# ON PURE QUASI QUANTUM QUADRATIC OPERATORS OF $\mathbb{M}_{2}(\mathbb{C})$ 

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#### Abstract

In the present paper we study quasi quantum quadratic operators (q.q.o) acting on the algebra of $2 \times 2$ matrices $\mathbb{M}_{2}(\mathbb{C})$. It is known that a channel is called pure if it sends pure states to pure ones. In this papers, we introduce a weaker condition, called $q$-purity, than purity of the channel. To study $q$-pure channels, we concentrate ourselves to quasi q.q.o. acting on $\mathbb{M}_{2}(\mathbb{C})$. We describe all trace-preserving quasi q.q.o. on $\mathbb{M}_{2}(\mathbb{C})$, which allowed us to prove that if a trace-preserving symmetric quasi q.q.o. such that the corresponding quadratic operator is linear, then its $q$-purity implies its positivity. If a symmetric quasi q.q.o. has a Haar state $\tau$, then its corresponding quadratic operator is nonlinear, and it is proved that such $q$-pure symmetric quasi q.q.o. cannot be positive. We think that such a result will allow to check whether a given mapping from $\mathbb{M}_{2}(\mathbb{C})$ to $\mathbb{M}_{2}(\mathbb{C}) \otimes \mathbb{M}_{2}(\mathbb{C})$ is pure or not. On the other hand, our study is related to construction of pure quantum nonlinear channels. Moreover, it is also considered that nonlinear dynamics associated with quasi pure q.q.o. may have differen kind of dynamics, i.e. it may behave chaotically or trivially, respectively.


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## 1. Introduction

It is known that entanglement is one of the essential features of quantum physics and is fundamental in modern quantum technologies [26]. One of the central problems in the entanglement theory is the discrimination between separable and entangled states. There are several tools which can be used for this purpose. The most general consists in applying the theory of linear positive maps [29]. In these studies one of the goal is to construct a map from the state space of a system to the state space of another system. In the literature on quantum information and communication systems, such a map is called a channel [26]. Note that the concept of state in a physical system is a powerful weapon to study the dynamical behavior of that system. One of the important class of channels is so-called pure ones, which map pure states to pure ones (see [1, 2]). For example, important examples of such kind of maps are conjugation of automorphisms of given algebra. But, if a channel acts from algebra to another one, then the description of pure channels is a tricky job. Therefore, it would be interesting characterize such kind of maps (or channels). Note that quantum mutual entropy of such kind of maps can be calculated easier way than others [27, 28].

In the present paper we are going to describe pure quasi quantum quadratic operators (see also [25]). On the other hand, such kind of operators define quadratic operators. We should stress that quadratic dynamical systems have been proved to be a rich source of analysis for the investigation of dynamical properties and modeling in different domains, such as population dynamics [5, 10, 12], physics [30, 34], economy [7], mathematics [13, 16, 35, 36]. The problem of studying the behavior of trajectories of quadratic stochastic operators was stated in [35]. The limit behavior and ergodic properties of trajectories of such operators were studied in [15, 16, 17, 19, 36]. However, such kind of operators do not cover the case of quantum systems. Therefore, in [8, 9] quantum quadratic operators acting on a von Neumann algebra were defined
and studied. Certain ergodic properties of such operators were studied in [21, 22]. In those papers basically dynamics of quadratic operators were defined according to some recurrent rule (an analog of Kolmogorov-Chapman equation) which makes a possibility to study asymptotic behaviors of such operators. However, with a given quadratic operator one can define also a non-linear operator whose dynamics (in non-commutative setting) is not well studied yet. Some class of such kind of operators defined on $M_{2}(\mathbb{C})$ has been studied in [24, 25]. Note that in [18] another construction of nonlinear quantum maps were suggested and some physical explanations of such nonlinear quantum dynamics were discussed. In all these investigations, the said quantum quadratic operators by definition are positive. But, in general, to study the nonlinear dynamics the positivity of the operator is strong condition. Therefore, in the present paper we are going to introduce a weaker than the positivity, and corresponding operators are called quasi quantum quadratic. In the paper we concentrate ourselves to trace-preserving operators acting on $\mathbb{M}_{2}(\mathbb{C})$. Each such kind of operator defines a quadratic operator acting on state space of $\mathbb{M}_{2}(\mathbb{C})$. It is known that a mapping is called pure if it sends pure states to pure ones. In this papers, we introduce a weaker condition, called $q$-purity, than purity of the mapping. To study $q$-pure channels, we concentrate ourselves to quasi q.q.o. acting on $\mathbb{M}_{2}(\mathbb{C})$. We call such an operator $q$-pure, if its corresponding quadratic operator maps pure state to pure ones. We first describe all trace-preserving quasi q.q.o. on $\mathbb{M}_{2}(\mathbb{C})$, which allowed us to describe all $q$-pure quadratic operators. Then we prove that if a trace-preserving symmetric quasi q.q.o. such that the corresponding quadratic operator is linear, then its $q$-purity implies its positivity. Moreover, if a symmetric quasi q.q.o. has a Haar state $\tau$, then its corresponding quadratic operator is nonlinear, and it is proved that such $q$-pure symmetric quasi q.q.o. cannot be positive. We think that such a result will allow to check whether a given mapping from $\mathbb{M}_{2}(\mathbb{C})$ to $\mathbb{M}_{2}(\mathbb{C}) \otimes \mathbb{M}_{2}(\mathbb{C})$ is pure or not. On the other hand, our study is related to construction of pure quantum nonlinear channels. Besides, it is also considered that nonlinear dynamics associated with quasi pure q.q.o. may have differen kind of dynamics, i.e. it may behave chaotically or trivially, respectively.

## 2. Preliminaries

Let $B(H)$ be the set of linear bounded operators from a complex Hilbert space $H$ to itself. By $B(H) \otimes B(H)$ we mean tensor product of $B(H)$ into itself. In the sequel $\mathbb{I}$ means an identity matrix. By $B(H)^{*}$ it is usually denoted the conjugate space of $B(H)$. We recall that a linear functional $\varphi \in B(H)^{*}$ is called positive if $\varphi(x) \geq 0$ whenever $x \geq 0$. The set of all positive linear functionals is denoted by $B(H)_{+}^{*}$. A positive functional $\varphi$ is called state if $\varphi(\mathbb{I})=1$. By $S(B(H))$ we denote the set of all states defined on $B(H)$.

Let $\Delta: B(H) \rightarrow B(H) \otimes B(H)$ be a linear operator. Then $\Delta$ defines a conjugate operator $\Delta^{*}:(B(H) \otimes B(H))^{*} \rightarrow B(H)^{*}$ by

$$
\Delta^{*}(f)(x)=f(\Delta x), f \in(B(H) \otimes B(H))^{*}, x \in B(H)
$$

One can define an operator $V_{\Delta}$ by

$$
V_{\Delta}(\varphi)=\Delta^{*}(\varphi \otimes \varphi), \varphi \in B(H)^{*}
$$

Let $U: B(H) \otimes B(H) \rightarrow B(H) \otimes B(H)$ be a linear operator such that $U(x \otimes y)=y \otimes x$ for all $x, y \in \mathbb{M}_{2}(\mathbb{C})$.

Definition 2.1. A linear operator $\Delta: B(H) \rightarrow B(H) \otimes B(H)$ is said to be
(a) - a quasi quantum quadratic operator (quasi q.q.o) if it is unital (i.e. $\Delta \mathbb{I}=\mathbb{I} \otimes \mathbb{1}$ ), *-preserving (i.e. $\left.\Delta\left(x^{*}\right)=\Delta(x)^{*}, \forall x \in B(H)\right)$ and

$$
V_{\Delta}(\varphi) \in B(H)_{+}^{*} \quad \text { whenever } \varphi \in B(H)_{+}^{*} ;
$$

(b) - a quantum quadratic operator (q.q.o.) if it is unital (i.e. $\Delta \mathbb{I}=\mathbb{I} \otimes \mathbb{I}$ ) and positive ( i.e. $\Delta x \geq 0$ whenever $x \geq 0$ );
(c) - a quantum convolution if it is a q.q.o. and satisfies coassociativity condition:

$$
(\Delta \otimes i d) \circ \Delta=(i d \otimes \Delta) \circ \Delta
$$

where $i d$ is the identity operator of $\mathbb{M}_{2}(\mathbb{C})$;
(d) - a symmetric if one has $U \Delta=\Delta$.

One can see that if $\Delta$ is q.q.o. then it is a quasi q.q.o. A state $h \in S(B(H))$ is called a Haar state for a quasi q.q.o. $\Delta$ if for every $x \in B(H)$ one has

$$
\begin{equation*}
(h \otimes i d) \circ \Delta(x)=(i d \otimes h) \circ \Delta(x)=h(x) \mathbb{I} . \tag{2.1}
\end{equation*}
$$

Remark 2.2. Note that if a quantum convolution $\Delta$ on $B(H)$ becomes a $*$-homomorphic map with a condition

$$
\overline{\operatorname{Lin}}((\mathbb{I} \otimes B(H)) \Delta(B(H)))=\overline{\operatorname{Lin}}((B(H) \otimes \mathbb{1}) \Delta(B(H)))=B(H) \otimes B(H)
$$

then a pair $(B(H), \Delta)$ is called a compact quantum group [37, 32]. It is known [37] that for given any compact quantum group there exists a unique Haar state w.r.t. $\Delta$.

Remark 2.3. In 21] it has been studied symmetric q.q.o., which was called quantum quadratic stochastic operator.

Remark 2.4. We note that there is another approach to nonlinear quantum operators on $C^{*}$ algebras (see [18]).

Note that from unitality of $\Delta$ we conclude that for any quasi q.q.o. $V_{\Delta}$ maps $S(B(H))$ into itself. In some literature operator $V_{\Delta}$ is called quadratic convolution (see for example [11]). In [25] certain dynamical properties of $V_{\Delta}$ associated with q.q.o. defined on $\mathbb{M}_{2}(\mathbb{C})$ are investigated. In [24] Kadison-Schwarz property of q.q.o. has been studied.

In quantum information, pure channels play important role, which can be defined as follows: a channel (i.e. positive and unital mapping) $T: B\left(H_{1}\right) \rightarrow B\left(H_{2}\right)$ is called pure if for any pure state $\varphi \in S\left(B\left(H_{1}\right)\right)$ the state $T^{*} \varphi$ is also pure (see [2]). It is essential to describe such channels. Of course, if $H_{1}=H_{2}$ then one can see that automorphisms of $B\left(H_{1}\right)$ are examples of pure channels. But, in general, the description of pure channels is a tricky job.

Now let us assume that $\Delta$ be a pure q.q.o. Then for any pure states $\varphi, \psi \in S(B(H))$ one concludes that $\Delta^{*}(\varphi \otimes \psi)$ is also pure. In particularly, for any pure $\varphi \in S(B(H)$ we have $\Delta^{*}(\varphi \otimes \varphi)$ is also pure. Note that the reverse is not true. Therefore, in this paper we are going to define more weaker notion than purity for quasi q.q.o.

Definition 2.5. A quasi q.q.o. $\Delta$ is called $q$-pure if for any pure state $\varphi$ the state $V_{\Delta}(\varphi)$ is also pure.

From this definition one can immediately see that purity of quasi q.q.o. implies its $q$-purity.

## 3. Quasi quantum quadratic operators on $\mathbb{M}_{2}(\mathbb{C})$

By $\mathbb{M}_{2}(\mathbb{C})$ be an algebra of $2 \times 2$ matrices over complex field $\mathbb{C}$. In this section we are going to describe quantum quadratic operators on $\mathbb{M}_{2}(\mathbb{C})$ as well as find necessary conditions for such operators to satisfy the Kadison-Schwarz property.

Recall [6] that the identity and Pauli matrices $\left\{\mathbb{I}, \sigma_{1}, \sigma_{2}, \sigma_{3}\right\}$ form a basis for $\mathbb{M}_{2}(\mathbb{C})$, where

$$
\sigma_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \sigma_{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right) \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

In this basis every matrix $x \in \mathbb{M}_{2}(\mathbb{C})$ can be written as $x=w_{0} \mathbb{I}+\mathbf{w} \sigma$ with $w_{0} \in \mathbb{C}$, $\mathbf{w}=\left(w_{1}, w_{2}, w_{3}\right) \in \mathbb{C}^{3}$, here $\mathbf{w} \sigma=w_{1} \sigma_{1}+w_{2} \sigma_{2}+w_{3} \sigma_{3}$. In what follows, we frequently use notation $\overline{\mathbf{w}}=\left(\overline{w_{1}}, \overline{w_{2}}, \overline{w_{3}}\right)$.
Lemma 3.1. 31] The following assertions hold true:
(a) $x$ is self-adjoint iff $w_{0}, \mathbf{w}$ are reals;
(b) $\operatorname{Tr}(x)=1$ iff $w_{0}=0.5$, here $\operatorname{Tr}$ is the trace of a matrix $x$;
(c) $x>0$ iff $\|\mathbf{w}\| \leq w_{0}$, where $\|\mathbf{w}\|=\sqrt{\left|w_{1}\right|^{2}+\left|w_{2}\right|^{2}+\left|w_{3}\right|^{2}}$;
(d) A linear functional $\varphi$ on $\mathbb{M}_{2}(\mathbb{C})$ is a state iff it can be represented by

$$
\begin{equation*}
\varphi\left(w_{0} \mathbf{I}+\mathbf{w} \sigma\right)=w_{0}+\langle\mathbf{w}, \mathbf{f}\rangle \tag{3.1}
\end{equation*}
$$

where $\mathbf{f}=\left(f_{1}, f_{2}, f_{3}\right) \in \mathbb{R}^{3}$ such that $\|\mathbf{f}\| \leq 1$. Here as before $\langle\cdot, \cdot\rangle$ stands for the scalar product in $\mathbb{C}^{3}$.
(e) A state $\varphi$ is a pure if and only if $\|\mathbf{f}\|=1$. So pure states can be seen as the elements of unit sphere in $\mathbb{R}^{3}$.

In the sequel we shall identify a state with a vector $\mathbf{f} \in \mathbb{R}^{3}$. By $\tau$ we denote a normalized trace, i.e. $\tau(x)=\frac{1}{2} \operatorname{Tr}(x), x \in \mathbb{M}_{2}(\mathbb{C})$.

Let $\Delta: \mathbb{M}_{2}(\mathbb{C}) \rightarrow \mathbb{M}_{2}(\mathbb{C}) \otimes \mathbb{M}_{2}(\mathbb{C})$ be a quasi q.q.o. Then we write the operator $\Delta$ in terms of a basis in $\mathbb{M}_{2}(\mathbb{C}) \otimes \mathbb{M}_{2}(\mathbb{C})$ formed by the Pauli matrices. Namely,

$$
\begin{align*}
& \Delta \mathbb{I}=\mathbb{I} \otimes \mathbb{I} ; \\
& \Delta\left(\sigma_{i}\right)=b_{i}(\mathbb{I} \otimes \mathbb{I})+\sum_{j=1}^{3} b_{j i}^{(1)}\left(\mathbb{I} \otimes \sigma_{j}\right)+\sum_{j=1}^{3} b_{j i}^{(2)}\left(\sigma_{j} \otimes \mathbb{I}\right)+\sum_{m, l=1}^{3} b_{m l, i}\left(\sigma_{m} \otimes \sigma_{l}\right), \tag{3.2}
\end{align*}
$$

where $i=1,2,3$.
In general, a description of positive operators is one of the main problems of quantum information. In the literature most tractable maps are positive and trace-preserving ones, since such maps arise naturally in quantum information theory (see [26]). Therefore, in the sequel we shall restrict ourselves to trace-preserving quasi q.q.o., i.e. $\tau \otimes \tau \circ \Delta=\tau$. So, we would like to describe all such kind of maps.
Proposition 3.2. Let $\Delta: \mathbb{M}_{2}(\mathbb{C}) \rightarrow \mathbb{M}_{2}(\mathbb{C}) \otimes \mathbb{M}_{2}(\mathbb{C})$ be a trace-preserving quasi q.q.o., then in (3.21) one has $b_{j}=0$, and $b_{i j}^{(1)}, b_{i j}^{(2)}, b_{i j, k}$ are real for every $i, j, k \in\{1,2,3\}$. Moreover, $\Delta$ has the following form:

$$
\begin{equation*}
\Delta(x)=w_{0} \mathbb{I} \otimes \mathbb{I}+\mathbb{I} \otimes \mathbf{B}^{(1)} \mathbf{w} \cdot \sigma+\mathbf{B}^{(2)} \mathbf{w} \cdot \sigma \otimes \mathbb{I}+\sum_{m, l=1}^{3}\left\langle\mathbf{b}_{m l}, \overline{\mathbf{w}}\right\rangle \sigma_{m} \otimes \sigma_{l} \tag{3.3}
\end{equation*}
$$

where $x=w_{0} \mathbb{I}+\mathbf{w} \sigma, \mathbf{b}_{m l}=\left(b_{m l, 1}, b_{m l, 2}, b_{m l, 3}\right)$, and $\mathbf{B}^{(k)}=\left(b_{i j}^{(k)}\right)_{i, j=1}^{3}, k=1,2$. Here as before $\langle\cdot, \cdot\rangle$ stands for the standard scalar product in $\mathbb{C}^{3}$.

Proof. From the *-preserving condition we get

$$
\Delta\left(\sigma_{i}^{*}\right)=\overline{b_{i}}(\mathbb{I} \otimes \mathbb{I})+\sum_{j=1}^{3} \overline{b_{j i}^{(1)}}\left(\mathbb{I} \otimes \sigma_{j}\right)+\sum_{j=1}^{3} \overline{b_{j i}^{(2)}}\left(\sigma_{j} \otimes \mathbb{I}\right)+\sum_{m, l=1}^{3} \overline{b_{m l, i}}\left(\sigma_{m} \otimes \sigma_{l}\right) .
$$

This yields that $b_{i}=\overline{b_{i}}, b_{j i}^{(k)}=\overline{b_{j i}^{(k)}}(k=1,2)$ and $b_{m l, i}=\overline{b_{m l, i}}$, i.e. all coefficients are real numbers.

Using the trace-preserving condition one finds

$$
\tau \otimes \tau\left(\Delta\left(\sigma_{i}\right)\right)=b_{i}=\tau\left(\sigma_{i}\right)
$$

Therefore, $b_{i}=0, i=1,2,3$. Hence, $\Delta$ has the following form

$$
\begin{equation*}
\Delta\left(\sigma_{i}\right)=\sum_{j=1}^{3} b_{j i}^{(1)}\left(\mathbb{I} \otimes \sigma_{j}\right)+\sum_{j=1}^{3} b_{j i}^{(2)}\left(\sigma_{j} \otimes \mathbb{I}\right)+\sum_{m, l=1}^{3} b_{m l, i}\left(\sigma_{m} \otimes \sigma_{l}\right), \quad i=1,2,3 \tag{3.4}
\end{equation*}
$$

Denoting

$$
\begin{equation*}
\mathbf{B}^{(k)}=\left(b_{i j}^{(k)}\right)_{i, j=1}^{3}, \quad k=1,2, \quad \mathbf{b}_{m l}=\left(b_{m l, 1}, b_{m l, 2}, b_{m l, 3}\right) \tag{3.5}
\end{equation*}
$$

and taking any $x=w_{0} \mathbb{I}+\mathbf{w} \sigma \in \mathbb{M}_{2}(\mathbb{C})$, from (3.4) we immediately find (3.3). This completes the proof.

One can rewrite (3.3) as follows

$$
\begin{equation*}
\Delta(x)=\lambda \Delta_{1}(x)+(1-\lambda) \Delta_{2}(x) \tag{3.6}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta_{1}(x)=w_{0} \mathbb{I} \otimes \mathbb{I}+\frac{1}{\lambda} \sum_{m, l=1}^{3}\left\langle\mathbf{b}_{m l}, \overline{\mathbf{w}}\right\rangle \sigma_{m} \otimes \sigma_{l},  \tag{3.7}\\
& \Delta_{2}(x)=w_{0} \mathbb{I} \otimes \mathbb{I}+\frac{1}{1-\lambda}\left(\mathbf{B}^{(2)} \mathbf{w} \cdot \sigma \otimes \mathbb{I}+\mathbb{I} \otimes \mathbf{B}^{(1)} \mathbf{w} \cdot \sigma\right) . \tag{3.8}
\end{align*}
$$

Now assume that $b_{i j, k}=0$ for all $i, j, k \in\{1,2,3\}$ and $\Delta$ is $q$-pure symmetric quasi q.q.o. In this case, $\Delta$ has the following form

$$
\begin{equation*}
\Delta\left(w_{0} \mathbb{I}+\mathbf{w} \sigma\right)=w_{0} \mathbb{I} \otimes \mathbb{I}+\mathbf{B w} \cdot \sigma \otimes \mathbb{I}+\mathbb{I} \otimes \mathbf{B w} \cdot \sigma . \tag{3.9}
\end{equation*}
$$

Let us take any $\varphi \in S\left(\mathbb{M}_{2}(\mathbb{C})\right)$ and $\mathbf{f} \in \mathbb{R}^{3}$ be the corresponding vector. Then we find

$$
\varphi \otimes \varphi\left(\Delta\left(w_{0} \mathbb{I}+\mathbf{w} \sigma\right)\right)=w_{0}+2\langle\mathbf{B w}, \mathbf{f}\rangle=w_{0}+\left\langle\mathbf{w}, 2 \mathbf{B}^{*} \mathbf{f}\right\rangle
$$

Hence, if $\varphi$ is pure, then $\|\mathbf{f}\|=1$. Denoting $\mathbf{U}=2 \mathbf{B}^{*}$ and the $q$-purity of $\Delta$ yields that $\|\mathbf{U f}\|=1$ for all $\mathbf{f}$ with $\|\mathbf{f}\|=1$. This means that $\mathbf{U}$ is isometry, so $\|\mathbf{U}\|=1$, i.e. $\|\mathbf{B}\|=1 / 2$. Consequently, one concludes that $\Delta$ is $q$-pure if and only if $2 \mathbf{B}$ is isometry.

Now we are interested, whether $q$-pure symmetric quasi q.q.o. is positive. To answer to this question we need some auxiliary facts.

Lemma 3.3. Let $x=w_{0} \mathbb{I} \otimes \mathbb{I}+\mathbf{w} \cdot \sigma \otimes \mathbb{I}+\mathbb{I} \otimes \mathbf{r} \cdot \sigma$. Then the following statements hold true:
(i) $x$ is self-adjoint if and only if $w_{0} \in \mathbb{R}$ and $\mathbf{w}, \mathbf{r} \in \mathbb{R}^{3}$;
(ii) $x$ is positive if and only if $w_{0}>0$ and $\|\mathbf{w}\|+\|\mathbf{r}\| \leq w_{0}$.

Proof. (i). One can see that

$$
x^{*}=\overline{w_{0}} \mathbb{I} \otimes \mathbb{I}+\overline{\mathbf{w}} \cdot \sigma \otimes \mathbb{I}+\mathbb{I} \otimes \overline{\mathbf{r}} \cdot \sigma
$$

So, self adjointness $x$ implies $\overline{w_{0}}=w_{0}, \overline{\mathbf{w}}=\mathbf{w}, \overline{\mathbf{r}}=\mathbf{r}$.
(ii). Let $x$ be self-adjoint. Then from the definition of Pauli matrices one finds

$$
x=\left(\begin{array}{cccc}
w_{0}+w_{3}+r_{3} & w_{1}-i w_{2} & r_{1}-i r_{2} & 0 \\
w_{1}+i w_{2} & w_{0}-w_{3}+r_{3} & 0 & r_{1}-i r_{2} \\
r_{1}+i r_{2} & 0 & w_{0}+w_{3}-r_{3} & w_{1}-i w_{2} \\
0 & r_{1}+i r_{2} & w_{1}+i w_{2} & w_{0}-w_{3}-r_{3}
\end{array}\right)
$$

It is easy to calculate that eigenvalues of last matrix are the followings

$$
\begin{array}{ll}
\lambda_{1}=w_{0}-\|\mathbf{r}\|+\|\mathbf{w}\|, & \lambda_{2}=w_{0}-\|\mathbf{r}\|-\|\mathbf{w}\|, \\
\lambda_{3}=w_{0}+\|\mathbf{r}\|+\|\mathbf{w}\|, & \lambda_{4}=w_{0}+\|\mathbf{r}\|-\|\mathbf{w}\|
\end{array}
$$

So, we can conclude that $x$ is positive if and only if the smallest eigenvalue is positive. This means $w_{0}-\|\mathbf{r}\|-\|\mathbf{w}\| \geq 0$, which completes the proof.
Proposition 3.4. The mapping $\Delta$ given by (3.9) is positive if and only if $\|\mathbf{B}\| \leq 1 / 2$.
Proof. Let $x=w_{0} \mathbb{I}+\mathbf{w} \cdot \sigma$ be positive, i.e. $w_{0}>0,\|\mathbf{w}\| \leq w_{0}$. Without lost of generality we may assume $w_{0}=1$. Now Lemma 3.3 yields that $\Delta(x)$ is positive if and only if $2\|\mathbf{B w}\| \leq 1$. This competes the proof.

From this Proposition and above made conclusions we immediately get the following
Theorem 3.5. Let $\Delta$ be given by (3.9). Then the following statements hold true:
(i) $\Delta$ is quasi q.q.o. if and only if $\Delta$ is positive, i.e. $\|\mathbf{B}\| \leq 1 / 2$;
(ii) $\Delta$ is $q$-pure if and only if $2 \mathbf{B}$ is isometry. Moreover, $\Delta$ is positive.

Note that using the methods of [23] one may study Kadison-Schwarz property of mappings given by (3.9). Now the question is what about the case when $b_{i j, k} \neq 0$. Therefore, the next section is devoted to this this question.

## 4. Q-PURE SYMMETRIC QUASI QUANTUM QUADRATIC OPERATORS ON $\mathbb{M}_{2}(\mathbb{C})$

In this section we are going to describe trace-preserving $q$-pure symmetric quasi q.q.o.
Denote

$$
\begin{aligned}
& \mathbf{D}=\left\{\mathbf{p}=\left(p_{1}, p_{2}, p_{3}\right) \in \mathbb{R}: p_{1}^{2}+p_{2}^{2}+p_{3}^{2} \leq 1\right\} \\
& \mathbf{S}=\left\{\mathbf{p}=\left(p_{1}, p_{2}, p_{3}\right) \in \mathbb{R}: p_{1}^{2}+p_{2}^{2}+p_{3}^{2}=1\right\}
\end{aligned}
$$

Let $\Delta$ be a trace-preserving symmetric quasi q.q.o. on $\mathbb{M}_{2}(\mathbb{C})$. Then due to Lemma 3.1 (d) and Proposition 3.2 the functional $\Delta^{*}(\varphi \otimes \psi)$ is a state if and only if the vector
$\mathbf{f}_{\Delta^{*}(\varphi, \psi)}=\left(\sum_{j=1}^{3} b_{j 1}\left(p_{j}+f_{j}\right)+\sum_{i, j=1}^{3} b_{i j, 1} f_{i} p_{j}, \sum_{j=1}^{3} b_{j 2}\left(p_{j}+f_{j}\right)+\sum_{i, j=1}^{3} b_{i j, 2} f_{i} p_{j}, \sum_{j=1}^{3} b_{j 3}\left(p_{j}+f_{j}\right)+\sum_{i, j=1}^{3} b_{i j, 3} f_{i} p_{j}\right)$.
satisfies $\left\|\mathbf{f}_{\Delta^{*}(\varphi, \psi)}\right\| \leq 1$.
Let us consider the quadratic operator, which is defined by $V_{\Delta}(\varphi)=\Delta^{*}(\varphi \otimes \varphi), \varphi \in$ $S\left(\mathbb{M}_{2}(\mathbb{C})\right)$. From the last expression we find that

$$
V_{\Delta}(\varphi)\left(\sigma_{k}\right)=\sum_{j=1}^{3} 2 b_{j k} f_{j}+\sum_{i, j=1}^{3} b_{i j, k} f_{i} f_{j}, \quad \mathbf{f} \in \mathbf{D}
$$

This suggests us the consideration of a nonlinear operator $V: \mathbf{D} \rightarrow \mathbf{D}$ defined by

$$
\begin{equation*}
V(\mathbf{f})_{k}=\sum_{j=1}^{3} 2 b_{j k} f_{j}+\sum_{i, j=1}^{3} b_{i j, k} f_{i} f_{j}, \quad k=1,2,3 . \tag{4.1}
\end{equation*}
$$

where $\mathbf{f}=\left(f_{1}, f_{2}, f_{3}\right) \in \mathbf{D}$.
From the definition and Lemma 3.1 (e) we conclude that the $\Delta$ is $q$-pure if and only if $V(\mathbf{S}) \subset \mathbf{S}$.

Example. Let us consider an example of pure symmetric quasi q.q.o. Let

$$
\begin{aligned}
\Delta_{0}(x)= & w_{0} \mathbb{I} \otimes \mathbb{I}+w_{1} \sigma_{1} \otimes \sigma_{2}+w_{1} \sigma_{2} \otimes \sigma_{1}+w_{2} \sigma_{1} \otimes \sigma_{1} \\
& -w_{2} \sigma_{2} \otimes \sigma_{2}-w_{2} \sigma_{3} \otimes \sigma_{3}+w_{3} \sigma_{1} \otimes \sigma_{3}+w_{3} \sigma_{3} \otimes \sigma_{1}
\end{aligned}
$$

Then the corresponding quadratic operator has the following form

$$
V_{0}(\mathbf{f})=\left\{\begin{array}{l}
2 f_{1} f_{2}  \tag{4.2}\\
f_{1}^{2}-f_{2}^{2}-f_{3}^{2} \\
2 f_{1} f_{3}
\end{array}\right.
$$

Let us show $V_{0}$ maps $\mathbf{S}$ to $\mathbf{S}$. Indeed, let $\mathbf{f}=\left(f_{1}, f_{2}, f_{3}\right) \in \mathbf{S}$, i.e. $f_{1}^{2}+f_{2}^{2}+f_{3}^{2}=1$. Then we have

$$
\begin{aligned}
\left(2 f_{1} f_{2}\right)^{2}+\left(f_{1}^{2}-f_{2}^{2}-f_{3}^{2}\right)^{2}+\left(2 f_{1} f_{3}\right)^{2} & =4 f_{1}^{2} f_{2}^{2}+\left(2 f_{1}^{2}-1\right)^{2}+4 f_{1}^{2} f_{3}^{2} \\
& =4 f_{1}^{2}\left(1-f_{1}^{2}-f_{3}^{2}\right)+4 f_{1}^{4}-4 f_{1}^{2}+1+4 f_{1}^{2} f_{3}^{2} \\
& =1
\end{aligned}
$$

This shows that $\Delta_{0}$ is q-pure.
Now let us rewrite the quadratic operator $V$ (see (4.1)) as follows
(4.3) $V(\mathbf{f})=$
where $\mathbf{f} \in \mathbf{D}$.

Theorem 4.1. The operator $V$ given by (4.3) maps $\mathbf{S}$ into itself if and only if the followings hold true
(i) $\|\mathbf{a}\|^{2}+\|\mathbf{d}\|^{2}=1,\|\mathbf{b}\|^{2}+\|\mathbf{e}\|^{2}=1,\|\mathbf{c}\|^{2}+\|\mathbf{g}\|^{2}=1$;
(ii) $\|A\|=\|\mathbf{a}-\mathbf{b}\|,\|\Gamma\|=\|\mathbf{a}-\mathbf{c}\|,\|B\|=\|\mathbf{b}-\mathbf{c}\|$;
(iii) $\langle\mathbf{a}, \mathbf{d}\rangle=0,\langle\mathbf{b}, \mathbf{e}\rangle=0,\langle\mathbf{c}, \mathbf{g}\rangle=0$;
(iv) $\langle\mathbf{a}, \Gamma\rangle=\langle\mathbf{c}, \Gamma\rangle,\langle\mathbf{b}, B\rangle=\langle\mathbf{c}, B\rangle,\langle\mathbf{a}, A\rangle=\langle\mathbf{b}, A\rangle$;
(v) $\langle\mathbf{c}, \Gamma\rangle+\langle\mathbf{d}, \mathbf{g}\rangle=0,\langle\mathbf{c}, B\rangle+\langle\mathbf{e}, \mathbf{g}\rangle=0,\langle\mathbf{c}, \mathbf{d}\rangle+\langle\Gamma, \mathbf{g}\rangle=0$,
$\langle\mathbf{c}, \mathbf{e}\rangle+\langle B, \mathbf{g}\rangle=0,\langle\mathbf{b}, \mathbf{d}\rangle+\langle A, \mathbf{e}\rangle=0,\langle\mathbf{b}, A\rangle+\langle\mathbf{d}, \mathbf{e}\rangle=0$,
$\langle\mathbf{b}, \mathbf{g}\rangle+\langle B, \mathbf{e}\rangle=0,\langle\mathbf{a}, \mathbf{e}\rangle+\langle A, \mathbf{d}\rangle=0,\langle\mathbf{a}, \mathbf{g}\rangle+\langle\Gamma, \mathbf{d}\rangle=0 ;$
(vi) $\langle\mathbf{a}, B\rangle-\langle\mathbf{c}, B\rangle+\langle A, \Gamma\rangle=0,\langle\mathbf{b}, \Gamma\rangle-\langle\mathbf{c}, \Gamma\rangle+\langle A, B\rangle=0$, $\langle A, \mathbf{g}\rangle+\langle B, \mathbf{d}\rangle+\langle\Gamma, \mathbf{e}\rangle=0,\langle\mathbf{c}, A\rangle+\langle\mathbf{d}, \mathbf{e}\rangle+\langle B, \Gamma\rangle=0$,
where $\mathbf{a}=\left(a_{1}, a_{2}, a_{3}\right), \mathbf{b}=\left(b_{1}, b_{2}, b_{3}\right), \mathbf{c}=\left(c_{1}, c_{2}, c_{3}\right), \mathbf{d}=\left(d_{1}, d_{2}, d_{3}\right), \mathbf{e}=\left(e_{1}, e_{2}, e_{3}\right), \mathbf{g}=$ $\left(g_{1}, g_{2}, g_{3}\right), \Gamma=\left(\gamma_{1}, \gamma_{2}, \gamma_{3}\right), \quad A=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right), B=\left(\beta_{1}, \beta_{2}, \beta_{3}\right)$.

Proof. "only if" part. It is enough to show

$$
\begin{equation*}
\left(V(\mathbf{f})_{1}\right)^{2}+\left(V(\mathbf{f})_{2}\right)^{2}+\left(V(\mathbf{f})_{3}\right)^{2}=1 \tag{4.4}
\end{equation*}
$$

for any $\mathbf{f}=\left(f_{1}, f_{2}, f_{3}\right)$ with $f_{1}^{2}+f_{2}^{2}+f_{3}^{2}=1$.
Let us rewrite (4.3) as follows

$$
(4.5) V(\mathbf{f})=\left\{\begin{array}{l}
\left(a_{1}-c_{1}\right) f_{1}^{2}+\left(b_{1}-c_{1}\right) f_{2}^{2}+c_{1}+\alpha_{1} f_{1} f_{2}+\beta_{1} f_{2} f_{3}+\gamma_{1} f_{1} f_{3}+d_{1} f_{1}+e_{1} f_{2}+g_{1} f_{3} \\
\left(a_{2}-c_{2}\right) f_{1}^{2}+\left(b_{2}-c_{2}\right) f_{2}^{2}+c_{2}+\alpha_{2} f_{1} f_{2}+\beta_{2} f_{2} f_{3}+\gamma_{2} f_{1} f_{3}+d_{2} f_{1}+e_{2} f_{2}+g_{2} f_{3} \\
\left(a_{3}-c_{3}\right) f_{1}^{2}+\left(b_{3}-c_{3}\right) f_{2}^{2}+c_{3}+\alpha_{3} f_{1} f_{2}+\beta_{3} f_{2} f_{3}+\gamma_{3} f_{1} f_{3}+d_{3} f_{1}+e_{3} f_{2}+g_{3} f_{3}
\end{array}\right.
$$

From (4.4), (4.5) we derive

$$
\begin{aligned}
& \left(\left(a_{1}-c_{1}\right) f_{1}^{2}+\left(b_{1}-c_{1}\right) f_{2}^{2}+c_{1}+\alpha_{1} f_{1} f_{2}+\beta_{1} f_{2} f_{3}+\gamma_{1} f_{1} f_{3}+d_{1} f_{1}+e_{1} f_{2}+g_{1} f_{3}\right)^{2} \\
+ & \left(\left(a_{2}-c_{2}\right) f_{1}^{2}+\left(b_{2}-c_{2}\right) f_{2}^{2}+c_{2}+\alpha_{2} f_{1} f_{2}+\beta_{2} f_{2} f_{3}+\gamma_{2} f_{1} f_{3}+d_{2} f_{1}+e_{2} f_{2}+g_{2} f_{3}\right)^{2} \\
+ & \left(\left(a_{3}-c_{3}\right) f_{1}^{2}+\left(b_{3}-c_{3}\right) f_{2}^{2}+c_{3}+\alpha_{3} f_{1} f_{2}+\beta_{3} f_{2} f_{3}+\gamma_{3} f_{1} f_{3}+d_{3} f_{1}+e_{3} f_{2}+g_{3} f_{3}\right)^{2}=1
\end{aligned}
$$

After some calculations we obtain the following

$$
\begin{aligned}
& \left(\|\mathbf{a}\|^{2}+\|\mathbf{c}\|^{2}-\|\Gamma\|^{2}-2\langle\mathbf{a}, \mathbf{c}\rangle\right) f_{1}^{4}+\left(\|\mathbf{b}\|^{2}+\|\mathbf{c}\|^{2}-\|B\|^{2}-2\langle\mathbf{b}, \mathbf{c}\rangle\right) f_{2}^{4} \\
+\quad & (2\langle\mathbf{a}, A\rangle-2\langle B, \Gamma\rangle-2\langle\mathbf{c}, A\rangle) f_{1}^{3} f_{2}+(2\langle\mathbf{a}, \Gamma\rangle-2\langle\mathbf{c}, \Gamma\rangle) f_{1}^{3} f_{3} \\
+\quad & (2\langle\mathbf{a}, \mathbf{d}\rangle-2\langle\mathbf{c}, \mathbf{d}\rangle-2\langle\Gamma, \mathbf{g}\rangle) f_{1}^{3}+(2\langle\mathbf{b}, A\rangle-2\langle B, \Gamma\rangle-2\langle\mathbf{c}, A\rangle) f_{1} f_{2}^{3} \\
+\quad & (2\langle\mathbf{b}, B\rangle-2\langle\mathbf{c}, B\rangle) f_{2}^{3} f_{3}+(2\langle\mathbf{b}, \mathbf{e}\rangle-2\langle\mathbf{c}, \mathbf{e}\rangle-2\langle B, \mathbf{g}\rangle) f_{2}^{3} \\
+\quad & \left(2\|\mathbf{c}\|^{2}+\|A\|^{2}-\|B\|^{2}-\|\Gamma\|^{2}+2\langle\mathbf{a}, \mathbf{b}\rangle-2\langle\mathbf{b}, \mathbf{c}\rangle-2\langle\mathbf{a}, \mathbf{c}\rangle\right) f_{1}^{2} f_{2}^{2} \\
+\quad & (2\langle\mathbf{a}, B\rangle+2\langle A, \Gamma\rangle-2\langle\mathbf{c}, B\rangle) f_{1}^{2} f_{2} f_{3}+(2\langle\mathbf{a}, \mathbf{e}\rangle+2\langle A, \mathbf{d}\rangle-2\langle\mathbf{c}, \mathbf{e}\rangle-2\langle B, \mathbf{g}\rangle) f_{1}^{2} f_{2} \\
+\quad & (2\langle\mathbf{a}, \mathbf{g}\rangle+2\langle\Gamma, \mathbf{d}\rangle-2\langle\mathbf{c}, \mathbf{g}\rangle) f_{1}^{2} f_{3}+\left(\|\Gamma\|^{2}+\|\mathbf{d}\|^{2}-2\|\mathbf{c}\|^{2}-\|\mathbf{g}\|^{2}+2\langle\mathbf{a}, \mathbf{c}\rangle\right) f_{1}^{2} \\
+\quad & (2\langle\mathbf{b}, \Gamma\rangle+2\langle A, B\rangle-2\langle\mathbf{c}, \Gamma\rangle) f_{1} f_{2}^{2} f_{3}+(2\langle\mathbf{b}, \mathbf{d}\rangle+2\langle A, \mathbf{e}\rangle-2\langle\mathbf{c}, \mathbf{d}\rangle-2\langle\Gamma, \mathbf{g}\rangle) f_{1} f_{2}^{2} \\
+\quad & (2\langle\mathbf{b}, \mathbf{g}\rangle+2\langle B, \mathbf{e}\rangle-2\langle\mathbf{c}, \mathbf{g}\rangle) f_{2}^{2} f_{3}+\left(\|B\|^{2}+\|\mathbf{e}\|^{2}-2\|\mathbf{c}\|^{2}-\|\mathbf{g}\|^{2}+2\langle\mathbf{b}, \mathbf{c}\rangle\right) f_{2}^{2} \\
+\quad & (2\langle A, \mathbf{g}\rangle+2\langle B, \mathbf{d}\rangle+2\langle\Gamma, \mathbf{e}\rangle) f_{1} f_{2} f_{3}+(2\langle\mathbf{c}, A\rangle+2\langle B, \Gamma\rangle+2\langle\mathbf{d}, \mathbf{e}\rangle) f_{1} f_{2} \\
+\quad & (2\langle\mathbf{c}, \Gamma\rangle+2\langle\mathbf{d}, \mathbf{g}\rangle) f_{1} f_{3}+(2\langle\mathbf{c}, B\rangle+2\langle\mathbf{e}, \mathbf{g}\rangle) f_{2} f_{3}+(2\langle\mathbf{c}, \mathbf{d}\rangle+2\langle\Gamma, \mathbf{g}\rangle) f_{1} \\
+\quad & (2\langle\mathbf{c}, \mathbf{e}\rangle+2\langle B, \mathbf{g}\rangle) f_{2}+2\langle\mathbf{c}, \mathbf{g}\rangle f_{3}+\|\mathbf{c}\|^{2}+\|\mathbf{g}\|^{2}-1=0
\end{aligned}
$$

which is satisfied (i)-(vi).
" if " part is obvious. This completes the proof.
In what follows, we are interested in the case when $\Delta_{2}=0$ in (3.6). This means that $\Delta$ has a Haar state $\tau$. Indeed, using the equality (2.1) with $h=\tau$ one gets

$$
(i d \otimes \tau)\left(\Delta\left(\sigma_{i}\right)\right)=\sum_{j=1}^{3} b_{j i} \sigma_{j}=\tau\left(\sigma_{i}\right) \mathbb{I}=0, \quad i=1,2,3
$$

Therefore, $b_{j i}=0$ for all $i, j \in\{1,2,3\}$. Hence, $\Delta$ has the following form

$$
\begin{equation*}
\Delta\left(w_{0} \mathbb{\Pi}+\mathbf{w} \sigma\right)=w_{0} \mathbb{I} \otimes \mathbb{I}+\sum_{m, l=1}^{3}\left\langle\mathbf{b}_{m l}, \overline{\mathbf{w}}\right\rangle \sigma_{m} \otimes \sigma_{l}, \tag{4.6}
\end{equation*}
$$

Then the corresponding quadratic operator $V$ has the form (4.3) with constrains $\mathbf{d}=\mathbf{e}=\mathbf{g}=0$.
From Theorem 4.1 one immediately gets
Corollary 4.2. Let the operator $V$ given by (4.3) with $\mathbf{d}=\mathbf{e}=\mathbf{g}=0$. Then $V(\mathbf{S}) \subset \mathbf{S}$ if and only if the followings hold true
(i) $\|\mathbf{a}\|=1,\|\mathbf{b}\|=1,\|\mathbf{c}\|=1$;
(ii) $\|A\|=\|\mathbf{a}-\mathbf{b}\|,\|\Gamma\|=\|\mathbf{a}-\mathbf{c}\|,\|B\|=\|\mathbf{b}-\mathbf{c}\|$;
(iii) $\langle\mathbf{a}, B\rangle+\langle A, \Gamma\rangle=0,\langle\mathbf{b}, \Gamma\rangle+\langle A, B\rangle=0,\langle\mathbf{c}, A\rangle+\langle B, \Gamma\rangle=0$;
(iv) $\langle\mathbf{a}, A\rangle=0,\langle\mathbf{a}, \Gamma\rangle=0,\langle\mathbf{b}, A\rangle=0,\langle\mathbf{b}, B\rangle=0,\langle\mathbf{c}, \Gamma\rangle=0,\langle\mathbf{c}, B\rangle=0$
where the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, A, B, \Gamma$ are given in Theorem 4.1.
Let us consider a symmetric quasi q.q.o. $\Delta$ with Haar state $\tau$, corresponding to (4.3). Then according to (4.6) the operator $\Delta$ has the following form

$$
\begin{aligned}
\Delta(x)=w_{0} \mathbb{\Pi} \otimes \mathbb{I} & +a_{1} w_{1} \sigma_{1} \otimes \sigma_{1}+\frac{\alpha_{1}}{2} w_{1} \sigma_{1} \otimes \sigma_{2}+\frac{\gamma_{1}}{2} w_{1} \sigma_{1} \otimes \sigma_{3} \\
& +\frac{\alpha_{1}}{2} w_{1} \sigma_{2} \otimes \sigma_{1}+b_{1} w_{1} \sigma_{2} \otimes \sigma_{2}+\frac{\beta_{1}}{2} w_{1} \sigma_{2} \otimes \sigma_{3} \\
& +\frac{\gamma_{1}}{2} w_{1} \sigma_{3} \otimes \sigma_{1}+\frac{\beta_{1}}{2} w_{1} \sigma_{3} \otimes \sigma_{2}+c_{1} w_{1} \sigma_{3} \otimes \sigma_{3} \\
& +a_{2} w_{2} \sigma_{1} \otimes \sigma_{1}+\frac{\alpha_{2}}{2} w_{2} \sigma_{1} \otimes \sigma_{2}+\frac{\gamma_{2}}{2} w_{2} \sigma_{1} \otimes \sigma_{3} \\
& +\frac{\alpha_{2}}{2} w_{2} \sigma_{2} \otimes \sigma_{1}+b_{2} w_{2} \sigma_{2} \otimes \sigma_{2}+\frac{\beta_{2}}{2} w_{2} \sigma_{2} \otimes \sigma_{3} \\
& +\frac{\gamma_{2}}{2} w_{2} \sigma_{3} \otimes \sigma_{1}+\frac{\beta_{2}}{2} w_{2} \sigma_{3} \otimes \sigma_{2}+c_{2} w_{2} \sigma_{3} \otimes \sigma_{3} \\
& +a_{3} w_{3} \sigma_{1} \otimes \sigma_{1}+\frac{\alpha_{3}}{2} w_{3} \sigma_{1} \otimes \sigma_{2}+\frac{\gamma_{3}}{2} w_{3} \sigma_{1} \otimes \sigma_{3} \\
& +\frac{\alpha_{3}}{2} w_{3} \sigma_{2} \otimes \sigma_{1}+b_{3} w_{3} \sigma_{2} \otimes \sigma_{2}+\frac{\beta_{3}}{2} w_{3} \sigma_{2} \otimes \sigma_{3} \\
& +\frac{\gamma_{3}}{2} w_{3} \sigma_{3} \otimes \sigma_{1}+\frac{\beta_{3}}{2} w_{3} \sigma_{3} \otimes \sigma_{2}+c_{3} w_{3} \sigma_{3} \otimes \sigma_{3}
\end{aligned}
$$

Calculating the last one, we obtain

$$
\Delta(x)=\left(\begin{array}{cccc}
w_{0}+R & N-i P & N-i P & L-2 i M-O  \tag{4.7}\\
N+i P & w_{0}-R & L+O & -N+i P \\
N+i P & L+O & w_{0}-R & -N+i P \\
L+2 i M-O & -N-i P & -N-i P & w_{0}+R
\end{array}\right)
$$

where

$$
\begin{aligned}
L & =\langle\mathbf{a}, \mathbf{w}\rangle, & M & =\frac{1}{2}\langle A, \mathbf{w}\rangle,
\end{aligned} \quad N=\frac{1}{2}\langle\Gamma, \mathbf{w}\rangle, ~ 子=\langle\mathbf{b}, \mathbf{w}\rangle, \quad P=\frac{1}{2}\langle B, \mathbf{w}\rangle, \quad R=\langle\mathbf{c}, \mathbf{w}\rangle .
$$

Theorem 4.3. Let $\Delta: \mathbb{M}_{2}(\mathbb{C}) \rightarrow \mathbb{M}_{2}(\mathbb{C}) \otimes \mathbb{M}_{2}(\mathbb{C})$ be a $q$-pure symmetric quasi q.q.o. with Haar state $\tau$. Then $\Delta$ is not positive.

Proof. Let us prove from the contrary. Assume that $\Delta$ is positive. This means that the matrix given by (4.7) should be positive, whenever $x$ is positive. The positivity of $x$ yields that $w_{0}, w_{1}, w_{2}, w_{3}$ are real numbers. In what follows, without loss of generality, we may assume that $w_{0}=1$, and therefore $\|\mathbf{w}\| \leq 1$. It is known that the positivity of the matrix $\Delta(x)$ is equivalent to the positivity of its eigenvalues, and it should be positive for any values of $\|\mathbf{w}\| \leq 1$.

We note that the $q$-purity of $\Delta$ implies that the conditions (i)-(iv) of Corollary 4.2 are satisfied.

Let us take $x=\mathbb{1}+\mathbf{a} \sigma$, then from (4.7) one gets

$$
\Delta(x)=\left(\begin{array}{cccc}
1+\langle\mathbf{c}, \mathbf{a}\rangle & -\frac{i}{2}\langle B, \mathbf{a}\rangle & -\frac{i}{2}\langle B, \mathbf{a}\rangle & 1-\langle\mathbf{b}, \mathbf{a}\rangle \\
\frac{i}{2}\langle B, \mathbf{a}\rangle & 1-\langle\mathbf{c}, \mathbf{a}\rangle & 1+\langle\mathbf{b}, \mathbf{a}\rangle & \frac{i}{2}\langle B, \mathbf{a}\rangle \\
\frac{i}{2}\langle B, \mathbf{a}\rangle & 1+\langle\mathbf{b}, \mathbf{a}\rangle & 1-\langle\mathbf{c}, \mathbf{a}\rangle & \frac{i}{2}\langle B, \mathbf{a}\rangle \\
1-\langle\mathbf{b}, \mathbf{a}\rangle & -\frac{i}{2}\langle B, \mathbf{a}\rangle & -\frac{i}{2}\langle B, \mathbf{a}\rangle & 1+\langle\mathbf{c}, \mathbf{a}\rangle
\end{array}\right) .
$$

A simple algebra shows us that all eigenvalues of $\Delta(x)$ can be written as follows

$$
\begin{aligned}
& \lambda_{1}=-\langle\mathbf{c}, \mathbf{a}\rangle-\langle\mathbf{b}, \mathbf{a}\rangle \\
& \lambda_{2}=\langle\mathbf{c}, \mathbf{a}\rangle+\langle\mathbf{b}, \mathbf{a}\rangle \\
& \lambda_{3}=2+\sqrt{\langle\mathbf{b}, \mathbf{a}\rangle^{2}-2\langle\mathbf{c}, \mathbf{a}\rangle\langle\mathbf{b}, \mathbf{a}\rangle+\langle\mathbf{c}, \mathbf{a}\rangle^{2}+\langle B, \mathbf{a}\rangle^{2}} \\
& \lambda_{4}=2-\sqrt{\langle\mathbf{b}, \mathbf{a}\rangle^{2}-2\langle\mathbf{c}, \mathbf{a}\rangle\langle\mathbf{b}, \mathbf{a}\rangle+\langle\mathbf{c}, \mathbf{a}\rangle^{2}+\langle B, \mathbf{a}\rangle^{2}} .
\end{aligned}
$$

Now using (ii) of Corollary 4.2 we rewrite $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ as follows

$$
\begin{aligned}
& \lambda_{1}=-2+\frac{\|\Gamma\|^{2}}{2}+\frac{\|A\|^{2}}{2} \\
& \lambda_{2}=2-\frac{\|\Gamma\|^{2}}{2}-\frac{\|A\|^{2}}{2} \\
& \lambda_{3}=2+\frac{1}{2} \sqrt{\|A\|^{4}-2\|\Gamma\|^{2}\|A\|^{2}+\|\Gamma\|^{4}+\langle B, \mathbf{a}\rangle^{2}} \\
& \lambda_{4}=2-\frac{1}{2} \sqrt{\|A\|^{4}-2\|\Gamma\|^{2}\|A\|^{2}+\|\Gamma\|^{4}+\langle B, \mathbf{a}\rangle^{2}} .
\end{aligned}
$$

Knowing $\lambda_{1} \geq 0, \lambda_{2} \geq 0$ we have

$$
\|A\|^{2}+\|\Gamma\|^{2}=4
$$

By considering elements $x=\mathbb{I}+\mathbf{b} \sigma, x=\mathbb{I}+\mathbf{c} \sigma$, respectively, and using the similar argument one finds

$$
\|B\|^{2}+\|A\|^{2}=4 \quad\|\Gamma\|^{2}+\|B\|^{2}=4
$$

Therefore, we conclude that

$$
\|A\|^{2}=2,\|B\|^{2}=2,\|\Gamma\|^{2}=2
$$

Hence, again taking into account (ii) of Corollary 4.2 we find that

$$
\langle\mathbf{a}, \mathbf{b}\rangle=0,\langle\mathbf{a}, \mathbf{c}\rangle=0,\langle\mathbf{b}, \mathbf{c}\rangle=0 .
$$

This means that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent. Therefore, one can write

$$
\begin{aligned}
A & =\eta_{1} \mathbf{a}+\mu_{1} \mathbf{b}+\tau_{1} \mathbf{c} \\
B & =\eta_{2} \mathbf{a}+\mu_{2} \mathbf{b}+\tau_{2} \mathbf{c} \\
\Gamma & =\eta_{3} \mathbf{a}+\mu_{3} \mathbf{b}+\tau_{3} \mathbf{c}
\end{aligned}
$$

where $\eta_{i}^{2}+\mu_{i}^{2}+\tau_{i}^{2}=2, i=\overline{1,3}$.
From (iv) of Corollary 4.2 we find that

$$
\eta_{1}=0, \eta_{3}=0, \mu_{1}=0, \mu_{2}=0, \tau_{2}=0, \tau_{3}=0
$$

This implies that

$$
A=\tau_{1} \mathbf{c}, \quad B=\eta_{2} \mathbf{a}, \quad \Gamma=\mu_{3} \mathbf{b}
$$

Hence, from (iii) of Corollary 4.2 we have

$$
\langle A, B\rangle+\langle\mathbf{b}, \Gamma\rangle=0 \quad \Rightarrow \quad \mu_{3}=0
$$

which contradicts to $\mu_{3} \neq 0$. This completes the proof.
This theorem implies that $q$-pure symmetric quasi q.q.o. with Haar state can not be q.q.o. Moreover, if one has pure quasi q.q.o., then it cannot be positive. As we have seen in the previous section a quasi q.q.o. with only "linear" term can be positive. But the last theorem shows the difference between Theorem 3.5. Namely, if one considers a quadratic operator $V$ which is linear (this corresponds to the case of Theorem 3.5), then $q$-pure quasi q.q.o. is positive. But Theorem 4.3 implies a different kind of statement, i.e. if $V$ contains a nonlinear term, i.e. quadratic term, then the $q$-purity of $\Delta$ does not imply its positivity.

## 5. On dynamics of q-Pure quasi quantum quadratic operator.

In this section we are going to make some remarks on dynamics of $q$-pure quasi q.q.o.
Let $\Delta$ be a $q$-pure quasi q.q.o. By $V$ we denote the corresponding quadratic operator. Now we want to study the dynamics of $V$.

Proposition 5.1. Let $V$ be a quadratic operator corresponding to $q$-pure quasi q.q.o. with Haar state $\tau$. Then for any $\mathbf{f} \in \mathbf{D} \backslash \mathbf{S}$ one has

$$
\lim _{n \rightarrow \infty} V^{n}(\mathbf{f})=0
$$

Proof. Let $\mathbf{f} \in \mathbf{D} \backslash \mathbf{S}$ then one can see $\|\mathbf{f}\|<1$. Denote $\mathbf{g}=\frac{\mathbf{f}}{\|\mathbf{f}\|}$ then $\mathbf{g} \in \mathbf{S}$. Therefore using purity of $\Delta$ we conclude $V(\mathbf{g}) \in \mathbf{S}$. This means

$$
1=\left\|V\left(\frac{\mathbf{f}}{\|\mathbf{f}\|}\right)\right\|=\frac{1}{\|\mathbf{f}\|^{2}}\|V(\mathbf{f})\|
$$

So

$$
\|V(\mathbf{f})\|=\|\mathbf{f}\|^{2}
$$

Hence, we find

$$
\left\|V^{n}(\mathbf{f})\right\|=\|\mathbf{f}\|^{2^{n}}
$$

which implies $V^{n}(\mathbf{f}) \rightarrow 0$ as $n \rightarrow \infty$.
Corollary 5.2. Let $V$ be as in Proposition 5.1, then any nonzero fixed point (if it exists) belongs to $\mathbf{S}$. Moreover, $(0,0,0)$ is unique fixed point in $\mathbf{D} \backslash \mathbf{S}$

Now to investigate dynamics of $V$ it remains to study it on $\mathbf{S}$. Next examples show how the dynamics could be unpredictable on $\mathbf{S}$.

1. Let us study the dynamics of the given operator $V_{0}$ given by (4.2). We consider several cases.

Now assume $\mathbf{f} \in \mathbf{S}$. Suppose that $f_{1}=0$. Then we have

$$
V_{0}(\mathbf{f})=(0,-1,0)
$$

Hence, $V_{0}^{n}(\mathbf{f})=(0,-1,0)$, for every $n \in \mathbb{N}$.
Suppose that $f_{2}=0$. Then

$$
V_{0}\left(f_{1}, 0, f_{3}\right)=\left(0, f_{1}^{2}-f_{3}^{2}, 2 f_{1} f_{3}\right)
$$

So we have $V_{0}^{k}(\mathbf{f}) \rightarrow(0,-1,0)$ as $n \rightarrow \infty$.
Suppose that $f_{3}=0$. Then

$$
V_{0}\left(f_{1}, f_{2}, 0\right)=\left(2 f_{1} f_{2}, f_{1}^{2}-f_{2}^{2}, 0\right)=\left( \pm 2 f_{1} \sqrt{1-f_{1}^{2}}, 2 f_{1}^{2}-1,0\right)
$$

To investigate the dynamics of $V_{0}$, let us consider the following function

$$
g(x)=2 x \sqrt{1-x^{2}},|x| \leq 1
$$

For us it is enough to study the dynamics of $g(x)$. It is clear that

$$
g[0,1] \subset[0,1], g[-1,0] \subset[-1,0]
$$

Since the function is odd it is sufficient to study the dynamics of $g$ on $[0,1]$. Denote $h(x)=\sqrt{x}$. One can see that

$$
h^{-1}(g(h(x)))=4 x(1-x)
$$

This means $g(x)$ and $\ell(x)=4 x(1-x)$ are conjugate on $[0,1]$. It is known that the function $\ell(x)$ is the logistic function which is chaotic. Hence, $g(x)$ is also chaotic. From this we conclude that the behavior of $V_{0}$ on $\mathbf{S}$ with $f_{3}=0$ is chaotic. Note that similar kind of dynamical system has been investigated in [3, 20, 33].
2. Let

$$
\Delta_{1}(x)=w_{0} \mathbb{I} \otimes \mathbb{I}+\langle\mathbf{t}, \mathbf{w}\rangle\left(\sigma_{1} \otimes \sigma_{1}+\sigma_{2} \otimes \sigma_{2}+\sigma_{3} \otimes \sigma_{3}\right)
$$

then the corresponding quadratic operator has the following form

$$
V_{1}\left(f_{1}, f_{2}, f_{3}\right)=\left\{\begin{array}{c}
t_{1}\left(f_{1}^{2}+f_{2}^{2}+f_{3}^{2}\right) \\
t_{2}\left(f_{1}^{2}+f_{2}^{2}+f_{3}^{2}\right) \\
t_{3}\left(f_{1}^{2}+f_{2}^{2}+f_{3}^{2}\right)
\end{array}\right.
$$

where $\|\mathbf{t}\|=1, \mathbf{t}=\left(t_{1}, t_{2}, t_{3}\right)$.
One can see $V_{1}$ has only two fixed points which are $(0,0,0),\left(t_{1}, t_{2}, t_{3}\right)$. It is easy to see that $V(\mathbf{S})=\{\mathbf{t}\}$, so $\Delta_{1}$ is $q$-pure quasi q.q.o. Therefore, we conclude that

$$
\lim _{n \rightarrow \infty} V_{1}^{n}(\mathbf{f})= \begin{cases}\mathbf{t}, & \mathbf{f} \in \mathbf{S} \\ \mathbf{0}, & \mathbf{f} \in \mathbf{D} \backslash \mathbf{S} .\end{cases}
$$

## 6. CONCLUSION

In the present paper we studied quasi quantum quadratic operators (q.q.o) acting on the algebra of $2 \times 2$ matrices $\mathbb{M}_{2}(\mathbb{C})$. We have introduced a weaker condition, called $q$-purity, than purity of the channel. To study $q$-pure channels, we have described all trace-preserving quasi q.q.o. acting on $\mathbb{M}_{2}(\mathbb{C})$, which allowed us to describe all $q$-pure quadratic operators. Then we prove that if a trace-preserving symmetric quasi q.q.o. such that the corresponding quadratic operator is linear, then its $q$-purity implies its positivity. Moreover, if a symmetric quasi q.q.o. has a Haar state $\tau$, then its corresponding quadratic operator is nonlinear, and it is proved that such $q$-pure symmetric quasi q.q.o. cannot be positive. Note that there are nontrivial q.q.o. such that their corresponding quadratic operators are nonlinear [24]. We think that such a result will allow to check whether a given mapping from $\mathbb{M}_{2}(\mathbb{C})$ to $\mathbb{M}_{2}(\mathbb{C}) \otimes \mathbb{M}_{2}(\mathbb{C})$ is pure or not. On the other hand, our study is related to construction of pure quantm nonlinear channels. We should stress that nonlinear channels appear in many branches of quantum information (see for example [4, 14, 28]). Moreover, one also established that nonlinear dynamics associated with quasi pure q.q.o. may have differen kind of dynamics, i.e. it may behave chaotically or trivially, respectively.

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