Remarks on the structure of Clifford quantum cellular automata

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Abstract. We report here on the structure of reversible quantum cellular automata with the additional restriction that these are also Clifford operations. This means that tensor products of Weyl operators (projective representation of a finite abelian symplectic group) are mapped to multiples of tensor products of Weyl operators. Therefore Clifford quantum cellular automata are induced by symplectic cellular automata in phase space. We characterize these symplectic cellular automata and find that all possible local rules must be, up to some global shift, reflection invariant with respect to the origin. In the one dimensional case we also find that all 1D Clifford quantum cellular automata are generated by a few elementary operations.

1. Introduction

A standard modeling technique for various complex systems are cellular automata. In fact they are ideally suited for models of diverse phenomena as coffee percolation, highway traffic and oil extraction from porous media. Cellular automata also provide an abstract computational model that can simulate Turing machines, and even explicit simple automata such as Conway's life game have been shown to support universal computation [2]. Quantum cellular automata provide model for analyzing quantum computational processes and quantum computational complexity. In his famous paper [3], Feynman discusses this idea in order to obtain a model for quantum computing which can be more powerful than a classical computer. Quantum cellular automata also play a role in view of quantum computational complexity. This aspect has been studied by Bernstein and Vazirani [1] by using the concept of quantum Turing machines. Watrous (see e.g. [11]) continued this discussion by relating quantum Turing machines to quantum cellular automata. Last but not least, there may also be interesting applications besides quantum information theory, for instance, quantum cellular automata could serve as ultra-violet regularized quantum field theories.

In this paper, we are concerned with the "quantized version" of cellular automata based on the concepts that are outlined in the article of Schumacher and Werner [9].

2. The general concept

In order to motivate the concept for "quantum" cellular automata, we briefly review here the idea of reversible classical cellular automata from an algebraic point of view. In this context a complex classical system consists of single cells that are labeled by a countable set X. For many applications this set is given by a regular cubical lattice $X = \mathbb{Z}^d$ of dimension d. In our later discussion, we restrict our considerations to this case. To each cell $x \in X$ a finite set of states Q is associated, so that the classical configuration space of the total system is given by all functions from X into the set Q. A further important aspect is concerned with the local action of cellular automata which means that when the automaton is applied, the updated state of a single cell $x \in X$ only depends on the states of a finite set $N(x) \subset X$ of "neighboring" cells. In the case of a regular cubic lattice $X = \mathbb{Z}^d$, the neighboring scheme is usually chosen to be translationally invariant. Here one takes some finite set $N \subset \mathbb{Z}^d$ and defines the neighbors of a cell x according to N(x) = N + x. In mathematical terms, a reversible classical quantum cellular automata is given a bijective map $T: Q^X \to Q^X$ such that for each $x \in X$ there exists a function $T_x: Q^{N(x)} \to Q$ which satisfies $T(q)(x) = T_x(q|_{N(x)})$. Here q is a function in Q^X and $q|_{N(x)}$ denotes the restriction of q to the neighborhood N(x).

In the case of a regular lattice structure $X = \mathbb{Z}^d$, translation invariance is an additional requirement for the automaton T. The translation group \mathbb{Z}^d acts naturally on the configuration space $Q^{\mathbb{Z}^d}$ according to $(\tau_x q)(y) = q(y - x)$. Translation invariance for T means that T commutes with all translations τ_x . As a consequence the cellular automaton T is completely determined by its "local rule" $T_0: Q^N \to Q$. Namely, the local rule T_x at any cell x can be calculated from the local rule T_0 at the origin by $T_x(q|_{N+x}) = T_0(\tau_{-x}q|_N)$.

In order to motivate the quantized concept, we reformulate the classical quantum cellular automata algebraically. If Q is a finite set, then the configuration space Q^X is compact in the Tychonov topology. The observable algebra of the classical system is given by the abelian C*-algebra of continuous functions $C(Q^X)$ which is canonically isomorphic to the tensor product $C(Q^X) = \bigotimes_{x \in X} C(Q)$, where $C(Q) \cong \mathbb{C}^Q$ is the abelian C*-algebra of functions on the single cell configuration space Q. In this picture, a reversible cellular automaton T induces an automorphism α on $C(Q^X)$ by the pullback $\alpha(f)(q) = f(T(q))$.

The functions in $C(Q^X)$ with values in the interval [0, 1] can be regarded as classical observables. Such an observable is localized in a subset $U \subset X$ if the corresponding function f only depends on the restriction $q|_U$ of a classical configuration q. With abuse of notation we express this fact as $f(q) = f(q|_U)$. Thus if we restrict the automorphism α to observables that are localized in a single cell xthe resulting observable is localized in the neighbor hood of x. This can be verified as follows: If an observable f is localized at x, then the value f(q) = f(q(x)) only depends on q(x). The application of the automorphism α therefore gives $\alpha(f)(q(x)) = f(T(q)(x)) = f(T_x(q|_{N(x)}))$, where T_x is the local rule at x. Thus the automorphism α propagates the localization region of an observable only into its neighboring cells.

The basic idea to quantize the concept of cellular automata is to replace the classical systems by quantum systems, i.e. the abelian C*-algebra $C(Q^X) = \bigotimes_{x \in X} C(Q)$ is replaced be a non-abelian one. The system under consideration is now described by a tensor product

$$\mathfrak{A} = \bigotimes_{x \in X} \mathfrak{A}(x) \tag{1}$$

where to each cell x there is a (finite dimensional) C*-algebra $\mathfrak{A}(x)$ associated with. We require here, that to each cell x an isomorphic copy of a fixed C*-algebra is assigned, i.e. $\mathfrak{A}(x) \cong \mathfrak{A}_0$. Since X can be any countable set, we have to deal here with infinite tensor products. However, this is well defined in terms of the so called inductive limit. The algebra \mathfrak{A} is usually called the "quasi-local" algebra of observables. For each finite subset $U \subset X$ there is a natural "local" subalgebra $\mathfrak{A}(U) = \bigotimes_{x \in U} \mathfrak{A}(x)$ of the quasi local algebra. The operators $a \in \mathfrak{A}(U)$ are identified with operators in $\mathfrak{A}(X)$ by filling the remaining tensor positions $X \setminus U$ with the unit operator. The concept of a reversible quantum cellular automaton is defined as follows:

DEFINITION 1. A reversible quantum cellular automata (QCA) is a *-automorphism α of the quasi local algebra \mathfrak{A} that fulfills the "locality condition": For each cell $x \in X$ and for each operator $A \in \mathfrak{A}(x)$ the operator $\alpha(A) \in \mathfrak{A}(N(x))$ is localized in the neighborhood of x.

As Definition 2. indicates, the concept of a QCA works for any type of lattice X with an appropriate neighborhood scheme where the locality requirement is the essential ingredient. In the subsequent analysis we focus on regular cubic lattices $X = \mathbb{Z}^d$ only. In this case we have a natural action of the lattice translation group \mathbb{Z}^d by automorphisms τ_x on the quasilocal algebra where τ_x is determined by

$$\tau_x \left(\bigotimes_{y \in X} A_y \right) = \bigotimes_{y \in X} A_{y-x} \tag{2}$$

with $A_x \in \mathfrak{A}(x)$. We now consider those QCAs that respect the symmetry of lattice translations.

DEFINITION 2. A translationally invariant reversible QCA is a reversible QCA α that commutes with the lattice translation group: $\alpha \circ \tau_x = \tau_x \circ \alpha$.

4

In the subsequent we always refere to the translationally invariant situation. The translation symmetry can be exploited for the structural analysis of QCAs. In fact, a translationally invariant QCA is completely determined by its local rule at the origin. Recall that the local rule α_0 at x = 0 is the restriction of the QCA α to the algebra $\mathfrak{A}(0)$. Due to the locality condition, there is a finite subset $N \subset \mathbb{Z}^d$ such that $\alpha(\mathfrak{A}(0)) \subset \mathfrak{A}(N)$. To be compatible with translation invariance, the neighborhood scheme can be chosen such that N(x) = N + x and the "global rule", which is just the automorphism α , can be expressed in terms of the local rule α_0 by

$$\alpha\left(\bigotimes_{x\in X} A_x\right) = \prod_{x\in X} \tau_x \alpha_0 \tau_{-x}(A_x) .$$
(3)

Thus a translationally invariant QCAs can be described in terms of its local rule only. In particular, if the single cell algebras are finite dimensional, the consruction of the QCA is a problem in finite dimensions.

A strategy for constructing a QCA is based on finding a valid local rule. One has to choose a *-homomorphism $\alpha_0: \mathfrak{A}(0) \to \mathfrak{A}(N)$ and has to check the commutator condition $[\tau_x(\alpha_0(A)), \alpha_0(B)] = 0$ for all $A, B \in \mathfrak{A}(0)$ and for all $x \in X$ with $N \cap N + x \neq \emptyset$. Assuming that the single cell algebra $\mathfrak{A}(0)$ is finite dimensional, there are only finitely many conditions to be tested. For a comprehensive review on this issue, we refere here to the work of Schumacher and Werner [9].

Although there are only finitely many conditions to check, a general systematic classification of QCAs is a highly non-trivial and still unsolved task. But there are particular classes of QCAs for which a complete and explicit classification is possible, as the class of Clifford (or quasifree) quantum cellular automata which we review here in the following. The results that we are presenting here are based on our previous article [8].

3. Clifford quantum cellular automata

To explain the concept of Clifford quantum cellular, we consider a regular cubic lattice \mathbb{Z}^d . To each cell we associate a full matrix algebra $\mathfrak{A}(x) = M_p(\mathbb{C})$ where pis a prime number. Moreover we choose a basis of Weyl operators in $M_p(\mathbb{C})$. These operators are generalizations of the Pauli operators and are constructed by shift and multiplier unitaries. To be more precise, we consider an orthonormal basis $|q\rangle$ of the Hilbert space \mathbb{C}^p that is labled by elements q of the finite field $\mathbb{F}_p = \mathbb{Z}_p$. One way to define the Weyl operators is to determine its action on the basis $|q\rangle$ according to

$$\mathbf{w}(\xi)|q\rangle = \mathbf{w}(\xi_+, \xi_-)|q\rangle = \varepsilon_p^{\xi_+ q}|q + \xi_-\rangle \tag{4}$$

where ε_p is the *p*th root of unity. As a consequence, the Weyl operators fulfill the relation

$$\mathbf{w}(\xi + \eta) = \varepsilon_p^{\xi - \eta} \mathbf{w}(\xi) \mathbf{w}(\eta) \tag{5}$$

which shows that the Weyl operators form a unitary projective representation of the additive group \mathbb{F}_p .

For the special case p = 2, which corresponds to qubits, the corresponding Weyl operators are related to the Pauli matrices X, Y, Z by $X = \mathbf{w}(0, 1), Z = \mathbf{w}(1, 0)$ and $Y = -\mathbf{i}\mathbf{w}(1, 1)$.

We are now concerned with the quasi local algebra \mathfrak{A} which is given by the infinite tensor product of single cell algebras $\mathfrak{A}(x) = M_p(\mathbb{C})$ over the regular lattice \mathbb{Z}^d . The Weyl operators for the infinite system are given by tensor products

$$\mathbf{w}(\xi) := \bigotimes_{x \in \mathbb{Z}} \mathbf{w}(\xi(x)) \tag{6}$$

where the "phase space" vector ξ is a function from the lattice \mathbb{Z}^d to the vector space \mathbb{F}_p^2 with finite support. Note that the finite support condition guarantees that only finitely many tensor factors are different from the identity which implies that $\mathbf{w}(\xi)$ is a well defined unitary operator that belongs to quasilocal algebra. Moreover, the complex linear hull of the Weyl operators is norm dense subalgebra of \mathfrak{A} . I this sense, the Weyl operators form a "basis" for the quasi local algebra.

Obviously, the Weyl operators of the infinite system fulfill the relation

$$\mathbf{w}(\xi+\eta) = \epsilon_p^{\beta(\xi,\eta)} \mathbf{w}(\xi) \mathbf{w}(\eta) \quad \text{with} \quad \beta(\xi,\eta) = \sum_{x \in \mathbb{Z}} \xi_+(x) \eta_-(x) \;. \tag{7}$$

which implies the commutation relation

$$\mathbf{w}(\eta)\mathbf{w}(\xi) = \epsilon_p^{\sigma(\xi,\eta)}\mathbf{w}(\xi)\mathbf{w}(\eta) \text{ with } \sigma(\xi,\eta) = \beta(\xi,\eta) - \beta(\eta,\xi) .$$
(8)

Note that the symplectic form σ for the infinite system is well defined since it is evaluated only for function with finite support. This relations justifies to interprete the functions ξ as vectors in a discrete phase space — denoted by $\Xi_{p,d}$ in the following — with symplectic form σ . We are now prepared to give a precise definition of Clifford quantum cellular automata.

DEFINITION 3. A Clifford quantum cellular automata (CQCA) α is a translationally invariant reversible QCA which maps Weyl operators to multiples of Weyl operators. Thus there exists a function $\mathbf{S}: \Xi_{p,d} \to \Xi_{p,d}$ as well as a phase-valued function $\varphi: \Xi_{p,d} \mapsto \mathrm{U}(1) = \{z \in \mathbb{C} | |z| = 1\}$ such that

$$\alpha(\mathbf{w}(\xi)) = \varphi(\xi)\mathbf{w}(\mathbf{S}\xi) \tag{9}$$

holds for all phase space vectors ξ .

A simple CQCA is given by lattice translations. The lattice translations act on the phase space vectors ξ by just translating the function $(\tau_y \xi)(x) = \xi(x - y)$. With

abuse of notation we use the same symbol for the action on phase space as for the action on the quasilocal algebra. By construction the covariance relation

$$\tau_x(\mathbf{w}(\xi)) = \mathbf{w}(\tau_x \xi) \tag{10}$$

follows immediately. Hence, the lattice translations τ_x are CQCAs in the sense of the definition given above.

As we will see, the translation invariance together with the condition to map Weyl operators to multiple of Weyl operators is sufficient to determine a CQCA. This means, that the locality is a consequence of these conditions. To deal with the translation symmetry in an appropriate way, we identify the phase space $\Xi p, d = D_{p,d}^2$ as a two dimensional module over the ring $D_{p,d}$ of functions from the lattice \mathbb{Z}^d into the finite field \mathbb{F}_p having finite support. The multiplication in the ring is the convolution of functions which is given by

$$f \star g = \sum_{x} f(x)\tau_{x}g.$$
(11)

We are now prepared to state the first structure theorem on CQCAs. We refere here the reader to our article [8] for a complete discussion of the proof in which uses techniques from the theory of projective representations of symplectic abelian groups (see e.g. [12]) as well as results from the theory of covariant completely positive maps [10, 5].

THEOREM 4. For each CQCA α , there exists a two-by-two matrix $\mathbf{s} \in M_2(D_{p,q})$ with entrees in the ring $D_{p,d}$ and a translationally invariant phase valued function $\varphi: D_{n,d}^2 \to U(1)$ such that

$$\alpha(\mathbf{w}(\xi)) = \varphi(\xi)\mathbf{w}(\mathbf{s} \star \xi) \tag{12}$$

and φ fulfills the cocycle condition

$$\varphi(\xi + \eta) = \epsilon_p^{\beta(\xi,\eta) - \beta(\mathbf{s} \star \xi, \mathbf{s} \star \eta)} \varphi(\xi) \varphi(\eta) .$$
(13)

Moreover, the map $\mathbf{s} \star$ *preserves the symplectic form* σ *, i.e.* $\sigma(\mathbf{s} \star \xi, \mathbf{s} \star \eta) = \sigma(\xi, \eta)$ *.*

We sketch here just the basic idea of the proof: It follows from the Weyl relations that each automorphism α that maps Weyl operator to multiples of Weyl operators according to (9) induces a \mathbb{F}_p -linear map **S** on phase space that preserves the symplectic form. Moreover, the condition to be an automorphism implies that the phase valued function φ fulfills (13) where, for this moment, we have to repace the operator s* by **S**.

By taking advantage of the translation invariance, the phase valued function φ is translationally invariant and the symplectic map **S** commutes with the lattice

6

translations, it follows that **S** is given by the convolution s^* with a matrix-valued function with finite support. Note that a two-by-two matrix **s** with entries in the ring $D_{p,d}$ can equivalently be seen as a function that maps a lattice site $x \in \mathbb{Z}^d$ to a two-by-two matrix with entries in the finite field \mathbb{F}_p . The convolution with a phase space ξ vector is given by $(\mathbf{s} \star \xi)(x) = \sum_y \mathbf{s}(y) \cdot \xi(x - y)$ where \cdot is usual matrix multiplication. The localization region of a Weyl operator $\mathbf{w}(\xi)$ is just the support of the function ξ . If the support of ξ is just a single site x, then $\mathbf{s} \star \xi$ has support in N + x, where N is the support of \mathbf{s} . Therefore, the application of the corresponding CQCA yields an operator $\alpha(\mathbf{w}(\xi)) = \varphi(\xi)\mathbf{w}(\mathbf{s}\star\xi)$ which is localized in N + x. As a consequence, the support of \mathbf{s} determines the neighborhood scheme of the QCA.

We also shown in [8] that each matrix-valued function s for which the convolution s* preserves the symplectic form σ a phase valued function φ can be found, such that the equation (12) defines a CQCA. Therefore, the classification of CQCAs is equivalent to characterize all two-by-two matrices $s \in M_2(D_{p,d})$ with entries in the ring $D_{p,d}$ whose convolution s* is symplectic. In accordance with [8], we call the convolution s* a "symplectic cellular automaton (SCA)".

4. On the structure of Clifford quantum cellular automata

For the further analysis of CQCAs, we have a closer look at the ring and module structure of the phase space $\Xi_{p,d} = D_{p,d}^2$. As already mentioned $\Xi_{p,d}$ is a two dimensional module over the ring $D_{p,d}$ where the product is the convolution. A function $f \in D_{p,d}$ acts on a phase space vector by $f \star \xi = f \star (\xi_+, \xi_-) = (f \star \xi_+, f \star \xi_-)$. A symplectic cellular automaton (SCA), which induces a CQCA, is then a module homomorphism. Namely, since the convolution in $D_{p,d}$ is commutative, we observe that $\mathbf{s} \star f \star \xi = f \star \mathbf{s} \star \xi$.

To take advantage of the translation symmetry in an appropriate manner, we have introduced the "algebraic Fourier transform", which identifies the ring $D_{p,d}$ with the commutative ring of Laurent polynomials

$$\hat{D}_{p,d} = \mathbb{F}_p[u_1, u_2, \cdots, u_d, u_1^{-1}, \cdots, u_d^{-1}]$$
(14)

generated by the variables u_1, \dots, u_d and its inverses $u_1^{-1}, \dots, u_d^{-1}$. For a function $f \in D_{p,d}$ the corresponding Laurent-polynomial is simply given by

$$\hat{f} = \sum_{x} f(x)u^x \tag{15}$$

where we write $u^x = u_1^{x_1} u_2^{x_2} \cdots u_d^{x_d}$ for a handy notation. To view the elements in $D_{p,d}$ as formal polynomials, gives us a convenient book-keeping at hand. Namely, the convolution turns into a product of polynomials, i.e. for two functions the identity

$$\widehat{f} \star \widehat{h} = \widehat{f}\widehat{h} \tag{16}$$

holds. We mention here that the ring $\hat{D}_{p,d}$ is a "divison ring" which means that fh = 0 implies that either f = 0 or h = 0. Moreover, the only invertible elements in $\hat{D}_{p,d}$ are the monomials u^x with $x \in \mathbb{Z}^d$.

With help of this ring isomorphism $f \mapsto \hat{f}$, the phase space can be identified with $\hat{D}_{p,d}^2$ and, since a SCA is a module homomorphism, its Fourier transform just acts by matrix multiplication. To be more precise, a SQCA is given by a two-by-two matrix \hat{s} with entries in the polynom ring $\hat{D}_{p,d}$ acting on a phase space vector by

$$\widehat{\mathbf{s} \star \xi} = \begin{pmatrix} \hat{\mathbf{s}}_{++} & \hat{\mathbf{s}}_{+-} \\ \hat{\mathbf{s}}_{-+} & \hat{\mathbf{s}}_{--} \end{pmatrix} \begin{pmatrix} \hat{\xi}_{+} \\ \hat{\xi}_{-} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{s}}_{++}\hat{\xi}_{+} + \hat{\mathbf{s}}_{+-}\hat{\xi}_{-} \\ \hat{\mathbf{s}}_{-+}\hat{\xi}_{+} + \hat{\mathbf{s}}_{--}\hat{\xi}_{-} \end{pmatrix} .$$
(17)

After applying the algebraic Fourier transform, the symplectic form is a $\hat{D}_{p,d}$ -valued bilinear from Σ on $\hat{D}_{p,d}^2$ that can be calculated according to

$$\Sigma(\xi,\eta) = \overline{\xi_+}\eta_- - \overline{\xi_-}\eta_+ , \qquad (18)$$

where $f \mapsto \overline{f}$ is the involution on the ring $\hat{D}_{p,d}$ which replaces in the polynomial f = f(u) the variable u_k by its inverse u_k^{-1} . In the lattice space, this corresponds to a reflection at the origin. The form Σ is related to the underlying symplectic form σ by

$$\Sigma(\xi,\eta) = \sum_{x} \sigma(\check{\xi}, \tau_x \check{\eta}) u^x \tag{19}$$

where $f \mapsto \check{f}$ is the inverse algebraic Fourier transform sending a polynomial f to a function \check{f} on the lattice. It is not difficult to observe, that Σ is a module homomorphism in the second argument, i.e. $\Sigma(\xi, f\eta) = \Sigma(\xi, \eta)f$ and that it fulfills the relation $\Sigma(\xi, \eta) = -\overline{\Sigma(\eta, \xi)} = -\Sigma(\overline{\eta}, \overline{\xi})$. Thus Σ is antisymmetric for reflection invariant polynomials. In this context, a helpful lemma for the characterization of SCAs is the following:

LEMMA 5. A two-by-two matrix $\mathbf{s} \in M_2(\hat{D}_{p,d})$ with entries in the polynom ring $\hat{D}_{p,d}$ is a symplectic cellular automaton, if and only if, it preserves the form Σ , i.e. the identity $\Sigma(\mathbf{s}\xi, \mathbf{s}\eta) = \Sigma(\xi, \eta)$ holds.

According to this lemma, the characterization of CQCAs (hence SCAs) reduces to the problem of finding two-by-two matrices with entries in the ring $\hat{D}_{p,d}$ which preserve Σ . Since we deal here with module homomorphism — a linear structure over the ring $\hat{D}_{p,d}$ — we have reduced a problem in infinitely many degrees of freedom to an effectively two-dimensional problem.

There is an important subring in $D_{p,d}$, which we denote here by $P_{p,d}$, which consists of all polynomials that are invariant under the reflection $u_k \mapsto u_k^{-1}$ which means $f = f(u) = f(u^{-1}) = \overline{f}$. For d = 1, the corresponding function \widetilde{f} in lattice

space is then given by a palindrome string $\check{f} = (q_n q_{n-1} \cdots q_1 q_0 q_1 q_2 \cdots q_n)$ starting at the left boundary of the support x = -n and ending at the right boundary x = n. For this reason, we call $P_{p,d}$ the polynom subring of palindromes in $\hat{D}_{p,d}$. If we look at the properties of the form, Σ , we see that it is a non-degenerate antisymmetric $P_{p,d}$ -bilinear form on $P_{p,d}^2$. In analogy, that the symplectic group of \mathbb{C}^2 is given by the special linear group $SL(2, \mathbb{C})$ a first guess is, that the group of SCA is given by all two-by-two matrices with entries in the palindrome subring $P_{p,d}$ having ringvalued determinant equal to one. Indeed, if we choose a matrix $\mathbf{s} \in SL(2, P_{p,d})$, then we observe by a straight forward calculation that \mathbf{s} preserves the form Σ . If we multiply \mathbf{s} with a monomial u^a , which corresponds to a lattice translation by $a \in \mathbb{Z}^d$, then we observe that $\Sigma(u^a \mathbf{s}\xi, u^a \mathbf{s}\eta) = \Sigma(\mathbf{s}\xi, \mathbf{s}\eta)u^{-a}u^a = \Sigma(\xi, \eta)$. Thus if \mathbf{s} is a SCA then $u^a \mathbf{s}$ is a SCA too. Indeed, all SCAs are of this type. The precise statement which we have established in [8, Theorem 3.4] is the following:

THEOREM 6. The group of Clifford quantum cellular automata acting on a ddimensional lattice with single cell algebras $M_p(\mathbb{C})$ is isomorphic to the direct product $\mathbb{Z}^d \times SL(2, P_{p,d})$ of the lattice translation group and the special linear group of two-by-two matrices with entries in in the palindrome subring $P_{p,d}$.

This theorem can be used to build up a simple cooking recipe for constructing CQCAs. Firstly, take two palindromes $f, h \in P_{p,d}$. Recall that palindromes are easy to get. Namely, for the case that g is not a palindrome you just make one by taking $h = g + \overline{g}$. Secondly, factorize the polynomial 1 - fg = f'h' in the subring $P_{p,d}$ into two palindromes f', h'. Finally, you get your CQCA by building the matrix

$$\mathbf{s} = \begin{pmatrix} f & f' \\ h' & h \end{pmatrix} \,. \tag{20}$$

For this type of recipe, to find all possible factorizations of the polynomial 1 - fh is the crucial problem which can be quite cumbersome. However, there is allways the trivial solution which is given by h' = 1 and f' = 1 - fh.

Concerning the factorization problem, at least for a one dimensional lattice d = 1 the situation can be tackled. Here one takes advantage of the fact that the ring $P_{1,p}$ is a so called "Euclidean ring" (see e.g. [6]) and an extended Euclidean algorithm for finding greatest common divisors can be applied. This yields in an even stronger classification result of one-dimensional CQCAs than provided by the dimension independent Theorem 4... We have shown the following [8, Theorem 3.11]:

THEOREM 7. Every Clifford quantum cellular automata s acting on a one dimensional lattice with single cell algebras $M_p(\mathbb{C})$ can be factorized into a product of a unique shift u^a and elementary CQCAs of the following two types: The first type is a shear transformation

$$\mathbf{g}_n = \begin{pmatrix} 1 & 0\\ u^{-n} + u^n & 1 \end{pmatrix}$$
(21)

depending on an integer $n \in \mathbb{N}$. The second type depends on a constant $c \in \mathbb{F}_p$ according to

$$\mathbf{f}_c = \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \,. \tag{22}$$

The automata \mathbf{f}_c are constant matrix valued polynomials which implies that they act on each lattice site independently. The have zero propagation speed since their neighborhood scheme only consits of the origin. On the other hand the propagation of the localization region of an observable is induced by the shear automata g_n . Their local rule propagate from the origin into the cells $\{-n, n\}$.

5. Concluding remarks

In this note, we have discussed some aspects on the structure of quantum cellular automata where we have mainly focused on our results on Clifford quantum cellular automata [8].

We have characterized the group of CQCAs in terms of symplectic cellular automata on a suitable phase space. With the help of the concept of algebraic Fourier transform, this phase space can be identified with two-dimensional vectors of Laurent-polynomials, and symplectic cellular automata can be described by two-by-two matrices with Laurent-polynomial entries. We have reported that these entries must be reflection invariant and that up to some global shift the determinant of the matrix must be one, so the group of CQCAs is isomorphic to the direct product of the lattice translation group with the special linear group of two-by-two matrices with reflection invariant polynomials as matrix elements.

Due to the specialty that for a 1D lattice we are faced with an Euclidean ring, each one-dimensional CQCA can be factorized into a product of elementary shear automata and local transforms.

Besides the core results, that we have presented here, there is a correspondence between 1D CQCAs and 1D translationally invariant stabilizer (graph) states (see e.g. [4, 7] for the notion of stabilizer (graph) states). For a fixed translationally invariant pure stabilizer state, which is in particular a product state, every other translationally invariant pure stabilizer state can be created by applying an appropriate CQCA.

A further natural question is concerned with lattices with periodic boundary conditions. Here the techniques from infinitely extended lattices can be applied to a certain extend. The technical problem is here, however, that the involved polynom ring is no longer a division ring.

10

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