Uncertainty product for Vilenkin groups

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Abstract

We study a localization of functions defined on Vilenkin groups. To measure the localization we introduce two uncertainty products UP_λ and UP_G that are similar to the Heisenberg uncertainty product. UP_λ and UP_G differ from each other by the metric used for the Vilenkin group G. We discuss analogs of a quantitative uncertainty principle. Representations for UP_λ and UP_G in terms of Walsh and Haar basis are given.

Keywords Vilenkin group; uncertainty product; Haar wavelet; modified Gibbs derivative; generalized Walsh function.

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1 Introduction

An uncertainty product for a function characterizes how concentrated is the function in time and frequency domain. Initially the notion of uncertainty product was introduced for $f \in L_2(\mathbb{R})$ by W. Heisenberg [6] and E. Schrödinger [12]. Later on extensions of this notion appeared for various algebraic and topological structures. For periodic functions, it was suggested by E. Breitenberger [1]. For some particular cases of locally compact groups (namely a euclidean motion groups, non-compact semisimple Lie groups, Heisenberg groups) the counterpart was derived in [11]. Uncertainty products on compact Riemannian manifolds was discussed in [4]. In [8], this concept was introduced for functions defined on the Cantor group. In this paper, we discuss localization of functions defined on Vilenkin groups.

To measure the localization we introduce a functional that is similar to the Heisenberg uncertainty product (see Definition 1). It depends on the metric used for the Vilenkin group G. Two equivalent metrics are in common use for the group G. So we discuss two uncertainty

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products UP_λ and UP_G. The first one is a strict counterpart of "dyadic uncertainty constant" introduced in [8] (see Theorems 1 and 2). Usage of another metric in the second uncertainty product allows for exploitation of a modified Gibbs derivative that plays a role of usual derivative for the Heisenberg uncertainty product. At the same time it turns out that usage of Haar basis is a good approach for evaluation of UP_G (see Theorem 3). In particular, it allows for an estimate of Fourier-Haar coefficients for functions defined on the Vilenkin group (see Corollary 2). The connection between UP_λ and UP_G is showed in Lemma 1.

2 Auxiliary results

We recall necessary facts about the Vilenkin group. More details can be found in [3, 13]. The Vilenkin group $G = G_p$, $p \in \mathbb{N}$, $p \neq 1$, is a set of the sequences

$$x = (x_j) = (\dots, 0, 0, x_{-k}, x_{-k+1}, x_{-k+2}, \dots),$$

where $x_j \in \{0, \dots, p-1\}$ for $j \in \mathbb{Z}$. The operation on G is denoted by $\oplus$ and defined as the coordinatewise addition modulo p :

$$(z_j) = (x_j) \oplus (y_j) \iff z_j = x_j + y_j \pmod p \quad \text{for} \quad j \in \mathbb{Z}.$$

The inverse operation of $\oplus$ is denoted by $\ominus$. The symbol $\ominus x$ denotes the inverse element of $x \in G$. The sequence $\mathbf{0} = (\dots, 0, 0, \dots)$ is a neutral element of G. If $x \neq \mathbf{0}$, then there exists a unique number $N = N(x)$ such that $x_N \neq 0$ and $x_j = 0$ for $j < N$. The Vilenkin group G_p, where $p = 2$ is called the Cantor group. In this case the inverse operation $\ominus$ coicides with the group operation $\oplus$.

Define a map $\lambda : G \to [0, +\infty)$

$$\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j-1}, \qquad x = (x_j) \in G.$$

The mapping $x \mapsto \lambda(x)$ is a bijection taking $G \setminus \mathbb{Q}_0$ onto $[0, \infty)$, where $\mathbb{Q}_0$ is a set of all elements terminating with $p-1$'s.

Two equivalent metrics are in common use for the group G. One metric is defined by $d_1(x, y) := \lambda(x \ominus y)$ for $x, y \in G$. To define another one d_2 we consider a map $\|\cdot\|_G : G \to [0, \infty)$, where $\|\mathbf{0}\|_G := 0$ and $\|x\|_G := p^{-N(x)}$ for $x \neq \mathbf{0}$. Then $d_2(x, y) := \|x \ominus y\|_G$, $x, y \in G$. Given $n \in \mathbb{Z}$ and $x \in G$, denote by $I_n(x)$ the ball of radius 2^{-n} with the center at x, i.e.

$$I_n(x) = \{y \in G : d(x, y) < 2^{-n}\}.$$

For brevity we set $I_j := I_j(\mathbf{0})$ and $I := I_0$.

We denote dilation on G by $D : G \to G$, and set $(Dx)_k = x_{k+1}$ for $x \in G$. Then $D^{-1} : G \to G$ is the inverse mapping $(D^{-1}x)_k = x_{k-1}$. Set $D^k = D \circ \dots \circ D$ (k times) if $k > 0$, and $D^k = D^{-1} \circ \dots \circ D^{-1}$ ($-k$ times) if $k < 0$; D^0 is the identity mapping.

We deal with functions taking G to $\mathbb{C}$. Denote $\mathbb{1}_E$ the characteristic function of a set $E \subset G$. Given a function $f : G \to \mathbb{C}$ and a number $h \geq 0$, for every $x \in G$ we define $f_{0,h}(x) = f(x \oplus \lambda^{-1}(h))$. Finally, we set for $j \in \mathbb{Z}$

$$f_{j,h}(x) = p^{j/2} f_{0,h}(D^j x), \quad x \in G.$$

The functional spaces $L_q(G)$ and $L_q(E)$, where E is a measurable subset of G, are derived using the Haar measure (see [7]).

Given $\xi \in G$, a group character of G is defined by

$$\chi_\xi(x) = \chi(x, \xi) := \exp\left(\frac{2\pi i}{p} \sum_{j\in\mathbb{Z}} x_j \, \xi_{-1-j}\right).$$

The functions $\mathrm{w}_n(x) := \chi(\lambda^{-1}(n), x)$ are called the generalized Walsh functions. If $p = 2$, than w_n are called the Walsh functions.

The Fourier transform of a function $f \in L^1(G)$ is defined by

$$Ff(\omega) = \int_G f(x)\overline{\chi(x,\omega)}d\mu(x), \quad \omega \in G. \tag{1}$$

The Fourier transform is extended to $L_2(G)$ in a standard way, and the Plancherel equality takes place

$$\langle f, g\rangle := \int\limits_G f(x)\overline{g(x)}\, dx = \int\limits_G Ff(\xi)\overline{Fg(\xi)}\, d\xi = \langle Ff, Fg\rangle, \quad f, g \in L_2(G).$$

The inversion formula is valid for any $f \in L_2(G)$

$$F^{-1}Ff(x) = \int_G Ff(\omega)\chi(x,\, \omega)d\mu(\omega) = f(x).$$

It is straightforward to see that

$$F(f_{j,n})(\xi) = p^{-j/2}\chi(k,\, D^{-j}\xi)Ff(D^{-j}\xi), \quad n \in \mathbb{Z}_+, j \in \mathbb{Z}. \tag{2}$$

The discrete Vilenkin-Chrestenson transform of a vector $x = (x_k)_{k=\overline{0,p^n-1}} \in \mathbb{C}^{p^n}$ is a vector $y = (y_k)_{k=\overline{0,p^n-1}} \in \mathbb{C}^{p^n}$, where

$$y_k = p^{-n} \sum_{s=0}^{p^n-1} x_s \mathrm{w}_k(\lambda^{-1}(s/p^n)), \quad 0 \leq k \leq p^n - 1. \tag{3}$$

The inverse transform is

$$x_k = \sum_{s=0}^{p^n-1} y_s \overline{\mathrm{w}_k(\lambda^{-1}(s/p^n))}. \quad 0 \leq k \leq p^n - 1. \tag{4}$$

Given $f : G_2 \to \mathbb{C}$, the function

$$f^{[1]}(x) := \lim_{n\to\infty} \sum_{j=-n}^{n} 2^{j-1}(f(x) - f_{0,2^{-j-1}}(x))$$

is called the Gibbs derivative of a function f. The following properties hold true

$$Ff^{[1]}(\xi) = \lambda(\xi)Ff(\xi), \quad \mathrm{w}_n^{[1]}(x) = n\mathrm{w}_n(x). \tag{5}$$

Set $\varphi = \mathbb{1}_I$. The Haar functions ψ^ν, $\nu = 1, \dots, p-1$ are defined by

$$\psi^\nu(x) = \sum_{n=0}^{p-1} \exp\left(\frac{2\pi i \nu n}{p}\right) \varphi(Dx \oplus \lambda^{-1}(n)). \tag{6}$$

The system $\psi^\nu_{j,k}$, $\nu = 1, \dots, p-1$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, forms an orthonormal basis (Haar basis) for $L_2(G)$, see [5, 9].

It follows from (1) that $F\varphi = \varphi = \mathbb{1}_I$ and $F\psi = \mathbb{1}_{I_0 \oplus \lambda^{-1}(p-\nu)}$. Taking into account (2), we get

$$F\psi^\nu_{j,k}(\xi) = p^{-j/2}\chi(k,\, D^{-j}\xi)\mathbb{1}_{I_{-j}\oplus\lambda^{-1}((p-\nu)p^j)}. \tag{7}$$

Given $f \in L_1(G)$, the modified Gibbs derivative $\mathcal{D}$ is defined by

$$F\mathcal{D}f = \|\cdot\|_G Ff. \tag{8}$$

It was introduced in [2] for $L_1(G_2)$. Such kind of operators are often called pseudo-differential.

Proposition 1. *Suppose g, Fg, $\|\cdot\|_G Fg$ are locally integrable on G, $j \in \mathbb{Z}$. Then the assertion* $\operatorname{supp} \widehat{g} \subset I_{-j-1} \setminus I_{-j}$ *is necessary and sufficient for g to be an eigenfunction of $\mathcal{D}$ corresponding to the eigenvalue p^j.*

The proof can be rewritten from Proposition 1 [10], where it is proved for the Cantor group.

Corollary 1. *Any Haar function $\psi^\nu_{j,k}$ is an eigenfunction of $\mathcal{D}^\alpha$ corresponding to the eigenvalue p^j.*

Proof. The statement follows from Proposition 1 and (7). □

3 Uncertainty product and metrics

Originally, the concept of an uncertainty product was introduced for the real line case in 1927. The Heisenberg uncertainty product of $f \in L_2(\mathbb{R})$ is the functional $UC_H(f) := \Delta_f \Delta_{\widehat{f}}$ such that

$$\Delta_f^2 := \|f\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} (x - x_f)^2 |f(x)|^2\, dx, \quad \Delta_{\widehat{f}}^2 := \|\widehat{f}\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} (t - t_{\widehat{f}})^2 |\widehat{f}(t)|^2\, dt,$$

$$x_f := \|f\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} x|f(x)|^2\,dx, \quad t_{\widehat{f}} := \|\widehat{f}\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} t|\widehat{f}(t)|^2\,dt,$$

where $\widehat{f}$ denotes the Fourier transform of $f \in L_2(\mathbb{R})$. It is well known that $UC_H(f) \geq 1/2$ for a function $f \in L_2(\mathbb{R})$ and the minimum is attained on the Gaussian. To motivate the definition of a localization characteristic for the Vilenkin group we note that on one hand x_f is the solution of the minimization problem

$$\min_{\tilde{x}} \int_{\mathbb{R}} (x - \tilde{x})^2 |f(x)|^2\,dx,$$

and on another hand the sense of the sign "-" in the definition of Δ_f is the distance between x and x_f. So we come to the main definition.

Definition 1. *Suppose $f : G \to \mathbb{C}$, $f \in L_2(G)$, and d is a metric on G, then a functional*

$$UP(f) := V(f)V(Ff), \qquad \textit{where}$$

$$V(f) := \frac{1}{\|f\|^2_{L_2(G)}} \min_{\tilde{x}} \int_G (d(x,\,\tilde{x}))^2 |f(x)|^2\,dx$$

is called the uncertainty product of a function f defined on the Vilenkin group.

Thus, we study two uncertainty products UP_λ and UP_G that corresponds to the metric $d_1(x,y) := \lambda(x \ominus y)$ and $d_2(x,y) := \|x \ominus y\|_G$. More precisely,

$$UP_\lambda(f) := V_\lambda(f)V_\lambda(Ff), \qquad \text{where}$$

$$V_\lambda(f) := \frac{1}{\|f\|^2_{L_2(G)}} \min_{\tilde{x}} \int_G (\lambda(x \ominus \tilde{x}))^2 |f(x)|^2\,dx.$$

The functional UP_G is defined as

$$UP_G(f) := V_G(f)V_G(Ff), \qquad \text{where}$$

$$V_G(f) := \frac{1}{\|f\|^2_{L_2(G)}} \min_{\tilde{x}} \int_G \|x \ominus \tilde{x}\|^2_G |f(x)|^2\,dx.$$

The functional UP_λ for functions defined on the Cantor group was introduced and studied in [8]. The following results are extended from the Cantor group to the Vilenkin group without any essential changes. So we omit the proofs.

Theorem 1. *Suppose $f : G \to \mathbb{C}$, $f \in L_2(G)$. Then the following inequality holds true*

$$UP_\lambda(f) \geq C, \ \textit{where}\ C \simeq 8.5 \times 10^{-5}.$$

Theorem 2. *Let $f(x) = \mathbb{1}_{\lambda^{-1}[0,1)}(x) \sum_{k=0}^{\infty} a_k \mathrm{w}_k(x)$ be a uniformly convergent series. Denote*

$$f_n(x) = \mathbb{1}_{\lambda^{-1}[0,1)}(x) \sum_{k=0}^{p^n-1} a_k \mathrm{w}_k(x).$$

Let $V_\lambda(f)<+\infty$, $V_\lambda(Ff)<+\infty$. *Then* $UP_\lambda(f)=\lim_{n\to\infty}V_\lambda(f_n)V_\lambda(Ff_n)$, *where*

$$V_\lambda(f_n)=\frac{\min_{k_0=\overline{0,p^n-1}}\sum_{k=0}^{p^n-1}p^{-n}|b_{\lambda(\lambda^{-1}(k)\oplus\lambda^{-1}(k_0))}|^2((k+1)^3-k^3)/3}{\sum_{k=0}^{p^n-1}|a_k|^2},$$

$$V_\lambda(Ff_n)=\frac{\min_{k_1=\overline{0,p^n-1}}\sum_{k=0}^{p^n-1}|a_{\lambda(\lambda^{-1}(k)\oplus\lambda^{-1}(k_1))}|^2((k+1)^3-k^3)/3}{\sum_{k=0}^{p^n-1}|a_k|^2},$$

and b_k, $0\le k\le p^n-1$, *is the inverse discrete Vilenkin-Chrestenson transform (4).*

The following Lemma shows that the functionals UP_λ and UP_G have the same order.

Lemma 1. *Suppose* $f\in L_2(G)$, *then* $p^{-4}UP_G(f)\le UP_\lambda(f)<UP_G(f)$.

Proof. It is sufficient to note that $p^{-1}\|x\|_G\le\lambda(x)<\|x\|_G$. □

Taking into account Theorem 1, we conclude that UP_G has a positive lower bound. So, UP_G satisfies the uncertainty principle.

Example 1. Let us illustrate a definition of UP_G for $p=2$ using functions f_1, g_1, f_2, and g_2 taken from [8, Example 1]. Recall $f_1(x)=\mathbb{1}_{\lambda^{-1}[0,\,1/4)}(x)$, $g_1(x)=\mathbb{1}_{\lambda^{-1}[3/4,\,1)}(x)$, $f_2(x)=\mathbb{1}_{\lambda^{-1}[0,\,3/8)}(x)$, and $g_2(x)=\mathbb{1}_{\lambda^{-1}[3/4,\,9/8)}(x)$. Their Walsh-Fourier transforms are $Ff_1=\mathbb{1}_{\lambda^{-1}[0,\,4)}/4$, $Fg_1=\mathrm{w}_3\,(\cdot/4)\,\mathbb{1}_{\lambda^{-1}[0,\,4)}/4$, $Ff_2=\mathbb{1}_{\lambda^{-1}[0,\,4)}/4+\mathrm{w}_1\,(\cdot/4)\,\mathbb{1}_{\lambda^{-1}[0,\,8)}/8$, and $Fg_2=\mathrm{w}_3\,(\cdot/4)\,\mathbb{1}_{\lambda^{-1}[0,\,4)}/4+\mathrm{w}_1(\cdot)\mathbb{1}_{\lambda^{-1}[0,\,8)}/8$. Given $\alpha\in[0,\,\infty)$, since the mapping $\alpha\mapsto\|\lambda^{-1}(\alpha)\|_G$ is increasing and a measure of the set $\lambda^{-1}[a,\,b)\ominus\tilde{x}$ does not depend on $\tilde{x}$, it follows that

$$\min_{\tilde{x}}\int_{\lambda^{-1}[0,\,\frac14)}\|x\ominus\tilde{x}\|_G\,dx=\min_{\tilde{x}}\int_{\lambda^{-1}[0,\,\frac14)\ominus\tilde{x}}\|\tau\|_G\,d\tau=\int_{\lambda^{-1}[0,\,\frac14)}\|\tau\|_G\,d\tau,$$

and $\lambda^{-1}[0,\,1/4)$ is a set of minimizing $\tilde{x}$'s as well. So, taking into account $\|f_1\|^2_{L_2(G)}=\|Ff_1\|^2_{L_2(G)}=1/4$, we get

$$V_G(f_1)=\frac{1}{\|f_1\|^2_{L_2(G)}}\min_{\tilde{x}}\int_G\|x\ominus\tilde{x}\|^2_G|f_1(x)|^2dx=4\min_{\tilde{x}}\int_{\lambda^{-1}[0,\frac14)}\|x\ominus\tilde{x}\|^2_Gdx$$

$$=4\int_{\lambda^{-1}[0,\frac14)}\|\tau\|^2_Gd\tau=4\sum_{i=2}^{\infty}\int_{\lambda^{-1}[\frac{1}{2^{i+1}},\frac{1}{2^i})}\|\tau\|^2_Gd\tau=4\sum_{i=2}^{\infty}\left(\frac{1}{2^i}-\frac{1}{2^{i+1}}\right)2^{-2i}=\frac{1}{28}.$$

Analogously, we obtain

$$V_G(Ff_1)=\frac{1}{\|Ff_1\|^2_{L_2(G)}}\min_{\tilde{x}}\int_G\|x\ominus\tilde{x}\|^2_G|Ff_1(x)|^2dx=\frac14\min_{\tilde{x}}\int_{\lambda^{-1}[0,4)}\|x\ominus\tilde{x}\|^2_Gdx$$

$$=\frac14\int_{\lambda^{-1}[0,4)}\|\tau\|^2_Gd\tau=\frac14\sum_{i=-2}^{\infty}\int_{\lambda^{-1}[\frac{1}{2^{i+1}},\frac{1}{2^i})}\|\tau\|^2_Gd\tau=\frac14\sum_{i=-2}^{\infty}\left(\frac{1}{2^i}-\frac{1}{2^{i+1}}\right)2^{-2i}=\frac{64}{7}.$$

Thus, $UP_G(f_1)=16/49$. Using the same arguments, we calculate UP_G for the remaining functions. We collect all the information in Table 1. Values of UP_λ we extract from [8, Example 1]. Columns named $\tilde{x}_0(f)$ and $\tilde{t}_0(f)$ contain sets of $\tilde{x}$ and $\tilde{t}$ minimizing the functionals $V_\lambda(f)$, $V_G(f)$ and $V_\lambda(Ff)$, $V_G(Ff)$ respectively. With respect both uncertainty products UP_G and UP_λ, functions f_1 and g_1 have the same localization, while function f_2 is more localized then g_2, that is adjusted with a naive idea of localization as a characteristic of a measure for a function support.

Table 1: UP_G and UP_λ: Example 1.

f	$\tilde{x}_0(f)$	$\tilde{t}_0(f)$	$V_\lambda(f)$	$V_\lambda(Ff)$	$UP_\lambda(f)$	$V_G(f)$	$V_G(Ff)$	$UP_G(f)$
f_1	[0, 1/4)	[0, 4)	1/48	16/3	1/9	1/28	64/7	16/49
g_1	[3/4, 1)	[0, 4)	1/48	16/3	1/9	1/28	64/7	16/49
f_2	[0, 1/8)	[0, 2)	3/64	8	3/8	4/21	96/7	128/49
g_2	[3/4, 7/8)	[0, 4)	71/64	32/3	71/6	19/14	255/14	4845/196

Example 2. Here we discuss a dependence of a localization for a fixed function on a parameter p of the Vilenkin group G_p. Let us consider a function $f_1(x) = \mathbb{1}_{\lambda^{-1}[0,1/4)}(x)$ and $p = 2^k$, $k \in \mathbb{N}$. We calculate $UP_G(f_1)$.

(1) If $k = 1$, then $UP_G(f_1) = \dfrac{16}{49}$ (see Example 1.);

(2) If $k = 2$, then

$$V_G(f_1) = \frac{1}{\|f_1\|^2_{L_2(G)}} \min_{\tilde{x}} \int_G \|x \ominus \tilde{x}\|^2_G |f_1(x)|^2 dx = 4 \min_{\tilde{x}} \int_{\lambda^{-1}[0,\frac{1}{4})} \|x \ominus \tilde{x}\|^2_G dx$$

$$= 4 \int_{\lambda^{-1}[0,\frac{1}{4})} \|\tau\|^2_G d\tau = 4 \sum_{i=1}^{\infty} \int_{\lambda^{-1}[\frac{1}{4^{i+1}},\frac{1}{4^i})} \|\tau\|^2_G d\tau = 4 \sum_{i=1}^{\infty} \left(\frac{1}{4^i} - \frac{1}{4^{i+1}} \right) 4^{-2i} = \frac{1}{21}.$$

$$V_G(Ff_1) = \frac{1}{\|Ff_1\|^2_{L_2(G)}} \min_{\tilde{x}} \int_G \|x \ominus \tilde{x}\|^2_G |Ff_1(x)|^2 dx = \frac{1}{4} \min_{\tilde{x}} \int_{\lambda^{-1}[0,4)} \|x \ominus \tilde{x}\|^2_G dx$$

$$= \frac{1}{4} \int_{\lambda^{-1}[0,4)} \|\tau\|^2_G d\tau = \frac{1}{4} \sum_{i=-1}^{\infty} \int_{\lambda^{-1}[\frac{1}{4^{i+1}},\frac{1}{4^i})} \|\tau\|^2_G d\tau = \frac{1}{4} \sum_{i=-1}^{\infty} \left(\frac{1}{4^i} - \frac{1}{4^{i+1}} \right) 4^{-2i} = \frac{256}{21}.$$

Hence, $UP_G(f_1) = \dfrac{256}{441}$.

(3) If $k > 2$, then

$$V_G(f_1) = \frac{1}{\|f_1\|^2_{L_2(G)}} \min_{\tilde{x}} \int_G \|x \ominus \tilde{x}\|^2_G |f_1(x)|^2 dx = 4 \min_{\tilde{x}} \int_{\lambda^{-1}[0,\frac{1}{4})} \|x \ominus \tilde{x}\|^2_G dx$$

$$= 4 \int_{\lambda^{-1}[0,\frac{1}{2^k}) \oplus [\frac{1}{2^k},\frac{1}{4})} \|\tau\|^2_G d\tau = 4 \left(\sum_{i=1}^{\infty} \left(\frac{1}{(2^k)^i} - \frac{1}{(2^k)^{i+1}} \right) (2^k)^{-2i} + \left(\frac{1}{4} - \frac{1}{2^k} \right) \right)$$

$$= 1 - \frac{4}{2^k} + \frac{4}{2^k(2^{2k} + 2^k + 1)}.$$

$$V_G(Ff_1) = \frac{1}{\|Ff_1\|^2_{L_2(G)}} \min_{\tilde{x}} \int_G \|x \ominus \tilde{x}\|^2_G |f_1(x)|^2 dx = \frac{1}{4} \min_{\tilde{x}} \int_{\lambda^{-1}[0,4)} \|x \ominus \tilde{x}\|^2_G dx$$

$$= \frac{1}{4} \int_{\lambda^{-1}[0,1)\oplus[1,4)} \|\tau\|^2_G d\tau = \frac{1}{4} \left(\sum_{i=0}^{\infty} \left(\frac{1}{(2^k)^i} - \frac{1}{(2^k)^{i+1}} \right) (2^k)^{-2i} + (4-1) \cdot 2^{2k} \right)$$

$$= \frac{3}{4} \cdot 2^{2k} + \frac{1}{4} \cdot \frac{2^{2k}}{2^{2k} + 2^k + 1}.$$

Therefore, $UP_G(f_1) = \left(1 - \frac{4}{2^k} + \frac{4}{2^k(2^{2k} + 2^k + 1)} \right) \left(\frac{3}{4} \cdot 2^{2k} + \frac{1}{4} \cdot \frac{2^{2k}}{2^{2k} + 2^k + 1} \right)$.

It is easy to see that time variance $V_G(f_1)$ goes to 1, and frequency variance $V_G(Ff_1)$ goes to infinity as $k \to \infty$.

4 Uncertainty product UP_G.

In this section we concentrate on the uncertainty product corresponding to the metric d_2. It turns out that the modified Gibbs derivative $\mathcal{D}$ plays a role of a usual derivative in this case. And since the Haar functions are the eigenfunctions of $\mathcal{D}$, it is possible to get representation for UP_G using the Haar coefficients.

Theorem 3. *Suppose $f \in L_2(G) \cap L_1(G)$, $\| \cdot \|_G f \in L_2(G)$, where "dot" $\cdot$ means the argument $x \in G$ of a function f, and $f(x) = \sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} c^\nu_{j,k} \psi^\nu_{j,k}(x)$. Then*

$$\int_G \|t\|^2_G |Ff(t)|^2 \, dt = \int_G |\mathcal{D}f(t)|^2 \, dt = \sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} |p^j c^\nu_{j,k}|^2 \tag{9}$$

$$\int_G \|x\|^2_G |f(x)|^2 \, dx = \int_G |\mathcal{D}Ff(x)|^2 \, dx = \sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} |p^j d^\nu_{j,k}|^2, \tag{10}$$

where $d^\nu_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, $\nu = 1, \dots, p-1$, are the coefficients in the Haar series for the function Ff, that is $Ff(t) = \sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} d^\nu_{j,k} \psi^\nu_{j,k}(t)$.

Proof. By the definition of the modified Gibbs derivative and the Plancherel equality we get

$$\int_G \|t\|^2_G |Ff(t)|^2 \, dt = \int_G |F\mathcal{D}f(t)|^2 \, dt = \int_G |\mathcal{D}f(t)|^2 \, dt.$$

Expanding a function in the Haar series and applying Corollary 1, we get

$$\int_G |\mathcal{D}f(t)|^2 \, dt = \int_G \left| \sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} c^\nu_{j,k} \mathcal{D}\psi^\nu_{j,k}(t) \right|^2 \, dt$$

$$= \int_G \left| \sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} c_{j,k}^{\nu} p^j \psi_{j,k}^{\nu}(t) \right|^2 dt = \sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} |p^j c_{j,k}^{\nu}|^2$$

The last equality follows from the orthonormality of the Haar system. Equality (10) is proved analogously to (9). □

Remark 1. *Formally, it is possible to write $\int_G \lambda^2(x)|Ff(x)|^2\,dx = \int_G |f^{[1]}(x)|^2\,dx$ and to try to represent UC_λ in terms of eigenfunctions of the Gibbs derivative $f^{[1]}$ in the case of the Cantor group. (The Gibbs derivative is defined for functions defined on the Cantor group only.) However, the Gibbs differentiation is not a local operation, that is $(f\mathbb{1}_E)^{[1]} \neq f^{[1]}\mathbb{1}_E$, see also discussion in [10]. So, usage of Walsh functions instead of Haar basis might give interesting results for periodic functions only.*

We did not found in the literature a formula expressing $d_{j,k}^{\mu}$ in terms of $c_{j,k}^{\nu}$. So we obtain this formula in the following lemma.

Lemma 2. *Suppose $f \in L_2(G)$ and the coefficients $c_{j,k}^{\nu}$, $d_{j,k}^{\mu}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, $\nu, \mu = 1, \ldots, p-1$, are defined in Theorem 3. Then*

$$d_{j,k}^{\mu} = \sum_{\nu=1}^{p-1} p^{q_0/2} b_k^{\nu} + p^{j/2} \sum_{\nu=1}^{p-1} c_{-j-1,0}^{\nu} \exp\left(-\frac{2\pi i \nu\mu}{p}\right) \delta_{k,0} + p^{j/2} \sum_{i=-\infty}^{-j-2} \sum_{\nu=1}^{p-1} c_{i,0}^{\nu} \delta_{k,0}, \qquad (11)$$

where $b_k^{\nu} = p^{-q_0} \sum_{n=0}^{p^{q_0}-1} c_{q_0-j,n+(p-\mu)p^{q_0}}^{\nu} \chi(\lambda^{-1}(n), D^{-q_0}\lambda^{-1}(k))$ is the k-th term of the discrete Vilenkin-Chrestenson transform of $(c_{q_0-j,n+(p-\mu)p_0^q}^{\nu})_{n=0}^{p_0^q-1}$, $q_0 = \left[\log_p \dfrac{k}{p-\nu}\right]$, and $\delta_{0,0} = 1$, and $\delta_{k,0} = 0$, if $k \neq 0$.

Proof. Using the Plancherel equality and (7), we get

$$d_{j,k}^{\mu} = \int_G Ff(x)\overline{\psi_{j,k}^{\mu}(x)}\,dx = \int_G f(x)\overline{F\psi_{j,k}^{\mu}(x)}\,dx = \sum_{\nu=1}^{p-1} \sum_{i\in\mathbb{Z}} \sum_{n\in\mathbb{Z}_+} c_{i,n}^{\nu} \int_G \psi_{i,n}^{\nu}(x)\overline{F\psi_{j,k}^{\mu}(x)}\,dx$$

$$= \sum_{\nu=1}^{p-1} \sum_{i\in\mathbb{Z}} \sum_{n\in\mathbb{Z}_+} c_{i,n}^{\nu} \int_G \psi_{i,n}^{\nu}(x) p^{-j/2} \overline{\chi(\lambda^{-1}(k), D^{-j}x)} \mathbb{1}_{I_{-j}\oplus\lambda^{-1}((p-\mu)p^j)}(x)\,dx.$$

Since $\operatorname{supp}\psi_{i,n}^{\nu} = \lambda^{-1}([np^{-i}, (n+1)p^{-i}))$, it follows that the last expression takes the form

$$\sum_{\nu=1}^{p-1} \sum_{i=-j}^{\infty} \sum_{n=(p-\mu)p^{i+j}}^{(p-\mu+1)p^{i+j}-1} c_{i,n}^{\nu} \int_G \psi_{i,n}^{\nu}(x) p^{-j/2} \overline{\chi(\lambda^{-1}(k), D^{-j}x)}\,dx$$

$$+p^{-j/2} \left(\sum_{\nu=1}^{p-1} c_{-j-1,0}^{\nu} \exp\left(-\frac{2\pi i \nu\mu}{p}\right) + \sum_{\nu=1}^{p-1} \sum_{i=-\infty}^{-j-2} c_{i,0}^{\nu} \right)$$

$$\times \int_G \overline{\chi(\lambda^{-1}(k), D^{-j}x)} \mathbb{1}_{I_{-j}\oplus\lambda^{-1}((p-\mu)p^j)}(x)\, dx =: S_1 + S_2.$$

For the first sum by (7) we note that

$$\int_G \psi_{i,n}^{\nu}(x)\overline{\chi(\lambda^{-1}(k), D^{-j}x)}\, dx = F\psi_{i,n}^{\nu}(D^{-j}\lambda^{-1}(k))$$

$$= p^{-i/2}\chi(n, D^{-i-j}\lambda^{-1}(k))\mathbb{1}_{I_{-i}\oplus\lambda^{-1}((p-\nu)p^i)}(D^{-j}\lambda^{-1}(k)).$$

Therefore, the first sum takes the form

$$S_1 = \sum_{\nu=1}^{p-1}\sum_{i=-j}^{\infty}\sum_{n=(p-\mu)p^{i+j}}^{(p-\mu+1)p^{i+j}-1} p^{-(j+i)/2}c_{i,n}^{\nu}\chi(\lambda^{-1}(n), D^{-i-j}\lambda^{-1}(k))\mathbb{1}_{I_{-i-j}\oplus\lambda^{-1}((p-\nu)p^(i+j))}(\lambda^{-1}(k))$$

$$= \sum_{\nu=1}^{p-1}\sum_{q=0}^{\infty} p^{-q/2}\sum_{n=0}^{p^q-1} c_{q-j,n+(p-\mu)p^q}^{\nu}\chi(\lambda^{-1}(n), D^{-q}\lambda^{-1}(k))\mathbb{1}_{I_{-q}\oplus\lambda^{-1}((p-\nu)p^q)}(\lambda^{-1}(k)).$$

Since $\mathbb{1}_{I_{-q}\oplus\lambda^{-1}((p-\nu)p^q)}(\lambda^{-1}(k)) = 1$ for $(p-\nu)p^q \le k < (p-\nu+1)p^q$ and $\mathbb{1}_{I_{-q}\oplus\lambda^{-1}((p-\nu)p^q)}(\lambda^{-1}(k)) = 0$ for the remaining k, and since the inequality $(p-\nu)p^q \le k < (p-\nu+1)p^q$, $q \in \mathbb{Z}_+$ is equivalent to $q = \left[\log_p \dfrac{k}{p-\nu}\right]$, it follows that the only nonzero term in the sum $\sum_{q=0}^{\infty}$ has the number $q_0 := \left[\log_p \dfrac{k}{p-\nu}\right]$. So

$$S_1 = \sum_{\nu=1}^{p-1} p^{-q_0/2}\sum_{n=0}^{p^{q_0}-1} c_{q_0-j,n+(p-\mu)p^{q_0}}^{\nu}\chi(\lambda^{-1}(n), D^{-q_0}\lambda^{-1}(k)).$$

By (3) we notice that up to the multiplication by a constant the inner sum in the last expression is the k-th term of the discrete Vilenkin-Chrestenson transform of the vector $(c_{q_0-j,n+(p-\mu)p_0^q}^{\nu})_{n=0}^{p_0^q-1}$. Denote this term by b_k^{ν}. Finally, for S_1 we get

$$S_1(x) = \sum_{\nu=1}^{p-1} p^{q_0/2}b_k^{\nu}.$$

Thus, the first sum takes the desired form. To conclude the proof it remains to calculate the following part of the second sum

$$\int_G \overline{\chi(k, D^{-j}x)}\mathbb{1}_{I_{-j}\oplus\lambda^{-1}((p-\mu)p^j)}(x)\, dx = p^j\int_G \overline{\chi(k,x)}\mathbb{1}_{I\oplus\lambda^{-1}(p-\mu)}(x)\, dx$$

$$= p^j\int_I \overline{\chi(k, x\ominus\lambda^{-1}(p-\mu))}\, dx = p^j\int_I \overline{\chi(k,x)}\, dx = p^j\delta_{k,0},$$

where $\delta_{0,0} = 1$, and $\delta_{k,0} = 0$, if $k \neq 0$. $\square$

It is easy to see from (9) that $\min \int_G \|t\|_G^2 |Ff(t)|^2\, dt = 0$ and $\max \int_G \|t\|_G^2 |Ff(t)|^2\, dt = \infty$ under the restriction $\|f\|_{L_2(G)} = 1$.

Formulas (9) and (10) allow for the following result on estimation of Fourier-Haar coefficients for functions defined on the Vilenkin group.

Corollary 2. *Suppose* $\|\cdot\|_G Ff \in L_2(G)$, *and* $f(x) = \sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} c_{j,k}^{\nu} \psi_{j,k}^{\nu}(x)$. *Then the series* $\sum_{\nu=1}^{p-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} |p^j c_{j,k}^{\nu}|^2$ *is convergent.*

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