{k} \frac{(G(\boldsymbol{\xi}))^{m}}{(P(\boldsymbol{\xi}))^{m}}(M(\boldsymbol{\xi}))^{m}=\frac{(G(\boldsymbol{\xi}))^{m}}{(P(\boldsymbol{\xi}))^{m-k}}(M(\boldsymbol{\xi}))^{m-k}=(G(\boldsymbol{\xi}))^{k} \hat{\phi}^{m-k}(\boldsymbol{\xi})
$$

So, in the $x$-domain, we have $\mathcal{G}^{k} \phi^{m-k}$, where $\mathcal{G}$ is a finite-difference operator corresponding to trigonometric polynomial (3.4).

Rewrite trigonometric polynomial (3.4) in exponent-wise form

$$
\begin{aligned}
G(\boldsymbol{\xi})=- & \sum_{1 \leq i \leq d} q_{i i}\left(e^{i \xi_{i}}+e^{-i \xi_{i}}-2\right) \\
& -\frac{1}{2} \sum_{\substack{1 \leq i, j \leq d \\
i<j}} q_{i j}\left(e^{i\left(\xi_{i}+\xi_{j}\right)}+e^{-i\left(\xi_{i}+\xi_{j}\right)}-e^{i\left(\xi_{i}-\xi_{j}\right)}-e^{-i\left(\xi_{i}-\xi_{j}\right)}\right) .
\end{aligned}
$$

It is obvious that, in the $x$-domain, the previous polynomial corresponds to the following finitedifference expression

$$
\begin{aligned}
& \mathcal{G} f=-\sum_{1 \leq i \leq d} q_{i i}\left(f\left(\cdot-\boldsymbol{e}_{j}\right)-2 f+f\left(\cdot+\boldsymbol{e}_{j}\right)\right) \\
&-\frac{1}{2} \sum_{\substack{1 \leq i, j \leq d \\
i<j}} q_{i j}\left(f\left(\cdot-\boldsymbol{e}_{i}-\boldsymbol{e}_{j}\right)+f\left(\cdot+\boldsymbol{e}_{i}+\boldsymbol{e}_{j}\right)-f\left(\cdot-\boldsymbol{e}_{i}+\boldsymbol{e}_{j}\right)-f\left(\cdot+\boldsymbol{e}_{i}-\boldsymbol{e}_{j}\right)\right) \\
&=-\sum_{1 \leq i \leq d} q_{i i} \mathcal{D}_{i}^{2} f-2 \sum_{\substack{1 \leq i, j \leq d \\
i<j}} q_{i j} \mathcal{D}_{i j} f=-P\left(\mathcal{D}_{1}, \ldots, \mathcal{D}_{d}\right) f
\end{aligned}
$$

Remark 4.3. Note that the finite-difference operator $\mathcal{G}:=-P\left(\mathcal{D}_{1}, \ldots, \mathcal{D}_{d}\right)$ is the simplest central-difference approximation of the differential operator with symbol $P(\boldsymbol{\xi})$.

The Fourier transform of the Green function of the operator $\left(\Delta^{\sharp}\right)^{m}$ (where $\widehat{\Delta^{\sharp}}(\boldsymbol{\xi})$ is given by (4.6)) is of the form

$$
\begin{equation*}
\hat{\rho}(\boldsymbol{\xi}):=\left(\frac{M(\boldsymbol{\xi})}{P(\boldsymbol{\xi})}\right)^{m}, \quad \boldsymbol{\xi} \in \mathbb{R}^{d} \tag{4.8}
\end{equation*}
$$

Indeed, in the Fourier domain, we have

$$
\mathcal{F}\left(\left(\Delta^{\sharp}\right)^{m} \rho\right)(\boldsymbol{\xi})=\left(\widehat{\Delta^{\sharp}}(\boldsymbol{\xi})\right)^{m} \hat{\rho}(\boldsymbol{\xi}) \equiv 1 .
$$

Thus, in the $x$-domain, we obtain: $\left(\Delta^{\sharp}\right)^{m} \rho(\boldsymbol{x})=\delta(\boldsymbol{x})$. And we can formulate the following proposition.
Proposition 4.3. Any elliptic scaling function $\phi^{m}$, see (4.1), (4.2), is a linear combination of shifted versions of the Green function whose Fourier transform is defined by (4.8). Moreover, we have the relation

$$
\left(\Delta^{\sharp}\right)^{m} \phi^{m}(\boldsymbol{x})=0, \quad \forall \boldsymbol{x} \in \mathbb{R}^{d} \backslash \mathbb{Z}^{d} .
$$

In the present paper, we shall not study the operators $\left(\Delta^{\sharp}\right)^{m}$ in more detail. In particular, we do not verify wether the operator defined by (4.6) is pseudo-differential. However we can state that the operators $\left(\Delta^{\sharp}\right)^{m}$ are elliptic operators. Note that the action of the operators $\left(\Delta^{\sharp}\right)^{m}$ can be explicitly obtained for some types of functions, for example, algebraic and trigonometric polynomials. Here note only that the algebraic and trigonometric polynomials under action of the operator $\left(\Delta^{\sharp}\right)^{m}$ are transformed to polynomials of lower degree: $\left(\Delta^{\sharp}\right)^{m} p=\Delta^{m} p+$ lower order terms, where $p$ is a polynomial and $\Delta$ is the elliptic operator corresponding to the quadratic form $P(\boldsymbol{\xi})$. Moreover, the elliptic scaling functions can supply the compactly supported wavelets adapted to operators (4.6), see for example Refs. [5, 12, 23. This will be the object of another paper.

Finally note that, using definition (4.6), the Fourier transform of the elliptic scaling function can be presented as

$$
\hat{\phi}^{m}(\boldsymbol{\xi})=\left(\frac{G(\boldsymbol{\xi})}{\widehat{\Delta^{\sharp}}(\boldsymbol{\xi})}\right)^{m}
$$

where $G(\boldsymbol{\xi})$ is a trigonometric polynomial such that its analytic continuation on $\mathbb{C}^{d}$ cancels all zeros of the analytic continuation of the symbol of the operator $\Delta^{\sharp}$. Actually the analytic continuation $\widehat{\Delta^{\sharp}}(\boldsymbol{z}), \boldsymbol{z} \in \mathbb{C}^{d}$, has only one zero at the origin. This will be discussed elsewhere.

### 4.3 Univariate case

The univariate cardinal B-splines correspond to the presented approach. Indeed, we see that, in the univariate case, the dilation 'matrix' is number 2 and $q:=2$. Thus $P(\xi)=\xi^{2}, \xi \in \mathbb{R}$; consequently, by (3.2), (3.4), we have

$$
G(\xi):=4 \sin ^{2} \frac{\xi}{2}=2(1-\cos \xi) \quad \text { and } \quad m_{0}(\xi):=\frac{G(\xi+\pi)}{G(\pi)}=\frac{1+\cos \xi}{2}
$$

Using (3.6), (3.8), we get

$$
\begin{aligned}
& \mu(\xi)=\frac{2(1+\cos \xi / 2)(1-\cos \xi / 2)}{1-\cos \xi} \equiv 1 \\
& \hat{\phi}(\xi)=\frac{2(1-\cos \xi)}{\xi^{2}}=\frac{\sin ^{2} \xi / 2}{(\xi / 2)^{2}}=: \hat{B}_{1}(\xi) .
\end{aligned}
$$

Hence the univariate elliptic scaling function $\phi$ is the cardinal B-spline of degree 1. Moreover, the univariate scaling function $\phi^{m}$ is the cardinal B-spline of odd degree $2 m-1$ and vice versa.

We can say that the univariate cardinal B-splines satisfy the properties of Theorem 4.1 In fact, the B-spline $B_{2 m-1}(x), x \in \mathbb{R}$, can be considered as the elliptic scaling function $\phi^{m}$ and $B_{2 m-1}$ represents all the polynomials of degree $\leq 2 m-1$. In the univariate case, all the polynomials of degree $\leq 2 m-1$ belong to the null-space of the operator $\frac{d^{2 m}}{d x^{2 m}}$, but there are not polynomials of degree greater than $2 m-1$ that belong to ker $\frac{d^{2 m}}{d x^{2 m}}$.

## 5 Examples

In this section, we present several bivariate isotropic dilation matrices and consider some properties of the corresponding elliptic scaling functions of the first order.

## $5.1 \operatorname{det} A=2$

In the bivariate case, the dilation matrix $A$ with determinant 2 can be isotropic if and only if $0 \leq \operatorname{Tr} A \leq 2$, see for example Ref. [25].

### 5.1.1 $\operatorname{Tr} A=2$

Consider a quincunx dilation matrix

$$
A_{1}:=\left(\begin{array}{cc}
1 & -1  \tag{5.1}\\
1 & 1
\end{array}\right)
$$

For $A_{1}, \mathcal{S}:=\{\mathbf{0},(1 / 2,1 / 2)\}$; and the matrix $A_{1}$ is isotropic. Thus the matrix $A_{1}^{-T}$ can be presented in the form (2.4), where $U$ is a rotation matrix by the angle $\pi / 4$ and $Q$ is the identity matrix. Hence the quadratic form $P(\boldsymbol{\xi})$ is $|\boldsymbol{\xi}|^{2}, \boldsymbol{\xi} \in \mathbb{R}^{2}$; consequently the corresponding differential operator is the Laplace operator. The trigonometric polynomial $G\left(\xi_{1}, \xi_{2}\right):=$ $4\left(\sin ^{2}\left(\xi_{1} / 2\right)+\sin ^{2}\left(\xi_{2} / 2\right)\right)$ and the mask is of the form

$$
\begin{equation*}
m_{0,1}\left(\xi_{1}, \xi_{2}\right):=\frac{1}{2}+\frac{1}{4} \cos \xi_{1}+\frac{1}{4} \cos \xi_{2}, \tag{5.2}
\end{equation*}
$$

and $\mathcal{B}$ (given by (3.15)) is equal to 1 .
Matrix (5.1) and the corresponding scaling function, denoted by $\phi_{1}$, are 'very nice' for several reasons.

- The factorization of the dilation matrix is very simple.
- The matrix is a quincunx dilation matrix, i.e., it defines the quincunx lattice.
- The quadratic form is $|\boldsymbol{\xi}|^{2}, \boldsymbol{\xi} \in \mathbb{R}^{2}$, which corresponds to the Laplace operator.
- Since $\mathcal{B}=1$, it follows that $\left|\hat{\phi}_{1}(\boldsymbol{\xi})\right| \leq C(1+|\boldsymbol{\xi}|)^{-2}, \forall \boldsymbol{\xi} \in \mathbb{R}^{2}$. Consequently $\phi_{1}$ forms a Riesz basis and belongs to $L^{2}\left(\mathbb{R}^{2}\right)$.
- The elliptic spline

$$
\mathcal{F}^{-1}\left(\frac{G(\boldsymbol{\xi})}{P(\boldsymbol{\xi})}\right)(\boldsymbol{x})=\mathcal{F}^{-1}\left(\frac{4\left(\sin ^{2}\left(\xi_{1} / 2\right)+\sin ^{2}\left(\xi_{2} / 2\right)\right)}{\xi_{1}^{2}+\xi_{2}^{2}}\right)(\boldsymbol{x})
$$

is the polyharmonic B -spline $B_{1}$, see (1.1).

- The elliptic operator with symbol $\widehat{\Delta^{\sharp}}(\boldsymbol{\xi})=\frac{|\boldsymbol{\xi}|^{2}}{M(\boldsymbol{\xi})}$ is uniformly elliptic.
- The scaling function is an interpolating scaling function, i.e., $\phi_{1}(\boldsymbol{k})=\delta_{\boldsymbol{k} \mathbf{0}}, \boldsymbol{k} \in \mathbb{Z}^{2}$. (Because $m_{0,1}\left(\xi_{1}, \xi_{2}\right)+m_{0,1}\left(\xi_{1}+\pi, \xi_{2}+\pi\right) \equiv 1$.)
- The scaling function $\phi_{1}(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}=\left(\frac{k_{1}}{2^{j_{1}}}, \frac{k_{2}}{2^{j 2}}\right), k_{1}, k_{2} \in \mathbb{Z}, j_{1}, j_{2} \in \mathbb{Z}_{\geq 0}$ (because all the coefficients $h_{\boldsymbol{k}}$ in the refinement relation, see (1.4), are not negative); and we can suppose that $\phi_{1}(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \mathbb{R}^{2}$.
- The more traditional quincunx dilation matrix

$$
\tilde{A}_{1}:=\left(\begin{array}{cc}
1 & 1  \tag{5.3}\\
1 & -1
\end{array}\right)
$$

gives the same scaling function. (Because mask (5.2) is even and invariant with respect to the change of the variables.)

Remark 5.1. Mask (5.2) has been considered in the book [19] in the context of dual masks construction (for the quincunx dilation matrix case).

### 5.1.2 $\operatorname{Tr} A=1$

In this subsection, we consider two very close matrices, but the corresponding elliptic scaling functions have different properties

$$
A_{2}:=\left(\begin{array}{cc}
0 & -2  \tag{5.4}\\
1 & 1
\end{array}\right), \quad A_{3}:=\left(\begin{array}{cc}
1 & -2 \\
1 & 0
\end{array}\right)
$$

Since the determinants and traces of dilation matrices (5.4) are equal; both the matrices are similar to the same orthogonal (actually, rotation) matrix, see for example Ref. [25],

$$
U_{2}=U_{3}:=\left(\begin{array}{cc}
\frac{1}{2 \sqrt{2}} & -\frac{\sqrt{7}}{2 \sqrt{2}}  \tag{5.5}\\
\frac{\sqrt{7}}{2 \sqrt{2}} & \frac{1}{2 \sqrt{2}}
\end{array}\right)
$$

The squares of the corresponding similarity transformation matrices are

$$
Q_{2}^{2}:=\left(\begin{array}{cc}
2 & -\frac{1}{2} \\
-\frac{1}{2} & 1
\end{array}\right), \quad Q_{3}^{2}:=\left(\begin{array}{cc}
2 & \frac{1}{2} \\
\frac{1}{2} & 1
\end{array}\right),
$$

respectively. Thus the masks are

$$
\begin{aligned}
& m_{0,2}\left(\xi_{1}, \xi_{2}\right):=\frac{1}{4}\left(3+2 \cos \xi_{1}-\cos \xi_{2}+\frac{1}{2} \sin \xi_{1} \sin \xi_{2}\right) \\
& m_{0,3}\left(\xi_{1}, \xi_{2}\right):=\frac{1}{6}\left(3+2 \cos \xi_{1}+\cos \xi_{2}+\frac{1}{2} \sin \xi_{1} \sin \xi_{2}\right)
\end{aligned}
$$

respectively.
Nevertheless, the decays of the Fourier transform of the corresponding elliptic scaling functions are very different. Indeed, $\mathcal{B}_{2}=2$ and $\mathcal{B}_{3}=\frac{25}{24}$; consequently $\hat{\phi}_{2}(\boldsymbol{\xi})=O(1)$ (see Remark (3.1) and $\left|\hat{\phi}_{3}(\boldsymbol{\xi})\right| \leq C(1+|\boldsymbol{\xi}|)^{-1.882}, C>0, \forall \boldsymbol{\xi} \in \mathbb{R}^{2}$. Thus the function $\phi_{3}$ forms a Riesz basis and belongs to $L^{2}\left(\mathbb{R}^{2}\right)$; and the function $\phi_{2}$ can be considered very likely only as a distribution.
Remark 5.2. The angle of rotation $\arccos \frac{1}{2 \sqrt{2}} \approx 69.2951889^{\circ}$ performed by rotation matrix (5.5) is incommensurable to $\pi$, see Ref. [25]. So we can suppose that if the corresponding scaling functions possess some angular selectivity; then the wavelet transform can detect arbitrary orientated features.

## $5.2 \operatorname{det} A=4$

Finally we consider a diagonal dilation matrix without any rotation (for this matrix, the orthogonal matrix is the identity matrix)

$$
A_{4}:=\left(\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right)
$$

Nevertheless, for this matrix, it is also possible to obtain the elliptic scaling function. Indeed, the similarity transformation matrix is the identity matrix, hence the quadratic form is $|\boldsymbol{\xi}|^{2}, \boldsymbol{\xi} \in \mathbb{R}^{2}$, and the mask is of the form

$$
m_{0,4}\left(\xi_{1}, \xi_{2}\right)=\frac{1}{16}\left(2+\cos \xi_{1}+\cos \xi_{2}\right)\left(2+\cos \xi_{1}-\cos \xi_{2}\right)\left(2-\cos \xi_{1}+\cos \xi_{2}\right)
$$

We have $\mathcal{B}_{4}:=\frac{9}{8}$, consequently the scaling function $\phi_{4}$ corresponding to $A_{4}$ and $m_{0,4}$ satisfies the estimation $\left|\hat{\phi}_{4}(\boldsymbol{\xi})\right| \leq C(1+|\boldsymbol{\xi}|)^{-1.8301}, C>0, \forall \boldsymbol{\xi} \in \mathbb{R}^{2}$. Thus $\phi_{4}$ belongs to $L^{2}\left(\mathbb{R}^{2}\right)$ and forms a Riesz basis.

Note that, in spite of the fact that the matrix $A_{4}=\tilde{A}_{1}^{2}$, where $\tilde{A}_{1}$ is given by (5.3), the mask $m_{0,4}(\boldsymbol{\xi}) \not \equiv m_{0,1}\left(\tilde{A}_{1}^{T} \boldsymbol{\xi}\right) m_{0,1}(\boldsymbol{\xi})$, where $m_{0,1}$ is given by (5.2); and consequently $\phi_{4}$ is not the convolution of the scaling functions from subsection 5.1.1.

## Conclusion

In conclusion, we can summarize that the elliptic scaling functions have all the basic properties of the univariate cardinal B-splines and can be considered as a generalization of the B-splines to several variables.

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