

Nonstationary Gabor Frames - Existence and Construction

Monika Dörfler, Ewa Matusiak*

Department of Mathematics, NuHAG, University of Vienna, Austria

Abstract

Nonstationary Gabor frames were recently introduced in adaptive signal analysis. They represent a natural generalization of classical Gabor frames by allowing for adaptivity of windows and lattice in either time or frequency. In this paper we show a general existence result for this family of frames. We then give a perturbation result for nonstationary Gabor frames and construct nonstationary Gabor frames with non-compactly supported windows from a related painless nonorthogonal expansion. Finally, the theoretical results are illustrated by two examples of practical relevance.

Keywords: adaptive representations, nonorthogonal expansions, irregular Gabor frames, existence

1. Introduction

The principal idea of Gabor frames was introduced in [14] with the aim to represent signals in a time-frequency localized manner. Since the work of Gabor himself, a lot of research has been done on the topic of atomic time-frequency representation. While it turned out that the original model proposed by Gabor does not yield stable representations in the sense of frames [4, 7, 10], the existence of Gabor frames was first established in the so called painless case, [7], which requires the use of compactly supported analysis windows. The existence of Gabor frames in more general situations was proved later [19, 23] and the proof often uses an argument invoking the invertibility of diagonally dominant matrices.

Various irregular and adaptive versions of Gabor frames have been introduced over the years, cf. [1, 5, 12, 21]. In these approaches, the irregularity usually concerns either the sampling set, which is allowed to deviate from a lattice, or the window, which is allowed to be modified. In [1], varying windows as well as irregular sampling points are allowed, however, existence of a local frame is assumed, from which a global frame is constructed. Nonstationary Gabor frames give up the strict regularity of the classical Gabor setting, but, as opposed to irregular frames, maintain enough structure to guarantee efficient implementation and, possibly approximate, efficient reconstruction. In analogy to the classical, regular case [7], painless nonstationary Gabor frames were introduced in [2], where the principal idea is described and illustrated in detail. The construction of painless nonstationary Gabor

*Corresponding author

Email addresses: monika.doerfler@univie.ac.at (Monika Dörfler), ewa.matusiak@univie.ac.at (Ewa Matusiak)

frames is similar to, but more flexible than the construction of windowed modified cosine transforms and other lapped transforms [17, 24] that allow for adaptivity of the window length. An efficient and perfectly invertible constant-Q transform was recently introduced using nonstationary Gabor transforms [22]. In this and similar situations, redundancy of the transform is crucial, since non-redundant versions of the constant-Q transform lead to dyadic wavelet transforms, which are often inappropriate for audio signal processing. Redundancy allows for good localization of both the analysis and synthesis windows, and their respective Fourier transforms and often promote sparse representations in adaptive processing.

Painless non-orthogonal expansions can only be devised if the involved analysis windows are either compactly supported or band limited. This requirement may sometimes be too restrictive. For instance, one may be interested in designing frequency-adaptive nonstationary Gabor frames with windows that are compactly supported in time, i.e. can be implemented as FIR filters, cf. [11] and lend themselves to real-time implementation, cp. [9].

The present contribution addresses the case of nonstationary Gabor frames with more general windows than used in the painless case. In Theorem 3.4, we derive the existence of nonstationary Gabor frames directly from a generalized Walnut representation: under mild uniform decay conditions on all windows involved, we show an existence result of nonstationary Gabor frames in parallel to the result given in [23] for regular Gabor frames, also compare [15, Theorem 6.5.1]. Note that the existence of a different class of nonstationary Gabor frames, the *quilted Gabor frames*, [8], was recently proved in the general context of spline type spaces in the remarkable paper [18].

This paper is organized as follows. In the next section, we introduce notation and state some auxiliary results. In Section 3, we first define nonstationary Gabor frames and recall known results for the painless case. In Section 3.2 a Walnut representation and a corresponding bound of the frame operator in the general setting is derived and Section 3.3 provides the existence of nonstationary Gabor frames. In Section 3.4 we pursue two basic approaches for the construction of nonstationary Gabor frames. Using tools from the theory of perturbation of frames, we construct nonstationary frames from an existing frame in Proposition 3.7. In Corollary 3.8 we design nonstationary Gabor frames by exploiting knowledge about a related painless frame, to obtain "almost painless nonstationary Gabor frames". In Section 4 we provide examples based on the two introduced construction principles.

2. Notation and Preliminaries

Given a non-zero function $g \in L^2(\mathbb{R})$, let $g_{k,l}(t) = M_{bl}T_{ak}g(t) := e^{2\pi i b l t}g(t - ak)$. M_{bl} is a modulation operator, or frequency shift, and T_{ak} is a time shift.

The set $\mathcal{G}(g, a, b) = \{g_{k,l} : k, l \in \mathbb{Z}\}$ is called a Gabor system for any real, positive a, b . $\mathcal{G}(g, a, b)$ is a Gabor frame for $L^2(\mathbb{R})$, if there exist frame bounds $0 < A \leq B < \infty$ such that for every $f \in L^2(\mathbb{R})$ we have

$$A\|f\|_2^2 \leq \sum_{k,l \in \mathbb{Z}} |\langle f, g_{k,l} \rangle|^2 \leq B\|f\|_2^2. \quad (1)$$

To every Gabor system, we associate the analysis operator C_g given by $(C_g f)_{k,l} = \langle f, g_{k,l} \rangle$, and the synthesis operator $U_\gamma = C_\gamma^*$, given by $U_\gamma c = \sum_{k,l \in \mathbb{Z}} c_{k,l} \gamma_{k,l}$ for $c \in \ell^2$. The operator

$S_{g,\gamma}$ associated to $\mathcal{G}(g, a, b)$ and $\mathcal{G}(\gamma, a', b')$, where $S_{g,\gamma} = U_\gamma C_g$ reads

$$S_{g,\gamma} f = \sum_{k,l \in \mathbb{Z}} \langle f, g_{k,l} \rangle \gamma_{k,l}.$$

The inequality (1) is equivalent to the invertibility and boundedness of the frame operator $S_{g,g}$ of $\mathcal{G}(g, a, b)$.

The analysis operator is the sampled short-time Fourier transform (STFT). For a fixed window $g \in L^2(\mathbb{R})$, the STFT of $f \in L^2(\mathbb{R})$ is

$$V_g f(x, \omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} \overline{g(t-x)} dt = \langle f, M_\omega T_x g \rangle.$$

Setting $(x, \omega) = (ak, bl)$, leads to $V_g f(ak, bl) = (C_g f)_{k,l}$.

When working with irregular grids, we assume that the sampling points form a separated set: a set of sampling points $\{a_k : k \in \mathbb{Z}\}$ is called δ -separated, if $|a_k - a_m| > \delta$ for a_k, a_m , whenever $k \neq m$. χ_I will denote the characteristic function of the interval I .

A convenient class of window functions for time-frequency analysis on $L^2(\mathbb{R})$ is the Wiener space.

Definition 2.1. A function $g \in L^\infty(\mathbb{R})$ belongs to the Wiener space $W(L^\infty, \ell^1)$ if

$$\|g\|_{W(L^\infty, \ell^1)} := \sum_{k \in \mathbb{Z}} \text{ess sup}_{t \in Q} |g(t+k)| < \infty, \quad Q = [0, 1].$$

For $g \in W(L^\infty, \ell^1)$ and $\delta > 0$ we have [15]

$$\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g(t - \delta k)| \leq (1 + \delta^{-1}) \|g\|_{W(L^\infty, \ell^1)}. \quad (2)$$

In dealing with polynomially decaying windows, we will repeatedly use the following lemma.

Lemma 2.2. For $p > 1$ the following estimates hold:

(a) Let $\delta > 0$, then

$$\sum_{k=1}^{\infty} (1 + \delta k)^{-p} \leq (1 + \delta)^{-p} (\delta^{-1} + p)(p-1)^{-1}.$$

(b) Let $\{a_k : k \in \mathbb{Z}\} \subset \mathbb{R}$ be a δ -separated set. Then

$$\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p} \leq 2 (1 + (1 + \delta)^{-p} (\delta^{-1} + p)(p-1)^{-1}).$$

Proof. To show (a) we write

$$\sum_{k=1}^{\infty} (1 + \delta k)^{-p} = (1 + \delta)^{-p} + \sum_{k=2}^{\infty} (1 + \delta k)^{-p} = (1 + \delta)^{-p} + \sum_{k=2}^{\infty} \int_{[0,1]+k} (1 + \delta k)^{-p} dt.$$

for $t \in [k, k+1]$, we have $\delta t \leq \delta(k+1)$ which implies that $1 + \delta(t-1) \leq 1 + \delta k$. Therefore,

$$\begin{aligned} \sum_{k=2}^{\infty} \int_{[0,1]+k} (1 + \delta k)^{-p} dt &\leq \sum_{k=2}^{\infty} \int_{[0,1]+k} (1 + \delta(t-1))^{-p} dt = \int_2^{\infty} (1 + \delta(t-1))^{-p} dt \\ &= (1 + \delta)^{-p+1} \delta^{-1} (p-1)^{-1}, \end{aligned}$$

and the estimate follows.

To prove (b), fix $t \in \mathbb{R}$. Since $|a_k - a_l| > \delta$ for $k \neq l$, each interval of length δ contains at most one point $t - a_k$, $k \in \mathbb{Z}$. Therefore we may write $t - a_k = \delta n_k + x_k$ for unique $n_k \in \mathbb{Z}$ and $x_k \in [0, \delta)$, and by the choice of δ , we have $n_k \neq n_l$ for $k \neq l$. We assume, without loss of generality, that $n_k = 0$ for $k = 0$, and we find that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p} &= \sum_{k \in \mathbb{Z}} (1 + |\delta n_k + x_k|)^{-p} \\ &\leq 1 + \sum_{k \in \mathbb{Z}; n_k > 0} (1 + \delta n_k + x_k)^{-p} + \sum_{k \in \mathbb{Z}; n_k < 0} (1 + \delta n_k - x_k)^{-p} \\ &\leq 1 + \sum_{k \in \mathbb{Z}; n_k > 0} (1 + \delta n_k)^{-p} + \sum_{k \in \mathbb{Z}; n_k < 0} (1 + \delta n_k - \delta)^{-p} \\ &\leq 1 + \sum_{k=1}^{\infty} (1 + \delta k)^{-p} + \sum_{k=1}^{\infty} (1 + \delta(k-1))^{-p} \\ &= 2 \left(1 + \sum_{k=1}^{\infty} (1 + \delta k)^{-p} \right) \leq 2 (1 + (1 + \delta)^{-p} (\delta^{-1} + p)(p-1)^{-1}). \end{aligned}$$

The last expression is independent of t , and the claim follows. \square

Remark 1. When the set $\mathcal{A} = \{a_k : k \in \mathbb{Z}\} \subset \mathbb{R}$ is relatively δ -separated, meaning

$$\text{rel}(\mathcal{A}) := \max_{t \in \mathbb{R}} \# \{ \mathcal{A} \cap ([0, \delta] + t) \} < \infty,$$

then the estimate (b) in Lemma 2.2 becomes

$$\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p} \leq 2 \text{rel}(\mathcal{A}) (1 + (1 + \delta)^{-p} (\delta^{-1} + p)(p-1)^{-1}).$$

Notice, that for a separated set, $\text{rel}(\mathcal{A}) = 1$.

3. Nonstationary Gabor frames

Nonstationary Gabor systems provide a generalization of the classical Gabor systems of time-frequency-shifted versions of a single window function.

Definition 3.1. Let $\mathbf{g} = \{g_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\}$ be a set of window functions and let $\mathbf{b} = \{b_k : k \in \mathbb{Z}\}$ be a corresponding sequence of frequency-shift parameters. Set $g_{k,l} = M_{b_k l} g_k$. Then, the set

$$\mathcal{G}(\mathbf{g}, \mathbf{b}) = \{g_{k,l} : k, l \in \mathbb{Z}\}$$

is called a *nonstationary Gabor system*.

Note that, conceptually, we assume that the windows g_k are centered at points $\{a_k : k \in \mathbb{Z}\}$, in direct generalization of the regular case, where $g_k(t) = g(t - ak)$ for some time-shift parameter a . In this sense, we have a two-fold generalization: the sampling points can be irregular *and* the windows can change for every sampling point. We are interested in conditions under which a nonstationary Gabor system forms a frame. We first recall the case of nonstationary Gabor frames with compactly supported windows, see [2] and <http://www.univie.ac.at/nonstatgab/> for further information.

3.1. Compactly supported windows: the painless case

Based on the support length of the windows g_k , we can easily determine frequency-shifts parameters b_k , for which we obtain a frame. The following result is the nonstationary version of the result given in [7].

Proposition 3.2 ([2]). *Let $\mathbf{g} = \{g_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\}$ be a collection of compactly supported functions with $|\text{supp } g_k| \leq 1/b_k$. Then $\mathcal{G}(\mathbf{g}, \mathbf{b})$ is a frame for $L^2(\mathbb{R})$ if there exist constants $A > 0$ and $B < \infty$ such that*

$$A \leq G_0(t) = \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 \leq B \quad \text{a.e. .}$$

The dual atoms are then $\gamma_{k,l}(t) = M_{lb_k} G_0^{-1}(t) g_k(t)$.

Remark 2. An analogous theorem holds for bandlimited functions g_k .

The above theorem follows from the fact that the frame operator associated to the collection of atoms described in the theorem can be written as

$$S_{g,g} f(t) = \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 f(t) \quad \text{a.e. .}$$

The diagonality of the frame operator in the painless case is derived from a generalized Walnut representation for the frame operator $S_{g,g}$ of nonstationary Gabor frames. In the next section we will see that this representation immediately implies diagonality of $S_{g,g}$ under the assumptions of Proposition 3.2.

3.2. A Walnut representation for nonstationary Gabor Frames

Let us now consider nonstationary Gabor systems $\mathcal{G}(\mathbf{g}, \mathbf{b})$ and $\mathcal{G}(\gamma, \mathbf{b})$, with all windows g_k and γ_k in $W(L^\infty, \ell^1)$. The operator associated to $\mathcal{G}(\mathbf{g}, \mathbf{b})$ and $\mathcal{G}(\gamma, \mathbf{b})$ reads

$$S_{g,\gamma} f = \sum_{k,l \in \mathbb{Z}} \langle f, M_{lb_k} g_k \rangle M_{lb_k} \gamma_k. \quad (3)$$

Proposition 3.3. *The operator $S_{g,\gamma}$ in (3) admits a Walnut representation*

$$S_{g,\gamma} f = \sum_{k,l \in \mathbb{Z}} G_{k,l}^{g,\gamma} \cdot T_{lb_k^{-1}} f, \quad \text{where} \quad G_{k,l}^{g,\gamma}(t) = b_k^{-1} \overline{g_k(t - lb_k^{-1})} \gamma_k(t), \quad (4)$$

for $f \in L^2(\mathbb{R})$. Moreover, its operator norm can be bounded by

$$\begin{aligned} |\langle S_{g,\gamma} f, h \rangle| &\leq \left(\sup_{k \in \mathbb{Z}} (1 + b_k^{-1}) \|\gamma_k\|_{W(L^\infty, \ell^1)} \right)^{1/2} \left(\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)| \right)^{1/2} \\ &\quad \cdot \left(\sup_{k \in \mathbb{Z}} (1 + b_k^{-1}) \|g_k\|_{W(L^\infty, \ell^1)} \right)^{1/2} \left(\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\gamma_k(t)| \right)^{1/2} \|f\|_2 \|h\|_2 \end{aligned} \quad (5)$$

for all $f, h \in L^2(\mathbb{R})$.

Proof. First assume that $f, h \in L^2(\mathbb{R})$ are compactly supported. Since $\langle f, M_{lb_k} g_k \rangle = \widehat{(f \bar{g}_k)}(lb_k)$ we can write $S_{g,\gamma}$ as

$$S_{g,\gamma} f(t) = \sum_{k,l \in \mathbb{Z}} \widehat{(f \bar{g}_k)}(lb_k) M_{lb_k} \gamma_k(t) = \sum_{k \in \mathbb{Z}} m_k(t) \gamma_k(t), \quad (6)$$

where $m_k(t) = \sum_{l \in \mathbb{Z}} \widehat{(f \bar{g}_k)}(lb_k) e^{2\pi i l b_k t}$, for every $k \in \mathbb{Z}$. The functions m_k are b_k^{-1} periodic and by the Poisson formula can be written as

$$m_k(t) = b_k^{-1} \sum_{l \in \mathbb{Z}} (f \bar{g}_k)(t - lb_k^{-1}). \quad (7)$$

Therefore, substituting (7) in (6) yields the Walnut representation.

We next prove the boundedness (5). In the following chain of inequalities, we will use Cauchy-Schwartz inequality for sums and integrals and, since all summands have absolute value, Fubini's theorem to justify changing the order of summation and integral. We thus find

$$\begin{aligned} |\langle S_{g,\gamma} f, h \rangle| &= \left| \left\langle \sum_{k,l \in \mathbb{Z}} b_k^{-1} \overline{g_k(\cdot - lb_k^{-1})} \gamma_k(\cdot) f(\cdot - lb_k^{-1}), h \right\rangle \right| \\ &\leq \sum_{k,l \in \mathbb{Z}} b_k^{-1} \int_{\mathbb{R}} |g_k(t - lb_k^{-1})| |\gamma_k(t)| |f(t - lb_k^{-1})| |h(t)| dt \\ &\leq \sum_{k,l \in \mathbb{Z}} b_k^{-1} \left[\int_{\mathbb{R}} |g_k(t - lb_k^{-1})| |\gamma_k(t)| |f(t - lb_k^{-1})|^2 dt \right]^{1/2} \left[\int_{\mathbb{R}} |g_k(t - lb_k^{-1})| |\gamma_k(t)| |h(t)|^2 dt \right]^{1/2} \\ &\leq \left[\sum_{k,l \in \mathbb{Z}} b_k^{-1} \int_{\mathbb{R}} |g_k(t)| |\gamma_k(t + lb_k^{-1})| |f(t)|^2 dt \right]^{1/2} \left[\sum_{k,l \in \mathbb{Z}} b_k^{-1} \int_{\mathbb{R}} |\gamma_k(t)| |g_k(t - lb_k^{-1})| |h(t)|^2 dt \right]^{1/2} \\ &= \left[\int_{\mathbb{R}} |f(t)|^2 \sum_{k,l \in \mathbb{Z}} b_k^{-1} |g_k(t)| |\gamma_k(t - lb_k^{-1})| dt \right]^{1/2} \left[\int_{\mathbb{R}} |h(t)|^2 \sum_{k,l \in \mathbb{Z}} b_k^{-1} |\gamma_k(t)| |g_k(t - lb_k^{-1})| dt \right]^{1/2}. \end{aligned} \quad (8)$$

The first term in the last expression can be bounded as follows

$$\begin{aligned}
\int_{\mathbb{R}} |f(t)|^2 \sum_{k,l \in \mathbb{Z}} b_k^{-1} |g_k(t)| |\gamma_k(t - lb_k^{-1})| dt &= \sum_{k \in \mathbb{Z}} b_k^{-1} \int_{\mathbb{R}} \sum_{l \in \mathbb{Z}} |\gamma_k(t - lb_k^{-1})| |f(t)|^2 |g_k(t)| dt \\
&\leq \sum_{k \in \mathbb{Z}} \left(b_k^{-1} \operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |\gamma_k(t - lb_k^{-1})| \right) \int_{\mathbb{R}} |f(t)|^2 |g_k(t)| dt \\
&\leq \sup_{k \in \mathbb{Z}} \left(b_k^{-1} \operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |\gamma_k(t - lb_k^{-1})| \right) \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |f(t)|^2 |g_k(t)| dt \\
&\leq \sup_{k \in \mathbb{Z}} \left(b_k^{-1} \operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |\gamma_k(t - lb_k^{-1})| \right) \left(\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)| \right) \|f\|_2^2. \quad (9)
\end{aligned}$$

Using relation (2) for $\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |\gamma_k(t - lb_k^{-1})|$ and substituting (9) into (8) yields (5). The estimate for the second term in (8) is obtained analogously. By the density of compactly supported functions in $L^2(\mathbb{R})$, the estimate holds for all of $L^2(\mathbb{R})$. \square

Remark 3. From (8) in the proof of Proposition 3.3, it follows that the operator $S_{g,\gamma}$ is also bounded by

$$\begin{aligned}
|\langle S_{g,\gamma} f, h \rangle| &\leq \left(\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{k,l \in \mathbb{Z}} b_k^{-1} |g_k(t - lb_k^{-1})| |\gamma_k(t)| \right)^{1/2} \\
&\quad \cdot \left(\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{k,l \in \mathbb{Z}} b_k^{-1} |\gamma_k(t - lb_k^{-1})| |g_k(t)| \right)^{1/2} \|f\|_2 \|h\|_2.
\end{aligned}$$

Remark 4. In the case of a frame operator $S_{g,g}$, the Walnut representation (4) becomes

$$S_{g,g} f(t) = \sum_{k,l \in \mathbb{Z}} b_k^{-1} \overline{g_k(t - lb_k^{-1})} g_k(t) f(t - lb_k^{-1}) \quad \text{a.e.}, \quad (10)$$

and the above bounds reduce to

$$\begin{aligned}
|\langle S_{g,g} f, h \rangle| &\leq \left(\sup_{k \in \mathbb{Z}} (1 + b_k^{-1}) \|g_k\|_{W(L^\infty, \ell^1)} \right) \left(\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)| \right) \|f\|_2 \|h\|_2 \\
|\langle S_{g,g} f, h \rangle| &\leq \left(\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{k,l \in \mathbb{Z}} b_k^{-1} |g_k(t - lb_k^{-1})| |g_k(t)| \right) \|f\|_2 \|h\|_2
\end{aligned}$$

Remark 5. Note that in the painless case of Theorem 3.2, the frame operator reduces to the multiplication operator $S_{g,g} f = \sum_{k \in \mathbb{Z}} G_{k,0}^{g,g} \cdot f = G_0 \cdot f$.

Remark 6. In the standard setting of Gabor frames, i.e. $g_k(t) = g(t - ak)$ for fixed $a > 0$, and $b_k = b$ for all $k \in \mathbb{Z}$, the above bound reduces to the well know bound

$$|\langle S_{g,g} f, h \rangle| \leq (1 + a^{-1})(1 + b^{-1}) \|g\|_{W(L^\infty, \ell^1)}^2 \|f\|_2 \|h\|_2$$

3.3. Existence of nonstationary Gabor frames

For windows g_k that are neither compactly supported nor bandlimited, we are interested in the existence of frames of the form $\mathcal{G}(\mathbf{g}, \mathbf{b})$ and in the construction of the involved parameters. The following theorem derives a sufficient condition for the existence of nonstationary Gabor frames and shows that this condition can be satisfied.

In this and the subsequent sections, $[b_L, b_U]$, $[p_L, p_U]$, $[C_L, C_U]$ are compact intervals of positive real numbers.

Theorem 3.4. *Let $\mathbf{g} = \{g_k \in W(L^\infty, \ell^1) : k \in \mathbb{Z}\}$ be a set of windows such that*

i) for some positive constants A_0, B_0

$$0 < A_0 \leq \sum_{k \in \mathbb{Z}} |g_k(t)|^2 \leq B_0 < \infty \quad \text{a.e.}; \quad (11)$$

ii) for all $k \in \mathbb{Z}$, the windows decay polynomially around a δ -separated set $\{a_k : k \in \mathbb{Z}\}$ of time-sampling points a_k

$$|g_k(t)| \leq C_k(1 + |t - a_k|)^{-p_k}, \quad (12)$$

where $p_k \in [p_L, p_U] \subset \mathbb{R}$, $p_L > 2$ and $C_k \in [C_L, C_U]$.

Then there exists a sequence $\{b_k^0\}_{k \in \mathbb{Z}}$, such that for $b_k \leq b_k^0$, $k \in \mathbb{Z}$, the nonstationary Gabor system $\mathcal{G}(\mathbf{g}, \mathbf{b})$ forms a frame for $L^2(\mathbb{R})$.

Proof. Let $f \in L^2(\mathbb{R})$. Applying (10), we write

$$\langle S_{g,g}f, f \rangle = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 |f(t)|^2 dt + \int_{\mathbb{R}} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} g_k(t) \overline{g_k(t - lb_k^{-1})} f(t - lb_k^{-1}) \overline{f(t)} dt$$

Using similar arguments as in the derivation of (8), we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} g_k(t) \overline{g_k(t - lb_k^{-1})} f(t - lb_k^{-1}) \overline{f(t)} dt \right| &\leq \\ &\leq \left(\text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)| |g_k(t - lb_k^{-1})| \right) \|f\|_2^2 \\ &\leq \underbrace{\max_{k \in \mathbb{Z}} \{b_k^{-1}\} \sum_{l \in \mathbb{Z} \setminus \{0\}} \left(\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)| |g_k(t - lb_k^{-1})| \right)}_R \|f\|_2^2. \end{aligned}$$

Therefore, lower and upper frame bounds are obtained from

$$\begin{aligned} \langle S_{g,g}f, f \rangle \|f\|_2^{-2} &\geq \min_{k \in \mathbb{Z}} \{b_k^{-1}\} \left(\text{ess inf}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)|^2 - \frac{\max_{k \in \mathbb{Z}} \{b_k^{-1}\}}{\min_{k \in \mathbb{Z}} \{b_k^{-1}\}} R \right) \\ \langle S_{g,g}f, f \rangle \|f\|_2^{-2} &\leq \max_{k \in \mathbb{Z}} \{b_k^{-1}\} \left(\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)|^2 + R \right), \end{aligned} \quad (13)$$

We need to construct a sequence of b_k , $k \in \mathbb{Z}$, such that for all $f \in L^2(\mathbb{R})$, (13) is bounded away from zero.

Let $\epsilon < C_L$ and consider the sequence of frequency shifts $b_k = (\frac{\epsilon}{C_k})^{1/p_k}$. Then $\min_{k \in \mathbb{Z}} \{b_k^{-1}\} \geq (C_L \epsilon^{-1})^{1/p_2}$, $\max_{k \in \mathbb{Z}} \{b_k^{-1}\} \leq (C_U \epsilon^{-1})^{1/p_1}$ and

$$\frac{\max_{k \in \mathbb{Z}} \{b_k^{-1}\}}{\min_{k \in \mathbb{Z}} \{b_k^{-1}\}} \leq C_U^{1/p_L} C_L^{-1/p_U} \epsilon^{1/p_U - 1/p_L}. \quad (14)$$

Since $(1 + |x + y|)^{-p} \leq (1 + |x|)^p (1 + |y|)^{-p}$ for $x, y \in \mathbb{R}$ and $p \geq 0$, using (12), we have, for some μ with $p_L - 2 > \mu > 0$:

$$\begin{aligned} |g_k(t)| |g_k(t - lb_k^{-1})| &\leq C_k^2 (1 + |t - a_k|)^{-p_k} (1 + |t - a_k - lb_k^{-1}|)^{-p_k + (1+\mu)} \\ &\leq C_k^2 (1 + |t - a_k|)^{(1+\mu)} (1 + |l| b_k^{-1})^{-p_k + (1+\mu)} \\ &\leq C_k^2 (1 + |t - a_k|)^{-(1+\mu)} |l|^{-p_k + (1+\mu)} b_k^{p_k - (1+\mu)} \\ &= \underbrace{C_k^{1+(1+\mu)/p_k}}_{E_k} (1 + |t - a_k|)^{-(1+\mu)} |l|^{-p_k + (1+\mu)} \epsilon^{1-(1+\mu)/p_k}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |g_k(t)| |g_k(t - lb_k^{-1})| &\leq \sum_{k \in \mathbb{Z}} E_k (1 + |t - a_k|)^{-(1+\mu)} |l|^{-p_k + (1+\mu)} \epsilon^{1-(1+\mu)/p_k} \\ &\leq \max_{k \in \mathbb{Z}} E_k |l|^{-p_L + (1+\mu)} \epsilon^{1-(1+\mu)/p_L} \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-(1+\mu)} \\ &\leq \max_{k \in \mathbb{Z}} E_k |l|^{-p_L + (1+\mu)} \epsilon^{1-(1+\mu)/p_L} 2 \left(1 + (1 + \delta)^{-(1+\mu)} (\delta^{-1} + 1 + \mu) \mu^{-1}\right), \quad (15) \end{aligned}$$

where the last estimate follows from Lemma 2.2 (b). Summing the expression (15) over $l \in \mathbb{Z} \setminus \{0\}$ using Lemma 2.2 i, we see that R , as a function of ϵ , behaves like $\epsilon^{1-(1+\mu)/p_L}$, i.e. $R \approx \epsilon^{1-(1+\mu)/p_L}$, and R tends to 0 for $\epsilon \rightarrow 0$. Moreover,

$$\frac{\max_{k \in \mathbb{Z}} \{b_k^{-1}\}}{\min_{k \in \mathbb{Z}} \{b_k^{-1}\}} R \approx \epsilon^{1-(2+\mu)/p_L + 1/p_U}$$

can be made arbitrarily small by choosing ϵ small since $1 - (2 + \mu)/p_L + 1/p_U > 0$. Therefore, if ϵ_0 is such that for $b_k^0 := (\frac{\epsilon_0}{C_k})^{1/p_k}$, $\frac{\max_{k \in \mathbb{Z}} \{b_k^{-1}\}}{\min_{k \in \mathbb{Z}} \{b_k^{-1}\}} R < A_0$, then $\{M_{lb_k} g_k\}_{k, l \in \mathbb{Z}}$ is a frame for all $b_k \leq b_k^0$. \square

For completeness, we state the equivalent result for analysis windows g_k with polynomial decay in the frequency domain.

Corollary 3.5. *Let $\mathbf{g} = \{g_k \in L^2(\mathbb{R}) : \hat{g}_k \in W(L^\infty, \ell^1), k \in \mathbb{Z}\}$ be a set of windows such that*

i) for some positive constants A_0, B_0

$$0 < A_0 \leq \sum_{k \in \mathbb{Z}} |\hat{g}_k(t)|^2 \leq B_0 < \infty \quad a.e. ; \quad (16)$$

ii) for all $k \in \mathbb{Z}$, the windows decay polynomially around a δ -separated set $\{b_k : k \in \mathbb{Z}\}$ of frequency-sampling points b_k :

$$|\hat{g}_k(t)| \leq C_k(1 + |t - b_k|)^{-p_k}, \quad (17)$$

where p_k and C_k are chosen as in Theorem 3.4.

Then there exists a sequence $\{a_k^0\}_{k \in \mathbb{Z}}$, such that for $a_k \leq a_k^0$, $k \in \mathbb{Z}$, the nonstationary Gabor system $\{T_{la_k}g_k : k, l \in \mathbb{Z}\}$ forms a frame for $L^2(\mathbb{R})$.

3.3.1. Nonstationary Gabor frames on modulation spaces

Modulation spaces, cf. [13, 15], are considered as the appropriate function spaces for time-frequency analysis and in particular, for the study of Gabor frames. By their definition, modulation spaces require decay in both time and frequency. Under additional assumptions on the windows g_k , the collection $\mathcal{G}(\mathbf{g}, \mathbf{b})$ is a frame for all modulation spaces M^p , $1 \leq p \leq \infty$.

Proposition 3.6. *Let $\mathcal{G}(\mathbf{g}, \mathbf{b})$ be a frame for $L^2(\mathbb{R})$ satisfying the uniform estimate*

$$|V_\phi g_k(x, \omega)| \leq C(1 + |x - a_k|)^{-r-2}(1 + |\omega|)^{-r-2}, \quad r > 2 \quad (18)$$

where ϕ is a Gaussian window. Then the frame operator S is invertible simultaneously on all modulation spaces M^p for $1 \leq p \leq \infty$.

Proof. Notice, that

$$|V_\phi g_{k,l}(x, \omega)| \leq C(1 + |(x - a_k, \omega - lb_k)|)^{-r-2},$$

since $(1 + |x| + |\omega|)^{-r} \geq (1 + |x|)^{-r}(1 + |\omega|)^{-r}$ and the weights $(1 + |(x, \omega)|)^r$ and $(1 + |x| + |\omega|)^r$ are equivalent.

A result on Gabor molecules [16] states that, if an L^2 -frame $\{g_z : z = (z_1, z_2) \in \mathcal{Z} \subseteq \mathbb{R}^2\}$, where \mathcal{Z} is separable, satisfies the uniform estimate $|V_\phi g_z(x, \omega)| \leq C(1 + |(x - z_1, \omega - z_2)|)^{-r-2}$, then the frame operator $Sf = \sum_{z \in \mathcal{Z}} \langle f, g_z \rangle g_z$ is invertible simultaneously on all M^p for each $1 \leq p \leq \infty$. The result hence follows from condition (18). \square

3.4. Constructing nonstationary Gabor frames

Theorem 3.4 shows that for windows with sufficient uniform decay, nonstationary Gabor frames can always be constructed by choosing sufficient density in the frequency samples. In the present section we assume the existence of a certain nonstationary Gabor frame and explicitly construct a new frame by exploiting the prior information about the original one. This is a situation of practical relevance, since we may often be interested in using windows that decay fast and are negligible outside a support of interest. In particular, we will use the fact that painless nonstationary Gabor frames are easily constructed and deduce new frames from painless frames. The new frames thus obtained will be called *almost painless nonstationary Gabor frames*.

We will subsequently assume $b_k \in [b_L, b_U]$ for frequency-shift parameters and we let $C_k \in [C_L, C_U]$ and $p_k \in [p_L, p_U]$ with $p_L > 1$ for the constants involved in the decay assumptions

for the analysis windows. We then work with the following constants that depend on the separation of the sampling points, the decay of the windows and the frequency-sampling parameters:

$$E_1 = 1 + \frac{\delta^{-1} + p_L}{(1 + \delta)^{p_L}(p_L - 1)} \quad \text{and} \quad E_2 = 1 + \frac{b_U + p_U}{(1 + b_U^{-1})^{p_L}(p_L - 1)} \quad (19)$$

We first consider nonstationary Gabor frames obtained by perturbation of a known frame. This result is in the spirit of similar results for regular Gabor frames [3, 6].

Proposition 3.7. *Let $\{a_k : k \in \mathbb{Z}\}$ be a δ -separated set and $\mathcal{G}(\mathbf{h}, \mathbf{b})$ a nonstationary Gabor frame with frame bounds A_h, B_h and frame operator $S_{h,h}$. Let $g_k \in L^2(\mathbb{R})$ be a set of windows such that for all $k \in \mathbb{Z}$ and for almost all $t \in \mathbb{R}$*

$$|g_k(t) - h_k(t)| \leq C_k(1 + |t - a_k|)^{-p_k}. \quad (20)$$

If $C_U < \sqrt{A_h \lambda^{-1}}$ for

$$\lambda = 4b_L^{-1} \cdot E_1 \cdot E_2, \quad (21)$$

then $\mathcal{G}(\mathbf{g}, \mathbf{b})$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A = A_h(1 - \sqrt{C_U^2 \lambda A_h^{-1}})^2$ and $B = B_h(1 + \sqrt{C_U^2 \lambda B_h^{-1}})^2$.

Proof. By applying Cauchy-Schwartz inequality, it is easy to see that, for $\mathcal{G}(\mathbf{g}, \mathbf{b})$ to be a frame for $L^2(\mathbb{R})$, it suffices to show that $\sum_{k,l \in \mathbb{Z}} |\langle f, g_{k,l} - h_{k,l} \rangle|^2 \leq R \|f\|_2^2$ for some $R < A_h$, also cf. [6, Proposition 4.1.]. Then, frame bounds of $\mathcal{G}(\mathbf{g}, \mathbf{b})$ can be taken as $A_h(1 - \sqrt{R/A_h})^2$ and $B_h(1 + \sqrt{R/B_h})^2$.

To obtain the required error bound, we let $\psi_k(t) := g_k - h_k$ and use the estimate given in (9):

$$\sum_{k,l \in \mathbb{Z}} |\langle f, \psi_{k,l} \rangle|^2 \leq \sup_{k \in \mathbb{Z}} \left(b_k^{-1} \text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |\psi_k(t - lb_k^{-1})| \right) \left(\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\psi_k(t)| \right) \|f\|_2^2. \quad (22)$$

The first term of the above in the above inequality is bounded by assumption (20) and Lemma 2.2 (b):

$$\sum_{k \in \mathbb{Z}} |\psi_k(t)| \leq C_U \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p_k} \leq C_U \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p_1} \leq 2C_U E_1.$$

To bound the second term, note that $\sum_{l \in \mathbb{Z}} (1 + |t - a_k - lb_k^{-1}|)^{-p_k}$ is b_k^{-1} -periodic, therefore we can simplify

$$\text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |\psi_k(t - lb_k^{-1})| \leq C_k \text{ess sup}_{t \in [0, b_k^{-1}]} \sum_{l \in \mathbb{Z}} (1 + |t - lb_k^{-1}|)^{-p_k}.$$

Hence, by Lemma 2.2, we obtain for $t \in [0, b_k^{-1}]$:

$$\begin{aligned} \sum_{l \in \mathbb{Z}} (1 + |t - lb_k^{-1}|)^{-p_k} &\leq 1 + \sum_{l=1}^{\infty} (1 + |t - lb_k^{-1}|)^{-p_k} + \sum_{l=1}^{\infty} (1 + t + lb_k^{-1})^{-p_k} \\ &\quad (1 + (l-1)b_k^{-1})^{-p_k} \\ &= 2 \sum_{l=0}^{\infty} (1 + lb_k^{-1})^{-p_k} = 2 \left(1 + \sum_{l=1}^{\infty} (1 + lb_k^{-1})^{-p_k} \right) \leq 2E_2. \end{aligned}$$

Gathering all the estimates, we obtain

$$\sum_{k,l \in \mathbb{Z}} |\langle f, \psi_{k,l} \rangle|^2 \leq C_U^2 4b_1^{-1} E_1 \cdot E_2 \|f\|_2^2 = C_U^2 \lambda \|f\|_2^2.$$

By assumption $C_U^2 \lambda < A_h$, and the proof is complete. \square

Using Proposition 3.7 we next construct a special class of nonstationary Gabor frames by relying on knowledge of a painless nonstationary Gabor frame. We construct new windows which are no more compactly supported, but coincide with the known, compact windows on their support. We call the resulting new systems *almost painless* nonstationary Gabor frames.

Corollary 3.8. *Let $\mathbf{g} = \{g_k \in W(L^\infty, \ell^1) : k \in \mathbb{Z}\}$ be a set of windows, and let I_k be the intervals $I_k = [a_k - (2b_k)^{-1}, a_k + (2b_k)^{-1}]$ where $\{a_k : k \in \mathbb{Z}\}$ forms a δ -separated set. Assume that $\mathcal{G}(\mathbf{h}, \mathbf{b})$, where $h_k = g_k \chi_{I_k}$, is a Gabor frame with lower frame bound A_h , and that for $\psi_k = g_k - h_k$, all $k \in \mathbb{Z}$ and almost all $t \in \mathbb{R}$*

$$|\psi_k(t)| \leq \begin{cases} C_k (1 + t - a_k - (2b_k)^{-1})^{-p_k}, & t > a_k + (2b_k)^{-1}; \\ 0, & t \in I_k; \\ C_k (1 - t + a_k - (2b_k)^{-1})^{-p_k}, & t < a_k - (2b_k)^{-1}. \end{cases} \quad (23)$$

If $C_U < \sqrt{A_h \lambda^{-1}}$ for $\lambda = 4b_L^{-2} \cdot \delta^{-1} \cdot E_1 \cdot E_2$, then $\mathcal{G}(\mathbf{g}, \mathbf{b})$ forms a nonstationary Gabor frame for $L^2(\mathbb{R})$.

Proof. The proof follows the steps of the proof of Proposition 3.7 with small changes on how to approximate the terms in (22). First observe, that for any $t \in \mathbb{R}$ and all k , we have

$$|\psi_k(t)| \leq C_k [(1 + |t - a_k - (2b_k)^{-1}|)^{-p_k} + (1 + |t - a_k + (2b_k)^{-1}|)^{-p_k}]$$

Since the frequency shifts b_k are taken from a finite interval and the set $\{a_k : k \in \mathbb{Z}\}$ is δ -separated, the sets $\Gamma^+ = \{a_k + (2b_k)^{-1} : k \in \mathbb{Z}\}$ and $\Gamma^- = \{a_k - (2b_k)^{-1} : k \in \mathbb{Z}\}$ are relatively δ -separated with $\text{rel}(\Gamma) = \text{rel}(\Gamma^+) = \text{rel}(\Gamma^-) = \lfloor (2b_L \delta)^{-1} \rfloor$. Therefore, by Remark 1, it follows that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |\psi_k(t)| &\leq C_U \sum_{k \in \mathbb{Z}} (1 + |t - a_k - (2b_k)^{-1}|)^{-p_L} + C_U \sum_{k \in \mathbb{Z}} (1 + |t - a_k + (2b_k)^{-1}|)^{-p_L} \\ &\leq 4C_U \text{rel}(\Gamma) (1 + (1 + \delta)^{-p_L} (\delta^{-1} + p_L)(p_L - 1)^{-1}). \end{aligned}$$

Now, the expression $\sum_{l \in \mathbb{Z}} |\psi_k(t - lb_k^{-1})|$ is b_k^{-1} -periodic. Let $t \in I_k$, then, by (23)

$$\begin{aligned} \sum_{l \in \mathbb{Z}} |\psi_k(t - lb_k^{-1})| &\leq C_k \left[\sum_{l > 0} (1 + a_k - (2b_k)^{-1} - t + lb_k^{-1})^{-p_k} \right. \\ &\quad \left. + \sum_{l < 0} (1 - a_k - (2b_k)^{-1} + t - lb_k^{-1})^{-p_k} \right] \\ &\leq C_k 2 \sum_{l=0}^{\infty} (1 + lb_k^{-1})^{-p_k} \leq 2C_k (1 + (1 + b_k^{-1})^{-p_k} (b_k + p_k)(p_k - 1)^{-1}), \end{aligned}$$

where the last estimate follows from Lemma 2.2(a). \square

4. Examples

We illustrate our theory with two examples. In both examples, we consider a basic window and dilations by 2 and $\frac{1}{2}$, respectively. Since the dilation parameters take only three different values, there are three kinds of windows, with support size $1/2$, 1 and 2, respectively. Note that, while theoretically possible, sudden changes in the shape and width of adjacent windows turn out to be undesirable for applications, hence we only allow for stepwise change in dilation parameters.

In the first example we consider a frame that arises as perturbation of a painless non-stationary Gabor frame. The perturbation consists in the application of a bandpass filter in order to obtain windows with compact support in the frequency domain.

Example 4.1. Let h be a Hann or raised cosine window, i.e. $h(t) = 0.5 + 0.5 \cos(2\pi t)$ for $t \in [-1/2, 1/2]$, and zero otherwise. We construct a painless nonstationary Gabor frame by dilating h by 2 or $\frac{1}{2}$, respectively: Let $\{s_k\}_{k \in \mathbb{Z}}$ be a sequence with values from the set $\{-1, 0, 1\}$ with the restriction that $|s_k - s_{k-1}| \in \{0, 1\}$ to avoid sudden changes between adjacent windows. We then define corresponding shift-parameters by setting $a_0 = 0$ and

$$\begin{aligned} a_{k+1} &= a_k + 2^{-s_k} \cdot \frac{5}{6} & \text{if } s_k > s_{k+1}, \\ a_{k+1} &= a_k + 2^{-s_k+1} \cdot \frac{1}{3} & \text{if } s_k = s_{k+1}, \\ a_{k+1} &= a_k + 2^{-s_{k+1}} \cdot \frac{5}{6} & \text{if } s_k < s_{k+1}. \end{aligned}$$

The points a_k , $k \in \mathbb{Z}$, form a separated set with minimum separation $\delta = 1/3$. Setting $b_k = 2^{s_k}$ and $h_k(t) = T_{a_k} \sqrt{2^{s_k}} h(2^{s_k} t)$, the system $\{M_{lb_k} h_k : k, l \in \mathbb{Z}\}$ forms a painless nonstationary Gabor frame with lower frame bound $A_h = 0.5$.

Let $\Omega = 0.02$ and ϕ be a bandlimited filter given by

$$\widehat{\phi}(\omega) = 0.5 + 0.5 \cos(2\pi \Omega^{-1} \omega) \text{ on its support } [-\Omega/2, \Omega/2].$$

We build new windows g_k by convolving ϕ with h_k . The windows $g_k := \phi * h_k$ are no more compactly supported. Since $|\phi(t)| \leq \Omega(1 + |t|)^{-3}$, we rely on [20, Theorem 9.9] to deduce

the following bound, with $C' = \|h_k\|_\infty \frac{\Omega}{2}$, $I_k = [a_k - 2^{-s'_k}, a_k + 2^{-s'_k}]$ and $s'_k = s_k + 1$:

$$|g_k(t) - h_k(t)| \leq C' \begin{cases} (1 + (t - a_k) - 2^{-s'_k})^{-2} - (1 + (t - a_k) + 2^{-s'_k})^{-2} & t > a_k + 2^{-s'_k} \\ 2 - (1 + (t - a_k) + 2^{-s'_k})^{-2} - (1 - (t - a_k) + 2^{-s'_k})^{-2} & t \in I_k \\ (1 - (t - a_k) - 2^{-s'_k})^{-2} - (1 - (t - a_k) + 2^{-s'_k})^{-2} & t < a_k - 2^{-s'_k} \end{cases}$$

We obtain the joint bound, $|g_k(t) - h_k(t)| \leq C_U(1 + |t - a_k|)^{-2}$ by setting $C_k = C' \cdot (1 + 2^{-s'_k})^2$ for all $k \in \mathbb{Z}$ and $C_U = \max_{k \in \mathbb{Z}} C_k = 0.0282 < \sqrt{A_h \lambda^{-1}} = 0.0768$, with λ as defined in (21). Thus, by Proposition 3.7, $\{M_{lb_k} g_k : k, l \in \mathbb{Z}\}$ is a Gabor frame with a lower frame bound $A = 0.2$.

Remark 7. Note that the construction presented in the Example 4.1 is of particular interest for constructing frequency-adaptive frames with windows that are compactly supported in time. This is a situation of considerable interest in applications, since it allows for real-time implementation with finite impulse response filters, cp. [11].

In the second example we construct a nonstationary Gabor frame by applying Corollary 3.8. The windows of the new frame coincide with the windows of a painless frame on their support. The windows in this example are constructed in analogy to the windows used in *scale frames*, introduced in [2] to automatically improve the resolution of transients in audio signals.

Example 4.2. As in the previous example, let $s_k \in \{-1, 0, 1\}$ with $|s_k - s_{k-1}| \in \{0, 1\}$ for all $k \in \mathbb{Z}$. We consider a sequence of windows g_k that are translated and dilated versions of the Gaussian window $g(t) = e^{-\pi(2.5t)^2}$: $g_k(t) = T_{a_k} \sqrt{2^{s_k}} g(2^{s_k} t)$ with $a_0 = 0$ and

$$\begin{aligned} a_{k+1} &= a_k + 2^{-s_{k+1}-1} & \text{if } s_k = s_{k+1}, \\ a_{k+1} &= a_k + \frac{1}{3} \cdot 2^{-s_{k+1}} & \text{if } s_k > s_{k+1}, \\ a_{k+1} &= a_k + \frac{1}{3} \cdot 2^{-s_k} & \text{if } s_k < s_{k+1}. \end{aligned}$$

Here, the $\{a_k : k \in \mathbb{Z}\}$ are separated with minimum distance $\delta = 1/4$. We arrange the windows as follows: after each change of window size, no change is allowed in the next step; in other words, each window has at least one neighbor of the same size. An example of the arrangement is shown in Figure 1.

Let $I_k = [a_k - 2^{-s_k-1}, a_k + 2^{-s_k-1}]$ and define a new set of windows by $h_k(t) = g_k(t) \chi_{I_k}$. Then $\{M_{lb_k} h_k : k, l \in \mathbb{Z}\}$ with $b_k = 2^{s_k}$ is a painless nonstationary Gabor frame. By numerical calculations, its lower frame bound is $A_h = 0.1609$ and $\psi_k(t) = g_k(t) - h_k(t)$ can be bounded by

$$|\psi_k(t)| \leq \begin{cases} \sqrt{2^{s_k}} g(1/2) (1 + t - a_k - 2^{-s_k-1})^{-19} & t > a_k + 2^{-s_k-1} \\ 0 & t \in I_k \\ \sqrt{2^{s_k}} g(1/2) (1 + a_k - 2^{-s_k-1} - t)^{-19} & t < a_k - 2^{-s_k-1}. \end{cases}$$

From Proposition 3.8 it follows that $4(\delta b_L^2)^{-1} \cdot C_U^2 \cdot E_1 \cdot E_2 = 0.0071 < A_h$, and $\{M_{lb_k} T_{a_k} g_k : k, l \in \mathbb{Z}\}$ is a nonstationary Gabor frame with a lower frame bound $A = 0.1538$.

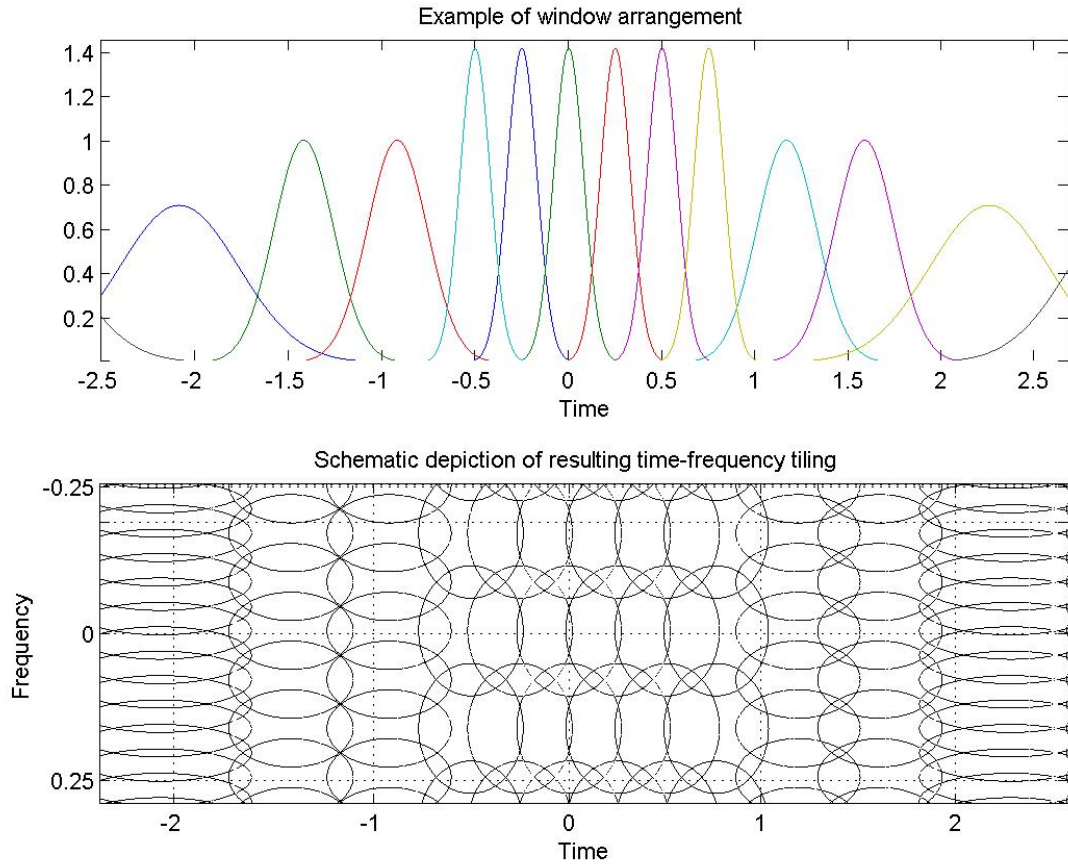


Figure 1: An example for the arrangement of dilated windows in Example 2

5. Acknowledgement

This work was supported by the WWTF project *Audiominer* (MA09-24) and the Austrian Science Fund (FWF):[T384-N13] *Locatif*.

References

- [1] A. Aldroubi, C. A. Cabrelli, and U. Molter. Wavelets on Irregular Grids with Arbitrary Dilation Matrices, and Frame Atoms for $L_2(\mathbb{R}^d)$. *Appl. Comput. Harmon. Anal.*, Special Issue on Frames II.:119–140, 2004.
- [2] P. Balazs, M. Dörfler, F. Jaillet, N. Holighaus, and G. A. Velasco. Theory, implementation and applications of nonstationary Gabor Frames. *J. Comput. Appl. Math.*, 236:1481–1496, 2011.
- [3] I. Carrizo and S. Favier. Perturbation of wavelet and Gabor frames. *Anal. Theory Appl.*, 19(3):238–254, 2003.

- [4] O. Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser, 2003.
- [5] O. Christensen, S. Favier, and F. Zó. Irregular wavelet frames and Gabor frames. *Approx. Theory Appl.*, 17(3):90–101, 2001.
- [6] O. Christensen and C. Heil. Perturbations of Banach frames and atomic decompositions. *Math. Nachr.*, 185:33–47, December 1997.
- [7] I. Daubechies, A. Grossmann, and Y. Meyer. Painless nonorthogonal expansions. *J. Math. Phys.*, 27(5):1271–1283, May 1986.
- [8] M. Dörfler. Quilted Gabor frames - A new concept for adaptive time-frequency representation. *Advances in Applied Mathematics*, 47(4):668 – 687, Oct. 2011.
- [9] N. Holighaus, M. Dörfler, G. Velasco, and T. Grill. A framework for invertible, real-time constant-Q transforms. *preprint*, submitted, <http://www.univie.ac.at/nonstatgab/slicq>, 2012.
- [10] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.
- [11] G. Evangelista, M. Dörfler, and E. Matusiak. Phase Vocoder With Arbitrary Frequency Band Selection. *Proceedings of the 9th Sound and Music Computing Conference, July 11-14th 2012 Copenhagen*, 2012.
- [12] S. J. Favier and R. A. Zalik. On the stability of frames and Riesz bases. *Appl. Comput. Harmon. Anal.*, 2(2):160–173, 1995.
- [13] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92:269–289, 1981.
- [14] D. Gabor. Theory of communication. *J. IEE*, 93(26):429–457, 1946.
- [15] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser Boston, 2001.
- [16] K. Gröchenig. *Localization of Frames, Banach Frames, and the invertibility of the frame operator*. Proceedings of SPIE, San Diego, 2004.
- [17] H. Malvar. *Signal Processing with Lapped Transforms*. Boston, MA: Artech House. xvi, 1992.
- [18] J. L. Romero. Surgery of spline-type and molecular frames. *J. Fourier Anal. Appl.*, 17:135 – 174, 2011.
- [19] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$. *Duke Math. J.*, 89(2):237–282, 1997.
- [20] W. Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, New York, 1966.

- [21] W. Sun and X. Zhou. Irregular wavelet/Gabor frames. *Appl. Comput. Harmon. Anal.*, 13(1):63–76, 2002.
- [22] G. A. Velasco, N. Holighaus, M. Dörfler, and T. Grill. Constructing an invertible constant-Q transform with non-stationary Gabor frames. In *Proceedings of DAFX11, Paris*, September 2011.
- [23] D. F. Walnut. Continuity properties of the Gabor frame operator. *J. Math. Anal. Appl.*, 165(2):479–504, 1992.
- [24] Z. Xiong and H. Malvar. A nonuniform modulated complex lapped transform. *IEEE Signal Processing Letters*, 8(9):257–260, September 2001.