ON CONTROLLED FRAMES IN HILBERT C*-MODULES

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ABSTRACT. In this paper, we introduce controlled frames in Hilbert C^* -modules and we show that they share many useful properties with their corresponding notions in Hilbert space. Next, we give a characterization of controlled frames in Hilbert C^* -module. Also multiplier operators for controlled frames in Hilbert C^* -modules will be defined and some of its properties will be shown. Finally, we investigate weighted frames in Hilbert C^* -modules and verify their relations to controlled frames and multiplier operators.

1. INTRODUCTION

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [11] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies, Grossmann and Meyer[10], and popularized from then on. For basic results on frames, see [8].

Hilbert C^* -modules form a wide category between Hilbert spaces and Banach spaces. Their structure was first used by Kaplansky [20] in 1952. They are an often used tool in operator theory and in operator algebra theory. They serve as a major class of examples in operator C^* -module theory.

The notions of frames in Hilbert C^* -modules were introduced and investigated in [14]. Frank and Larson [14, 15] defined the standard frames in Hilbert C^* -modules in 1998 and got a series of result for standard frames in finitely or countably generated Hilbert C^* -modules over unital C^* -algebras. Extending the results to this more general framework is not a routine generalization, as there are essential differences between Hilbert C^* -modules and Hilbert spaces. For example, any closed subspace in a Hilbert space has an orthogonal complement, but this fails in Hilbert C^* -module. Also there is no explicit analogue of the Riesz representation theorem of continuous functionals in Hilbert C^* -modules. We refer the readers to [24] and [19] for more details on Hilbert C^* -modules and to [15, 28, 30, 29] for a discussion of basic properties of frame in Hilbert C^* -modules and their generalizations.

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Controlled frames have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [5], however they are used earlier in [7] for spherical wavelets.

In this paper, we introduce controlled frames in Hilbert C^* -modules and we show that they share many useful properties with their corresponding notions in Hilbert space. Next, we give a characterization of controlled frames in Hilbert C^* -module. Also multiplier operators for controlled frames in Hilbert C^* -modules will be defined and some of its properties will be shown. Finally, we investigate weighted frames in Hilbert C^* -modules and verify their relation to controlled frames and multiplier operators.

The paper is organized as follows. In section 2, we review the concept Hilbert C^* -modules, frames and multiplier operators in Hilbert C^* -modules. Also the analysis, synthesis, frame operator and dual frames be reviewed. In section 3, we introduce controlled frames in Hilbert C^* -modules and characterize them. In section 4, we investigate weighted frames in Hilbert C^* -modules and verify their relation to controlled frames and multiplier operators.

2. Preliminaries

In this section, we collect the basic notations and some preliminary results. We denote by I the identity operator on \mathcal{H} . Let $B(\mathcal{H}_1, \mathcal{H}_2)$ be the set of all bounded linear operators from \mathcal{H}_1 to \mathcal{H}_2 . This set is a Banach space for the operator norm $||A|| = \sup_{\|\mathcal{H}_1\|_{\mathcal{H}_1} \leq 1} ||Ax||_{\mathcal{H}_2}$. The adjoint of the operator A is denoted by A^* and the spectrum of A by $\sigma(A)$. We define $GL(\mathcal{H}_1, \mathcal{H}_2)$ as the set of all bounded linear operators with a bounded inverse, and similarly for $GL(\mathcal{H})$. Our standard reference for Hilbert space and operator theory is [9].

Controlled frames introduced in [5] as follows. Let $C \in GL(\mathcal{H})$. A frame controlled by the operator C or C-controlled frame is a family of vectors $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$, such that there exist two constants m > 0 and $M < \infty$ satisfying

$$m\|f\|^2 \le \sum_j \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \le M\|f\|^2,$$

for all $f \in \mathcal{H}$.

Also weighted frames are defined as follows. Let $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ be a sequence of elements in \mathcal{H} and $\{\omega_j : j \in J\} \subseteq \mathbb{R}^+$ a sequence of positive weights. This pair is called a weighted frame of \mathcal{H} if there exist constants m > 0 and $M < \infty$ such that

$$m\|f\|^2 \le \sum_j \omega_j |\langle f, \psi_j \rangle|^2 \le M\|f\|^2,$$

for all $f \in \mathcal{H}$.

Hilbert C^* -modules form a wide category between Hilbert spaces and Banach spaces. Hilbert C^* -modules are generalizations of Hilbert spaces by

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allowing the inner product to take values in a C^* -algebra rather than in the field of complex numbers.

Let A be a C*-algebra with involution *. An inner product A-module (or pre Hilbert A-module) is a complex linear space \mathcal{H} which is a left A-module with an inner product map $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to A$ which satisfies the following properties:

- (1) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ for all $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$;
- (2) $\langle af, g \rangle = a \langle f, g \rangle$ for all $f, g \in \mathcal{H}$ and $a \in A$;
- (3) $\langle f, g \rangle = \langle g, f \rangle^*$ for all $f, g \in \mathcal{H}$;
- (4) $\langle f, f \rangle \ge 0$ for all $f \in \mathcal{H}$ and $\langle f, f \rangle = 0$ iff f = 0.

For $f \in \mathcal{H}$, we define a norm on \mathcal{H} by $||f||_{\mathcal{H}} = ||\langle f, f \rangle||_A^{1/2}$. If \mathcal{H} is complete with this norm, it is called a (left) Hilbert C^* -module over A or a (left) Hilbert A-module.

An element a of a C^* -algebra A is positive if $a^* = a$ and its spectrum is a subset of positive real numbers. In this case, we write $a \ge 0$. By condition (4) in the definition $\langle f, f \rangle \ge 0$ for every $f \in \mathcal{H}$, hence we define $|f| = \langle f, f \rangle^{1/2}$. We call $Z(A) = \{a \in A : ab = ba, \forall b \in A\}$, the center of A. If $a \in Z(A)$, then $a^* \in Z(A)$, and if a is an invertible element of Z(A), then $a^{-1} \in Z(A)$, also if a is a positive element of Z(A), then $a^{\frac{1}{2}} \in Z(A)$. Let $Hom_A(M, N)$ denotes the set of all A-linear operators from M to N. Let

$$\ell^{2}(A) = \left\{ \{a_{j}\} \subseteq A : \sum_{j \in J} a_{j}^{*}a_{j} \text{ converges } in \|.\| \right\}$$

with inner product

$$\langle \{a_j\}, \{b_j\} \rangle = \sum_{j \in J} a_j^* b_j, \quad \{a_j\}, \{b_j\} \in \ell^2(A)$$

and

$$\|\{a_j\}\| := \sqrt{\|\sum a_j^* a_j\|},$$

it was shown that [33], $\ell^2(A)$ is Hilbert A-module.

Note that in Hilbert C^* -modules the Cauchy-Schwartz inequality is valid. Let $f, g \in \mathcal{H}$, where \mathcal{H} is a Hilbert C^* -module , then

$$\|\langle f,g\rangle\|^2 \le \|\langle f,f\rangle\| \times \|\langle g,g\rangle\|.$$

We are focusing in finitely and countably generated Hilbert C^* - modules over unital C^* -algebra A. A Hilbert A-module \mathcal{H} is finitely generated if there exists a finite set $\{x_1, x_2, ..., x_n\} \subseteq \mathcal{H}$ such that every $x \in \mathcal{H}$ can be expressed as $x = \sum_{i=1}^{n} a_i x_i, a_i \in A$. A Hilbert A-module \mathcal{H} is countably generated if there exits a countable set of generators.

The notion of (standard) frames in Hilbert C^* -modules is first defined by Frank and Larson [15]. Basic properties of frames in Hilbert C^* -modules are discussed in [16, 17, 21, 22]. Let \mathcal{H} be a Hilbert C^* -module, and J a set which is finite or countable. A system $\{f_j : j \in J\} \subseteq \mathcal{H}$ is called a frame for \mathcal{H} if there exist constants C, D > 0 such that

(2.1)
$$C\langle f, f \rangle \le \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle \le D \langle f, f \rangle$$

for all $f \in \mathcal{H}$. The constants C and D are called the frame bounds. If C = D it called a tight frame and in the case C = D = 1, it called Parseval frame. It is called a Bessel sequence if the second inequality in (2.1) holds.

Unlike Banach spaces, it is known [15] that every finitely generated or countably generated Hilbert C^* - modules admits a frame.

The following characterization of frames in Hilbert C^* - modules, which was obtained independently in [1] and [18], enables us to verify whether a sequence is a frames in Hilbert C^* - modules in terms of norms. It also allows us to characterize frames in Hilbert C^* - modules from the operator theory point of view.

Theorem 2.1. Let \mathcal{H} be a finitely or countably generated Hilbert A-module over a unital C^* -algebra A and $\{f_j : j \in J\} \subseteq \mathcal{H}$ a sequence. Then $\{f_j : j \in J\}$ is a frame for \mathcal{H} if and only if there exist constants C, D > 0 such that

$$C \|f\|^2 \le \left\| \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle \right\| \le D \|f\|^2, \quad f \in \mathcal{H}.$$

Let $\{f_j : j \in J\}$ be a frame in Hilbert A-module \mathcal{H} and $\{g_j : j \in J\}$ be a sequence of \mathcal{H} . Then $\{g_j : j \in J\}$ is called a dual sequence of $\{f_j : j \in J\}$ if

$$f = \sum_{j \in J} \langle f, g_j \rangle f_j$$

for all $f \in \mathcal{H}$. The sequences $\{f_j : j \in J\}$ and $\{g_j : j \in J\}$ are called a dual frame pair when $\{g_j : j \in J\}$ is also a frame.

For the frame $\{f_j : j \in J\}$ in Hilbert A-module \mathcal{H} , the operator S defined by

$$Sf = \sum_{j \in J} \langle f, f_j \rangle f_j, \ f \in \mathcal{H}$$

is called the frame operator. It is proved that [15], S is invertible, positive, adjointable and self-adjoint. Since

$$\langle Sf, f \rangle = \langle \sum_{j \in J} \langle f, f_j \rangle f_j, f \rangle = \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle, \quad f \in \mathcal{H}$$

it follows that

$$C\langle f, f \rangle \le \langle Sf, f \rangle \le D\langle f, f \rangle, \quad f \in \mathcal{H}$$

and the following reconstruction formula holds

$$f = SS^{-1}f = S^{-1}Sf = \sum_{j \in J} \langle S^{-1}f, f_j \rangle f_j = \sum_{j \in J} \langle f, S^{-1}f_j \rangle f_j$$

for all $f \in \mathcal{H}$. Let $\tilde{f}_j = S^{-1} f_j$, then

$$f = \sum_{j \in J} \langle f, \tilde{f}_j \rangle f_j = \sum_{j \in J} \langle f, f_j \rangle \tilde{f}_j$$

for any $f \in \mathcal{H}$. The sequence $\{\tilde{f}_j : j \in J\}$ is also a frame for \mathcal{H} which is called the canonical dual frame of $\{f_j : j \in J\}$.

In [31], R. Schatten provided a detailed study of ideals of compact operators using their singular decomposition. He investigated the operators of the form $\sum_j \lambda_j \varphi_j \otimes \psi_j$ where (ϕ_j) and (ψ_j) are orthonormal families. In [4], the orthonormal families were replaced with Bessel and frame sequences to define Bessel and frame multipliers.

Definition 2.2. Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces, let $(\psi_j) \subseteq \mathcal{H}_1$ and $(\phi_j) \subseteq \mathcal{H}_2$ be Bessel sequences. Fix $m = (m_j) \in l^{\infty}$. The operator

$$\mathbf{M}_{m,(\phi_j),(\psi_j)}: \mathcal{H}_1 \to \mathcal{H}_2$$

defined by

$$\mathbf{M}_{m,(\phi_j),(\psi_j)}(f) = \sum_{j \in J} m_j \langle f, \psi_j \rangle \phi_j$$

for all $f \in \mathcal{H}_1$ is called the Bessel multiplier for the Bessel sequences (ψ_j) and (ϕ_j) . The sequence *m* is called the symbol of **M**. For frames, this operator is called frame multiplier, for Riesz sequences a Riesz multiplier.

Basic properties and some applications of this operator for Bessel sequences, frames and Riesz basis have been proved by Peter Balazs in his Ph.D habilation [3]. Recently, the concept of multipliers extended and introduced for continuous frames [6], fusion frames [2], *p*-Bessel sequences [26], generalized frames [25], controlled frames [27], Banach frames [12, 13], Hilbert C^* -modules [23] and etc.

Definition 2.3. Let A be a unital C^* -algebra, J be a finite or countable index set and $\{f_j : j \in J\}$ and $\{g_j : j \in J\}$ be Hilbert C^* -modules Bessel sequences for \mathcal{H} . For $m = \{m_j\}_{j \in J} \in \ell^{\infty}(A)$ with $m_j \in Z(A)$, for each $j \in J$, the operator $M_{m,\{f_j\},\{g_j\}} : \mathcal{H} \to \mathcal{H}$ defined by

$$M_{m,\{f_j\},\{g_j\}}f := \sum_{j \in J} m_j \langle f, f_j \rangle g_j, \quad f \in \mathcal{H}$$

called the multiplier operator of $\{f_j : j \in J\}$ and $\{g_j : j \in J\}$. The sequence $m = \{m_j\}_j \in J$ called the symbol of $M_{m,\{f_i\},\{g_j\}}$.

The symbol of m has important role in the studying of multiplier operators. In this paper m is always a sequence $m = \{m_j\}_{j \in J} \in \ell^{\infty}(A)$ with $m_j \in Z(A)$, for each $j \in J$.

3. Controlled Frames In Hilbert C^* -modules

In this section, we introduce controlled frames in Hilbert C^* -modules and we show that they share many useful properties with their corresponding notions in Hilbert space. We also give a characterization of controlled frames in Hilbert C^* -module.

Definition 3.1. Let \mathcal{H} be a Hilbert C^* -module and $C \in GL(\mathcal{H})$. A frame controlled by the operator C or C-controlled frame in Hilbert C^* -module \mathcal{H} is a family of vectors $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$, such that there exist two constants m > 0 and $M < \infty$ satisfying

$$m\langle f, f \rangle \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \leq M \langle f, f \rangle,$$

for all $f \in \mathcal{H}$.

Likewise, $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ is called a *C*-controlled Bessel sequence with bound *M* if there exists $M < \infty$ such that

$$\sum_{j \in J} \langle f, \psi_j \rangle \langle C \psi_j, f \rangle \le M \langle f, f \rangle,$$

for every $f \in \mathcal{H}$, where the sum in the inequality is convergent in norm.

If m = M, we call this C-controlled frame a tight C-controlled frame, and if m = M = 1 it is called a Parseval C-controlled frame.

Every frame is a I-controlled frame. Hence controlled frames are generalizations of frames.

The proof of the following lemma is straightforward.

Lemma 3.2. Let \mathcal{H} be a Hilbert C^* -module and $C \in GL(\mathcal{H})$. A sequence $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ is C-controlled Bessel sequence in Hilbert C*-module \mathcal{H} if and only if the operator

$$S_C f = \sum_{j \in J} \langle f, \psi_j \rangle C \psi_j$$

is well defined and there exists constant $M < \infty$ such that

$$\sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \le M \langle f, f \rangle,$$

for every $f \in \mathcal{H}$.

By using Lemma 3.2 we have the following definition.

Definition 3.3. Let \mathcal{H} be a Hilbert C^* -module and $C \in GL(\mathcal{H})$. Assume the sequence $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ is C-controlled Bessel sequence in Hilbert C*-module \mathcal{H} . The operator

$$S_C f = \sum_{j \in J} \langle f, \psi_j \rangle C \psi_j$$

is called *C*-controlled frame operator.

According to the following proposition, the main properties of controlled frame operators in Hilbert C^* -modules are the same as controlled frame operators in Hilbert spaces.

Proposition 3.4. Let \mathcal{H} be a Hilbert C^* -module on C^* -algebra A and $C \in End^*_A(U, V)$. Assume $\{\psi_j : j \in J\}$ is a C-controlled frame in Hilbert C^* -module \mathcal{H} with bounds m, M > 0 Then C-controlled frame operator S_C is invertible, positive, adjointable and self-adjoint.

Proof. Let $f, g \in \mathcal{H}$ and $Sf = \sum_{j \in J} \langle f, \psi_j \rangle \psi_j$ be frame operator of $\{\psi_j : j \in J\}$, then

$$\langle S_C f, g \rangle = \left\langle \sum_{j \in J} \langle f, \psi_j \rangle C \psi_j, g \right\rangle = \sum_{j \in J} \langle f, \psi_j \rangle \langle C \psi_j, g \rangle$$
$$= \sum_{j \in J} \langle f, \psi_j \rangle \langle \psi_j, C^* g \rangle = \langle S f, C^* g \rangle = \langle f, S C^* g \rangle.$$

Therefore controlled frame operator S_C is adjointable and $S_C^* = SC^*$. Since

$$\langle S_C f, f \rangle = \left\langle \sum_{j \in J} \langle f, \psi_j \rangle C \psi_j, f \right\rangle = \sum_{j \in J} \langle f, \psi_j \rangle \langle C \psi_j, f \rangle$$

It follows that

$$m\langle f, f \rangle \le \langle S_C f, f \rangle \le M \langle f, f \rangle,$$

and

$$mId_{\mathcal{H}} \leq S_C \leq MId_{\mathcal{H}}.$$

So S_C is positive and invertible. By regard to this fact that every positive operator in Banach space is self-adjoint, *C*-controlled frame operator S_C is self-adjoint.

Now, by using the following lemma in [1], we give a characterization of controlled frames.

Lemma 3.5. [1] Let A be a C^{*}-algebra, U and V two Hilbert A-modules, and $T \in End^*_A(U, V)$. Then the following statements are equivalent:

- (1) T is surjective;
- (2) T^* is bounded below with respect to norm, that is, there is m > 0such that $||T^*f|| \ge m||f||$ for all $f \in U$;
- (3) T^* is bounded below with respect to the inner product, that is, there is m' > 0 such that $\langle T^*f, T^*f \rangle \ge m' \langle f, f \rangle$ for all $f \in U$.

Theorem 3.6. Let \mathcal{H} be a Hilbert C^* -module and $C \in GL(\mathcal{H})$. A sequence $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ is C-controlled frame in Hilbert C^* -module \mathcal{H} if and only if there exists constants m > 0 and $M < \infty$ such that

(3.1)
$$m\|f\|^2 \le \left\|\sum_{j\in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \right\| \le M\|f\|^2,$$

for all $f \in \mathcal{H}$.

Proof. Let the sequence $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ is *C*-controlled frame in Hilbert C^* -module \mathcal{H} . By the definition of *C*-controlled frame inequality (3.1) holds.

Now suppose that the inequality (3.1) holds. From Proposition 3.4 *C*controlled frame operator S_C is positive, self-adjoint and invertible, hence $\langle S_C^{\frac{1}{2}}f, S_C^{\frac{1}{2}}f \rangle = \langle S_C f, f \rangle = \sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle$. So we have $\sqrt{m} ||f|| \leq ||S_C^{\frac{1}{2}}f|| \leq \sqrt{M} ||f||$ for any $f \in \mathcal{H}$. According to Lemma 3.5 there are constants C, D > 0 such that

$$C||f||^{2} \leq \left\| \sum_{j \in J} \langle f, \psi_{j} \rangle \langle C\psi_{j}, f \rangle \right\| \leq D||f||^{2},$$

which implies that $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ is C-controlled frame in Hilbert C^* -module \mathcal{H} .

We prove the following theorems to show that every controlled frame is a classical frame.

Theorem 3.7. Let \mathcal{H} be a Hilbert C^* -module on C^* -algebra A and $\{\psi_j\}_{j\in J}$ be a sequence in \mathcal{H} . Assume S is the frame operator associated with $\{\psi_j\}_{j\in J}$ i.e. $Sf = \sum_{j\in J} \langle f, \psi_j \rangle \psi_j$. Then the following conditions are equivalent:

- (1) $\{\psi_j\}_{j\in J}$ is a frame with frame bounds C and D;
- (2) We have $CId_{\mathcal{H}} \leq S \leq DId_{\mathcal{H}}$.

Proof. $(1) \Rightarrow (2)$. This is implied by proof of Proposition 3.4.

 $(2) \Rightarrow (1)$. Let T be the analysis operator associated with $\{\psi_j\}_{j\in J}$. Since $S = T^*T$ and hence $||T||^2 = ||S||$ for any $f \in \mathcal{H}$ we obtain

$$\left|\sum_{j\in J} \langle f, \psi_j \rangle \langle \psi_j, f \rangle \right\| = \|\langle Sf, f \rangle\| \le \|Sf\| \|f\|^2 \le D \|f\|^2.$$

Also, for all $f \in \mathcal{H}$,

$$\|\langle Sf, f \rangle\| \ge \|\langle Cf, f \rangle\| = C \|f\|^2.$$

Therefor for all $f \in \mathcal{H}$

$$C||f||^{2} \leq ||\sum_{j \in J} \langle f, \psi_{j} \rangle \langle \psi_{j}, f \rangle || \leq D||f||^{2}.$$

Now, by Theorem 2.1 proof is complete.

Theorem 3.8. [8] Let X be a Banach space, $U : X \to X$ a bounded operator and ||I - U|| < 1. Then U is invertible.

Theorem 3.9. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. Then the following conditions are equivalent:

(1) There exist m > 0 and $M < \infty$ such that $mI \leq T \leq MI$;

(2) $T \in GL^+(\mathcal{H})$ i.e.T is invertible and positive.

Proof. (1) \Rightarrow (2) Since $mI \leq T$, then 0 < (T - mI), therefore

$$0 < \langle (T - mI)f, f \rangle = \langle Tf, f \rangle - \langle mf, f \rangle$$

. Hence

$$0 < m\langle f, f \rangle \le \langle Tf, f \rangle,$$

therefore T is positive.

Since $mI \leq T \leq MI$, then $0 \leq I - M^{-1}T \leq \left(\frac{M-m}{M}\right)$. Therefore $||I - M^{-1}T|| \leq \left(\frac{M-m}{M}\right) < 1$. Now by Theorem 3.8 the operator T is invertible.

(1) \Rightarrow (2) Let ||T|| = M. Since $||Th|| \leq ||T|| ||h||$ for every $h \in \mathcal{H}$, then $||Th|| \leq M ||h||$ for every $h \in \mathcal{H}$, therefore T < MI. Also,

$$||h|| = ||T^{-1}Th|| \le ||T^{-1}|| ||Th||,$$

hence

$$||T^{-1}||^{-1}||h|| \le ||Th||$$

for every $h \in \mathcal{H}$. Suppose that $m = ||T^{-1}||^{-1}$. Since T is positive, $mI \leq T$.

The following proposition shows that any C-controlled frame is a controlled frame.

Proposition 3.10. Let sequence $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ be *C*-controlled frame in Hilbert C^{*}-module \mathcal{H} for $C \in GL(\mathcal{H})$. Then Ψ is a frame in Hilbert C^{*}-module \mathcal{H} . Furthermore $CS = SC^*$ and so

$$\sum_{j \in J} \langle f, \psi_j \rangle C \psi_j = \sum_{j \in J} \langle f, C \psi_j \rangle \psi_j.$$

Proof. Define $S := C^{-1}S_C$. Then for every $f \in \mathcal{H}$

$$Sf = C^{-1}S_C f = C^{-1}\sum_{j\in J} \langle f, \psi_j \rangle C\psi_j = \sum_{j\in J} \langle f, \psi_j \rangle \psi_j.$$

The operator $S : \mathcal{H} \to \mathcal{H}$ is positive, invertible and self-adjoint. Now by Theorem 3.7 and Theorem 3.9 Ψ is a frame.

Since the operator S_C is self-adjoint and $S_C = CS$, then $CS = S_C = S_C^* = S_C^* = SC^*$.

According to the following proposition, if the operator C is self-adjoint, then C-controlled frames are equivalent to classical frames. this is a generalization of Proposition 3.3 in [5] for Hilbert C^* -module setting.

Proposition 3.11. Let \mathcal{H} be a Hilbert C^* -module and $C \in GL(\mathcal{H})$ be selfadjoint. Then $\{\psi_j \in \mathcal{H} : j \in J\}$ is a C- controlled frame for \mathcal{H} if and only if Ψ is a frame for \mathcal{H} and C is positive and commutes with frame operator S. *Proof.* Let $\{\psi_j : j \in J\}$ be a *C*-controlled frame for \mathcal{H} . Then from Proposition 3.10 $\{\psi_j : j \in J\}$ is a frame \mathcal{H} and *C* is commutes with frame operator *S*. Therefore $C = S_C C^{-1}$ is positive.

For the converse, we note that, if $\{\psi_j : j \in J\}$ is a frame, then the frame operator S is positive and invertible. Therefore $CS = S_C$ is positive and invertible. Now, by Theorem 3.9 $\{\psi_j : j \in J\}$ is a C-controlled frame for \mathcal{H} .

4. Multipliers of controlled frames and weighted frames in Hilbert C^* -modules

In this section, we generalize the concept of multipliers of frames for controlled frames in Hilbert C^* -module. Then we investigate weighted frames in Hilbert C^* -modules and verify their relation to controlled frames and multiplier operators.

Proposition 4.1. Let \mathcal{H} be a Hilbert C^* -module and $C \in GL(\mathcal{H})$. Assume $\Phi = \{\phi_j \in \mathcal{H} : j \in J\}$ and $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ are C-controlled Bessel sequence for \mathcal{H} . Then the operator

$$M_{m,\Phi,\Psi}:\mathcal{H}\to\mathcal{H}$$

defined by

$$M_{m,\Phi,\Psi}f = \sum_{j\in J} m_j \langle f, \psi_j \rangle C\phi_j$$

is a well defined bounded operator.

Proof. Let $\Phi = \{\phi_j \in \mathcal{H} : j \in J\}$ and $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ be *C*-controlled Bessel sequences for \mathcal{H} with bounds *D* and *D'*, respectively. For any $f, g \in \mathcal{H}$ and finite subset $I \subset J$,

$$\begin{split} \left\| \sum_{i \in I} m_i \langle f, \psi_i \rangle C \phi_i \right\| &= \sup_{g \in \mathcal{H}, \|g\|=1} \left\| \sum_{i \in I} m_i \langle f, \psi_i \rangle \langle C \phi_i, g \rangle \right\| \\ &\leq \sup_{g \in \mathcal{H}, \|g\|=1} \left\| \left(\sum_{i \in I} |m_i|^2 |\langle f, \psi_i \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I} |\langle C \phi_i, g \rangle|^2 \right)^{\frac{1}{2}} \right\| \\ &\leq \sup_{g \in \mathcal{H}, \|g\|=1} \|m\|_{\infty} \left\| \left(\sum_{i \in I} |\langle f, \psi_i \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I} |\langle C \phi_i, g \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \|m\|_{\infty} \sqrt{DD'} \|f\| \end{split}$$

This show that $M_{m,\Phi,\Psi}$ is well defined and

$$\|M_{m,\Phi,\Psi}\| \le \|m\|_{\infty}\sqrt{BB'}.$$

Above lemma is a motivation to define the following definition.

Definition 4.2. Let \mathcal{H} be a Hilbert C^* -module and $C \in GL(\mathcal{H})$. Assume $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ and $\Phi = \{\phi_j \in \mathcal{H} : j \in J\}$ are *C*-controlled Bessel sequences for \mathcal{H} . Then the operator

$$M_{m,\Phi,\Psi}:\mathcal{H}\to\mathcal{H}$$

defined by

$$M_{m,\Phi,\Psi}f = \sum_{j\in J} m_j \langle f, \psi_j \rangle C\phi_j$$

is called the C-controlled multiplier operator with symbol m.

The following definition is a generalization of weighted frames in Hilbert space to Hilbert C^* -module.

Definition 4.3. Let $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ be a sequence of elements in Hilbert C^* -module \mathcal{H} and $\{\omega_j\}_{j\in J} \subseteq Z(A)$ a sequence of positive weights. This pair is called a *w*-frame of Hilbert C^* -module \mathcal{H} if there exist constants C, D > 0 and such that

$$C\langle f,f\rangle \leq \sum_{j\in J} \omega_j \langle f,\psi_j\rangle \langle \psi_j,f\rangle \leq D\langle f,f\rangle.$$

for all $f \in \mathcal{H}$.

A sequence $\{c_j : j \in J\} \in Z(A)$ is called semi-normalized if there are bounds $b \ge a > 0$, such that $a \le |c_n| \le b$.

The following proposition gives a relation between controlled frames, weighted frames and multiplier operators.

Proposition 4.4. Let \mathcal{H} be a Hilbert C^* -module and $C \in GL(\mathcal{H})$ be selfadjoint and diagonal on $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ and assume it generates a controlled frame. Then the sequence $W = \{\omega_j\}_{j \in J} \subseteq Z(A)$, which verifies the relations $C\psi_n = \omega_n\psi_n$, is semi-normalized and positive. Furthermore $C = M_{W,\tilde{\Psi},\Psi}$.

Proof. By Theorem 3.9, we get the following result for $C^{1/2}$:

$$m||f||^2 \le ||C^{1/2}f||^2 \le M||f||^2.$$

As $C\psi_j = \omega_j\psi_j$, clearly $C^{1/2}\psi_j = \sqrt{\omega_j}\psi_j$. Applying the above inequalities to the elements of the sequence, we get $0 < m \le \omega_j \le M$.

Clearly, for any $f \in \mathcal{H}$

$$\begin{split} Cf &= C\left(\sum_{j \in J} \langle f, \tilde{\psi}_j \rangle \psi_j\right) = \sum_{j \in J} \langle f, \tilde{\psi}_j \rangle C\psi_j \\ &= \sum_{j \in J} \langle f, \tilde{\psi}_j \rangle \omega_j \psi_j = M_{W, \tilde{\Psi}, \Psi} f. \end{split}$$

As a to the first part of Proposition 4.4 a frame weighted by seminormalized sequence is always a frame. Indeed, we have the following lemma.

Lemma 4.5. Let $\{\omega_j : j \in J\}$ be a semi-normalized sequence with bounds a and b. If $\{\psi_j : j \in J\}$ is a frame with bounds C and D in Hilbert C*-module \mathcal{H} , then $\{\omega_j\psi_j : j \in J\}$ is also a frame with bounds a^2C and b^2D . The sequence $\{\omega_j^{-1}\tilde{\psi}_j : j \in J\}$ is a dual frame of $\{\omega_j\psi_j : j \in J\}$.

Proof. Since for any $f \in \mathcal{H} |\langle f, \omega_j \psi_j \rangle|^2 = |\omega_j|^2 |\langle f, \psi_j \rangle|^2$, we get

$$\Delta := \sum_{j \in J} |\langle f, \omega_j \psi_j \rangle|^2 = \sum_{j \in J} |\omega_j|^2 |\langle f, \psi_j \rangle|^2.$$

Thus $\Delta \leq b^2 \sum_{j \in J} |\langle f, \omega_j \psi_j \rangle|^2 \leq b^2 D ||f||^2$. In addition,

$$\Delta \ge a^2 \sum_{j \in J} \langle f, \psi_j \rangle |^2 \ge a^2 C ||f||^2.$$

As $\sum_{j \in J} \langle f, \omega_j \psi_j \rangle \omega_j^{-1} \tilde{\psi}_j = \sum_{j \in J} \langle f.\psi_j \rangle \tilde{\psi}_j = f$, these two sequences are dual. Since ω_j^{-1} is bounded, $\{\omega_j^{-1} \tilde{\psi}_j : j \in J\}$ is a Bessel sequence dual to a frame. Therefore, it is a dual frame of $\{\omega_j \psi_j : j \in J\}$.

The following results give a connection between weighted frames and frame multipliers.

Lemma 4.6. Let $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ be a frame for Hilbert C^{*}-module \mathcal{H} . Let $m = \{m_j\}_{j \in J}$ be a positive and semi-normalized sequence. Then the multiplier $M_{m,\Psi}$ is the frame operator of the frame $\{\sqrt{m_j}\psi_j : j \in J\}$ and therefore it is positive, self-adjoint and invertible. If $\{m_j\}_{j \in J}$ is negative and semi-normalized, then $M_{m,\Psi}$ is negative, self-adjoint and invertible.

Proof.

$$M_{m,\Psi}f = \sum_{j \in J} m_j \langle f, \psi_j \rangle \psi_j = \sum_{j \in J} \langle f, \sqrt{m_j}\psi_j \rangle \sqrt{m_j}\psi_j.$$

By Lemma 4.5, $\{\sqrt{m_j}\psi_j : j \in J\}$ is a frame. Therefore $M_{m,\Psi} = S_{(\sqrt{m_j}\psi_j)}$ is positive and invertible.

Let $m_j < 0$ for all $j \in J$, then $m_j = -\sqrt{|m_j|^2}$. Therefore

$$M_{m,\Psi} = -\sum_{j \in J} \langle f, \sqrt{m_j} \psi_j \rangle \sqrt{m_j} \psi_j = -S_{(\sqrt{m_j} \psi_j)}.$$

Theorem 4.7. Let $\{\psi_j : j \in J\}$ be a sequence of elements in Hilbert C^* module \mathcal{H} . Let $W = \{\omega_j : j \in J\}$ be a sequence of positive and seminormalized weights. Then the following properties are equivalent:

- (1) $\{\psi_j : j \in J\}$ is a frame;
- (2) $M_{w,\Psi}$ is a positive and invertible operator;

(3) There are constants C, D > 0 such that for all $f \in \mathcal{H}$

$$C\langle f, f \rangle \leq \sum_{j \in J} \omega_j \langle f, \psi_j \rangle \langle \psi_j, f \rangle \leq D \langle f, f \rangle.$$

- *i.e.* the pair $\{\omega_j : j \in J\}, \{\psi_j : j \in J\}$ forms a weighted frame;
- (4) $\{\sqrt{\omega_j}\psi_j : j \in J\}$ is a frame;
- (5) $M_{w,\Psi}$ is a positive and invertible operator for any positive, seminormalized sequence $W' = \{\omega'_i\}_{j \in J};$
- (6) $(\omega_j \psi_j)$ is a frame, i.e. the pair $\{\omega_j : j \in J\}, \{\psi_j : j \in J\}$ forms a weighted frame.

Proof. It is similar to Theorem 4.5 of [5].

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