# Random quantum codes from Gaussian ensembles and an uncertainty relation 

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#### Abstract

Using random Gaussian vectors and an information-uncertainty relation, we give a proof that the coherent information is an achievable rate for entanglement transmission through a noisy quantum channel. The codes are random subspaces selected according to the Haar measure, but distorted as a function of the sender's input density operator. Using large deviations techniques, we show that classical data transmitted in either of two Fourier-conjugate bases for the coding subspace can be decoded with low probability of error. A recently discovered information-uncertainty relation then implies that the quantum mutual information for entanglement encoded into the subspace and transmitted through the channel will be high. The monogamy of quantum correlations finally implies that the environment of the channel cannot be significantly coupled to the entanglement, and concluding, which ensures the existence of a decoding by the receiver.


## I. PROBLEM AND BACKGROUND

For a bipartite quantum state $\rho^{A B}$, the coherent information is defined to be

$$
I(A\rangle B)_{\rho}=H\left(\rho^{B}\right)-H\left(\rho^{A B}\right)
$$

where $H$ denotes the von Neumann entropy. Sometimes, if the state is clear from context, we omit the subscript and simply write $H(A), I(A\rangle B)$, etc. By way of notation, we adopt the habit of writing the (Hilbert space) dimension of $A$ as $|A|$.

The hashing inequality [3] is the statement that asymptotically many copies of $\rho$ have a yield of $I(A\rangle B)$ ebits per copy under entanglement distillation procedures with only local operations and one-way classical communication from Alice to Bob.

Closely related, for a quantum channel (i.e. a completely positive, trace preserving - cptp - map on density operators)

$$
\mathcal{N}: \mathcal{B}\left(A^{\prime}\right) \longrightarrow \mathcal{B}(B)
$$

and a reference state $\rho^{A^{\prime}}$ on $A^{\prime}$, we can define the coherent information $I_{c}(\rho ; \mathcal{N})$ of the channel with respect to $\rho$ as follows: Consider a purification $|\phi\rangle^{A A^{\prime}}$ of $\rho^{A^{\prime}}$, and letting $\omega^{A B}:=(\operatorname{id} \otimes \mathcal{N})|\phi\rangle\langle\phi|$, define

$$
\left.I_{c}(\rho ; \mathcal{N})=I(A\rangle B\right)_{\omega}
$$

Introducing an isometric Stinespring dilation

$$
V: A^{\prime} \hookrightarrow B \otimes E
$$

for $\mathcal{N}$ mapping the input Hilbert space $A$ into the combined output and environment spaces, we can re-express this quantity as follows: introduce the three-party state

$$
|\psi\rangle^{A B E}=(\mathbb{1} \otimes V)|\phi\rangle^{A A^{\prime}},
$$

which is a purification of $\omega^{A B}$. Then

$$
I_{c}(\rho ; \mathcal{N})=H(B)_{\psi}-H(E)_{\psi}
$$

Finally, we need the concept of quantum code: for a channel $\widetilde{\mathcal{N}}: \mathcal{B}\left(\widetilde{A^{\prime}}\right) \rightarrow \mathcal{B}(\widetilde{B})$, this is given by a pair of cptp encoding and decoding maps

$$
\begin{aligned}
& \mathcal{E}: \mathcal{B}\left(\mathbb{C}^{N}\right) \rightarrow \mathcal{B}\left(\widetilde{A}^{\prime}\right), \\
& \mathcal{D}: \mathcal{B}(\widetilde{B}) \rightarrow \mathcal{B}\left(\mathbb{C}^{N}\right) .
\end{aligned}
$$

The important parameters of a code are the dimension $N$ of the encoded system, and the error, given by the trace distance

$$
P_{\mathrm{err}}^{q}:=\left\|(\mathcal{D} \circ \tilde{\mathcal{N}} \circ \mathcal{E} \otimes \mathrm{id}) \Phi_{N}-\Phi_{N}\right\|_{1}
$$

where $\Phi_{N}=\frac{1}{N} \sum_{j, k}|j j\rangle\langle k k|$ is the maximally entangled state on $\mathbb{C}^{N} \otimes \mathbb{C}^{N}$. For more on the history of these concepts, motivation, etc., we refer the reader to the companion papers (12] and 16] ; see also 20].

The main results we are going to prove are the following two:

Theorem 1 Let $\widetilde{\mathcal{N}}: \mathcal{B}\left(\widetilde{A^{\prime}}\right) \rightarrow \mathcal{B}(\widetilde{B})$ be a quantum channel with Stinespring dilation $V: \widetilde{A}^{\prime} \hookrightarrow \widetilde{B} \widetilde{E}, \widetilde{\rho}$ an input density operator, and $P^{B}, P^{E}$ projections in $\widetilde{B}$, $\widetilde{E}$, respectively, with the following properties (for some $1 / 3 \geq \epsilon>0$ and $D, \Delta>0)$ :

$$
\begin{aligned}
\operatorname{Tr}\left(\left(V \widetilde{\rho} V^{\dagger}\right)\left(P^{B} \otimes P^{E}\right)\right) & \geq 1-\epsilon \\
P^{B} \widetilde{\mathcal{N}}(\widetilde{\rho}) P^{B} & \leq D^{-1} P^{B} \\
\widetilde{\rho} & \leq \Delta^{-1} \mathbb{1}
\end{aligned}
$$

[^0]Then, for $0<\eta<1$, there exists a quantum code with encoded dimension

$$
N \leq \min \left\{\eta \frac{D}{\operatorname{rank} P^{E}}, \eta \Delta\right\},
$$

and error $P_{e r r}^{q} \leq 2 \sqrt{2 H_{2}(2 \lambda)+4 \lambda \log N}$, where $H_{2}(x)=$ $-x \log x-(1-x) \log (1-x)$ is the binary entropy, and

$$
\lambda=9 \sqrt{\epsilon}+7 \sqrt{\eta}+3 N \exp \left(-N \epsilon^{2} / 4\right)
$$

Assuming $N \geq 2$, one obtains the simplified error bound

$$
P_{e r r}^{q} \leq 7 \sqrt{\log N} \sqrt[4]{\lambda}
$$

A particular case is that of a memoryless channel $\widetilde{\mathcal{N}}=\mathcal{N}^{\otimes n}$. We call $Q$ an achievable quantum rate for $\mathcal{N}$ if there exists a sequence of codes $\left(\mathcal{E}_{n}, \mathcal{D}_{n}\right)$ with input dimensions $N_{n}$ and error $P_{\text {err }}^{q} \rightarrow 0$ as $n \rightarrow \infty$, such that

$$
\liminf _{n \rightarrow \infty} \frac{1}{n} \log N_{n} \geq Q
$$

Theorem 2 (Lloyd [21], Shor [26] and Devetak [7]) Consider a quantum channel $\mathcal{N}: \mathcal{B}(A) \rightarrow \mathcal{B}(B)$, and an input state $\rho$ on $A^{\prime}$. Then, the coherent information $I_{c}(\rho, \mathcal{N})$ is an achievable quantum rate.

In fact, using the concept of typical subspace, the second theorem follows easily from the first. We will prove Theorem 1 in section IV] after introducing Gaussian random vectors in section III and describing the random codes we are going to look at in section III The great conceptual significance of Theorem 2 is that it makes it possible to express the quantum capacity of $\mathcal{N}$, i.e. the largest achievable rate, in terms of the coherent information; thanks to a matching upper bound by Schumacher and Nielsen [25], the capacity is thus given by

$$
Q(\mathcal{N})=\lim _{n \rightarrow \infty} \frac{1}{n} \max _{\rho^{(n)}} I_{c}\left(\rho^{(n)} ; \mathcal{N}^{\otimes n}\right)
$$

Deducing Theorem 2 from Theorem 1 is a straightforward application of typical subspace techniques [24] see appendix A: choose projectors $P_{\delta}^{A}, P_{\delta}^{B}, P_{\delta}^{E}$ in $A^{n}$, $B^{n}, E^{n}$, respectively, according to Lemma 11 (appendix A). Furthermore, let $\widetilde{A}=A_{\delta}$ be the support of $P_{\delta}^{A}$, $\widetilde{B}=B^{n}, \widetilde{E}=E^{n}$ and $\widetilde{\rho}=\frac{1}{\operatorname{Tr} \rho^{\otimes n} P_{\delta}^{A}} P_{\delta}^{A} \rho^{\otimes n} P_{\delta}^{A}$. Then the conditions of Theorem 1 are satisfied, with $\operatorname{rank} P^{E}=$ $2^{n H(E)+n \delta}, D=2^{n H(B)-n \delta}$ and $\Delta=2^{n H(A)-n \delta}$, for $\epsilon=2 \cdot 2^{-c n \delta^{2}}$ and all sufficiently large $n$. Letting $\gamma=2^{-c n \delta^{2}}$, we see that we may take $N=2^{n I(A\rangle B)-3 n \delta}$, and the get a code of encoded dimension $N$ and with error exponentially small in $n$. In other words, the rate $I(A\rangle B)-3 \delta$ is achievable; since $\delta>0$ is arbitrary, Theorem 2 follows.

The strategy we will use to prove Theorem 1 will be familiar from various Shannon-style proofs; we shall find
a subspace of the input space by an appropriate random selection, However, the analysis of the code differs from the approaches of the companion papers [12] and 16].

Both these and the present proof hinge on the demonstration that the input and environment of the channel decouple when used with the appropriate code. Once this decoupling is established, the existence of a decoding/error correction procedure for the receiver follows by a standard argument.

So, all three proofs proceed via decoupling of the channel environment or, equivalently, by forcing the quantum mutual information between input and environment to be (close to) zero. This is shown by direct calculation in [12]. In 16], following 7], one first shows that the code subspace has a basis such that the receiver can successfully measure-decode the basis state while the environment learns (almost) nothing about it - after which one "makes the decoding coherent". Here, it is done by not involving the environment at all: instead, we show that both a special orthonormal basis of the subspace as well as the Fourier conjugate basis can be decoded at the output. This means that the Holevo quantities of the two state ensembles, basis and Fourier-conjugate, are close to maximal, implying, via a recent information-uncertainty relation, that the quantum mutual information down the channel is close to maximal. This finally yields the conclusion that the crucial mutual information between the input and the environment is close to zero.

We think that this analysis is closest (among the three proofs collected in this issue) to the original idea in [26]. It is still not the same, as there an explicit description of a quantum decoder is given, without recourse to decoupling the input from the environment. See however the recent paper [19] for an alternative argument.

The rest of the paper is organised as follows: in section (II) we introduce the notion of Gaussian distributed random vectors ("Gaussian vectors" for short) and review some of their properties, mostly cited from [4], except for a tail bound on the quantum expectation of random states with an arbitrary observable. Then, in section III] we define the quantum codes which we show to be good quantum transmission codes achieving the bound of Theorem 1 in section IV. Two appendices serve to collect various auxiliary results about states, measurements, and typical subspaces used throughout the paper, in addition to miscellaneous proofs.

## II. GAUSSIAN VECTORS

We take the following definitions in abridged form from appendix A of [4]; the interested reader is encouraged to consult the referenced paper.

A Gaussian complex number with mean 0 and variance $\sigma^{2}>0$ is a random variable $X+i Y$, where $X$ and $Y$ are independent real random variables with $X \sim N\left(0, \frac{\sigma^{2}}{2}\right)$ and $Y \sim N\left(0, \frac{\sigma^{2}}{2}\right)$. Its distribution is denoted $N_{\mathbb{C}}\left(0, \sigma^{2}\right)$.

For any orthonormal basis $\{|1\rangle, \ldots,|D\rangle\}$ of $\mathbb{C}^{D}$, a Gaussian vector is defined to be a random variable $|g\rangle \in \mathbb{C}^{D}$ whose distribution is described as follows:

$$
|g\rangle=\sum_{i=1}^{D} c_{i}|i\rangle
$$

with $N$ independent Gaussian complex numbers $c_{1}, \ldots, c_{D} \sim N_{\mathbb{C}}(0,1 / D)$. It is a fundamental property of the above sum that the resulting distribution is independent of the basis chosen. I.e., the distribution is unitarily invariant, and in particular, its density depends only on the length $\||g\rangle \|_{2}=\sqrt{\langle g \mid g\rangle}=\sqrt{\sum_{i}\left|c_{i}\right|^{2}}$. Indeed, we defined the Gaussian vectors in just such a way that $\mathbb{E}\langle g \mid g\rangle=1$. And according to Lemma 3 below the distribution is strongly concentrated around this value.

Lemma 3 Let $|g\rangle$ and $\left|g_{1}\right\rangle, \ldots,\left|g_{K}\right\rangle$ be independent Gaussian vectors in $\mathbb{C}^{D}$. Then, for $0 \leq \epsilon \leq 1$,

$$
\operatorname{Pr}\{|\operatorname{Tr}| g\rangle\langle g|-1 \mid>\epsilon\} \leq 2 \exp \left(-\epsilon^{2} d / 6\right)
$$

and, for a projector $P$ of rank $r$,
$\left.\left.\operatorname{Pr}\left\{\left|\sum_{k=1}^{K} \operatorname{Tr}\right| g_{k}\right\rangle\left\langle g_{k}\right| P-\frac{r K}{D} \right\rvert\,>\epsilon \frac{r K}{D}\right\} \leq 2 \exp \left(-r K \frac{\epsilon^{2}}{6}\right)$.
Furthermore, for $\epsilon \leq 1 / 3$, and $0 \leq A \leq \mathbb{1}$ an operator,

$$
\begin{align*}
& \operatorname{Pr}\left\{\operatorname{Tr}|g\rangle\langle g| A>(1+\epsilon) \frac{\operatorname{Tr} A}{d}\right\} \leq \exp \left(-\frac{\epsilon^{2}}{4} \operatorname{Tr} A\right),  \tag{1}\\
& \operatorname{Pr}\left\{\operatorname{Tr}|g\rangle\langle g| A<(1-\epsilon) \frac{\operatorname{Tr} A}{d}\right\} \leq \exp \left(-\frac{\epsilon^{2}}{4} \operatorname{Tr} A\right) \tag{2}
\end{align*}
$$

Proof. The first and second statement, about the lengths of Gaussian vectors and average inner products, is from Lemma 3 in [4] - see also appendix A there - or Lemma II. 3 in (13].

The third is a generalisation of Lemma 3 in [4] (Lemma II. 3 in [13]). It is proved in appendix B.

## III. RANDOM SUBSPACE PROJECTORS

For an input space $\widetilde{A}$ of dimension $|\widetilde{A}|$, and reference state $\widetilde{\rho}$, the code will be chosen as follows: pick a subspace $S_{0}$ of dimension $N$ according to the Haar measure, denoting its corresponding subspace projector $P_{0}$. Then, let $S=\sqrt{\widetilde{\rho}} S_{0}$, so its subspace projection $P$ projects onto supp $\sqrt{\widetilde{\rho}} P_{0} \sqrt{\widetilde{\rho}}$, the support of the projector $\sqrt{\widetilde{\rho}} P_{0} \sqrt{\widetilde{\rho}}$; this will be our random code for Theorem 1

Our preferred way of describing this random selection is via a spanning set of vectors drawn independently as follows. For $j=1, \ldots, N$, let $\left|g_{j}\right\rangle$ be i.i.d. Gaussian vectors in $\widetilde{A}$. With probability one, these are linearly independent, so they span an $N$-dimensional subspace $S_{0}$,
which, by the unitary invariance of the Gaussian measure, is itself distributed according to the unitarily invariant measure. Now let

$$
\left|\gamma_{j}\right\rangle:=\sqrt{|\widetilde{A}| \widetilde{\rho}}\left|g_{j}\right\rangle
$$

These vectors will turn out to be almost normalised, with high probability. They clearly span $S=\sqrt{\widetilde{\rho}} S_{0}$, but we are after more; we need an orthogonal basis of $S$. To get this, we follow the recipe of the "square root" or "pretty good" measurement: with the (random) operator $\Gamma:=\sum_{j=1}^{N}\left|\gamma_{j}\right\rangle\left\langle\gamma_{j}\right|$, we finally define

$$
\left|\phi_{j}\right\rangle:=\Gamma^{-1 / 2}\left|\gamma_{j}\right\rangle,
$$

which is an orthogonal basis of $S$ (if the $\left|\gamma_{j}\right\rangle$ are linearly independent) because the subspace projector is $P=\sum_{j}\left|\phi_{j}\right\rangle\left\langle\phi_{j}\right|$.

As outlined in the introduction, we will aim to show that this basis, sent through the channel with equal probabilities, will yield an output ensemble of states $\sigma_{j}=\mathcal{N}\left(\phi_{j}\right)$ with Holevo information close to $\log N$. In fact, we have to show this for the basis $\left\{\left|\phi_{j}\right\rangle\right\}$ as well as for its Fourier-conjugate basis consisting of the vectors

$$
\left|\widehat{\phi}_{k}\right\rangle=\frac{1}{\sqrt{N}} \sum_{j} e^{2 \pi i j k / N}\left|\phi_{j}\right\rangle
$$

On the face of it, this set of vectors could have a peculiar, perhaps hard to describe, distribution. This is not at all the case thanks to the particular properties of the Gaussian distribution and the Fourier transform.

Definition 4 We call a family $\left\{\left|w_{1}\right\rangle, \ldots,\left|w_{N}\right\rangle\right\}$ of vectors formally Fourier-conjugate to the family of vectors $\left\{\left|v_{1}\right\rangle, \ldots,\left|v_{N}\right\rangle\right\}$, if for all $k$,

$$
\left|w_{k}\right\rangle=\frac{1}{\sqrt{N}} \sum_{j} e^{2 \pi i j k / N}\left|v_{j}\right\rangle
$$

Note that we do not demand normalisation or orthogonality of the vectors in either family. Also, the dimension $D$ of the space may be different from $N$.

Lemma 5 If the family $\left\{\left|w_{1}\right\rangle, \ldots,\left|w_{N}\right\rangle\right\}$ of vectors is the formal Fourier-conjugate of the family $\left\{\left|v_{1}\right\rangle, \ldots,\left|v_{N}\right\rangle\right\}$, then for all $j$,

$$
\left|v_{j}\right\rangle=\frac{1}{\sqrt{N}} \sum_{k} e^{-2 \pi i j k / N}\left|w_{k}\right\rangle
$$

Furthermore,

$$
\sum_{j}\left|v_{j}\right\rangle\left\langle v_{j}\right|=\sum_{k}\left|w_{k}\right\rangle\left\langle w_{k}\right| .
$$

Finally, if $\left\{\left|v_{1}\right\rangle, \ldots,\left|v_{N}\right\rangle\right\}$ are independent Gaussian vectors with $N \leq D$, then so are $\left\{\left|w_{1}\right\rangle, \ldots,\left|w_{N}\right\rangle\right\}$.

Proof. Straightforward calculations.
This means that there is another, equivalent, way of arriving at the basis $\left\{\left|\widehat{\phi}_{k}\right\rangle\right\}$ of $S$ : namely, start with the set of (by Lemma 5, Gaussian!) vectors

$$
\left|\widehat{g}_{k}\right\rangle=\frac{1}{\sqrt{N}} \sum_{j} e^{2 \pi i j k / N}\left|g_{j}\right\rangle
$$

formally Fourier-conjugate to the $\left|g_{j}\right\rangle$. Then we can form the vectors $\left|\widehat{\gamma}_{k}\right\rangle=\sqrt{|\widetilde{A}| \widetilde{\rho}}\left|\widehat{g}_{k}\right\rangle$, and they are clearly formally Fourier-conjugate to the $\left|\gamma_{j}\right\rangle$. Finally, by Lemma 5 above, the normalisation operator $\widehat{\Gamma}=\sum_{k}\left|\widehat{\gamma}_{k}\right\rangle\left\langle\widehat{\gamma}_{k}\right|$ equals $\Gamma$, so we find that

$$
\left|\widehat{\phi}_{k}\right\rangle=\widehat{\Gamma}^{-1 / 2}\left|\widehat{\gamma}_{k}\right\rangle=\Gamma^{-1 / 2}\left|\widehat{\gamma}_{k}\right\rangle
$$

In other words, we have arrive at the
Proposition 6 The distribution of the set $\left\{\left|\widehat{\phi}_{k}\right\rangle\right\}_{k}$ is exactly the same as that of the set $\left\{\left|\phi_{j}\right\rangle\right\}_{j}$.

## IV. PERFORMANCE ANALYSIS

In the previous section we have described a random subspace $S$ of $\widetilde{A}^{\prime}$. The encoder of the code will simply be the isometric identification of $\mathbb{C}^{N}$ with $S: \mathcal{E}=U \cdot U^{\dagger}$, with

$$
\begin{gathered}
U: \mathbb{C}^{N} \longrightarrow S \hookrightarrow \widetilde{A}^{\prime} \\
|j\rangle \longmapsto\left|\phi_{j}\right\rangle .
\end{gathered}
$$

Following Devetak [7] - see Lemma 1.1 in [12] - we do not worry about the decoding map; it will exist once the "decoupling from the environment" condition holds. Namely, denoting $R=\mathbb{C}^{N}, \tau^{R}$ the maximally mixed state on $R$, and

$$
|\Psi\rangle^{R \widetilde{B} \widetilde{E}}:=(\mathbb{1} \otimes V U)\left|\Phi_{N}\right\rangle
$$

we know that a decoder $\mathcal{D}$ with error $p$ exists once we ascertain that

$$
\left\|\Psi^{R \widetilde{E}}-\tau^{R} \otimes \vartheta^{\widetilde{E}}\right\|_{1} \leq p
$$

for an arbitrary state $\vartheta^{\widetilde{E}}$ of the environment.
By Pinsker's inequality [23] for the relative entropy, applied to $\Psi^{R \widetilde{E}}$ and $\tau^{R} \otimes \Psi^{\widetilde{E}}$,

$$
\begin{aligned}
I(R: \widetilde{E}) & =D\left(\Psi^{R \widetilde{E}} \| \tau^{R} \otimes \Psi^{\widetilde{E}}\right) \\
& \geq\left(\frac{1}{2}\left\|\Psi^{R \widetilde{E}}-\tau^{R} \otimes \vartheta^{\widetilde{E}}\right\|_{1}\right)^{2}
\end{aligned}
$$

so it is enough to show $I(R: \widetilde{E}) \leq p^{2} / 4$. Here, $I(R: \widetilde{E})=H(R)+H(\widetilde{E})-H(R \widetilde{E})$ is the quantum
mutual information, and $D(\rho \| \sigma)=\operatorname{Tr} \rho(\log \rho-\log \sigma)$ is the quantum relative entropy.

By the elementary identity

$$
2 H(R)=I(R: \widetilde{E})+I(R: \widetilde{B})
$$

which holds for any pure state on $R \tilde{B} \tilde{E}$, and with $H(R)=\log N$ in our case, we will be done as soon as we show $I(R: \widetilde{B}) \geq 2 \log N-p^{2} / 4$. The proof that this inequality holds for a random subspace is based on the following "information-uncertainty relation":

## Lemma 7 (Information-uncertainty [5], Lemma 1)

 Let $\mathcal{E}_{0}=\{1 / N,|j\rangle\langle j|\}$ be the uniform ensemble for an arbitrary fixed orthonormal basis $\{|j\rangle\}$ of an $N$-dimensional Hilbert space $S$, and $\mathcal{E}_{1}=\left\{1 / N, \mathrm{QFT}|j\rangle\langle j| \mathrm{QFT}^{\dagger}\right\}$, where QFT is the Fourier transform in dimension $N$.Then, for any quantum channel $\mathcal{M}$ with input space $S$ and output $B$,

$$
\chi\left(\mathcal{M}\left(\mathcal{E}_{0}\right)\right)+\chi\left(\mathcal{M}\left(\mathcal{E}_{1}\right)\right) \leq I(R: B)_{\omega}
$$

Here, the right hand side is the quantum mutual information of the state $\omega^{R B}=(\mathrm{id} \otimes \mathcal{M}) \Phi_{d}$, where $\Phi_{d}$ is the maximally entangled state on $R S$. On the left hand side, we have two Holevo informations [14] of the ensembles $\mathcal{M}\left(\mathcal{E}_{i}\right)$ of channel output states; for an arbitrary ensemble $\mathcal{E}=\left\{p_{x}, \sigma_{x}\right\}$ of states,

$$
\chi(\mathcal{E}):=H\left(\sum_{x} p_{x} \sigma_{x}\right)-\sum_{x} p_{x} H\left(\sigma_{x}\right)
$$

Of course, the assumption of this lemma is just our situation: we have a subspace $S$ of dimension $N$ in $\widetilde{A}$, and consider two Fourier-conjugate bases.

Hence, in the light of Proposition 66 all we need to show is the following:

Proposition 8 Under the assumptions of Theorem 1, consider independent Gaussian vectors $\left|g_{1}\right\rangle, \ldots,\left|g_{N}\right\rangle \in$ $\widetilde{A^{\prime}}$, as well as

$$
\begin{aligned}
\left|\gamma_{j}\right\rangle & :=\sqrt{|\widetilde{A}| \widetilde{\rho}}\left|g_{j}\right\rangle, \\
\Gamma & :=\sum_{j}\left|\gamma_{j}\right\rangle\left\langle\gamma_{j}\right| \\
\left|\phi_{j}\right\rangle & :=\Gamma^{-1 / 2}\left|\gamma_{j}\right\rangle
\end{aligned}
$$

Then, for the output ensemble

$$
\mathcal{E}=\left\{1 / N, \sigma_{j}:=\tilde{\mathcal{N}}\left(\left|\phi_{j}\right\rangle\left\langle\phi_{j}\right|\right)\right\}
$$

it holds with probability $>1 / 2$ that

$$
\chi(\mathcal{E}) \geq \log N-H_{2}(2 \lambda)-2 \lambda \log N
$$

where

$$
\lambda=9 \sqrt{\epsilon}+7 \sqrt{\eta}+3 N \exp \left(-N \epsilon^{2} / 6\right)
$$

As a consequence, we have that with positive probability both $\mathcal{E}$ and the ensemble obtained from the Fourierconjugate inputs,

$$
\widehat{\mathcal{E}}=\left\{1 / N, \tilde{\mathcal{N}}\left(\left|\widehat{\phi}_{k}\right\rangle\left\langle\widehat{\phi}_{k}\right|\right)\right\}
$$

have $\chi(\mathcal{E}), \quad \chi(\widehat{\mathcal{E}}) \geq \log N-H_{2}(2 \lambda)-2 \lambda \log N$. By Lemma 7 this means $I(R: \widetilde{B}) \geq 2 \log N-2 H_{2}(2 \lambda)-$ $4 \lambda \log N$, hence $I(R: \widetilde{E}) \leq 2 H_{2}(2 \lambda)+4 \lambda \log N$, and we are done. Observing that $H_{2}(x) \leq 2 \sqrt{x(1-x)}$, the right hand side can be further upper bounded by $6 \sqrt{\lambda}+4 \lambda \log N$, which is $\leq 10 \sqrt{\lambda} \log N$ as long as $N \geq 2$.

To conclude, we use Pinsker's inequality, as described at the start of this section, to relate $P_{\text {err }}^{q}$ and (the upper bounds on) the mutual information $I(R: \widetilde{E})$.

Proof of Proposition 8. What we shall show is that there exists a classical decoder for the ensemble achieving small error probability; i.e. we need to find a POVM $\left(\Lambda_{j}\right)_{j=1}^{N}$ such that

$$
P_{\mathrm{err}}^{c}:=\frac{1}{N} \sum_{j} \operatorname{Tr}\left[\sigma_{j}\left(\mathbb{1}-\Lambda_{j}\right)\right]
$$

is small, at least in expectation. Then, denoting the random output of the measurement $j^{\prime}$, we have that by the monotonicity of the Holevo quantity under postprocessing and the classic Fano inequality [6],

$$
\chi(\mathcal{E}) \geq I\left(j: j^{\prime}\right) \geq \log N-P_{\mathrm{err}}^{c} \log N-H_{2}\left(P_{\mathrm{err}}^{c}\right)
$$

Looking at this, we are done once we show that

$$
\mathbb{E} P_{\mathrm{err}}^{c} \leq 9 \sqrt{\epsilon}+7 \sqrt{\eta}+3 N \exp \left(-N \epsilon^{2} / 6\right)=: \lambda
$$

The reason is Markov's inequality, telling us that the probability of a random random code having $P_{\text {err }}^{c}>2 \lambda$ is strictly smaller than $1 / 2$.

For this, we first analyse random codes drawn from the ensemble $|\gamma\rangle=\sqrt{|\widetilde{A}| \widetilde{\rho}}|g\rangle$, with Gaussian $|g\rangle$. The states $\gamma=|\gamma\rangle\langle\gamma|$ and so the $\sigma_{g}:=\widetilde{\mathcal{N}}(\gamma)$ are of course not generally normalised, but we can still apply the Packing Lemma (Lemma 9 in appendix A). There, we let $\Pi=$ $P^{B}$, and the $\Pi_{g}$ for the individual ensemble states $\widetilde{\mathcal{N}}(\gamma)$ are constructed as follows: observe that $\left|\gamma^{\prime}\right\rangle:=(\mathbb{1} \otimes$ $\left.P^{E}\right) V|\gamma\rangle \in \widetilde{B} \otimes \widetilde{E}$ is a vector of Schmidt rank at most $d=\operatorname{rank} P^{E}$, so we may choose $\Pi_{g}$ to be the projector onto the support of $\operatorname{Tr}_{E}\left|\gamma^{\prime}\right\rangle\left\langle\gamma^{\prime}\right|$. The conditions of the Packing Lemma are easily verified - observing that $\mathbb{E} \gamma=$ $\widetilde{\rho}$, so $\mathbb{E} \sigma_{g}=\widetilde{\mathcal{N}}(\widetilde{\rho})=: \sigma$.

We conclude that for i.i.d. $\left\{\left|\gamma_{1}\right\rangle, \ldots,\left|\gamma_{N}\right\rangle\right\}$ there is a $\operatorname{POVM}\left\{\Lambda_{1}, \ldots, \Lambda_{N}\right\}$ such that

$$
\mathbb{E} P_{\mathrm{err}}^{c}\left(\left\{\mathcal{N}\left(\gamma_{j}\right), \Lambda_{j}\right\}_{j=1}^{N}\right) \leq 6 \sqrt{\epsilon}+4 \eta
$$

Now, if we use the same decoder instead for the states $\left|\phi_{j}\right\rangle=\Gamma^{-1 / 2}\left|\gamma_{j}\right\rangle$, we incur additional errors, as follows:

First of all, by Lemma 3 applied to $A=\Delta \widetilde{\rho}$ we have, except with probability $\leq 2 N \exp \left(-\Delta \epsilon^{2} / 4\right)$, that

$$
\begin{equation*}
\forall j \quad 1-\epsilon \leq\left\langle\gamma_{j} \mid \gamma_{j}\right\rangle \leq 1+\epsilon \tag{3}
\end{equation*}
$$

which we shall assume to hold from now on.
Furthermore, we have, using the elementary inequality $\|\phi-\gamma\|_{1} \leq \sqrt{2}\|\phi-\gamma\|_{2}$ for rank one projectors $\phi$ and $\gamma$, and eq. (3), that

$$
\begin{align*}
& \frac{1}{N} \sum_{j} \frac{1}{2}\left\|\phi_{j}-\gamma_{j}\right\|_{1} \leq \frac{1}{N} \sum_{j} \frac{\sqrt{2}}{2}\left\|\phi_{j}-\gamma_{j}\right\|_{2} \\
& \quad=\frac{1}{N} \sum_{j} \frac{1}{\sqrt{2}} \sqrt{\left\langle\phi_{j} \mid \phi_{j}\right\rangle^{2}+\left\langle\gamma_{j} \mid \gamma_{j}\right\rangle^{2}-2\left|\left\langle\phi_{j} \mid \gamma_{j}\right\rangle\right|^{2}} \\
& \quad \leq \frac{1}{N} \sum_{j} \sqrt{(1+\epsilon)^{2}-\left|\left\langle\phi_{j} \mid \gamma_{j}\right\rangle\right|^{2}} \\
& \quad \leq \sqrt{\frac{1}{N} \sum_{j}\left((1+\epsilon)^{2}-\left|\left\langle\phi_{j} \mid \gamma_{j}\right\rangle\right|^{2}\right)} \\
& \quad \leq \sqrt{\frac{1}{N} \sum_{j} 2(1+\epsilon)\left((1+\epsilon)-\left|\left\langle\phi_{j} \mid \gamma_{j}\right\rangle\right|\right)} \tag{4}
\end{align*}
$$

where the second-to-last line follows by the concavity of the square root function, and the last involves the Cauchy-Schwarz inequality.

We shall concentrate for the moment on the average under the square root:

$$
\begin{align*}
& \frac{1}{N} \sum_{j}\left((1+\epsilon)-\left|\left\langle\phi_{j} \mid \gamma_{j}\right\rangle\right|\right)=\epsilon+\frac{1}{N} \sum_{j}\left(1-\left|\left\langle\phi_{j} \mid \gamma_{j}\right\rangle\right|\right) \\
& \quad=\epsilon+\frac{1}{N} \sum_{j}\left(1-\left\langle\gamma_{j}\right| \Gamma^{-1 / 2}\left|\gamma_{j}\right\rangle\right) \\
& \quad=\epsilon+1-\frac{1}{N} \operatorname{Tr} \sqrt{\Gamma} \tag{5}
\end{align*}
$$

where we have inserted the definition of the $\left|\phi_{j}\right\rangle$, and noted that the inner products $\left\langle\phi_{j} \mid \gamma_{j}\right\rangle$ are non-negative. Now, we use a trick from [10]: for the positive semidefinite operator $\Gamma$,

$$
\sqrt{\Gamma} \geq \frac{3}{2} \Gamma-\frac{1}{2} \Gamma^{2},
$$

so we can continue upper bounding as follows, using the abbreviation $S_{j k}=\left\langle\gamma_{j} \mid \gamma_{k}\right\rangle$ :

$$
\begin{aligned}
1-\frac{1}{N} \sqrt{\Gamma} & \leq 1-\frac{1}{N}\left(\frac{3}{2} \Gamma-\frac{1}{2} \Gamma^{2}\right) \\
& =\frac{1}{N}\left(N-\frac{3}{2} \sum_{j} S_{j j}+\frac{1}{2} \sum_{j k}\left|S_{j k}\right|^{2}\right) \\
& =\frac{1}{N} \sum_{j}\left(1-\frac{3}{2} S_{j j}+\frac{1}{2} S_{j j}^{2}\right)+\frac{1}{N} \sum_{j \neq k}\left|S_{j k}\right|^{2} \\
& =\frac{1}{N} \sum_{j}\left(1-S_{j j}\right)\left(1-\frac{1}{2} S_{j j}\right)+\frac{1}{N} \sum_{j \neq k}\left|S_{j k}\right|^{2} .
\end{aligned}
$$

Here, the first term is bounded above by $\epsilon \frac{1+\epsilon}{2}$. The second term consists of an average of $N$ expressions, one for each $j$, of the form

$$
\begin{aligned}
\sum_{k \neq j}\left|\left\langle\gamma_{j} \mid \gamma_{k}\right\rangle\right|^{2} & \left.=\sum_{k \neq j}\left|\left\langle\gamma_{j}\right| \sqrt{|\widetilde{A}| \widetilde{\rho}}\right| g_{k}\right\rangle\left.\right|^{2} \\
& \leq(1+\epsilon) \frac{|\widetilde{A}|}{\Delta} \sum_{k \neq j} \operatorname{Tr}\left|g_{k}\right\rangle\left\langle g_{k}\right| P_{j}
\end{aligned}
$$

with a rank one projector $P_{j}$. So we can apply Lemma 3 once more to find that, except with probability $\leq$ $N \exp \left(-N \epsilon^{2} / 6\right)$, the latter expressions are all upper bounded by

$$
(1+\epsilon) \frac{|\widetilde{A}|}{\Delta}(1+\epsilon) \frac{N}{|\widetilde{A}|} \leq(1+\epsilon)^{2} \eta
$$

Inserting all this into eq. (5), we find

$$
\frac{1}{N} \sum_{j}\left((1+\epsilon)-\left|\left\langle\phi_{j} \mid \gamma_{j}\right\rangle\right|\right) \leq \epsilon+\epsilon \frac{1+\epsilon}{2}+(1+\epsilon)^{2} \eta
$$

In turn plugging that into eq. (4), we arrive at

$$
\begin{aligned}
\frac{1}{N} \sum_{j} \frac{1}{2}\left\|\phi_{j}-\gamma_{j}\right\|_{1} & \leq \sqrt{2(1+\epsilon)\left(\epsilon \frac{3+\epsilon}{2}+(1+\epsilon)^{2} \eta\right)} \\
& \leq \sqrt{9 \epsilon+9 \eta} \leq 3 \sqrt{\epsilon}+3 \sqrt{\eta}
\end{aligned}
$$

remembering $\epsilon \leq 1 / 3$.
Putting all this together, with the monotonicity of the trace norm under cptp maps and using $\operatorname{Tr}((\rho-\sigma) \Lambda) \leq$ $\frac{1}{2}\|\rho-\sigma\|_{1}$ for states $\rho, \sigma$ and $0 \leq \Lambda \leq \mathbb{1}$, leads to

$$
\begin{aligned}
& \mathbb{E} P_{\text {err }}^{c}\left(\left\{\mathcal{N}\left(\phi_{j}\right), \Lambda_{j}\right\}_{j=1}^{N}\right) \\
& \leq \mathbb{E} P_{\text {err }}^{c}\left(\left\{\mathcal{N}\left(\gamma_{j}\right), \Lambda_{j}\right\}_{j=1}^{N}\right) \\
& \quad+3 \sqrt{\epsilon}+3 \sqrt{\eta}+3 N \exp \left(-N \epsilon^{2} / 6\right) \\
& \leq 6 \sqrt{\epsilon}+4 \eta+3 \sqrt{\epsilon}+3 \sqrt{\eta}+3 N \exp \left(-N \epsilon^{2} / 6\right)
\end{aligned}
$$

and we are done.

## V. CONCLUSION

We have given yet another proof of the direct part of the quantum channel coding theorem, in the sense of showing the achievability of the coherent information rate.

The present proof is distinguished from other approaches in that it is shown that the classical information in two Fourier-conjugate bases of the code subspace can be recovered at the output. Application of a recent information-uncertainty relation then ensures that the quantum information in the subspace can in fact be decoded.

It is tempting to speculate that the role of the pair of measurement-decoders for the two conjugate bases is to implement the measurement of the familiar basis and phase errors of a conventional quantum error correcting code, or their equivalents. To give more substance to this idea, it would be necessary to show how to build the quantum decoder directly from the two measurementdecoders. We leave this as an open problem.

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## APPENDIX A: MISCELLANEOUS LEMMAS

Lemma 9 (Packing [18]) Consider an ensemble $\left\{p_{m}, \sigma_{m}\right\}$ of positive semidefinite operators (not necessarily states!) with average $\sigma=\sum_{m} p_{m} \sigma_{m}$, which is assumed to be a density operator; in particular, $\sum_{m} p_{m} \operatorname{Tr} \sigma_{m}=1$. Assume the existence of projectors $\Pi$ and $\Pi_{m}$ with the following properties:

$$
\begin{aligned}
\sum_{m} p_{m} \operatorname{Tr} \sigma_{m} \Pi_{m} & \geq 1-\epsilon \\
\sum_{m} p_{m} \operatorname{Tr} \sigma_{m} \Pi & \geq 1-\epsilon \\
\operatorname{Tr} \Pi_{m} & \leq d \\
\Pi \sigma \Pi & \leq D^{-1} \Pi
\end{aligned}
$$

for all $m$. Let $N=\lfloor\eta D / d\rfloor$ for some $0<\eta<1$, and pick $m_{1}, \ldots, m_{N}$ independently at random according to the distribution $p_{m}$.

Then there exists a corresponding $\operatorname{POVM}\left\{\Lambda_{k}\right\}_{k=1}^{N}$ which reliably distinguishes between the states $\left\{\sigma_{m_{k}}\right\}_{k=1}^{N}=1$ in the sense that the expectation of the (average) error probability of the code $\left\{\sigma_{m_{k}}, \Lambda_{k}\right\}_{k=1}^{N}$,

$$
P_{e r r}^{c}=P_{e r r}^{c}\left(\left\{\sigma_{m_{k}}, \Lambda_{k}\right\}\right):=\frac{1}{N} \sum_{k} \operatorname{Tr}\left[\sigma_{m_{k}}\left(\mathbb{1}-\Lambda_{k}\right)\right],
$$

satisfies

$$
\mathbb{E} P_{e r r}^{c} \leq 2 \epsilon+4 \sqrt{\epsilon}+4 \eta \leq 6 \sqrt{\epsilon}+4 \eta
$$

(In particular, there exists a code with error bounded by the above quantity.)

The same statements hold for continuous ensembles the above formulation with a discrete probability distribution was chosen only for notational convenience.

Proof. It is almost the same statement and proof as Lemma 2 in [18], which itself is an adaptation of a result by Hayashi and Nagaoka 11.

Note that we demand state normalisation of the $\sigma_{m}$ not individually, but only in the ensemble average - which makes the lemma more suitable to be applied with the, generally unnormalised, Gaussian input states. Inspecting the proof in [18], it is evident that in fact only that is required.

There are only the following two other differences. We use the slightly better "Gentle measurement Lemma" of Ogawa and Nagaoka 22] instead of [27] - see Lemma 10 below. And whereas [18] demands that for all $m$,

$$
\operatorname{Tr} \sigma_{m} \Pi_{m}, \operatorname{Tr} \sigma_{m} \Pi \geq 1-\epsilon
$$

our conditions on $\Pi$ and the $\Pi_{m}$ require this to hold only on average over the ensemble $\left\{p_{m}, \sigma_{m}: m \in \mathcal{M}\right\}$. Looking at the proof in [18], it is evident that this condition is indeed enough for the conclusion.

## Lemma 10 (Gentle measurement [27] and [22])

Let $\rho$ be positive semidefinite, and $0 \leq X \leq \mathbb{1}$ be an operator on some Hilbert space, such that $\operatorname{Tr}(\rho(\mathbb{1}-X)) \leq \epsilon \operatorname{Tr} \rho$. Then,

$$
\|\rho-\sqrt{X} \rho \sqrt{X}\|_{1} \leq 2 \sqrt{\epsilon} \operatorname{Tr} \rho
$$

Here follow some properties of typical subspaces as defined in 24]; we quote directly from [12]. Consider a density matrix with spectral decomposition $\rho^{A}=$ $\sum_{x} p_{x}|x\rangle\left\langle\left. x\right|^{A}\right.$. Its $n$th tensor power can be written as

$$
\left(\rho^{A}\right)^{\otimes n}=\sum_{x^{n}} p_{x^{n}}\left|x^{n}\right\rangle\left\langle\left. x^{n}\right|^{A^{n}}\right.
$$

where $p_{x^{n}}=p_{x_{1}} \cdots p_{x^{n}}$ and $\left|x^{n}\right\rangle^{A^{n}}=\left|x_{1}\right\rangle^{A} \cdots\left|x_{n}\right\rangle^{A}$. The $\delta$-(entropy) typical subspace $A_{\delta}<A^{n}$ is defined as

$$
A_{\delta}=\operatorname{span}\left\{\left|x^{n}\right\rangle^{A^{n}}:\left|-\frac{1}{n} \log p_{x^{n}}-H\left(\rho^{A}\right)\right| \leq \delta\right\}
$$

and the $\delta$-typical projection $P_{\delta}^{A}$ is defined to project $A^{n}$ onto $A_{\delta}$. We shall need the following lemma:

Lemma 11 (Typicality) Let a tripartite pure state $|\psi\rangle^{A B C}$ be given. For every $\delta>0$ and all sufficiently large $n$ there are $\delta$-typical projections $P_{\delta}^{A}, P_{\delta}^{B}$ and $P_{\delta}^{E}$
onto $\delta$-typical subspaces $A_{\delta} \subseteq A^{n}, B_{\delta} \subseteq B^{n}$ and $E_{\delta} \subseteq$ $E^{n}$, respectively, such that the states

$$
\begin{aligned}
|\psi\rangle^{A^{n}} B^{n} E^{n} & :=\left(|\psi\rangle^{A B E}\right)^{\otimes n} \\
\left|\psi_{\delta}\right\rangle^{A^{n} B^{n} E^{n}} & :=\left(P_{\delta}^{A} \otimes P_{\delta}^{B} \otimes P_{\delta}^{E}\right)|\psi\rangle^{A^{n} B^{n} E^{n}}
\end{aligned}
$$

satisfy

$$
\begin{aligned}
&\left|A_{\delta}\right| \leq 2^{n H(A)+n \delta}, \\
&\left|B_{\delta}\right| \leq 2^{n H(B)+n \delta}, \\
&\left|E_{\delta}\right| \leq 2^{n H(E)+n \delta}, \\
& P_{\delta}^{B} \psi^{B^{n}} P_{\delta}^{B} \leq 2^{-n H(B)+n \delta} P_{\delta}^{B} \\
& \| \psi^{A^{n} B^{n} E^{n}-\psi_{\delta}^{A^{n} B^{n} E^{n}} \|_{1}} \leq,
\end{aligned}
$$

where $\epsilon=2^{-c n \delta^{2}}$ for some constant $c>0$ independent of $\delta$ and $n$.

Proof. See 17.

## APPENDIX B: PROOF OF LEMMA 3, EQS. (1) AND (2)

We shall use the following easy lemma:
Lemma 12 Let $\delta<1$. Then:

$$
\begin{aligned}
& \text { for }-\delta \leq x \leq 0, \quad \ln (1+x) \geq x-\frac{x^{2}}{2} \frac{1}{1-\delta} \\
& \quad \text { for } 0 \leq x \leq 1, \quad \ln (1+x) \geq x-\frac{x^{2}}{2}
\end{aligned}
$$

Proof. By Taylor expansion, $\ln (1+x)=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-$ $\frac{x^{4}}{4}+\frac{x^{5}}{5} \mp \ldots$.

The second bound is the easier one: just group each (positive) odd term with its immediately consecutive (negative) even term, i.e.

$$
\frac{x^{3}}{3}-\frac{x^{4}}{4}, \frac{x^{5}}{5}-\frac{x^{6}}{6}, \text { etc. }
$$

all of which are clearly non-negative, and we are done.
For the first bound, write $y=-x \leq \delta$, and observe

$$
\begin{aligned}
\ln (1+x) & =\ln (1-y)=-y-\frac{y^{2}}{2}-\frac{y^{3}}{3}-\ldots \\
& =-y-\frac{y^{2}}{2}\left(1+\frac{2}{3} y+\frac{2}{4} y^{2}+\ldots\right) \\
& \geq-y-\frac{y^{2}}{2}\left(1+y+y^{2}+y^{3}+\ldots\right) \\
& =x-\frac{x^{2}}{2} \frac{1}{1-y} \geq x-\frac{x^{2}}{2} \frac{1}{1-\delta}
\end{aligned}
$$

Proof of the probability bounds (1) and (2). Write $A$ in its eigenbasis, $A=\sum_{i} a_{i}|i\rangle\langle i|$, with $0 \leq a_{i} \leq 1$. The

Gaussian vector is $|g\rangle=\sum_{i} c_{i}|i\rangle$, with $c_{i} \sim \mathcal{N}_{\mathbb{C}}(0,1 / D)$. Then $\operatorname{Tr}(|g\rangle\langle g| A)=\sum_{i} a_{i}\left|c_{i}\right|^{2}$ is a weighted sum of independent random variables - which is where the large deviation behaviour will come from.

The "Bernstein trick" is the realisation that (for $t>0$ )

$$
\begin{aligned}
\operatorname{Pr} & \left\{\sum_{i} a_{i}\left|c_{i}\right|^{2}>(1+\epsilon) \frac{\operatorname{Tr} A}{D}\right\} \\
& =\operatorname{Pr}\left\{e^{t \sum_{i} a_{i}\left|c_{i}\right|^{2}}>e^{t(1+\epsilon)(\operatorname{Tr} A) / D}\right\} \\
& \leq\left(\mathbb{E} e^{t \sum_{i} a_{i}\left|c_{i}\right|^{2}}\right) e^{-t(1+\epsilon)(\operatorname{Tr} A) / D} \\
& =\prod_{i}\left(\mathbb{E} e^{t a_{i}\left|c_{i}\right|^{2}}\right) e^{-t(1+\epsilon) a_{i} / D}
\end{aligned}
$$

the second line by Markov's inequality, and the third by independence of the $c_{i}$. We take the evaluation of the expectation above (known as "moment generating function") from [4], Lemma 23 (appendix A): for $t<D / a_{i}$,

$$
\mathbb{E} e^{t a_{i}\left|c_{i}\right|^{2}}=\frac{1}{1-t \frac{a_{i}}{D}}
$$

Plugging this in and letting $t=D \frac{\epsilon}{1+\epsilon}$, we get the upper bound on the probability in question, of

$$
\prod_{i} e^{-\epsilon a_{i}-\ln \left(1-\frac{\epsilon a_{i}}{1+\epsilon}\right)}
$$

The exponents can be upper bounded using Lemma 12 , because we assume $\epsilon \leq 1 / 3$, the argument $\frac{\epsilon a_{i}}{1+\epsilon}$ is bounded above by $(1 / 3) /(1+1 / 3)=1 / 4$, so we get

$$
\begin{aligned}
-\epsilon a_{i}-\ln & \left(1-\frac{\epsilon a_{i}}{1+\epsilon}\right) \\
& \leq-\epsilon a_{i}+\frac{\epsilon a_{i}}{1+\epsilon}+\frac{1}{2} \frac{1}{1-1 / 4}\left(\frac{\epsilon a_{i}}{1+\epsilon}\right)^{2} \\
& =-\frac{\epsilon^{2} a_{i}}{1+\epsilon}+\frac{2}{3} \frac{\epsilon^{2} a_{i}^{2}}{(a+\epsilon)^{2}} \\
& \leq-\frac{1}{3(1+\epsilon)} \epsilon^{2} a_{i} \leq-\frac{1}{4} \epsilon^{2} a_{i} .
\end{aligned}
$$

So, we finally get that the probability in (1) is upper bounded by

$$
\prod_{i} e^{-\frac{1}{4} \epsilon^{2} a_{i}}=e^{-\frac{\epsilon^{2}}{4} \operatorname{Tr} A}
$$

which is what we wanted.

The bound in the other direction is fairly similar: here we have, for $t>0$, and pretty much as before (noting that the extra minus sign reverses the direction of the inequality),

$$
\begin{aligned}
\operatorname{Pr} & \left\{\sum_{i} a_{i}\left|c_{i}\right|^{2}<(1-\epsilon) \frac{\operatorname{Tr} A}{D}\right\} \\
& =\operatorname{Pr}\left\{e^{-t \sum_{i} a_{i}\left|c_{i}\right|^{2}}>e^{-t(1-\epsilon)(\operatorname{Tr} A) / D}\right\} \\
& \leq\left(\mathbb{E} e^{-t \sum_{i} a_{i}\left|c_{i}\right|^{2}}\right) e^{t(1-\epsilon)(\operatorname{Tr} A) / D} \\
& =\prod_{i}\left(\mathbb{E} e^{-t a_{i}\left|c_{i}\right|^{2}}\right) e^{t(1-\epsilon) a_{i} / D} \\
& =\prod_{i} \frac{1}{1+t \frac{a_{i}}{D}} e^{t(1-\epsilon) a_{i} / D} \\
& =\prod_{i} e^{t(1-\epsilon) a_{i} / D-\ln \left(1+t \frac{a_{i}}{D}\right)}
\end{aligned}
$$

Now, choosing $t=D \frac{\epsilon}{1-\epsilon}$, the exponent for each $i$ is

$$
\begin{aligned}
\epsilon a_{i}-\ln \left(1+\frac{\epsilon a_{i}}{1-\epsilon}\right) & \leq \epsilon a_{i}-\frac{\epsilon a_{i}}{1-\epsilon}+\frac{1}{2}\left(\frac{\epsilon a_{i}}{1-\epsilon}\right)^{2} \\
& =-\frac{\epsilon^{2} a_{i}}{1-\epsilon}+\frac{1}{2} \frac{\epsilon^{2} a_{i}^{2}}{(1-\epsilon)^{2}} \\
& \leq-\epsilon^{2} a_{i}\left(1-\frac{1}{2(1-1 / 3)}\right)=-\frac{1}{4} \epsilon^{2} a_{i}
\end{aligned}
$$

where we have once more invoked Lemma 12 and used $\epsilon \leq 1 / 3$.

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