# Entanglement Generation and Evolution in Open Quantum Systems

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#### Abstract

In the framework of the theory of open systems based on completely positive quantum dynamical semigroups, we study the continuous variable entanglement for a system consisting of two independent harmonic oscillators interacting with a general environment. We solve the Kossakowski-Lindblad master equation for the time evolution of the considered system and describe the entanglement in terms of the covariance matrix for an arbitrary Gaussian input state. Using Peres–Simon necessary and sufficient criterion for separability of two-mode Gaussian states, we show that for certain values of diffusion and dissipation coefficients describing the environment, the state keeps for all times its initial type: separable or entangled. In other cases, entanglement generation, entanglement sudden death or a periodic collapse and revival of entanglement take place. We analyze also the time evolution of the logarithmic negativity, which characterizes the degree of entanglement of the quantum state.

# 1 Introduction

The rapid development of the theory of quantum information has revived the interest in open quantum systems in connection, on one side, to the decoherence phenomenon and, on the other side, to their capacity of generating entanglement in multi-partite systems interacting with their environments. Quantum entanglement represents the physical resource in quantum information science which is indispensable for the description and performance of such tasks like teleportation, superdense coding, quantum cryptography and quantum computation [1]. Therefore the generation, detection and manipulation of the entanglement continues to be presently a problem of intense investigation. When two systems are immersed in an environment, then, besides and at the same time with the quantum decoherence, the environment can also generate a quantum entanglement of the two systems and therefore an additional mechanism to correlate them [2, 3]. In certain circumstances, the environment enhances entanglement and in others it suppresses the entanglement and the state describing the two systems becomes separable. The structure and properties of the environment may be such that not only the two systems become entangled, but also such that a certain amount of entanglement survives in the asymptotic long-time regime. The reason is that even if not directly coupled, the two systems immersed in the same environment can interact through the environment itself and it depends on how strong this indirect interaction is with respect to the quantum decoherence, whether entanglement can be generated at the beginning of the evolution and, in the case of an affirmative answer, if it can be maintained for a definite time or it survives indefinitely in time [2].

In this work we study, in the framework of the theory of open quantum systems based on completely positive dynamical semigroups, the dynamics of the continuous variable entanglement for a subsystem composed of two identical harmonic oscillators interacting with its environment. We are interested in discussing the correlation effect of the environment, therefore we assume that the two systems are independent, i.e. they do not interact directly. The initial state of the subsystem is taken of Gaussian form and the evolution under the quantum dynamical semigroup assures the preservation in time of the Gaussian form of the state.

The organizing of the paper is as follows. In Sect. 2 the notion of the quantum dynamical semigroup is defined using the concept of a completely positive map. Then we give the general form of the Kossakowski-Lindblad quantum mechanical master equation describing the evolution of open quantum systems in the Markovian approximation. We mention the role of complete positivity in connection with the quantum entanglement of systems interacting with an external environment. In Sec. 3 we write and solve the equations of motion in the Heisenberg picture for two independent harmonic oscillators interacting with a general environment. Then, by using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states [4, 5], we investigate in Sec. 4 the dynamics of entanglement for the considered subsystem. In particular, with the help of the asymptotic covariance matrix, we determine the behaviour of the entanglement in the limit of long times. We show that for certain classes of environments the initial state evolves asymptotically to an equilibrium state which is entangled, while for other values of the parameters describing the environment, the entanglement is suppressed and the asymptotic state is separable. We analyze also the time evolution of the logarithmic negativity, which characterizes the degree of entanglement of the quantum state. A summary and conclusions are given in Sec. 5.

# 2 Axiomatic theory of open quantum systems

The time evolution of a closed physical system is given by a dynamical group  $U_t$ , uniquely determined by its generator H, which is the Hamiltonian operator of the system. The action of the dynamical group  $U_t$  on any density matrix  $\rho$  from the set  $\mathcal{D}(\mathcal{H})$  of all density matrices in the Hilbert space  $\mathcal{H}$  of the quantum system is defined by

$$\rho(t) = U_t(\rho) = e^{-\frac{i}{\hbar}Ht}\rho e^{\frac{i}{\hbar}Ht} \tag{1}$$

for all  $t \in (-\infty, \infty)$ . According to von Neumann, density operators  $\rho \in \mathcal{D}(\mathcal{H})$  are trace class (Tr  $\rho < \infty$ ), self-adjoint ( $\rho^{\dagger} = \rho$ ), positive ( $\rho > 0$ ) operators with Tr  $\rho = 1$ . All these properties are conserved by the time evolution defined by  $U_t$ .

In the case of open quantum systems, the time evolution  $\Phi_t$  of the density operator  $\rho(t) = \Phi_t(\rho)$  has to preserve the von Neumann conditions for all times. It follows that  $\Phi_t$  must have the following properties:

(i)  $\Phi_t(\lambda_1\rho_1 + \lambda_2\rho_2) = \lambda_1\Phi_t(\rho_1) + \lambda_2\Phi_t(\rho_2)$  for  $\lambda_1, \lambda_2 \ge 0, \lambda_1 + \lambda_2 = 1$ , i. e.  $\Phi_t$  must preserve the convex structure of  $\mathcal{D}(\mathcal{H})$ ,

- (ii)  $\Phi_{\rm t}(\rho^{\dagger}) = \Phi_{\rm t}(\rho)^{\dagger}$ ,
- (iii)  $\Phi_{\rm t}(\rho) > 0$ ,
- (iv) Tr  $\Phi_t(\rho) = 1$ .

The time evolution  $U_t$  for closed systems must be a group  $U_{t+s} = U_t U_s$ . We have also  $U_0(\rho) = \rho$  and  $U_t(\rho) \to \rho$  in the trace norm when  $t \to 0$ . The dual group  $\tilde{U}_t$ acting on the observables  $A \in \mathcal{B}(\mathcal{H})$ , i.e. on the bounded operators on  $\mathcal{H}$ , is given by

$$\widetilde{U}_t(A) = e^{\frac{i}{\hbar}Ht} A e^{-\frac{i}{\hbar}Ht}.$$
(2)

Then  $\tilde{U}_t(AB) = \tilde{U}_t(A)\tilde{U}_t(B)$  and  $\tilde{U}_t(I) = I$ , where I is the identity operator on  $\mathcal{H}$ . Also,  $\tilde{U}_t(A) \to A$  ultraweakly when  $t \to 0$  and  $\tilde{U}_t$  is an ultraweakly continuous mapping [6, 7, 8]. These mappings have a strong positivity property called complete positivity:

$$\sum_{i,j} B_i^{\dagger} \widetilde{U}_t(A_i^{\dagger} A_j) B_j \ge 0, \quad A_i, B_i \in \mathcal{B}(\mathcal{H}).$$
(3)

In the axiomatic approach to the description of the evolution of open quantum systems [6, 7, 8], one supposes that the time evolution  $\Phi_t$  of open systems is not very different from the time evolution of closed systems. The simplest dynamics  $\Phi_t$  which introduces a preferred direction in time, characteristic for dissipative processes, is that in which the group condition is replaced by the semigroup condition [6, 9, 10]

$$\Phi_{t+s} = \Phi_t \Phi_s, \ t, s \ge 0. \tag{4}$$

The complete positivity condition has the form:

$$\sum_{i,j} B_i^{\dagger} \tilde{\Phi}_t(A_i^{\dagger} A_j) B_j \ge 0, \quad A_i, B_i \in \mathcal{B}(\mathcal{H}),$$
(5)

where  $\tilde{\Phi}_t$  denotes the dual of  $\Phi_t$  acting on  $\mathcal{B}(\mathcal{H})$  and is defined by the duality condition

$$\operatorname{Tr}(\Phi_t(\rho)A) = \operatorname{Tr}(\rho\tilde{\Phi}_t(A)).$$
(6)

Then the conditions  $\operatorname{Tr}\Phi_t(\rho) = 1$  and  $\tilde{\Phi}_t(I) = I$  are equivalent. Also the conditions  $\tilde{\Phi}_t(A) \to A$  ultraweakly when  $t \to 0$  and  $\Phi_t(\rho) \to \rho$  in the trace norm when  $t \to 0$ , are equivalent. For the semigroups with these properties and with a more weak property of positivity than Eq. (5), namely

$$A \ge 0 \to \Phi_t(A) \ge 0,\tag{7}$$

it is well known that there exists a (generally unbounded) mapping  $\tilde{L}$  – the generator of  $\tilde{\Phi}_t$ , and  $\tilde{\Phi}_t$  is uniquely determined by  $\tilde{L}$ . The dual generator of the dual semigroup  $\Phi_t$  is denoted by L:

$$\operatorname{Tr}(L(\rho)A) = \operatorname{Tr}(\rho \widetilde{L}(A)).$$
(8)

The evolution equations by which L and  $\tilde{L}$  determine uniquely  $\Phi_t$  and  $\tilde{\Phi}_t$ , respectively, are given in the Schrödinger and Heisenberg picture as

$$\frac{d\Phi_t(\rho)}{dt} = L(\Phi_t(\rho)) \tag{9}$$

and

$$\frac{d\tilde{\Phi}_t(A)}{dt} = \tilde{L}(\tilde{\Phi}_t(A)).$$
(10)

These equations replace in the case of open systems the von Neumann-Liouville equations

$$\frac{dU_t(\rho)}{dt} = -\frac{i}{\hbar}[H, U_t(\rho)] \tag{11}$$

and

$$\frac{d\tilde{U}_t(A)}{dt} = \frac{i}{\hbar} [H, \tilde{U}_t(A)], \qquad (12)$$

respectively. For applications, Eqs. (9) and (10) are only useful if the detailed structure of the generator  $L(\tilde{L})$  is known and can be related to the concrete properties of the open systems described by such equations. For the class of dynamical semigroups which are completely positive and norm continuous, the generator  $\tilde{L}$  is bounded. In many applications the generator is unbounded.

According to Lindblad [8], the following argument can be used to justify the complete positivity of  $\tilde{\Phi}_t$ : if the open system is extended in a trivial way to a larger system described in a Hilbert space  $\mathcal{H} \otimes \mathcal{K}$  with the time evolution defined by

$$\widetilde{W}_t(A \otimes B) = \widetilde{\Phi}_t(A) \otimes B, \quad A \in \mathcal{B}(\mathcal{H}), \ B \in \mathcal{B}(\mathcal{K}), \tag{13}$$

then the positivity of the states of the compound system will be preserved by  $\widetilde{W}_t$ only if  $\widetilde{\Phi}_t$  is completely positive. With this observation a new equivalent definition of the complete positivity is obtained:  $\widetilde{\Phi}_t$  is completely positive if  $\widetilde{W}_t$  is positive for any finite dimensional Hilbert space  $\mathcal{K}$ . The physical meaning of complete positivity can mainly be understood in relation to the existence of entangled states, the typical example being given by a vector state with a singlet-like structure that cannot be written as a tensor product of vector states. Positivity property guarantees the physical consistency of evolving states of single systems, while complete positivity prevents inconsistencies in entangled composite systems and therefore the existence of entangled states makes the request of complete positivity necessary [2].

A bounded mapping  $\tilde{L} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$  which satisfies  $\tilde{L}(I) = 0$ ,  $\tilde{L}(A^{\dagger}) = \tilde{L}(A)^{\dagger}$ and

$$\widetilde{L}(A^{\dagger}A) - \widetilde{L}(A^{\dagger})A - A^{\dagger}\widetilde{L}(A) \ge 0$$
(14)

is called dissipative. The 2-positivity property of the completely positive mapping  $\widetilde{\Phi}_t$ :

$$\widetilde{\Phi}_t(A^{\dagger}A) \ge \widetilde{\Phi}_t(A^{\dagger})\widetilde{\Phi}_t(A), \tag{15}$$

with equality at t = 0, implies that  $\tilde{L}$  is dissipative. Conversely, the dissipativity of  $\tilde{L}$  implies that  $\tilde{\Phi}_t$  is 2-positive.  $\tilde{L}$  is called completely dissipative if all trivial extensions of  $\tilde{L}$  to a compound system described by  $\mathcal{H} \otimes \mathcal{K}$  with any finite dimensional Hilbert space  $\mathcal{K}$  are dissipative. There exists a one-to-one correspondence between the completely positive norm continuous semigroups  $\tilde{\Phi}_t$  and completely dissipative generators  $\tilde{L}$ . The following structural theorem gives the most general form of a completely dissipative mapping  $\tilde{L}$  [8].

**Theorem.**  $\tilde{L}$  is completely dissipative and ultraweakly continuous if and only if it is of the form

$$\tilde{L}(A) = \frac{i}{\hbar}[H, A] + \frac{1}{2\hbar} \sum_{j} (V_{j}^{\dagger}[A, V_{j}] + [V_{j}^{\dagger}, A]V_{j}),$$
(16)

where  $V_j$ ,  $\sum_j V_j^{\dagger} V_j \in \mathcal{B}(\mathcal{H}), \ H \in \mathcal{B}(\mathcal{H})_{\text{s.a.}}$ .

The dual generator on the state space (Schrödinger picture) is of the form

$$L(\rho) = -\frac{i}{\hbar}[H,\rho] + \frac{1}{2\hbar} \sum_{j} ([V_{j}\rho, V_{j}^{\dagger}] + [V_{j}, \rho V_{j}^{\dagger}]).$$
(17)

Eqs. (9) and (17) give the explicit form of the Kossakowski-Lindblad master equation, which is the most general time-homogeneous quantum mechanical Markovian master equation with a bounded Liouville operator [8, 10, 11, 12]:

$$\frac{d\Phi_t(\rho)}{dt} = -\frac{i}{\hbar} [H, \Phi_t(\rho)] + \frac{1}{2\hbar} \sum_j ([V_j \Phi_t(\rho), V_j^{\dagger}] + [V_j, \Phi_t(\rho) V_j^{\dagger}]).$$
(18)

The assumption of a semigroup dynamics is only applicable in the limit of weak coupling of the subsystem with its environment, i.e. for long relaxation times [13]. We mention that the majority of Markovian master equations found in the literature are of this form after some rearrangement of terms, even for unbounded generators. It is also an empirical fact for many physically interesting situations that the time evolutions  $\Phi_t$  drive the system towards a unique final state  $\rho(\infty) = \lim_{t\to\infty} \Phi_t(\rho(0))$ for all  $\rho(0) \in \mathcal{D}(\mathcal{H})$ .

# 3 Time evolution of two independent harmonic oscillators interacting with the environment

We are interested in the dynamics of entanglement in a subsystem composed of two identical non-interacting (independent) harmonic oscillators in weak interaction with a general environment, so that their reduced time evolution can be described by a Markovian, completely positive quantum dynamical semigroup. If  $\tilde{\Phi}_t$  is the dynamical semigroup describing the irreversible time evolution of the open quantum system in the Heisenberg picture, then the Kossakowski-Lindblad master equation has the following form for an operator A (see Eqs. (10), (16)) [8, 10, 11, 12]:

$$\frac{d\tilde{\Phi}_t(A)}{dt} = \frac{i}{\hbar} [H, \tilde{\Phi}_t(A)] + \frac{1}{2\hbar} \sum_j (V_j^{\dagger}[\tilde{\Phi}_t(A), V_j] + [V_j^{\dagger}, \tilde{\Phi}_t(A)]V_j).$$
(19)

Here, H denotes the Hamiltonian of the open system and the operators  $V_j, V_j^{\dagger}$ , defined on the Hilbert space of H, represent the interaction of the open system with the environment. We are interested in the set of Gaussian states, therefore we introduce such quantum dynamical semigroups that preserve this set. Consequently H is taken to be a polynomial of second degree in the coordinates x, y and momenta  $p_x, p_y$  of the two quantum oscillators and  $V_j, V_j^{\dagger}$  are taken polynomials of first degree in these canonical observables. Then in the linear space spanned by the coordinates and momenta there exist only four linearly independent operators  $V_{j=1,2,3,4}$  [14]:

$$V_j = a_{xj}p_x + a_{yj}p_y + b_{xj}x + b_{yj}y,$$
(20)

where  $a_{xj}, a_{yj}, b_{xj}, b_{yj} \in \mathbf{C}$ . The Hamiltonian H of the two uncoupled identical harmonic oscillators of mass m and frequency  $\omega$  is given by

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\omega^2}{2}(x^2 + y^2).$$
 (21)

The fact that  $\tilde{\Phi}_t$  is a dynamical semigroup implies the positivity of the following matrix formed by the scalar products of the four vectors  $\mathbf{a}_x$ ,  $\mathbf{b}_x$ ,  $\mathbf{a}_y$ ,  $\mathbf{b}_y$ , whose entries are the components  $a_{xj}$ ,  $b_{xj}$ ,  $a_{yj}$ ,  $b_{yj}$ , respectively:

$$\frac{1}{2}\hbar \begin{pmatrix} (\mathbf{a}_x \mathbf{a}_x) & (\mathbf{a}_x \mathbf{b}_x) & (\mathbf{a}_x \mathbf{a}_y) & (\mathbf{a}_x \mathbf{b}_y) \\ (\mathbf{b}_x \mathbf{a}_x) & (\mathbf{b}_x \mathbf{b}_x) & (\mathbf{b}_x \mathbf{a}_y) & (\mathbf{b}_x \mathbf{b}_y) \\ (\mathbf{a}_y \mathbf{a}_x) & (\mathbf{a}_y \mathbf{b}_x) & (\mathbf{a}_y \mathbf{a}_y) & (\mathbf{a}_y \mathbf{b}_y) \\ (\mathbf{b}_y \mathbf{a}_x) & (\mathbf{b}_y \mathbf{b}_x) & (\mathbf{b}_y \mathbf{a}_y) & (\mathbf{b}_y \mathbf{b}_y) \end{pmatrix}.$$

$$(22)$$

Its matrix elements have to be chosen appropriately to suit various physical models of the environment. For a quite general environment able to induce noise and damping effects, we take this matrix of the following form, where all the coefficients  $D_{xx}$ ,  $D_{xp_x}$ ,... and  $\lambda$  are real quantities, representing the diffusion coefficients and, respectively, the dissipation constant:

$$\begin{pmatrix} D_{xx} & -D_{xp_x} - i\hbar\lambda/2 & D_{xy} & -D_{xp_y} \\ -D_{xp_x} + i\hbar\lambda/2 & D_{p_xp_x} & -D_{yp_x} & D_{p_xp_y} \\ D_{xy} & -D_{yp_x} & D_{yy} & -D_{yp_y} - i\hbar\lambda/2 \\ -D_{xp_y} & D_{p_xp_y} & -D_{yp_y} + i\hbar\lambda/2 & D_{p_yp_y} \end{pmatrix}.$$
(23)

It follows that the principal minors of this matrix are positive or zero. From the Cauchy-Schwarz inequality the following relations for the coefficients defined in Eq. (23) hold (from now on we put, for simplicity,  $\hbar = 1$ ):

$$D_{xx}D_{p_xp_x} - D_{xp_x}^2 \ge \frac{\lambda^2}{4}, \ D_{yy}D_{p_yp_y} - D_{yp_y}^2 \ge \frac{\lambda^2}{4}, D_{xx}D_{yy} - D_{xy}^2 \ge 0, \ D_{p_xp_x}D_{p_yp_y} - D_{p_xp_y}^2 \ge 0, D_{xx}D_{p_yp_y} - D_{xp_y}^2 \ge 0, \ D_{yy}D_{p_xp_x} - D_{yp_x}^2 \ge 0.$$
(24)

The matrix of the coefficients (23) can be conveniently written as (T denotes the transposed matrix)

$$\left(\begin{array}{cc} C_1 & C_3 \\ C_3^{\mathrm{T}} & C_2 \end{array}\right),\tag{25}$$

in terms of  $2 \times 2$  matrices  $C_1 = C_1^{\dagger}$ ,  $C_2 = C_2^{\dagger}$  and  $C_3$ . This decomposition has a direct physical interpretation: the elements containing the diagonal contributions  $C_1$  and  $C_2$  represent diffusion and dissipation coefficients corresponding to the first, respectively the second, system in absence of the other, while the elements in  $C_3$ represent environment generated couplings between the two oscillators, taken initially independent.

We introduce the following  $4 \times 4$  bimodal covariance matrix:

$$\sigma(t) = \begin{pmatrix} \sigma_{xx}(t) & \sigma_{xp_x}(t) & \sigma_{xy}(t) & \sigma_{xp_y}(t) \\ \sigma_{xp_x}(t) & \sigma_{pxp_x}(t) & \sigma_{yp_x}(t) & \sigma_{pxp_y}(t) \\ \sigma_{xy}(t) & \sigma_{yp_x}(t) & \sigma_{yy}(t) & \sigma_{yp_y}(t) \\ \sigma_{xp_y}(t) & \sigma_{pxp_y}(t) & \sigma_{ypy}(t) & \sigma_{pypy}(t) \end{pmatrix},$$
(26)

with the correlations of operators  $A_1$  and  $A_2$ , defined by using the density operator  $\rho$  of the initial state of the quantum system, as follows:

$$\sigma_{A_1A_2}(t) = \frac{1}{2} \operatorname{Tr}(\rho(A_1A_2 + A_2A_1)(t)) - \operatorname{Tr}(\rho A_1(t)) \operatorname{Tr}(\rho A_2(t)).$$
(27)

By using Eq. (19) we obtain by direct calculation the following systems of equations for the quantum correlations of the canonical observables [14]:

$$\frac{d\sigma(t)}{dt} = Y\sigma(t) + \sigma(t)Y^{\mathrm{T}} + 2D, \qquad (28)$$

where

$$Y = \begin{pmatrix} -\lambda & 1/m & 0 & 0\\ -m\omega^2 & -\lambda & 0 & 0\\ 0 & 0 & -\lambda & 1/m\\ 0 & 0 & -m\omega^2 & -\lambda \end{pmatrix},$$
 (29)

$$D = \begin{pmatrix} D_{xx} & D_{xp_x} & D_{xy} & D_{xp_y} \\ D_{xp_x} & D_{p_xp_x} & D_{yp_x} & D_{p_xp_y} \\ D_{xy} & D_{yp_x} & D_{yy} & D_{yp_y} \\ D_{xp_y} & D_{p_xp_y} & D_{yp_y} & D_{p_yp_y} \end{pmatrix}.$$
 (30)

Introducing the notation  $\sigma(\infty) \equiv \lim_{t\to\infty} \sigma(t)$ , the time-dependent solution of Eq. (28) is given by [14]

$$\sigma(t) = M(t)(\sigma(0) - \sigma(\infty))M^{\mathrm{T}}(t) + \sigma(\infty), \qquad (31)$$

where the matrix  $M(t) = \exp(Yt)$  has to fulfill the condition  $\lim_{t\to\infty} M(t) = 0$ . In order that this limit exists, Y must only have eigenvalues with negative real parts. The values at infinity are obtained from the equation

$$Y\sigma(\infty) + \sigma(\infty)Y^{\mathrm{T}} = -2D.$$
(32)

### 4 Dynamics of entanglement

The two-mode Gaussian state is entirely specified by its covariance matrix (26), which is a real, symmetric and positive matrix with the following block structure:

$$\sigma(t) = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}, \tag{33}$$

where A, B and C are  $2 \times 2$  matrices. Their entries are correlations of the canonical operators  $x, y, p_x$  and  $p_y$ ; A and B denote the symmetric covariance matrices for the individual reduced one-mode states, while the matrix C contains the cross-correlations between modes. The elements of the covariance matrix depend on Y and D and can be calculated from Eqs. (31), (32). Since the two oscillators are identical, it is natural to consider environments for which the two diagonal submatrices in Eq. (25) are equal,  $C_1 = C_2$ , and the matrix  $C_3$  is symmetric, so that in the following we take  $D_{xx} = D_{yy}, D_{xp_x} = D_{yp_y}, D_{p_xp_x} = D_{p_yp_y}, D_{xp_y} = D_{yp_x}$ . Then both unimodal covariance matrices are equal, A = B, and the entanglement matrix C is symmetric.

#### 4.1 Time evolution of entanglement

It is interesting that the general theory of open quantum systems allows couplings via the environment between uncoupled oscillators. According to the definitions of the environment parameters, the diffusion coefficients above can take non-zero values and therefore can simulate an interaction between the uncoupled oscillators. Consequently, the cross-correlations between modes can have non-zero values. In this case the Gaussian states with det  $C \ge 0$  are separable states, but for det C < 0 it may be possible that the states are entangled, as will be shown next.

In order to investigate whether an external environment can actually entangle the two independent systems, we use the partial transposition criterion [4, 5]: a state is entangled if and only if the operation of partial transposition does not preserve its positivity. For the particular case of Gaussian states, Simon [5] obtained the following necessary and sufficient criterion for separability:  $S \ge 0$ , where

$$S \equiv \det A \det B + (\frac{1}{4} - |\det C|)^2 - \text{Tr}[AJCJBJC^{\mathrm{T}}J] - \frac{1}{4}(\det A + \det B)$$
(34)

and J is the  $2 \times 2$  symplectic matrix

$$J = \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix}. \tag{35}$$

In the following we consider such environment diffusion coefficients, for which

$$m^2 \omega^2 D_{xx} = D_{p_x p_x}, \ D_{x p_x} = 0, \ m^2 \omega^2 D_{xy} = D_{p_x p_y}.$$
 (36)

This corresponds to the case when the asymptotic state is a Gibbs state [12].

In order to describe the dynamics of entanglement, we have to analyze the time evolution of the Simon function S(t) (34). We consider two cases, according to the type of the initial Gaussian state: separable or entangled.

1) To illustrate a possible generation of the entanglement, we represent in Figure 1 the function S(t) versus time t and diffusion coefficient  $D_{xp_y} \equiv d$  for a separable initial Gaussian state with initial correlations  $\sigma_{xx}(0) = 1$ ,  $\sigma_{p_xp_x}(0) = 1/2$ ,  $\sigma_{xp_x}(0) = 0$ ,  $\sigma_{xy}(0) = 0$ ,  $\sigma_{p_xp_y}(0) = 0$ ,  $\sigma_{xp_y}(0) = 0$ . We notice that, according to Peres-Simon criterion, for relatively small values of the coefficient d, the initial separable state remains separable for all times. For larger values of d, at some finite moment of time, when S(t) becomes negative, the state becomes entangled. In some cases the entanglement is only temporarily generated, that is the state becomes again separable. After that, at a certain moment of time, one can notice again a revival of entanglement. In these cases the generated entangled state remains entangled forever, including the asymptotic final state.

2) The evolution of an entangled initial state is illustrated in Figure 2, where we represent the function S(t) versus time and diffusion coefficient d for an initial entangled Gaussian state with initial correlations  $\sigma_{xx}(0) = 1$ ,  $\sigma_{p_xp_x}(0) = 1/2$ ,  $\sigma_{xp_x}(0) = 0$ ,  $\sigma_{xy}(0) = 1/2$ ,  $\sigma_{p_xp_y}(0) = -1/2$ ,  $\sigma_{xp_y}(0) = 0$ . We notice that for relatively small



Figure 1: Simon separability function S versus time t and environment coefficient d, for  $\lambda = 0.2$ , D = 0.115 and a separable initial Gaussian state with initial correlations  $\sigma_{xx}(0) = 1$ ,  $\sigma_{p_xp_x}(0) = 1/2$ ,  $\sigma_{xp_x}(0) = \sigma_{xy}(0) = \sigma_{p_xp_y}(0) = \sigma_{xp_y}(0) = 0$ . We have taken  $m = \omega = \hbar = 1$ .

values of d, at some finite moment of time S(t) takes non-negative values and therefore the state becomes separable. This is the so-called phenomenon of entanglement sudden death. Depending on the values of the coefficient d, it is also possible to have a repeated collapse and revival of the entanglement. One can also show that for relatively large values of the coefficients  $D_{xx}$  and d, the initial entangled state remains entangled for all times.

#### 4.2 Asymptotic entanglement

On general grounds, one expects that the effects of decoherence, counteracting entanglement production, is dominant in the long-time regime, so that no quantum correlation (entanglement) is expected to be left at infinity. Nevertheless, there are situations in which the environment allows the existence of entangled asymptotic equilibrium states. From Eq. (32) we obtain the following elements of the asymptotic entanglement matrix  $C(\infty)$ :

$$\sigma_{xy}(\infty) = \frac{m^2 (2\lambda^2 + \omega^2) D_{xy} + 2m\lambda D_{xp_y} + D_{p_x p_y}}{2m^2 \lambda (\lambda^2 + \omega^2)},$$
(37)

$$\sigma_{xp_y}(\infty) = \sigma_{yp_x}(\infty) = \frac{-m^2 \omega^2 D_{xy} + 2m\lambda D_{xp_y} + D_{p_x p_y}}{2m(\lambda^2 + \omega^2)},$$
(38)



Figure 2: Same as in Fig. 1, for an entangled initial Gaussian state with initial correlations  $\sigma_{xx}(0) = 1$ ,  $\sigma_{p_xp_x}(0) = 1/2$ ,  $\sigma_{xp_x}(0) = 0$ ,  $\sigma_{xy}(0) = 1/2$ ,  $\sigma_{p_xp_y}(0) = -1/2$ ,  $\sigma_{xp_y}(0) = 0$ .

$$\sigma_{p_x p_y}(\infty) = \frac{m^2 \omega^4 D_{xy} - 2m\omega^2 \lambda D_{xp_y} + (2\lambda^2 + \omega^2) D_{p_x p_y}}{2\lambda(\lambda^2 + \omega^2)}.$$
(39)

The elements of matrices  $A(\infty)$  and  $B(\infty)$  are obtained by putting x = y in the previous expressions. We calculate the determinant of the entanglement matrix and obtain:

$$\det C(\infty) = \frac{1}{4\lambda^2(\lambda^2 + \omega^2)} \times \left[ (m\omega^2 D_{xy} + \frac{1}{m} D_{p_x p_y})^2 + 4\lambda^2 (D_{xy} D_{p_x p_y} - D_{x p_y}^2) \right].$$
(40)

With the chosen coefficients (36), the Simon expression (34) takes the following form in the limit of large times:

$$S(\infty) = \left(\frac{m^2\omega^2(D_{xx}^2 - D_{xy}^2)}{\lambda^2} + \frac{D_{xp_y}^2}{\lambda^2 + \omega^2} - \frac{1}{4}\right)^2 - 4\frac{m^2\omega^2 D_{xx}^2 D_{xp_y}^2}{\lambda^2(\lambda^2 + \omega^2)}.$$
 (41)

For environments characterized by such coefficients that the expression  $S(\infty)$  (41) is strictly negative, the asymptotic final state is entangled. Just to give an example, without altering the general features of the system, we consider the particular case of  $D_{xy} = 0$ . Then we obtain that  $S(\infty) < 0$ , i.e. the asymptotic final state is entangled, for the following range of values of the coefficient  $D_{xp_y}$  characterizing the environment [15, 16]:

$$\frac{m\omega D_{xx}}{\lambda} - \frac{1}{2} < \frac{D_{xp_y}}{\sqrt{\lambda^2 + \omega^2}} < \frac{m\omega D_{xx}}{\lambda} + \frac{1}{2},\tag{42}$$



Figure 3: Logarithmic negativity L versus time t and environment coefficient d, for  $\lambda = 0.2, D = 0.115$  and a separable initial Gaussian state with initial correlations  $\sigma_{xx}(0) = 1, \ \sigma_{p_xp_x}(0) = 1/2, \ \sigma_{xp_x}(0) = \sigma_{xy}(0) = \sigma_{p_xp_y}(0) = \sigma_{xp_y}(0) = 0$ . We have taken  $m = \omega = \hbar = 1$ .

where the diffusion coefficient  $D_{xx}$  satisfies the condition  $m\omega D_{xx}/\lambda \ge 1/2$ , equivalent with the unimodal uncertainty relation. We remind that, according to inequalities (24), the coefficients have to fulfill also the constraint  $D_{xx} \ge D_{xp_y}$ . If the coefficients do not fulfil the inequalities (42), then  $S(\infty) \ge 0$  and the asymptotic state of the considered system is separable. These results show that, irrespective of the initial conditions, we can obtain either an separable or an inseparable asymptotic entangled state, for a suitable choice of the diffusion and dissipation coefficients.

#### 4.3 Logarithmic negativity

We apply the measure of entanglement based on negative eigenvalues of the partial transpose of the subsystem density matrix. For a Gaussian density operator, the negativity is completely defined by the symplectic spectrum of the partial transpose of the covariance matrix. The logarithmic negativity  $L(t) = -\frac{1}{2}\log_2[4f(\sigma(t))]$  determines the strength of entanglement for L(t) > 0. If  $L(t) \leq 0$ , then the state is separable. Here

$$f(\sigma(t)) = \frac{1}{2} (\det A + \det B) - \det C$$
$$-\left(\left[\frac{1}{2} (\det A + \det B) - \det C\right]^2 - \det \sigma(t)\right)^{1/2}.$$
(43)



Figure 4: Same as in Fig. 3, for an entangled initial Gaussian state with initial correlations  $\sigma_{xx}(0) = 1$ ,  $\sigma_{p_x p_x}(0) = 1/2$ ,  $\sigma_{xp_x}(0) = 0$ ,  $\sigma_{xy}(0) = 1/2$ ,  $\sigma_{p_x p_y}(0) = -1/2$ ,  $\sigma_{xp_y}(0) = 0$ .

In Figures 3 and 4 we represent the logarithmic negativity L(t) versus time t and diffusion coefficient  $D_{xp_y} \equiv d$  for the two types of the initial Gaussian state, separable or entangled, with the same initial correlations, previously considered when we analyzed the time evolution of the Simon function S(t). As expected, we remark that the logarithmic negativity has a behaviour similar to that one of the Simon function in what concerns the characteristics of the state of being separable or entangled. Depending on the values of the environment coefficients, the initial state can preserve for all times its initial property – separable or entangled, but we can also notice the generation of entanglement or the collapse of entanglement (entanglement sudden death) at those finite moments of time when the logarithmic negativity L(t) reaches zero value. One can also observe a repeated collapse and revival of the entanglement. In the case of an entangled initial state, the logarithmic negativity is a fluctuating function and decreases asymptotically in time.

In our case the asymptotic logarithmic negativity has the form

$$L(\infty) = -\log_2 \left[ 2 \left| \frac{m\omega D_{xx}}{\lambda} - \frac{D_{xp_y}}{\sqrt{\lambda^2 + \omega^2}} \right| \right].$$
(44)

It depends only on the diffusion and dissipation coefficients characterizing the environment and does not depend on the initial Gaussian state. One can easily see that the double inequality (42), determining the existence of asymptotic entangled states  $(S(\infty) < 0)$  is equivalent with the condition of positivity of the expression (44) of the logarithmic negativity,  $L(\infty) > 0$ .

### 5 Summary

We have given a brief review of the theory of open quantum systems based on completely positive quantum dynamical semigroups and mentioned the necessity of the complete positivity for the existence of entangled states of systems interacting with an external environment. In the framework of this theory we investigated the dynamics of the quantum entanglement for a subsystem composed of two uncoupled identical harmonic oscillators interacting with a common environment.

By using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states, we have described the generation and evolution of entanglement in terms of the covariance matrix for an arbitrary Gaussian input state. For some values of diffusion and dissipation coefficients describing the environment, the state keeps for all times its initial type: separable or entangled. In other cases, entanglement generation or entanglement suppression (entanglement sudden death) take place or even one can notice a repeated collapse and revival of entanglement. We have also shown that, independent of the type of the initial state, for certain classes of environments the initial state evolves asymptotically to an equilibrium state which is entangled, while for other values of the coefficients describing the environment, the asymptotic state is separable. We described also the time evolution of the logarithmic negativity, which characterizes the degree of entanglement of the quantum state.

The existence of quantum correlations between the two considered harmonic oscillators interacting with a common environment is the result of the competition between entanglement and quantum decoherence. From the formal point of view, the generation of entanglement or its suppression (entanglement sudden death) correspond to the finite time vanishing of the Simon separability function or, respectively, of the logarithmic negativity. Presently there is a large debate relative to the physical interpretation existing behind these fascinating phenomena. Due to the increased interest manifested towards the continuous variables approach [17] to quantum information theory, these results, in particular the possibility of maintaining a bipartite entanglement in a diffusive-dissipative environment for asymptotic long times, might be useful in controlling the entanglement in open systems and also for applications in the field of quantum information processing and communication.

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