Relative and Discrete Utility Maximising Entropy

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Abstract

The notion of utility maximising entropy (u-entropy) of a probability density, which was introduced and studied in [SZ04], is extended in two directions. First, the relative *u*-entropy of two probability measures in arbitrary probability spaces is defined. Then, specialising to discrete probability spaces, we also introduce the absolute *u*-entropy of a probability measure. Both notions are based on the idea, borrowed from mathematical finance, of maximising the expected utility of the terminal wealth of an investor. Moreover, *u*-entropy is also relevant in thermodynamics, as it can replace the standard Boltzmann-Shannon entropy in the Second Law. If the utility function is logarithmic or isoelastic (a power function), then the well-known notions of the Boltzmann-Shannon and Rényi relative entropy are recovered. We establish the principal properties of relative and discrete *u*-entropy and discuss the links with several related approaches in the literature.

1 Introduction

The notion of utility maximising entropy (or *u*-entropy for brevity) of a density f of a probability distribution with respect to a given probability measure μ was introduced and studied by two of the present authors in [SZ04].

The work in [SZ04] was motivated, on the one hand, by problems in mathematical finance concerned with a trader with a concave utility function u who wants to maximise the expected utility $\mathbb{E}_{\nu}(u(w))$ under the true market probability measure ν over all contingent claims w (where w is a non-negative random variable representing the final value of a contingent claim) whose initial value $\mathbb{E}_{\mu}(w)$ under a pricing measure μ is equal to the initial wealth of the trader, taken to be 1 for simplicity, so that $\mathbb{E}_{\mu}(w) = 1$. A value $c \in \mathbb{R}$, called the certainty equivalent, can be assigned to each contingent claim w so that $u(c) = \mathbb{E}_{\nu}(u(w))$. If ν is absolutely continuous with respect to μ with density f, then the u-entropy $H_u(f)$ is defined as the highest possible value of the logarithm of the certainty equivalent c over all contingent claims w with initial value $\mathbb{E}_{\mu}(w) = 1$.

Expected utility maximisation problems in mathematical finance have been studied extensively, for example, in [PK96], [AIS98], [REK00], [GR01], [BF02], [FB05], [Gun05], [GF06]. Some of the most general and elegant results, which have provided much inspiration for our work, belong to Kramkov and Schachermayer [KS99], [KS03], [Sch04], [HKS05].

On the other hand, further motivation for u-entropy comes from thermodynamics and statistical mechanics. When $u(x) = \ln x$, then the u-entropy $H_u(f)$ is equal to the classical Boltzmann-Gibbs entropy $H(f) = \mathbb{E}_{\mu}(f \ln f)$ (note the sign convention typical of mathematical literature; the opposite sign for entropy would normally be used in physics). The properties of the Boltzmann-Gibbs entropy H(f) and, in particular, its role in the Second Law of thermodynamics provided a fertile ground for generalisation to the case of u-entropy. As is very well known, a physical system in state f evolves towards equilibrium whenever H(f) tends to zero. An extension of this and other properties of entropy was achieved in [SZ04] by replacing the Boltzmann-Gibbs entropy H(f) with the u-entropy $H_u(f)$ for an arbitrary u from a broad class of utility functions.

In the present paper the concept of *u*-entropy is extended further to include relative entropy of two arbitrary probability measures ν and μ . It will be called the relative *u*-entropy and denoted by $H_u(\nu \parallel \mu)$. We also introduce the *u*entropy $h_u(p)$ of a probability measure p (rather than the relative entropy of one measure with respect to another or that of a density with respect to a given probability measure), but to do so need to specialise to the case of a discrete probability space, where p is a probability vector.

We establish some properties of relative *u*-entropy and discrete *u*-entropy, and study their relationships with other similar approaches in the literature. In particular, we discuss a link with the recent work by Friedman, Huang and Sandow [FHS07], and with a much older approach by Arimoto [Ari71], which does not refer to utility maximisation explicitly but is based on a similar concept. These two approaches work in the discrete case only. Moreover, returning once again to general probability spaces, we also establish a connection of relative *u*entropy with Frittelli's generalised distance between two probability measures, introduced in [Fri00] to solve the dual convex problem in a utility maximisation framework for asset pricing in an incomplete market.

It will prove convenient to adopt the convention $\infty \cdot 0 = -\infty \cdot 0 = 0$ throughout this paper.

2 Utility maximising relative entropy

2.1 Utility functions

Definition 2.1 Let $u : (0, \infty) \rightarrow \mathbb{R}$. We call u a **utility function** whenever u satisfies the **Inada conditions**, that is, u is a strictly concave strictly increasing continuously differentiable function such that

$$u'(0) := \lim_{x \searrow 0} u'(x) = \infty, \quad u'(\infty) := \lim_{x \nearrow \infty} u'(x) = 0.$$

We shall also use the notation

$$u\left(0\right):=\lim_{x\searrow0}u\left(x\right),\quad u\left(\infty\right):=\lim_{x\nearrow\infty}u\left(x\right)\;.$$

Proposition 2.2 The function $I := (u')^{-1} : (0, \infty) \to (0, \infty)$ is strictly decreasing and satisfies

$$I\left(0\right):=\lim_{x\searrow0}I\left(x\right)=\infty,\quad I\left(\infty\right):=\lim_{x\nearrow\infty}I\left(x\right)=0\ .$$

Definition 2.3 Let $u : (0, \infty) \to \mathbb{R}$ be a utility function. The **convex dual** $u^* : (0, \infty) \to \mathbb{R}$ is defined by

$$u^{*}(y) = \sup_{x>0} \left(u(x) - yx \right)$$
(2.1)

for any $y \in (0, \infty)$. We also put

$$u^{*}(0) := \lim_{x \searrow 0} u^{*}(x) = u(\infty), \quad u^{*}(\infty) := \lim_{x \nearrow \infty} u^{*}(x) = u(0).$$
(2.2)

If $\Lambda > 0$ and s = 0, we put $I(\Lambda/s) := 0$ and $u^*(\Lambda/s) := u(0)$, consistently with the adopted notation $I(\infty) = 0$ and $u^*(\infty) = u(0)$.

Example 2.4 Let $\gamma \in (-\infty, 1)$. Define $u : (0, \infty) \to \mathbb{R}$ by

$$u(t) = \begin{cases} \frac{1}{\gamma} (t^{\gamma} - 1) & \text{for } t \in (0, \infty) \text{ and } \gamma \in (-\infty, 0) \cup (0, 1) \\ \ln t & \text{for } t \in (0, \infty) \text{ and } \gamma = 0 \end{cases}$$

We call u the isoelastic utility of order γ if $\gamma \neq 0$, and the logarithmic utility if $\gamma = 0$.

The following definition is due to Kramkov and Schachermayer [KS99].

Definition 2.5 The asymptotic elasticity of a utility function $u: (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$AE(u) = \limsup_{x \nearrow \infty} \frac{xu'(x)}{u(x)} .$$

A utility function u is said to have reasonable asymptotic elasticity if AE(u) < 1.

Under the assumption of reasonable asymptotic elasticity, duality theory for utility maximisation works in a similar manner as in the finite-dimensional case. See [Sch04] for equivalent formulations of this assumption and a discussion of its economic meaning.

2.2 Relative *u*-entropy and *u*-entropy

2.2.1 Definition

Notation 2.6 Let (Ω, Σ) be a measurable space. We denote by $M_1(\Omega, \Sigma)$ the space of all probability measures on (Ω, Σ) . For any $\mu \in M_1(\Omega, \Sigma)$ we denote by $D(\mu)$ the set of all **densities** on the probability space (Ω, Σ, μ) , that is,

$$D\left(\mu\right) := \left\{ w \in L^{1}\left(\mu\right) : w \ge 0 \text{ and } \int_{\Omega} w d\mu = 1 \right\} \ .$$

By $B(\Omega, \Sigma)$ we denote the set of all bounded measurable real-valued functions on (Ω, Σ) . In the sequel we shall write simply M_1 and B whenever the measurable space (Ω, Σ) is unambiguous. For any $\mu \in M_1$ and $f \in D(\mu)$ we shall write $f\mu$ to denote the measure in M_1 with density f with respect to μ .

Definition 2.7 Let $u : (0, \infty) \to \mathbb{R}$ be a utility function. Let (Ω, Σ) be a measurable space and let $\nu, \mu \in M_1(\Omega, \Sigma)$. We put

$$N_{u}\left(\nu \parallel \mu\right) := \sup_{w \in \mathcal{A}(\nu,\mu)} \int_{\Omega} u\left(w\right) d\nu ,$$

where

$$\mathcal{A}(\nu,\mu) := \left\{ w \in D(\mu) : u(w)^{-} \in L^{1}(\nu) \right\} .$$

Here $x^- = \max(-x, 0)$ denotes the negative part of $x \in \mathbb{R}$. Note that $\int_{\Omega} u(w) d\nu \in (-\infty, \infty]$ for each $w \in \mathcal{A}(\nu, \mu)$. We define

$$H_u\left(\nu \parallel \mu\right) := \ln u^{-1}\left(N_u\left(\nu \parallel \mu\right)\right)$$

and call it the relative *u*-entropy (or relative utility maximising entropy) of ν with respect to μ .

Definition 2.8 (from [SZ04]) Let $u : (0, \infty) \to \mathbb{R}$ be a utility function and let $\mu \in M_1$. For any $f \in D(\mu)$ we put

$$N_{u}(f) := \sup_{w \in \mathcal{A}(f)} \int_{\Omega} u(w) f d\mu ,$$

where

$$\mathcal{A}(f) := \left\{ w \in D(\mu) : u(w)^{-} \in L^{1}(f\mu) \right\}$$

Note that $\int_{\Omega} u(w) f \, d\mu \in (-\infty, \infty]$ for each $w \in \mathcal{A}(f)$. We define

$$H_u(f) := \ln u^{-1} \left(N_u(f) \right)$$

and call it the *u*-entropy (utility maximising entropy) of f.

The next proposition follows immediately from the definitions.

Proposition 2.9 Let $\mu \in M_1$ and $f \in D(\mu)$. Then

$$N_u(f) = N_u(f\mu \parallel \mu) ,$$

$$H_u(f) = H_u(f\mu \parallel \mu) .$$

2.2.2 Properties

Proposition 2.10 The following inequalities hold:

$$u(1) \le N_u(\nu \parallel \mu) \le u(\infty) ,$$

$$0 \le H_u(\nu \parallel \mu) \le \infty .$$

Proof Taking $w \equiv 1 \in \mathcal{A}(\nu, \mu)$, we obtain the lower bound. The upper bound follows immediately from the definition.

Proposition 2.11 Let $\mu, \nu_1, \nu_2 \in M_1$ and $a \in [0, 1]$. Then

$$N_u (a\nu_1 + (1-a)\nu_2 \parallel \mu) \le aN_u (\nu_1 \parallel \mu) + (1-a)N_u (\nu_2 \parallel \mu)$$

Proof Put $\nu := a\nu_1 + (1-a)\nu_2$. First observe that for $w \in D(\mu)$ we have $\int_{\Omega} u^-(w) d\nu = a \int_{\Omega} u^-(w) d\nu_1 + (1-a) \int_{\Omega} u^-(w) d\nu_2$, and so $\mathcal{A}(\nu,\mu) = \mathcal{A}(\nu_1,\mu) \cap \mathcal{A}(\nu_2,\mu)$. Hence

$$\begin{split} N_{u} (a\nu_{1} + (1-a)\nu_{2} \parallel \mu) \\ &= \sup \left\{ \int_{\Omega} u(w) \, d\nu : w \in \mathcal{A}(\nu,\mu) \right\} \\ &= \sup \left\{ a \int_{\Omega} u(w) \, d\nu_{1} + (1-a) \int_{\Omega} u(w) \, d\nu_{2} : w \in \mathcal{A}(\nu,\mu) \right\} \\ &\leq a \sup \left\{ \int_{\Omega} u(w) \, d\nu_{1} \in \mathcal{A}(\nu,\mu) \right\} + (1-a) \sup \left\{ \int_{\Omega} u(w) \, d\nu_{2} : w \in \mathcal{A}(\nu,\mu) \right\} \\ &\leq a \sup \left\{ \int_{\Omega} u(w) \, d\nu_{1} \in \mathcal{A}(\nu_{1},\mu) \right\} + (1-a) \sup \left\{ \int_{\Omega} u(w) \, d\nu_{2} : w \in \mathcal{A}(\nu_{2},\mu) \right\} \\ &= a N_{u} (\nu_{1} \parallel \mu) + (1-a) N_{u} (\nu_{2} \parallel \mu) \ , \end{split}$$

as desired.

Next we show that relative *u*-entropy can be reduced to the case when $\nu \ll \mu$.

Theorem 2.12 Let $\mu, \nu \in M_1$. Then

$$N_{u}\left(\nu \parallel \mu\right) = \nu_{\perp}\left(\Omega\right) u\left(\infty\right) + \nu_{\ll}\left(\Omega\right) N_{u}\left(\frac{\nu_{\ll}}{\nu_{\ll}\left(\Omega\right)} \parallel \mu\right) ,$$

where $\nu_{\perp} + \nu_{\ll} = \nu$ is the Lebesgue decomposition of ν into the singular part ν_{\perp} and absolutely continuous part ν_{\ll} with respect to μ .

Proof Let $A \in \Sigma$ be such that $\mu(A) = 0$ and $\nu_{\perp}(A) = \nu_{\perp}(\Omega)$.

Step 1. If $\nu \ll \mu$, that is, $\nu_{\perp}(\Omega) = 0$, then the assertion is trivial. Suppose that $\nu \perp \mu$, i.e., $\nu_{\perp}(\Omega) = 1$. Then $\nu(A) = 1$ and $w_n :\equiv n1_A + 1_{A^{\neg}} \in \mathcal{A}(\nu, \mu)$ for $n \in \mathbb{N}$, and $\int_{\Omega} u(w_n) d\nu = u(n)$. Hence and from Proposition 2.10 we get $N_u(\nu \parallel \mu) = u(\infty)$, as required.

Step 2. Now we assume that $0 < \nu_{\perp}(\Omega) < 1$. Note that $\nu = \nu_{\perp}(\Omega) \frac{\nu_{\perp}}{\nu_{\perp}(\Omega)} + \nu_{\ll}(\Omega) \frac{\nu_{\ll}}{\nu_{\ll}(\Omega)}$, and from Proposition 2.11 and from Step 1 we get

$$N_{u}\left(\nu \parallel \mu\right) \leq \nu_{\perp}\left(\Omega\right) u\left(\infty\right) + \nu_{\ll}\left(\Omega\right) N_{u}\left(\frac{\nu_{\ll}}{\nu_{\ll}\left(\Omega\right)} \parallel \mu\right)$$

Let now $w \in \mathcal{A}\left(\frac{\nu_{\ll}}{\nu_{\ll}(\Omega)}, \mu\right)$. Put $w_n := n\mathbf{1}_A + w\mathbf{1}_{A^{\neg}}$ for $n \in \mathbb{N}$. Clearly, $w_n \in \mathcal{A}(\nu, \mu)$ and

$$\nu_{\perp}(\Omega) u(n) + \nu_{\ll}(\Omega) \int_{\Omega} u(w) d\frac{\nu_{\ll}}{\nu_{\ll}(\Omega)} = \int_{\Omega} u(w_n) d\nu \leq N_u(\nu \parallel \mu) .$$

Taking $n \to \infty$ completes the proof.

Corollary 2.13 In particular, if $u(\infty) = \infty$ and ν is not absolutely continuous with respect to μ , or if $\nu \perp \mu$, then $N_u(\nu \parallel \mu) = u(\infty)$ and $H_u(\nu \parallel \mu) = \infty$.

Proposition 2.14 Let $\mu, \nu \in M_1$. Then the following conditions are equivalent:

- (1) $N_u(\nu \parallel \mu) < u(\infty)$;
- (2) $N_u(\nu \parallel \mu) < \infty$;
- (3) $H_u(\nu \parallel \mu) < \infty$.

In particular, all three conditions are satisfied for any utility function u such that $u(\infty) < \infty$.

Proof The implications $(1) \Rightarrow (3) \Rightarrow (2)$ are obvious, as is $(2) \Rightarrow (1)$ when $u(\infty) = \infty$.

We shall prove 2) \Rightarrow 1) when $u(\infty) < \infty$. Put $A_n := \{w \ge n\}$ for any $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$. Consequently, there exists an $n \in \mathbb{N}$ such that $\nu(A_n) =: \gamma < 1$. Hence

$$\int_{\Omega} u(w) d\nu = \int_{A_n} u(w) d\nu + \int_{(A_n)^{\neg}} u(w) d\nu$$
$$\leq u(\infty) \gamma + u(n) (1 - \gamma)$$

for any $w \in \mathcal{A}(\nu, \mu)$. Thus $N_u(\nu \parallel \mu) \leq u(\infty)\gamma + u(n)(1-\gamma) < u(\infty)$, as required.

Proposition 2.15 The following conditions are equivalent:

- (1) $H_u(\nu \parallel \mu) = 0$;
- (2) $\nu = \mu$.

Proof (1) \Rightarrow (2). Let $\nu = \mu$. Take $w \in \mathcal{A}(\nu, \mu)$. By Jensen's inequality $\int_{\Omega} u(w) d\nu \leq u \left(\int_{\Omega} w d\nu \right) = u$ (1). Hence, by Proposition 2.10, $N_u(\nu \parallel \mu) = u(1)$, and so $H_u(\nu \parallel \mu) = 0$.

 $(2) \Rightarrow (1)$. Suppose that $\nu \neq \mu$. Then there is an $A \in \Sigma$ such that $\mu(A) \neq \nu(A)$ and $\mu(A) < 1$. We put

$$\begin{split} w_a &:= a \mathbf{1}_A + \frac{1 - a\mu(A)}{\mu(A^{\neg})} \mathbf{1}_{A^{\neg}} ,\\ \varphi\left(a\right) &:= \int_{\Omega} u(w_a) d\nu = u(a)\nu(A) + u\left(\frac{1 - a\mu(A)}{\mu(A^{\neg})}\right)\nu(A^{\neg}) \end{split}$$

for any $a \in (0, 1/\mu(A))$. Clearly, $w_a \in \mathcal{A}(\nu, \mu)$ and $w_1 \equiv 1$. Moreover, $\varphi'(1) = u'(1) \frac{\nu(A) - \mu(A)}{\mu(A')} \neq 0$. Hence there exists an $a \in (0, 1/\mu(A))$ such that $\int_{\Omega} u(w_a) d\nu = \varphi(a) > \varphi(1) = \int_{\Omega} u(w_1) d\nu = u(1)$. Thus $H_u(\nu \parallel \mu) > 0$.

Proposition 2.16 (linear transformation) Let $u : (0, \infty) \to \mathbb{R}$ be a utility function, let a > 0 and let $b \in \mathbb{R}$. Then $\tilde{u} = au + b$ is a utility function, and for any $\nu, \mu \in M_1$

$$\begin{split} N_{\widetilde{u}} \left(\nu \parallel \mu \right) &= a N_u \left(\nu \parallel \mu \right) + b \ , \\ H_{\widetilde{u}} \left(\nu \parallel \mu \right) &= H_u \left(\nu \parallel \mu \right) \ . \end{split}$$

Proof This follows immediately from the definition.

Remark 2.17 It has recently been proved by Urbański [Urb07] that in probability spaces without atoms $\tilde{u} = au + b$ is not only a sufficient condition, but in fact an equivalent condition for $H_{\tilde{u}} = H_u$. The equivalence can fail in a probability space with atoms.

In [SZ01, Theorem 20] we established a formula for *u*-entropy by convex duality methods. Namely, under the reasonable asymptotic elasticity assumption, if $f \in D(\mu)$, then

$$N_u(f) = \int_{\Omega} u(I(\Lambda_f/f)) f d\mu = \int_{\Omega} u^*(\Lambda_f/f) f d\mu + \Lambda_f ,$$

$$H_u(f) = \ln u^{-1} \left(\int_{\Omega} u(I(\Lambda_f/f)) f d\mu \right) ,$$

where $\Lambda_f > 0$ is given implicitly as the unique solution of

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$$\int_{\Omega} I\left(\Lambda_f/f\right) \, d\mu = 1 \; .$$

Combined with Theorem 2.12, this makes it possible to evaluate the relative *u*-entropy $H_u(\nu \parallel \mu)$ for any $\nu, \mu \in M_1$.

Example 2.18 (logarithmic utility) Let $u : (0, \infty) \to \mathbb{R}$ be given by $u(x) = \ln x$ for $x \in (0, \infty)$. Then H_u is equal to the **Boltzmann-Shannon relative entropy**

$$H_1(\nu \parallel \mu) = \begin{cases} \int_{\Omega} \frac{d\nu}{d\mu} \ln \frac{d\nu}{d\mu} d\mu & \text{if } \nu \ll \mu ,\\ \infty & \text{otherwise} \end{cases}$$

for $\mu, \nu \in M_1$.

Example 2.19 (isoelastic utility) Let $u : (0, \infty) \to \mathbb{R}$ be given by $u(x) = \frac{1}{\gamma}(x^{\gamma}-1)$ for $\gamma \in (-\infty, 0) \cup (0, 1)$ and $x \in (0, \infty)$. Then H_u is equal to the **Rényi relative entropy of order** $\alpha = (1 - \gamma)^{-1} \in (0, 1) \cup (1, \infty)$ given by

$$H_{\alpha}\left(\nu \parallel \mu\right) = \begin{cases} \frac{1}{\alpha - 1} \ln \int_{\Omega} \left(\frac{d\nu_{\ll}}{d\mu}\right)^{\alpha} d\mu & \text{ if } \gamma \in (-\infty, 0) \ ,\\ \frac{1}{\alpha - 1} \ln \int_{\Omega} \left(\frac{d\nu}{d\mu}\right)^{\alpha} d\mu & \text{ if } \gamma \in (0, 1) \text{ and } \nu \ll \mu \ ,\\ \infty & \text{ otherwise} \end{cases}$$

for $\mu, \nu \in M_1$.

Remark 2.20 The Boltzmann-Shannon relative entropy was introduced in [KL51] under the name of directed divergence. It is also called the Kullback-Leibler divergence, relative information, conditional entropy, information gain or function of discrimination. The definition of the Rényi relative entropy (or divergence) of order α was proposed in [Rén61].

3 Discrete *u*-entropy

Let $\Omega = \{\omega_1, \ldots, \omega_k\}$ be a finite probability space equipped with the sigmafield $\Sigma = 2^{\Omega}$ of all subsets of Ω . The family of probability measures on (Ω, Σ) will be denoted by S_k . For any $p \in S_k$ we shall write $p_i = p(\omega_i)$ for $i = 1, \ldots, k$. Thus, we can identify S_k with the set of probability vectors $\{p \in \mathbb{R}^k : \sum_{i=1}^k p_i = 1 \text{ and } p_i \ge 0 \text{ for } i = 1, \ldots, k\}$. Our definition of the relative *u*-entropy covers also the discrete case. In this situation (though not necessarily in the general case) it is also possible to define the (non-relative) *u*-entropy as follows.

Definition 3.1 Let $u: (0, \infty) \to \mathbb{R}$ be a utility function, and let $p \in S_k$. Then we put

$$n_{u}(p) := \sup_{w \in S_{k}} \sum_{i=1}^{k} u(w_{i}) p_{i}$$

and define the **discrete** u-entropy of p by

$$h_u(p) := -\ln u^{-1}(n_u(p))$$

Remark 3.2 Note that h_u depends only on the restriction of u to (0, 1].

Proposition 3.3 Let $u: (0, \infty) \to \mathbb{R}$ be a utility function. Let $p \in S_k$ and let $p_{(k)} \in S_k$ be the uniform probability vector, that is, $(p_{(k)})_i = 1/k$ for each $i = 1, \ldots, k$. Then

$$H_u\left(p \parallel p_{(k)}\right) = \ln k - h_{u_k}\left(p\right) ,$$

where $u_k: (0, \infty) \rightarrow \mathbb{R}$ is the rescaled utility function

$$u_k\left(x\right) := u\left(kx\right) \tag{3.1}$$

for $x \in (0, \infty)$.

Proof Since $u^{-1} = k u_k^{-1}$ and

$$N_u(p \parallel p_{(k)}) = \sup_{w \in D(p_{(k)})} \sum_{i=1}^k u(w_i) p_i$$

= $\sup_{w \in S_k} \sum_{i=1}^k u(kw_i) p_i = \sup_{w \in S_k} \sum_{i=1}^k u_k(w_i) p_i = n_{u_k}(p) ,$

it follows that

$$H_u(p \parallel p_{(k)}) = \ln u^{-1}(N_u(p \parallel p_{(k)})) = \ln[ku_k^{-1}(n_{u_k}(p))] = \ln k - h_{u_k}(p) .$$

Using the above statement we can deduce many properties of discrete *u*entropy from the respective properties of relative *u*-entropy. However, one can also prove them straightforwardly without assuming anything about the behaviour of the function *u* outside the interval (0, 1]. We could assume that $u: (0, 1] \to \mathbb{R}$ is a strictly concave strictly increasing continuously differentiable function such that $\lim_{x \to 0} u'(x) = \infty$. In this case $I := (u')^{-1}$ would be defined on the interval $[u'(1), \infty)$. The proofs of the following properties of discrete *u*-entropy are elementary.

Proposition 3.4 Let $p = (p_1, \ldots, p_k) \in S_k$. Then

- (1) $0 \le h_u(p) \le \ln k$.
- (2) $h_u(p) = 0$ iff $p_i = 1$ for some i = 1, ..., k.
- (3) $h_u(p) = \ln k \text{ iff } p = p_{(k)}.$
- (4) $h_u(p) = h_u((p_{\pi(1)}, \ldots, p_{\pi(k)}))$ for every permutation π .
- (5) $h_u(p) = h_u((p_1, \dots, p_k, 0)).$
- (6) For a > 0 and $b \in \mathbb{R}$ we have $h_{au+b} = h_u$.

The proof of the formula for u-entropy in the discrete case is also elementary and, by contrast to the general case, it does not require any further assumptions on u. **Proposition 3.5** Let $p \in S_k$. Then:

(1) There exists a unique $\Lambda_p \ge \alpha(p) := u'(1) \max_{j=1,\dots,k} p_j > 0$ such that

$$\sum_{i=1}^{k} I\left(\Lambda_p/p_i\right) = 1$$

(2) The following formulae hold:

$$n_{u}(p) = \sum_{i=1}^{k} u\left(I\left(\Lambda_{p}/p_{i}\right)\right) p_{i} = \sum_{i=1}^{k} u^{*}\left(\Lambda_{p}/p_{i}\right) p_{i} + \Lambda_{p} , \qquad (3.2)$$
$$h_{u}(p) = \ln u^{-1}\left(\sum_{i=1}^{k} u\left(I\left(\Lambda_{p}/p_{i}\right)\right) p_{i}\right) .$$

Proof To prove (1) consider the function $\phi_p : [\alpha(p), \infty) \to (0, \infty)$ given by $\phi_p(\Lambda) = \sum_{i=1}^k I(\Lambda/p_i)$ for $\Lambda \ge \alpha(p)$. Clearly, ϕ_p is continuous, strictly decreasing and satisfies $\phi_p(\alpha(p)) = \sum_{i=1}^k I(u'(1) \max_{j=1,\dots,k} p_j/p_i) \ge 1$ and $\lim_{\Lambda\to\infty} \phi_p(\Lambda) = 0$. As a result, there is a unique $\Lambda_p \ge \alpha(p)$ such that $\phi_p(\Lambda_p) =$ 1, as required. It follows from (1) that $n_u(p) \ge \sum_{i=1}^k u(I(\Lambda_p/p_i))p_i$. To prove the reverse inequality take $w \in S_k$. Let $i = 1, \dots, k$. From the well-known formula $u^*(y) = u(I(y)) - yI(y)$ for the convex dual we get

$$u(w_i) - (\Lambda_p/p_i) w_i \le u^* \left((\Lambda_p/p_i) \right) = u\left(I\left((\Lambda_p/p_i) \right) \right) - (\Lambda_p/p_i) I\left((\Lambda_p/p_i) \right) .$$
(3.3)

Multiplying (3.3) by p_i , summing over i = 1, ..., k, and adding Λ_p , we obtain

$$\sum_{i=1}^{k} u(w_i) p_i \le \sum_{i=1}^{k} u^* ((\Lambda_p/p_i)) p_i + \Lambda_p = \sum_{i=1}^{k} u(I((\Lambda_p/p_i))) p_i$$

Taking the supremum of the left-hand side over all such w's, we obtain the assertion.

Proposition 3.6 The function $h_u: S_k \to [0, \ln k]$ is continuous.

Proof According to (3.2), it is enough to prove that $S_k \ni p \to \Lambda_p \in [\alpha(p), \infty)$ is continuous. Define $F : \{(p, \Lambda)\} : p \in S_k, \Lambda \in [\alpha(p), \infty)\} \to \mathbb{R}$ by $F(p, \Lambda) = \sum_{i=1}^k I(\Lambda/p_i) - 1$. Clearly, F is continuous, $F(p, \Lambda_p) = 0$ for $p \in S_k$, and $[\alpha(p), \infty) \ni \Lambda \to F(p, \Lambda) \in \mathbb{R}$ is strictly decreasing for each $p \in S_k$. Now, the assertion follows from the implicit function theorem for continuous functions.

Example 3.7 (logarithmic utility) Let $u : (0, \infty) \to \mathbb{R}$ be given by $u(x) = \ln x$ for $x \in (0, \infty)$. Then the relative *u*-entropy $H_u(p \parallel q)$ is equal to the

discrete Boltzmann-Shannon relative entropy (Kullback-Leibler divergence)

$$H_1(p \parallel q) = \begin{cases} \sum_{\substack{i=1,\dots,k \\ q_i \neq 0 \\ \infty}} p_i \ln \frac{p_i}{q_i} & \text{if } p \ll q , \\ p_i \neq 0 & \text{otherwise} \end{cases}$$

for $p,q \in S_k$, and the discrete *u*-entropy $h_u(p)$ is equal to the **discrete** Boltzmann-Shannon entropy

$$h_1(p) = -\sum_{i=1}^k p_i \ln p_i$$

for $p \in S_k$.

Example 3.8 (isoelastic utility) Let $u : (0, \infty) \to \mathbb{R}$ be given by $u(x) = \frac{1}{\gamma}(x^{\gamma}-1)$ for $\gamma \in (-\infty,0) \cup (0,1)$ and $x \in (0,\infty)$. Then the discrete relative *u*-entropy $H_u(p \parallel p_{(k)})$ is equal to the **discrete Rényi relative entropy** (divergence) of order $\alpha = (1-\gamma)^{-1} \in (0,1) \cup (1,\infty)$

$$h_{\alpha}\left(p \parallel q\right) = \begin{cases} \frac{1}{\alpha - 1} \ln \sum_{\substack{i=1, \dots, k \\ i = 1, \dots, k}} p_i^{\alpha} q_i^{1 - \alpha} & \text{if } \gamma \in (-\infty, 0) \ ,\\ \frac{1}{\alpha - 1} \ln \sum_{\substack{i=1, \dots, k \\ q_i \neq 0}} p_i^{\alpha} q_i^{1 - \alpha} & \text{if } \gamma \in (0, 1) \text{ and } p \ll q \ ,\\ \infty & \text{otherwise} \end{cases}$$

for $p, q \in S_k$, and the discrete *u*-entropy $h_u(p)$ is equal to the **discrete Rényi** entropy of order α

$$h_{\alpha}\left(p\right) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{k} p_{i}^{\alpha}$$

for $p \in S_k$.

4 Relationships to other utility based concepts of entropy

4.1 Friedman-Huang-Sandow U-entropy

In [FHS07] (see also [FHS05]) the authors defined two quantities, which they called the U-entropy and U-relative entropy, noting their similarity to the u-entropy defined (in a much more general setting) in [SZ04]. In fact the U-entropy and U-relative entropy of Friedman, Huang and Sandow [FHS07], [FHS05] can be reduced by a simple transformation to the relative u-entropy discussed in the present paper, and so to the u-entropy defined in [SZ04]. As a result, the properties of U-entropy and U-relative entropy claimed in [FHS05], [FHS07] turn out to be immediate corollaries of the results of [SZ04], as shown below.

In the notation of the present paper the definitions in $[{\rm FHS07}]$ take the following form.

Definition 4.1 (Definition 5 from [FHS07]) Let $u : (0, \infty) \to \mathbb{R}$ be a utility function and let $p, q \in S_k$. If $p \ll q$, then the Friedman-Huang-Sandow *U*-relative entropy of p with respect to q is defined by

$$D_{u}\left(p \parallel q\right) := \sup_{w \in S_{k}} \sum_{i=1}^{k} u\left(\frac{w_{i}}{q_{i}}\right) p_{i} - u\left(1\right)$$

Remark 4.2 By contrast to [FHS07], it is not assumed here that u(1) = 0. To compensate, we subtract u(1) on the right-hand side of the formula defining $D_u(p || q)$. The same applies to the formula defining $H_u(p)$ below. Moreover, instead of $w \in S_k$ it is only assumed in [FHS07] that $\sum_{i=1}^k w_i = 1$, but presumably there is also a silent assumption that w_i/q_i belongs to the domain of u for each i. In our case this means that, additionally, $w \ge 0$, so that $w \in S_k$. The definitions and results easily extend to utility functions defined on an interval (a, b) other than $(0, \infty)$. If p is not absolutely continuous with respect to q, then $D_u(p || q)$ is undefined.

Remark 4.3 The relative entropy defined in [FHS07] coincides with the **decision maker's optimal expected utility** introduced in [JNW07, p.13].

Definition 4.4 (Definition 6 from [FHS07]) Let $u : (0, \infty) \to \mathbb{R}$ be a utility function and let $p \in S_k$. Then the Friedman-Huang-Sandow U-entropy of p is defined by

$$H_u(p) = u(k) - u(1) - D_u(p || p_{(k)})$$
.

Proposition 4.5 Let $u : (0, \infty) \rightarrow \mathbb{R}$ be a utility function. Then the following properties hold:

(1) For any $p, q \in S_k$ such that $p \ll q$

$$D_{u}(p || q) = u\left(e^{H_{u}(p || q)}\right) - u(1)$$
.

(2) For any $p \in S_k$

$$H_u(p) = u_k(1) - u_k\left(e^{-h_{u_k}(p)}\right) ,$$

where u_k is the rescaled utility function defined by (3.1).

Proof (1) If $p \ll q$, then $wq \in S_k$ is equivalent to $w \in \mathcal{A}(p,q)$. Hence

$$D_{u}(p \parallel q) = \sup_{w \in S_{k}} \sum_{i=1}^{k} u(w_{i}/q_{i}) p_{i} - u(1)$$
$$= \sup_{w \in \mathcal{A}(p,q)} \sum_{i=1}^{k} u(w_{i}) p_{i} - u(1) = N_{u}(p \parallel q) - u(1) .$$

The claim follows since $N_u(p \parallel q) = u(e^{H_u(p \parallel q)}).$

(2) This follows immediately from (1) and Proposition 3.3:

$$H_{u}(p) = u(k) - u(1) - D_{u}(p || p_{(k)}) = u(k) - u\left(e^{H_{u}(p || p_{(k)})}\right)$$
$$= u(k) - u\left(e^{\ln k - h_{u_{k}}(p)}\right) = u_{k}(1) - u_{k}\left(e^{-h_{u_{k}}(p)}\right) .$$

l	

The U-relative entropy $D_u(p \parallel q)$ and the U-entropy $H_u(p)$ of Friedman Huang and Sandow [FHS07] are therefore related to the u-entropy of [SZ04] by

$$D_{u}\left(p \parallel q\right) = u\left(e^{H_{u}\left(\frac{dp}{dq}\right)}\right) - u\left(1\right)$$

for each $p, q \in S_k$ such that $p \ll q$, and by

$$H_{u}(p) = u(k) - u\left(e^{H_{u}\left(\frac{dp}{dp_{(k)}}\right)}\right)$$

for each $p \in S_k$. Because of this, the following results in [FHS07] are immediate consequences of the corresponding earlier results in [SZ04]:

[SZ04]		[FHS07]
Theorem 20	\Rightarrow	Lemma 1
Propositions 8 and 10	\Rightarrow	Corollary 1.(i)
Proposition 13	\Rightarrow	Corollary 1.(ii), (iv)
Theorem 23	\Rightarrow	Corollaries 3, 6 and 7

Remark 4.6 The results of [SZ04] are valid in a much more general situation of arbitrary probability spaces, which requires the asymptotic elasticity assumption

$$AE(u) := \limsup_{x \nearrow \infty} \frac{xu'(x)}{u(x)} < 1$$

to hold. In the discrete case this assumption is unnecessary and all the arguments in [SZ04] work without it.

4.2 Arimoto entropy

A similar construction of entropy was first proposed by Arimoto [Ari71] without any explicit reference to the notion of utility.

Definition 4.7 (Arimoto [Ari71], see also [Tan07]) For a non-negative function $f : (0,1] \rightarrow \mathbb{R}$ continuously differentiable on (0,1] and such that f(1) = 0 Arimoto's entropy is defined by

$$H_f^A(p) := \inf_{w \in S_k} \sum_{i=1}^k f(w_i) p_i$$

for $p \in S_k$.

This was further generalized in [SS74] and also interpreted in [MPV96, Example 6] in terms of *prior Bayes risk*, where f plays the role of an individual uncertainty function. Arimoto's entropy is related to the entropy H_u defined in [FHS07] (see Definition 4.4 above) and to h_u (Definition 3.1) as follows.

Proposition 4.8 Let $u : (0, \infty) \rightarrow \mathbb{R}$ be a utility function such that u(1) = 0, and let $p \in S_k$. Then

$$H_{-u}^{A}(p) = H_{u_{1/k}}(p) = -u\left(e^{-h_{u}(p)}\right).$$

Proof This follows immediately from the definitions and Proposition 4.5 (2):

$$H_{-u}^{A}(p) = \inf_{w \in S_{k}} \sum_{i=1}^{k} \left[-u(w_{i}) \right] p_{i} = -\sup_{w \in S_{k}} \sum_{i=1}^{k} u(w_{i}) p_{i}$$
$$= -n_{u}(p) = -u\left(e^{-h_{u}(p)}\right) = H_{u_{1/k}}(p) .$$

Example 4.9 (logarithmic utility) Let u be the logarithmic utility. For $p, q \in S_k$ and $p \ll q$ we have $D_u(p \parallel q) = h_1(p \parallel q)$ and $H_u(p) = H^A_{-u}(p) = h_1(p)$.

Example 4.10 (isoelastic utility) Let u be the isoelastic utility of order $\gamma \in (-\infty, 0) \cup (0, 1)$ and let $\alpha = (1 - \gamma)^{-1}$.

1. For $p, q \in S_k$ with $p \ll q$

$$D_u\left(p \parallel q\right) = \frac{\alpha}{\alpha - 1} \left(\left(\sum_{i=1,\dots,k} p_i^{\alpha} q_i^{1-\alpha}\right)^{\frac{1}{\alpha}} - 1 \right)$$

is proportional to the Sharma-Mittal relative entropy of order α and degree 2 - 1/a;

2. For $p \in S_k$

$$\begin{aligned} H_u\left(p\right) &= k^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{1-\alpha} \left(\left(\sum_{i=1}^k p_i^{\alpha}\right)^{\frac{1}{\alpha}} - 1 \right) \\ &= k^{\frac{\alpha-1}{\alpha}} H_{-u}^A \;, \end{aligned}$$

where H_{-u}^A is called the **Arimoto entropy of kind** $1/\alpha$.

Remark 4.11 The Sharma-Mittal relative entropy was introduced in [SM75]. In [JNW07] the Sharma-Mittal relative entropy of order α and degree 2 - 1/a is called the **pseudospherical divergence of order** α . The Arimoto entropy was introduced by Arimoto [Ari71] and further elaborated in [BvdL80], see also [Tan07].

4.3 Frittelli generalised distance

This notion of generalised distance in the set of probability measures was introduced in [Fri00] as a tool for solving the convex dual problem to that of computing the value of a financial security consistent with the no-arbitrage principle in an incomplete market model in a utility maximisation framework.

Definition 4.12 (Definition 9 and formula (9) from [Fri00]) Let $\mu, \nu \in M^1$ be such that $\mu \ll \nu$. Then put

$$\Delta_u(\mu,\nu) := \sup_{\Lambda>0} \left(\Lambda + \int_{\Omega} u^* \left(\Lambda \frac{d\mu}{d\nu}\right) d\nu\right) ,$$

where u^* is the convex dual to the utility function u given by (2.1) and (2.2), and define the **Frittelli generalised distance** by

$$\delta_u(\mu,\nu) := u^{-1} \left(\Delta_u(\mu,\nu) \right) - 1 .$$

Remark 4.13 Note the different (but equivalent) conventions as compared to [Fri00]. The differences lie in using the supremum rather than infimum coupled with different signs of certain expressions in the definitions of u^* and $\Delta_u(\mu,\nu)$. The quantities $\Delta_u(\mu,\nu)$ and $\delta_u(\mu,\nu)$ are denoted by $\Delta(\mu,\nu;1)$ and $\delta(\mu,\nu;1)$ in [Fri00].

Proposition 4.14 For any $\mu, \nu \in M^1$ such that $\mu \ll \nu$

$$\Delta_u(\mu,\nu) = u(\infty)\nu_{\perp}(\Omega) + \nu_{\ll}(\Omega)\Delta_u\left(\mu,\frac{\nu_{\ll}}{\nu_{\ll}(\Omega)}\right)$$

where $\nu_{\perp} + \nu_{\ll} = \nu$ is the Lebesgue decomposition of ν into the singular part ν_{\perp} and absolutely continuous part ν_{\ll} with respect to μ .

Proof Because $\mu \ll \nu$, it follows that $\frac{d\mu}{d\nu} = 0$ a.s. with respect to ν_{\perp} and $\nu_{\ll}(\Omega) > 0$. We can assume that $\nu_{\perp}(\Omega) > 0$, since otherwise the assertion is obvious. Put $\tilde{\nu}_{\ll} = \frac{\nu_{\ll}}{\nu_{\ll}(\Omega)}$. As a result,

$$\begin{split} \Delta_{u}(\mu,\nu) &= \sup_{\Lambda>0} \left(\Lambda + \int_{\Omega} u^{*} \left(\Lambda \frac{d\mu}{d\nu} \right) d\nu \right) \\ &= \sup_{\Lambda>0} \left(\Lambda + \int_{\Omega} u^{*} \left(\Lambda \frac{d\mu}{d\nu} \right) d\nu_{\perp} + \int_{\Omega} u^{*} \left(\Lambda \frac{d\mu}{d\nu} \right) d\nu_{\ll} \right) \\ &= \nu_{\perp}(\Omega) u^{*}(0) + \nu_{\ll}(\Omega) \sup_{\Lambda>0} \left(\frac{\Lambda}{\nu_{\ll}(\Omega)} + \int_{\Omega} u^{*} \left(\frac{\Lambda}{\nu_{\ll}(\Omega)} \frac{d\mu}{d\tilde{\nu}_{\ll}} \right) d\tilde{\nu}_{\ll} \right) \\ &= \nu_{\perp}(\Omega) u(\infty) + \nu_{\ll}(\Omega) \sup_{\Lambda>0} \left(\Lambda + \int_{\Omega} u^{*} \left(\Lambda \frac{d\mu}{d\tilde{\nu}_{\ll}} \right) d\tilde{\nu}_{\ll} \right) \\ &= \nu_{\perp}(\Omega) u(\infty) + \nu_{\ll}(\Omega) \Delta_{u}(\mu, \tilde{\nu}_{\ll}) \;. \end{split}$$

Proposition 4.15 If u has reasonable asymptotic elasticity, then for any $\mu, \nu \in M^1$ such that $\mu \ll \nu$

$$\Delta_u(\mu, \nu) = N_u(\nu \parallel \mu) ,$$

$$\delta_u(\nu, \mu) = e^{H_u(\nu \parallel \mu)} - 1$$

Proof First we shall prove the proposition in the case when $\mu, \nu \in M^1$ are equivalent measures. Let $f = \frac{d\nu}{d\mu}$. By Proposition 2.9 above and by Lemma 17 and Theorem 20.4 in [SZ04], we then have

$$N_{u}(\nu \parallel \mu) = N_{u}(f) = \sup_{\Lambda > 0} \left(\Lambda + \int_{\Omega} u^{*}(\Lambda/f) f d\mu\right)$$
$$= \sup_{\Lambda > 0} \left(\Lambda + \int_{\Omega} u^{*}\left(\Lambda \frac{d\mu}{d\nu}\right) d\nu\right) = \Delta_{u}(\mu, \nu)$$

Now for any $\mu, \nu \in M^1$ such that $\mu \ll \nu$ we take the Lebesgue decomposition $\nu = \nu_{\perp} + \nu_{\ll}$ into the singular part ν_{\perp} and absolutely continuous part ν_{\ll} with respect to μ . Then μ and ν_{\ll} are equivalent measures. It follows by Propositions 2.12 and 4.14 that

$$N_{u}(\nu \parallel \mu) = \nu_{\perp}(\Omega) u(\infty) + \nu_{\ll}(\Omega) N_{u}\left(\frac{\nu_{\ll}}{\nu_{\ll}(\Omega)} \parallel \mu\right)$$
$$= \nu_{\perp}(\Omega) u(\infty) + \nu_{\ll}(\Omega) \Delta_{u}\left(\mu, \frac{\nu_{\ll}}{\nu_{\ll}(\Omega)}\right) = \Delta_{u}(\mu, \nu)$$

The equality $\delta_u(\nu,\mu) = e^{H_u(\nu \parallel \mu)} - 1$ now follows immediately from the definitions of $\delta_u(\nu,\mu)$ and $H_u(\nu \parallel \mu)$.

5 Concluding remarks

The notion of u-entropy of a probability density, based on the concept of expected utility maximisation in finance, was first introduced in [SZ04] and linked with the Second Law of thermodynamics. In this paper the definition of u-entropy has been extended, on the one hand, to the case of relative u-entropy of one probability measure with respect to another, and, on the other hand, in the discrete case, to absolute u-entropy of a probability measure. Having established the basic properties of these notions, we have studied the relationships with other entropy-like quantities of a similar kind that can be found in the literature. In particular, although all these approaches yield the Boltzmann-Shannon entropy when the logarithmic utility is used, it is only the relative u-entropy introduced in Definition 2.7 that is consistent with the Rényi entropy for isoelastic utility functions. The relationships between the various approaches are summarized in the diagram below. In this context, relative u-entropy emerges as the general unifying quantity among the various approaches related to expected utility maximisation.

 $H_u(f) \qquad \underbrace{\frac{\text{Thm.2.12}}{\text{Prop.2.9}}}_{\text{Prop.2.9}} \qquad H_u(\nu \parallel \mu)$ $\xrightarrow{\text{Prop.4.15}}$ general $\delta_u(\nu,\mu)$ case $\operatorname{Prop.4.5(1)}$ $H_u\left(p \parallel q\right)$ $D_u(p \parallel q)$ discrete Prop.3.3 [FHS07] case Prop.4.8 Prop.4.5(2) $h_u(p)$ $H_{-u}^{A}(p)$ $H_u(p)$

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