# Unital Positive Maps and Quantum States 

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#### Abstract

We analyze the structure of the subset of states generated by unital completely positive quantum maps, A witness that certifies that a state does not belong to the subset generated by a given map is constructed. We analyse the representations of positive maps and their relation to quantum Perron-Frobenius theory.


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## 1 Introduction

A genuine feature of quantum physics is the existence of relations between dynamical maps and quantum states. The analysis of such relations permits to relate entanglement properties of quantum states of composite systems and positivity properties of quantum maps. The interest on positive maps arises from its broad spectrum of mathematical and physical applications [1]-[36]. Although, from the viewpoint of dynamical maps (quantum channels) most of the interest has focused on completely positive maps, because of their physical interest, positive maps which are not completely positive have also revealed very useful as entanglement witness. Their classification of positive maps which are not completely positive is a challenging problem which still remains open.

In general, positive maps which are not completely positive might be useful as discriminators between quantum states which belong to subsets included into the images of completely positive maps. The aim of the present paper is to further explore the relations between quantum states and dynamical maps from this perspective. In particular, we use the Perron-Frobenius theory to show that all eigenvalues $\lambda$ of unital positive maps remain inside to the unit disk $|\lambda| \leq 1$.

In the abstract approach to quantum mechanics a physical system is characterized by an unital $\mathbb{C}^{*}$-algebra $A$, c.f. [37]. In the algebra $A$ there is a distinguished cone $A^{+}$generated by positive elements of $A$, i.e. elements which can be written in the form $a^{*} a, a \in A$. The unit $e$ is a very special element of $A^{+}$. The set of all states $S(A)=S$ is the convex set of normalized $(\omega(e)=1)$ linear continuous functionals $\omega: A \rightarrow \mathbb{C}$ which are positive, i.e. $\omega(a) \geq 0$ for all $a \in A^{+}$.

A linear map $\varphi: A \rightarrow A$ is called self-adjoint if $\varphi\left(a^{*}\right)=\varphi(a)^{*}$, positive if $\varphi: A^{+} \rightarrow A^{+}$, and unital if $\varphi(e)=e$.

Unital positive maps $\varphi$ on $A$ are also called dynamical maps [1]. Any dynamical map $\varphi$ define, by duality, an affine map in the space of states $\varphi^{*}: S \rightarrow S$, i.e., $\omega(\varphi(a))=\varphi^{*}(\omega)(a)$, for all $a \in A$ and $\omega \in S$.

The interest on positive maps in $\mathbb{C}^{*}$-algebras arises for different reasons [1]-[36]:
i) unital positive maps are generalizations of states, *-homomorphisms, Jordan homomorphisms, and conditional expectations,
ii) dual maps corresponds to some physical operations which can be performed on the physical systems (measurement, time evolution),
iii) there exists a strong relation between the classification of entanglement and positive maps [32]

Let $M_{k}(A)$ be the algebra of $k \times k$ matrices whose entries are elements of $A$ and $M_{k}^{+}(A)$ be the positive cone in $M_{k}(A)$. For $k \in \mathbb{N}$ and $\varphi: A \rightarrow A$ we define the maps $\varphi_{k}: M_{k}(A) \rightarrow M_{k}(A)$ and $\bar{\varphi}_{k}: M_{k}(A) \rightarrow M_{k}(A)$ by $\varphi_{k}\left(\left[a_{i j}\right]\right)=\left[\varphi\left(a_{i j}\right)\right]$ and $\bar{\varphi}_{k}\left(\left[a_{i j}\right]\right)=\left[\varphi\left(a_{j i}\right)\right]$. A map $\varphi$ is said to be $k$-positive ( $k$-copositive) if the map
$\varphi_{k}\left(\bar{\varphi}_{k}\right)$ is positive. A map is $\varphi$ is said to be $k$-decomposable if $\varphi_{k}$ and $\bar{\varphi}_{k}$ are positive for matrices $\left[a_{i j}\right] \in M_{k}(A)^{+}$with transposes $\left[a_{j i}\right]$ also in $M_{k}(A)^{+}$.

A map $\varphi: A \rightarrow A$ is said to be completely positive, completely copositive and decomposable if it is $k$-positive, $k$-copositive and $k$-decomposable for all $k \in \mathbb{N}$, respectively. According to Størmer's theorem [11] any decomposable map $\varphi: A \rightarrow$ $\mathcal{B}(\mathcal{H})$ into the $\mathbb{C}^{*}$-algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a complex Hilbert $\mathcal{H}$, can be decomposed as the sum $\varphi(a)=\varphi_{1}(a)+\varphi_{2}(a)$, of a completely positive $\varphi_{1}$ and a completely copositive $\varphi_{2}$ maps.

A positive map which is not decomposable is called indecomposable. In the case $A=M_{n}(\mathbb{C})$ every decomposable map $\varphi: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ can be written in the form

$$
\begin{equation*}
\varphi(a)=\sum_{k} a_{k}^{*} a a_{k}+\sum_{l} b_{l}^{*} a^{T} b_{l}, \tag{1}
\end{equation*}
$$

where $a_{1}, a_{2}, \cdots ; b_{1}, b_{2}, \cdots \in M_{n}(\mathbb{C})$. It is well known $[3,13,16]$ that for twodimensional systems every positive map $\varphi: M_{2}(\mathbb{C}) \rightarrow M_{2}(\mathbb{C})$ is decomposable. The first example of an indecomposable map in $M_{3}(\mathbb{C})$ was given by Choi $[26,9]$ for three-dimensional systems (see [17]-[31] for generalizations).

The structure of positive maps in $M_{d}(\mathbb{C})$ for $d \geq 3$ is not well known, except for completely positive, completely copositive and decomposable maps. It is interesting to note that in the case of unital positive projections $\pi: A \rightarrow A$, i.e. maps $\pi: A^{+} \rightarrow A^{+}$with $\pi(e)=e$ and $\pi^{2}=\pi$, positivitiy properties can be characterized in terms of the properties of the image $\pi(A)$, i.e. without referring to ancilla systems [34, 35, 36, 10].

Let us assume that $A$ is an unital $\mathbb{C}^{*}$-algebra and $\pi: A \rightarrow A$ is a unital positive projection. It is well know $[34,35]$ that the image $\pi(A)$ is a $\mathbb{C}^{*}$-subalgebra of $A$ under the product

$$
\begin{equation*}
\pi(a) \circ \pi(b)=\pi(a b) \tag{2}
\end{equation*}
$$

if and only if $\pi$ is completely positive, Similar results hold for decomposable unital projections. Unfortunately, the problem of similar characterization of positive unital maps is still open.

## 2 Quantum states and Unital Positive Maps

In quantum information theory (see e.g. [38]) certain classes of quantum states play a relevant role, e.g. separable, PPT, NPT, etc. These states span convex subsets of the set of physical states.

An interesting feature is the existence of relations between some classes of quantum states and physical operations which can be performed on a physical
system. The simplest construction that we shall analyze in this paper can be formulated in terms of unital positive maps.

Let $A$ be a unital $\mathbb{C}^{*}$-algebra, and $S=S(A)$ the set of quantum states, i.e. continuous, linear, positive and normalized functionals on $A$. Let us assume that $\varphi: A \rightarrow A$ is a unital positive map and let $\varphi^{*} S \rightarrow S$ be its dual map. We can associate to any positive unital map $\varphi$ on $A$ the subset of states given by $\varphi^{*}(S) \subseteq S$.

In terms of $\varphi^{*}(S)$ one can define a new cone $A^{+}(\varphi)$ in $A$ :

$$
\begin{equation*}
A^{+}(\varphi)=\left\{a=a^{*} \in A ; \rho(a) \geq 0 \text { for all } \rho \in \varphi^{*}(S)\right\} . \tag{3}
\end{equation*}
$$

It is clear that $A^{+} \subseteq A^{+}(\varphi)$. Let us assume that $\varphi^{*}(S) \subset S$, i.e. $A^{+} \subset A^{+}(\varphi)$, and let $\rho \in S$ be an state which does not belong to the subset $\varphi^{*}(S)$, i.e. $\rho \notin \varphi^{*}(S)$. Then, there exists $a_{\rho} \in A^{+}(\varphi)$ and $a_{\rho} \notin A^{+}$such that

$$
\begin{equation*}
\rho\left(a_{\rho}\right)<0 \tag{4}
\end{equation*}
$$

The element $a_{\rho} \in A^{+}(\varphi)$ plays a role of witness that $\rho \notin \varphi^{*}(S)$.
In the case where $\varphi$ is a unital positive projection $\pi$ on $A$, i.e., $\pi(e)=e, \pi^{2}=\pi$, the structure of the cone $A^{+}(\pi)$ is very simple. One has $A^{+}(\pi)=A^{+}+(\mathbb{I}-\pi)(A)=$ 0 since $A=\pi(A)+(\mathbb{I}-\pi)(A)$ and $\pi(\mathbb{I}-\pi)(A)=0$. In this case the image $(\mathbb{I}-\pi)(A)$ is the set of witnesses for the set $\pi^{*}(S)$. A state $\rho \in S$ does not belong to $\pi^{*}(S)$ iff there exist an element $a \in(\mathbb{I}-\pi) A$ such that $\rho(a) \neq 0$ since for any $\sigma \in \pi^{*}(S)$, $\sigma(a)=0$ for any $a \in(\mathbb{I}-\pi) A$.

It is commonly accepted that only completely positive unital maps are related to physical operations which can be performed on a physical system cf. [33] (see also counterexamples in [39]). Consequently the subsets of states that are generated by completely positive unital maps are physically relevant. It is, thus, interesting to provide some examples of subsets generated by completely positive unital maps.

Let $\mathcal{H}$ be a complex finite dimensional Hilbert space with $\operatorname{dim} \mathcal{H}=d, M=$ $M(\mathcal{H})$ the $\mathbb{C}^{*}$-algebra of linear operators on $\mathcal{H}$, and $M^{+} \equiv M(\mathcal{H})^{+}$the cone of positive operators in $M=M(\mathcal{H})$.

The set of quantum states $S=S(M)$ can be identified with the set of all density operators of $\mathcal{H}$, by the relation $\rho(a)=\operatorname{tr}(\rho a)$.

Let $p=\left(p_{1}, \cdots, p_{N}\right)$ be a family of projection operators on mutually orthogonal subspaces of $\mathcal{H}$ such that $p_{1}+\cdots+p_{N}=\mathbb{I}$, and $\omega=\left(\omega_{1}, \cdots, \omega_{N}\right)$ a family of states $\omega_{1}, \cdots, \omega_{N} \in S$ such that

$$
\begin{equation*}
\operatorname{tr}\left(\omega_{\alpha} p_{\beta}\right)=\delta_{\alpha \beta} ; \quad \alpha, \beta=1, \cdots, N \tag{5}
\end{equation*}
$$

It is easy to verify that the map $\pi(p, \omega): M(\mathcal{H}) \rightarrow M(\mathcal{H})$ defined by the relation

$$
\begin{equation*}
\pi(p, \omega)(a)=\sum_{\alpha=1}^{N} p_{\alpha} \operatorname{tr}\left(\omega_{\alpha} a\right) \tag{6}
\end{equation*}
$$

is a completely positive unital projection on $M(\mathcal{H})$, and its dual $\pi^{*}(p, \omega): S \rightarrow S$ has the form

$$
\begin{equation*}
\pi^{*}(p, \omega)(\rho)=\sum_{\alpha=1}^{N} \omega_{\alpha} \operatorname{tr}\left(p_{\alpha} \rho\right) \tag{7}
\end{equation*}
$$

Let us observe that invariant states of $\pi^{*}(p, \omega)$, i.e. those states $\rho \in S$ such that $\pi^{*}(p, \omega)(\rho)=\rho$, have the form

$$
\begin{equation*}
\rho=\sum_{\alpha=1}^{N} c_{\alpha} \bar{\omega}_{\alpha} \tag{8}
\end{equation*}
$$

where $c_{1}, \cdots, c_{N} \geq 0, \sum_{\alpha=1}^{N} c_{\alpha}=1$.
Choosing $\omega_{\alpha}=\widetilde{p}_{\alpha}=p_{\alpha} / \operatorname{tr} p_{\alpha}$, the relations (5) are automatically satisfied and the corresponding projections (6) will be denoted by $\pi(p)$, i.e.,

$$
\begin{equation*}
\pi(p)(a)=\sum_{\alpha=1}^{N} p_{\alpha} \operatorname{tr}\left(\tilde{p}_{\alpha} a\right) \tag{9}
\end{equation*}
$$

Moreover, $\pi^{*}(p)=\pi(p)$. The subset of $S$ generated by $\pi(p)$, i.e. the subset $\pi^{*}(p)(S)$ is given by

$$
\begin{equation*}
\pi^{*}(p)(S)=\left\{\sum_{\alpha=1}^{N} p_{\alpha} \operatorname{tr}\left(\tilde{p}_{\alpha} \rho\right) ; \rho \in S\right\} . \tag{10}
\end{equation*}
$$

Let us now consider the case $\mathcal{H}=\mathbb{C}^{n} \otimes \mathbb{C}^{n}$, with an orthonormal basis $e_{1}, \cdots, e_{n}$ in $\mathbb{C}^{n}$, and $e_{i j}=e_{i}\left(e_{j}.\right)$ in $\mathbb{C}^{n} \otimes \mathbb{C}^{n}$. Define the orthogonal projections in $\mathbb{C}^{n} \otimes \mathbb{C}^{n}$

$$
\begin{gather*}
p_{1}^{\prime}=\frac{1}{n} \sum_{i j=1}^{n} e_{i j} \otimes e_{i j}  \tag{11}\\
p_{0}^{\prime}=\mathbb{I}_{n} \otimes \mathbb{I}_{n}-p_{1}^{\prime} \tag{12}
\end{gather*}
$$

and the corresponding $\pi\left(p^{\prime}\right)$, then the subset $\pi^{*}\left(p^{\prime}\right)(S)$ is the class of Horodecki isotropic states [40]. On the other hand the orthogonal projections

$$
\begin{equation*}
p_{0}^{\prime \prime}=\frac{1}{2}\left(\mathbb{I}_{n} \otimes \mathbb{I}_{n}+F\right) \tag{13}
\end{equation*}
$$

$$
\begin{equation*}
p_{1}^{\prime \prime}=\frac{1}{2}\left(\mathbb{I}_{n} \otimes \mathbb{I}_{n}-F\right) \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
F=\sum_{i j=1}^{n} e_{i j} \otimes e_{j i} \tag{15}
\end{equation*}
$$

is the flip operator, define another unital projector $\pi\left(p^{\prime \prime}\right)$. The set $\pi^{*}\left(p^{\prime \prime}\right)(S)$ coincides with the set of Werner states [41].

It is easy to check that all classes of multipartite states constructed in [42, 43] are generated by completely positive unital projections $\pi(p)$ with appropriate choice of $\mathcal{H}$ and projectors $p_{1}, \cdots, p_{N}$. There exists a second class of completely positive unital projections which are of the form

$$
\begin{equation*}
\pi(a)=\sum_{\alpha=1}^{N} p_{\alpha} a p_{\alpha} \tag{16}
\end{equation*}
$$

and correspond to a measurement process. These maps are self-dual, i.e. $\pi^{*}=\pi$, and in the case that $p_{1}, \cdots, p_{N}$ are one-dimensional, then (16) reduces to (9).

Let us consider the completely positive unital map $\varphi: M(\mathcal{H}) \rightarrow M(\mathcal{H})$ such that

$$
\begin{equation*}
\varphi(a)=\frac{1}{d-1} a+\left(1-\frac{1}{d-1}\right) \mathbb{I}_{d} \operatorname{tr}\left(\omega_{0} a\right) \tag{17}
\end{equation*}
$$

where $\omega_{0}=\mathbb{I}_{d} / d$ is the maximally mixed state.
The map $\varphi$ is self-dual, i.e. $\varphi^{*}=\varphi$, and it is easy to verify that

$$
\begin{equation*}
\varphi^{*}(S)=\left\{\rho \in S ; \operatorname{tr} \rho^{2} \leq \frac{1}{d-1}\right\} \tag{18}
\end{equation*}
$$

On the other hand the cone $M^{+}(\varphi)$ defined by the set $\varphi^{*}(S)$, i.e.

$$
\begin{equation*}
M^{+}(\varphi)=\left\{a=a^{*} \in M ; \operatorname{tr}(a \rho) \geq 0 \text { for all } \rho \in \varphi^{*}(S)\right\} \tag{19}
\end{equation*}
$$

has the following structure

$$
\begin{equation*}
M^{+}(\varphi)=\left\{a=a^{*} \in M ; \operatorname{tr} a \geq 0, \quad \operatorname{tr} a^{2} \leq(\operatorname{tr} a)^{2}\right\} \supset M^{+} \tag{20}
\end{equation*}
$$

since $\varphi^{*}(S)$ is a ball.
In the case $\mathcal{H}=\mathbb{C}^{d}$ the set $\varphi^{*}(S)$ has been considered in Ref. [44] while in the case $\mathcal{H}=\mathbb{C}^{d_{1}} \otimes \cdots \otimes \mathbb{C}^{d_{k}}$ with $d=d_{1}, \cdots, d_{k}, \varphi^{*}(S)$ is the largest separable ball around the maximally mixed state [45].

## 3 Representations of linear maps

Let $\mathcal{L}\left(M_{d}, M_{d}\right)$ be the vector space of linear maps $\varphi: M_{d} \rightarrow M_{d}$, and $M_{d} \otimes M_{d}$ the vector space of linear maps $\widehat{\varphi}: \mathbb{C}^{d} \otimes \mathbb{C}^{d} \rightarrow \mathbb{C}^{d} \otimes \mathbb{C}^{d}$. Given two orthonormal bases $\left\{f_{\alpha}\right\}$ and $\left\{g_{\alpha}\right\}$ in $M_{d}$, the relation

$$
\begin{equation*}
\mathcal{L}\left(M_{d}, M_{d}\right) \ni \varphi \longrightarrow \widehat{\varphi}=\sum_{\alpha=1}^{d^{2}} f_{\alpha} \otimes \varphi\left(g_{\alpha}\right) \tag{21}
\end{equation*}
$$

defines an isomorphism between $\mathcal{L}\left(M_{d}, M_{d}\right)$ and $M_{d} \otimes M_{d}$, c.f. [46]-[54]. The isomorphism is an isometry which maps the hermitian product

$$
\begin{equation*}
(\varphi, \psi)=\sum_{\alpha=1}^{d^{2}} \operatorname{tr}_{\mathbb{C}^{n}}\left[\varphi\left(f_{\alpha}\right)^{*} \psi\left(f_{\alpha}\right)\right] \tag{22}
\end{equation*}
$$

of $\mathcal{L}\left(M_{d}, M_{d}\right)$ into that

$$
\begin{equation*}
\langle\hat{\varphi}, \widehat{\psi}\rangle=\operatorname{tr}_{\mathbb{C}^{d} \mathbb{C}^{d}}\left[\hat{\varphi}^{*} \hat{\psi}\right] . \tag{23}
\end{equation*}
$$

of $\mathbb{C}^{d} \otimes \mathbb{C}^{d}$, i.e.

$$
\begin{equation*}
(\varphi, \psi)=<\widehat{\varphi}, \widehat{\psi}> \tag{24}
\end{equation*}
$$

It is easy to show that the maps

$$
\begin{equation*}
\gamma_{\alpha \beta}(a)=f_{\alpha} a g_{\beta}^{*}, \quad \alpha, \beta=1,2, \cdots, d^{2} \tag{25}
\end{equation*}
$$

and

$$
\begin{equation*}
\epsilon_{\alpha \beta}(a)=f_{\alpha} \operatorname{tr}\left(g_{\beta}^{*} a\right), \quad \alpha, \beta=1,2, \cdots, d^{2} \tag{26}
\end{equation*}
$$

define two orthonormal bases of $\mathcal{L}\left(M_{d}, M_{d}\right)$, i.e.

$$
\begin{equation*}
\left(\gamma_{\alpha \beta}, \gamma_{\mu \nu}\right)=\delta_{\alpha \mu} \delta_{\beta \nu} \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\epsilon_{\alpha \beta}, \epsilon_{\mu \nu}\right)=\delta_{\alpha \mu} \delta_{\beta \nu} . \tag{28}
\end{equation*}
$$

Any map $\varphi \in \mathcal{L}\left(M_{d}, M_{d}\right)$ can be written as

$$
\begin{equation*}
\varphi(a)=\sum_{\alpha, \beta=1}^{d^{2}} A_{\alpha \beta} \gamma_{\alpha \beta}(a) \tag{29}
\end{equation*}
$$

or

$$
\begin{equation*}
\varphi(a)=\sum_{\alpha, \beta=1}^{d^{2}} B_{\alpha \beta} \epsilon_{\alpha \beta}(a), \tag{30}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{\alpha \beta}=\left(\gamma_{\alpha \beta}, \varphi\right), \quad B_{\alpha \beta}=\left(\epsilon_{\alpha \beta}, \varphi\right) . \tag{31}
\end{equation*}
$$

Using (25)-(31) one finds the relations

$$
\begin{equation*}
A_{\alpha \beta}=\sum_{\mu, \nu=1}^{d^{2}} \operatorname{tr}\left(f_{\alpha}^{*} f_{\mu} g_{\beta} g_{\nu}^{*}\right) B_{\mu \nu} \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{\alpha \beta}=\sum_{\mu, \nu=1}^{d^{2}} \operatorname{tr}\left(f_{\alpha}^{*} f_{\mu} g_{\beta} g_{\nu}^{*}\right) A_{\mu \nu} . \tag{33}
\end{equation*}
$$

A special case is provided by the choice $f_{\alpha}=g_{\alpha}=e_{i j}$, where $e_{i j}=e_{i}\left(e_{j}, \cdot\right)$ are the projectors defined by an orthonormal basis $e_{i}, \cdots, e_{d}$ of $\mathbb{C}^{d}$. In this case the correspondence (21) is known as Jamiolkowski isomorphism [15] and the corresponding matrices $A_{i j, k l}, B_{i j, k l}$ were introduced by Sudarshan [1], while the relations (32)(33) have the form

$$
\begin{equation*}
A_{i j, k l}=B_{i k, j l} . \tag{34}
\end{equation*}
$$

On the other hand choosing $f_{\alpha}=g_{\alpha}$ yields the following representations for $\varphi$,

$$
\begin{equation*}
\varphi(a)=\sum_{\alpha=1}^{d^{2}} A_{\alpha \beta} f_{\alpha} a f_{\beta}^{*} \tag{35}
\end{equation*}
$$

or

$$
\begin{equation*}
\varphi(a)=\sum_{\alpha=1}^{d^{2}} B_{\alpha \beta} f_{\alpha} \operatorname{tr}\left(f_{\beta}^{*} a\right) . \tag{36}
\end{equation*}
$$

The advantage of using the representation (35) is due to the fact that $A_{\alpha \beta}=$ $\bar{A}_{\beta \alpha}$ for any selfadjoint map $\varphi$, i.e. a map which satisfies that $\varphi\left(a^{*}\right)=\varphi(a)^{*}$, and $\left[A_{\alpha \beta}\right]$ is a semi-definite positive matrix if $\varphi$ is a completely positive map [33].

On the other hand, the composition of maps in the representation (36) corresponds to the product of $B$ matrices, while the completely positivity condition takes the form

$$
\begin{equation*}
\sum_{\alpha, \beta=1}^{d^{2}} B_{\alpha \beta} f_{\alpha}^{T} \otimes f_{\beta}^{*} \geq 0 \tag{37}
\end{equation*}
$$

Let us now consider the eigenvalue problem for the map $\varphi: M_{d} \rightarrow M_{d}$, i.e.

$$
\begin{equation*}
\varphi(a)=\lambda a . \tag{38}
\end{equation*}
$$

In the basis $\left\{f_{\alpha} ; \alpha=1, \cdots d^{2}\right\}$, the matrix $a$ can be written as

$$
a=\sum_{\alpha=1}^{d^{2}} a_{\alpha} f_{\alpha}
$$

and using (36) one gets that (38) is equivalent to

$$
\begin{equation*}
\sum_{\beta=1}^{d^{2}} B_{\alpha \beta} a_{\beta}=\lambda a_{\alpha} . \tag{39}
\end{equation*}
$$

For a positive map $\varphi$, the eigenvalue problem (38) (39) is related to the quantum version of Frobenius theory [50][51], and one has the following result.

Theorem
If $\varphi: M_{d} \longrightarrow M_{d}$ is positive all its eigenvalues satisfy the inequality

$$
\begin{equation*}
|\lambda| \leq\left\|\varphi\left(\mathbb{I}_{d}\right)\right\|_{\infty}, \tag{40}
\end{equation*}
$$

where $\|\cdot\|_{\infty}$ is the operator norm in $M_{d}$.
Proof
Since $\varphi$ is positive it follows from the theorem by Russo and Dye [52] (see also [53]) that $\|\varphi\|_{\infty}=\left\|\varphi\left(\mathbb{I}_{d}\right)\right\|_{\infty}$ and, then

$$
\begin{equation*}
|\lambda|\|a\|_{\infty}=\|\varphi(a)\|_{\infty} \leq\|a\|_{\infty}\|\varphi\|_{\infty}=\|a\|_{\infty}\left\|\varphi\left(\mathbb{I}_{d}\right)\right\|_{\infty}, \tag{41}
\end{equation*}
$$

which proves the theorem.
Corollary
If $\varphi$ is positive and unital, i.e. $\varphi\left(\mathbb{I}_{d}\right)=\mathbb{I}_{d}$, all eigenvalues of $\varphi$ lie in the unit circle.
Let $\varphi$ be a completely positive unital map on $M_{d}$. Then, there exist two bi-orthonormal bases $\left\{f_{\alpha}\right\}\left\{g_{\alpha}\right\}$ in $M_{d}$ such that

$$
\begin{equation*}
\varphi(a)=\sum_{\alpha=1}^{d^{2}} \lambda_{\alpha} f_{\alpha} \operatorname{tr}\left(g_{\alpha} a\right) \tag{42}
\end{equation*}
$$

where

$$
\begin{gather*}
\operatorname{tr}\left(f_{\alpha} g_{\beta}\right)=\delta_{\alpha \beta}  \tag{43}\\
\left|\lambda_{\alpha}\right| \leq 1, \quad \alpha, \beta=1, \cdots, d^{2} \tag{44}
\end{gather*}
$$

and

$$
\begin{equation*}
\sum_{\alpha=1}^{d^{2}} \lambda_{\alpha} f_{\alpha}^{T} \otimes g_{\alpha} \geq 0 \tag{45}
\end{equation*}
$$

such that

$$
\begin{equation*}
\varphi\left(f_{\alpha}\right)=\lambda_{\alpha} f_{\alpha} \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
\varphi^{*}\left(g_{\alpha}\right)=\lambda_{\alpha} g_{\alpha} \tag{47}
\end{equation*}
$$

$\varphi^{*}$ being the dual map.
Let us assume that $\varphi^{*}(\omega)=\omega$, i.e. $\omega$ is an invariant state. Then $\varphi$ has the following representation

$$
\begin{equation*}
\varphi(a)=\sum_{\alpha=1}^{d^{2}} \lambda_{\alpha} f_{\alpha} \operatorname{tr}\left(g_{\alpha} a\right), \tag{48}
\end{equation*}
$$

in terms of the bi-orthonormal bases $\left\{f_{\alpha} ; \alpha=1, \cdots, d^{2}\right\}$ and $\left\{g_{\alpha} ; \alpha=1, \cdots, d^{2}\right\}$ of $M_{d}$ given by

$$
\begin{equation*}
g_{d^{2}}=\omega, \quad f_{d^{2}}=\mathbb{I}_{d} \tag{49}
\end{equation*}
$$

and

$$
\begin{equation*}
g_{\alpha}=h_{\alpha}, f_{\alpha}=h_{\alpha}^{*}-\mathbb{I}_{d} \operatorname{tr}\left(\omega h_{\alpha}^{*}\right) \quad \alpha=1, \cdots, d^{2}-1, \tag{50}
\end{equation*}
$$

where $\left\{h_{\alpha} \in M_{d} ; \alpha=1, \cdots, d^{2}-1\right\}$ is a family of orthonormal traceless matrices, i.e.

$$
\begin{equation*}
\operatorname{tr} h_{\alpha}=0 ; \operatorname{tr} h_{\alpha} h_{\beta}^{*}=\delta_{\alpha \beta} . \tag{51}
\end{equation*}
$$

To illustrate the theorem let us consider the following example of unital map $\varphi: M_{d} \rightarrow M_{d}$ given by

$$
\begin{equation*}
\varphi(a)=\sum_{\substack{i, j=1 \\ i \neq j}}^{d-1} \alpha_{j-i} e_{i j}^{*} a e_{i j}+\alpha_{0} \sum_{i, j=1}^{d-1} \beta_{i j} e_{i i}^{*} a e_{j j} \tag{52}
\end{equation*}
$$

where the indices $j-i$ are understood $\bmod d$, i.e. if $j<i, \alpha_{j-i}:=\alpha_{d-j+i}$, and the coefficients $\alpha_{i}$ satisfy the constraints

$$
\begin{equation*}
\alpha_{0}, \alpha_{1}, \cdots, \alpha_{d-1} \geq 0 \tag{53}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=0}^{d-1} \alpha_{j}=1 \tag{54}
\end{equation*}
$$

The matrix $\beta=\left[\beta_{i j}\right] \geq 0$ is positive with $\beta_{i i}=1$ for all $i=0, \cdots, d-1$, which implies that all non-diagonal entries verify the inequality $\left|\beta_{i j}\right| \leq 1$.

It is easy to show that $\varphi$ is a completely positive bi-stochastic map and satisfies the following relations

$$
\begin{equation*}
\varphi\left(e_{i j}\right)=\alpha_{0} \beta_{i j} e_{i j} \quad i \neq j \tag{55}
\end{equation*}
$$

$$
\begin{equation*}
\varphi\left(u_{m}\right)=\rho_{m} u_{m}, \quad m=0, \cdots, d-1, \tag{56}
\end{equation*}
$$

where

$$
\begin{gather*}
u_{m}=\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \lambda^{j m} e_{j j},  \tag{57}\\
\operatorname{tr}\left(u_{m} u_{n}^{*}\right)=\delta_{m n}  \tag{58}\\
\lambda=\exp \left(\frac{2 \pi}{d} i\right) \tag{59}
\end{gather*}
$$

and

$$
\begin{equation*}
\rho_{m}=\sum_{j=0}^{d-1} \alpha_{j} \lambda^{-j m} . \tag{60}
\end{equation*}
$$

Indeed, the eigenvalues of $\varphi: \rho_{0}, \cdots, \rho_{d-1}$ and $\alpha_{0} \beta_{i j}, i \neq j=0, \cdots, d-1$ are contained in the unit disk. The map $\varphi$ can also be represented in the form

$$
\begin{equation*}
\varphi(a)=\sum_{j=0}^{d-1} \rho_{j} u_{j} \operatorname{tr}\left(u_{j}^{*} a\right)+\alpha_{0} \sum_{\substack{i, j=0 \\ i \neq j}}^{d-1} \beta_{i j} e_{i j} \operatorname{tr}\left(e_{i j}^{*} a\right) \tag{61}
\end{equation*}
$$

in terms of the orthonormal basis $\left\{f_{m n}, m, n=0 \cdots d-1 ; f_{m m}=u_{m}, f_{m n}=\right.$ $\left.e_{m n}(m \neq n)\right\}$ of $M_{d}$, which provides the spectral decomposition of the map $\varphi$.

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