On time-local generators of quantum evolution^{*}

Dariusz Chruściński Institute of Physics, Faculty of Physics, Astronomy and Informatics Nicolaus Copernicus University Grudziadzka 5, 87–100 Torun, Poland

(Processed: October 5, 2018; Received: _____.)

Abstract. We present a basic introduction to the dynamics of open quantum systems based on local-in-time master equations. We characterize the properties of time-local generators giving rise to legitimate completely positive trace preserving quantum evolutions. The analysis of Markovian and non-Markovian quantum dynamics is presented as well. The whole discussion is illustrated by the family of many instructive examples.

1. Introduction

The dynamics of open quantum systems attracts nowadays considerable attention. It is relevant not only for a better understanding of quantum theory but it is fundamental in various modern applications of quantum mechanics such as quantum communication, cryptography, computation and quantum metrology. Any realistic quantum system is an *open system* since it always interacts with its environment or "rest of the world". The proper descriptions of such systems is therefore of fundamental importance. There are several excellent books [1, 2, 3, 4, 5, 6] and review articles [7, 8] devoted to this subject. In this paper we provide an introduction to the dynamics of open quantum systems based on local-in-time master equation. There are several approaches to the dynamics of open quantum systems. It is clear that each description presents a departure from the standard von Neumann equation

$$i\dot{\rho}_t = [H, \rho_t] , \qquad (1)$$

which governs the evolution of the *closed system* fully characterized by the system Hamiltonian H (by ρ_t we denote the density operator (density matrix) at time t – see the next section for the notation) and $\hbar = 1$ throughout the paper. Introducing a linear map acting on the space of density operators

Lecture presented at the school Open Quantum Systems, Toruń, June 20-24, 2012; http://tv.umk.pl/#movie=1743

(sometimes called a superoperator) $L(\rho) = -i[H, \rho]$ the von Neumann equation may be rewritten in a more compact way as $\dot{\rho}_t = L(\rho_t)$. L is called a generator of the evolution. A natural way to depart from the world of *closed* systems is to change the corresponding equation of motion by changing the generator. The aim of this paper is to provide basic introduction to so called time-local master equation

$$\dot{\rho}_t = L_t(\rho_t) , \qquad (2)$$

which is fully characterized by the time-dependent generator L_t . Clearly, von Neumann equation is a special case of this general scheme.

There is an alternative approach (see e.g. [1]) which is based on the following non-local equation

$$\dot{\rho}_t = \int_0^t \mathcal{K}_{t-\tau}(\rho_\tau) \, d\tau \,\,, \tag{3}$$

fully characterized by the so called *memory kernel* \mathcal{K}_t . Usually, the non-local character of (3) is attributed to the presence of quantum memory effects: this simply means that the rate of change of the state ρ_t at time t depends on its history (starting at t = 0). Note that time-local master equation (2) with constant generator L is reobtained when $\mathcal{K}_t = 2\delta(t)L$.

One of the fundamental problems in the theory of open quantum systems is to find conditions on L_t and \mathcal{K}_t that ensure that solutions to (2) and (3) are physically legitimate. In the present paper we deal with conditions for time-local generator L_t only. Surprisingly, it turns out that the problem of necessary and sufficient conditions for L_t is open. Our aim is to show why this problem is difficult, what is already known and what are the perspectives.

The paper is organized as follows: in the next Section we introduce basic notation and recall basic notions we use in this paper like states, linear positive maps, quantum channels and all that. Section 3. provides the description of quantum evolution in terms of time-local master equation. Sections 4., 5. and 6. characterize classes of local generators for which the corresponding conditions for L_t may be easily formulated. We analyze time independent generators characterizing Markovian semigroups in Section 4., so called commutative dynamics in Section 5., and Markovian evolution in Section 6.. Section 7. provides a discussion on the general structure of time-local generator: we analyze a simple qubit dynamics to show an intricate structure of the corresponding local generator. Final conclusions are collected in the last section.

2. Preliminaries: quantum states and quantum channels

We begin by introducing basic notation and terminology.

2.1. The structure of quantum states

In this paper we consider a quantum system living in finite-dimensional Hilbert space \mathcal{H} isomorphic to \mathbb{C}^n . Fixing an orthonormal basis $\{e_1, \ldots, e_n\}$ in \mathcal{H} any linear operator in \mathcal{H} may be identified with an $n \times n$ complex matrix, i.e. an element from $M_n(\mathbb{C})$. A mixed state of such system is represented by a density matrix, i.e. a matrix ρ from $M_n(\mathbb{C})$ such that $\rho \geq 0$ and $\operatorname{Tr} \rho = 1$. A space of states \mathfrak{S}_n of *n*-level quantum system defines an $(n^2 - 1)$ -dimensional convex set. Pure states correspond to rank-1 projectors $|\psi\rangle\langle\psi|$ and define extremal elements of \mathfrak{S}_n . Any density matrix may be therefore decomposed as follows

$$\rho = \sum_{k} w_k |\psi_k\rangle \langle \psi_k| , \qquad (4)$$

with $w_k > 0$ and $\sum_k w_k = 1$, i.e. w_k provides a probability distribution. It should be stressed that the above decomposition is highly non-unique.

To illustrate a concept of density operators let us consider the following

Example 1 A 2-level system (qubit) living in \mathbb{C}^2 . Any hermitian operator ρ may be decomposed as follows

$$\rho = \frac{1}{2} (\mathbb{I}_2 + \sum_{k=1}^3 x_k \sigma_k) , \qquad (5)$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\{\sigma_1, \sigma_2, \sigma_3\}$ are Pauli matrices. As usual \mathbb{I}_n denotes a unit matrix in $M_n(\mathbb{C})$. It is, therefore, clear that ρ is entirely characterized by the Bloch vector \mathbf{x} . This representation already guaranties that $\operatorname{Tr} \rho = 1$. Hence, ρ represents density operator if and only if the corresponding eigenvalues $\{\lambda_-, \lambda_+\}$ are non-negative. One easily finds

$$\lambda_{-} = \frac{1}{2}(1 - |\mathbf{x}|) , \quad \lambda_{+} = \frac{1}{2}(1 + |\mathbf{x}|) , \qquad (6)$$

and hence $\rho \geq 0$ if and only if $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq 1$. This condition defines a unit ball in \mathbb{R}^3 known as a Bloch ball. A state is pure if ρ defines rank-1 projector, i.e. $\lambda_- = 0$ and $\lambda_+ = 1$. It shows that pure states belong to Bloch sphere corresponding to $|\mathbf{x}| = 1$. Unfortunately, this simple geometric picture is much more complicated if n > 2.

For any $A \in M_n(\mathbb{C})$ we denote by $||A||_1 := \operatorname{Tr} |A| = \operatorname{Tr} \sqrt{AA^{\dagger}}$ the tracenorm of A. If $\lambda_1, \ldots, \lambda_n$ are (necessarily nonnegative) eigenvalues of AA^{\dagger} , then

$$||A||_1 = \sqrt{\lambda_1} + \ldots + \sqrt{\lambda_n}$$
.

The space of states \mathfrak{S}_n is equipped with a natural metric structure: given two states $\rho, \sigma \in \mathfrak{S}_n$ one defines the corresponding distance

$$D[\rho,\sigma] = \frac{1}{2} ||\rho - \sigma||_1 .$$
(7)

This quantity measurers distinguishability between ρ and σ . It is clear that $D[\rho, \sigma] = 0$, i.e. ρ and σ are indistinguishable, if and only if $\rho = \sigma$. Note, that if ρ and σ are orthogonally supported, then

$$D[\rho,\sigma] = \frac{1}{2}(||\rho||_1 + ||\sigma||_1) = 1$$

since $||\rho||_1 = 1$ for any density matrix ρ . In this case ρ and σ are perfectly distinguishable. Hence

$$0 \le D[\rho, \sigma] \le 1 . \tag{8}$$

In particular, if ρ and σ are two states of a qubit and **x** and **y** are corresponding Bloch vectors then

$$D[\rho,\sigma] = \frac{1}{2} |\mathbf{x} - \mathbf{y}| , \qquad (9)$$

reproduces standard Euclidean distance in \mathbb{R}^3 . For more information about the structure of quantum states see [9, 10].

2.2. Positive and completely positive maps [11, 12, 13]

Consider now a linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and let $M_n^+(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid A \ge 0\} \subset M_n(\mathbb{C})$ be a convex subset of positive matrices. One calls a linear map Φ

- Hermicity-preserving iff $\Phi(A^{\dagger}) = [\Phi(A)]^{\dagger}$,
- positive iff $\Phi(M_n^+(\mathbb{C})) \subset M_n^+(\mathbb{C})$,
- trace-preserving iff $\operatorname{tr} \Phi(A) = \operatorname{tr} A$,
- unital iff $\Phi(\mathbb{I}_n) = \mathbb{I}_n$.

It is easy to show that positive map is necessarily Hermicity-preserving. Moreover, observing that $\mathfrak{S}_n = \{A \in M_n^+(\mathbb{C}) \mid \text{tr } A = 1\}$ it is clear that if Φ is positive and trace preserving than it maps density matrices into density matrices, i.e. $\Phi(\mathfrak{S}_n) \subset \mathfrak{S}_n$. If Φ is a linear map then one defines a dual map $\Phi^* : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$Tr[A\Phi^*(B)] = Tr[\Phi(A)B] , \qquad (10)$$

for all $A, B \in M_n(\mathbb{C})$. Φ is trace-preserving iff Φ^* is unital.

Example 2 Consider a transposition $T_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, i.e. $T_n(\rho) = \rho^T$. Since transposition does not affect the spectrum of A it clear that $A^T \ge 0$ whenever $A \ge 0$. Note that T_n is trace-preserving and unital. As another example consider $R_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by

$$R_n(A) = \frac{1}{n-1} (\mathbb{I}_n \operatorname{Tr} A - A) .$$
 (11)

One calls R_n a reduction map. Let $|x\rangle \in \mathbb{C}^n$ and $P_x := |x\rangle\langle x|$ be the corresponding rank-1 projector (we assume that $\langle x|x\rangle = 1$). One finds $R_n(P_x) = \frac{1}{n-1}P_x^{\perp}$, where P_x^{\perp} is a projector complementary to P_x , i.e. $P_x + P_x^{\perp} = \mathbb{I}_n$. Since any $A \ge 0$ is a convex combination of rank-1 projectors it proves the positivity of R_n . Moreover, R_n is trace-preserving and unital.

Remark 1 Note that fixing an orthonormal basis $\{e_k\}$ in \mathcal{H} and defining $P_k = |e_k\rangle\langle e_k|$ one easily shows that if Φ is a positive trace-preserving map, then the following $n \times n$ matrix

$$T_{ij} = \operatorname{Tr}(P_i \Phi(P_j)) , \qquad (12)$$

is stochastic. Recall, that $A \in M_n(\mathbb{R})$ is a stochastic matrix iff $A_{ij} \ge 0$ and $\sum_{i=1}^n A_{ij} = 1$ for all j = 1, ..., n. Indeed, $T_{ij} \ge 0$ and

$$\sum_{i=1}^{n} T_{ij} = \operatorname{Tr}\left(\left[\sum_{i=1}^{n} P_i\right] \Phi(P_j)\right) = \operatorname{Tr}\left[\Phi(P_j)\right] = 1 , \qquad (13)$$

for all j = 1, ..., n. It is clear, that if Φ is also unital, then T_{ij} is doubly stochastic.

Positive trace-preserving maps possess the following fundamental property

Proposition 1 ([11, 12, 13]) If Φ is positive and trace-preserving, then

$$||\Phi(X)||_1 \le ||X||_1 , \qquad (14)$$

for all $X \in M_n(\mathbb{C})$, that is, Φ is a contraction in trace-norm. Hence

$$D[\Phi(\rho), \Phi(\sigma)] \le D[\rho, \sigma] , \qquad (15)$$

which means that distinguishability of ρ and σ never increases under the action of positive and trace-preserving map.

It turns out that positivity property is not sufficient for quantum physics. It is connected with the notion of composed systems. Composing two systems

living in \mathcal{H}_1 and \mathcal{H}_2 , respectively, one obtains a system living in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let dim $\mathcal{H}_1 = n$, dim $\mathcal{H}_2 = m$ and consider two linear maps

$$\Phi_1: M_n(\mathbb{C}) \to M_n(\mathbb{C}) , \quad \Phi_2: M_m(\mathbb{C}) \to M_m(\mathbb{C})$$

Recalling that $M_{n \times m}(\mathbb{C}) = M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ one defines a tensor product

$$\Phi_1 \otimes \Phi_2 : M_{n \times m}(\mathbb{C}) \to M_{n \times m}(\mathbb{C}) ,$$

as follows: for a fixed orthonormal basis $\{e_1, \ldots, e_n\}$ in \mathcal{H}_1 let us define $e_{ij} := |e_i\rangle\langle e_j| \in M_n(\mathbb{C})$. Elements $\{e_{ij}\}$ for $i, j = 1, \ldots, n$ define an orthonormal basis in $M_n(\mathbb{C})$ with respect to the standard inner product $(A, B) = \operatorname{tr}(A^{\dagger}B)$. Now, any matrix $A \in M_{n \times m}(\mathbb{C})$ may be represented in the following block form

$$A = \sum_{i,j=1}^{n} e_{ij} \otimes A_{ij} , \qquad (16)$$

with $A_{ij} \in M_m(\mathbb{C})$. For example if n = 2 one has

$$A = \sum_{i,j=1}^{2} e_{ij} \otimes A_{ij} = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) .$$
(17)

Hence the action of $\Phi_1 \otimes \Phi_2$ is given by

$$[\Phi_1 \otimes \Phi_2](A) := \sum_{i,j=1}^n \Phi_1(e_{ij}) \otimes \Phi_2(A_{ij}) .$$
 (18)

In particular if n = 2 and $\Phi_1 = \mathbb{1}_2$, where $\mathbb{1}_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ denotes an identity map defined by $\mathbb{1}_n(X) = X$, then

$$[\mathbb{1}_2 \otimes \Phi](A) = \sum_{i,j=1}^2 e_{ij} \otimes \Phi(A_{ij}) = \left(\begin{array}{c|c} \Phi(A_{11}) & \Phi(A_{12}) \\ \hline \Phi(A_{21}) & \Phi(A_{22}) \end{array}\right) .$$
(19)

Now comes a surprise: even if Φ_1 and Φ_2 are positive $\Phi_1 \otimes \Phi_2$ needs not be a positive map.

Example 3 Interestingly, both maps considered in Example 2 loose their positivity when tensoring with other positive maps. This map is evidently positive and trace-preserving. Consider $\mathbb{1}_2 \otimes T_2$ and $\mathbb{1}_2 \otimes R_2$. It turns out that these maps are not positive in $M_4(\mathbb{C})$. Indeed, let

$$P_2^+ = \frac{1}{2} \sum_{i,j=1}^2 e_{ij} \otimes e_{ij} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} , \qquad (20)$$

be a state in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Note that $P_2^+ = |\psi_2^+\rangle \langle \psi_2^+|$ with $\psi_2^+ = (e_1 \otimes e_1 + e_2 \otimes e_2)/\sqrt{2}$ being one of the well-known Bell states of two qubits. One finds

$$[\mathbb{1}_2 \otimes T_2](P_2^+) = \frac{1}{2} \sum_{i,j=1}^2 e_{ij} \otimes e_{ji} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad (21)$$

and

$$[\mathbb{1}_2 \otimes R_2](P_2^+) = \frac{1}{2} \sum_{i,j=1}^2 e_{ij} \otimes R_2(e_{ij}) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} .$$
(22)

Note that both matrices $[\mathbb{1}_2 \otimes T_2](P_2^+)$ and $[\mathbb{1}_2 \otimes R_2](P_2^+)$ have one negative eigenvalue and hence neither $\mathbb{1}_2 \otimes T_2$ nor $\mathbb{1}_2 \otimes R_2$ is a positive map.

This example proves that quantum physics of composed systems needs a more refined notion of positivity. Consider again a linear map $\Phi: M_n(\mathbb{C}) \to M_n(\mathbb{C})$. One calls Φ k-positive if

$$\mathbb{1}_k \otimes \Phi : M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) , \qquad (23)$$

is positive. Clearly 1-positive is just positive and k-positivity implies ℓ positivity for $\ell < k$. Finally, Φ is called *completely positive* (CP) if it is k-positive for $k = 1, 2, \ldots$. Interestingly, one has the following characterization

Proposition 2 (Choi [14]) If dim $\mathcal{H} = n$, then Φ is CP if and only if Φ is *n*-positive.

Denoting by \mathcal{P}_k a convex set of k-positive maps one has the following chain of inclusions

$$CP \equiv \mathcal{P}_n \subset \ldots \subset \mathcal{P}_2 \subset \mathcal{P}_1 \equiv Positive maps .$$

Let $\{e_1, \ldots, e_n\}$ be a fixed orthonormal basis in \mathcal{H} and let

$$|\psi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n e_k \otimes e_k , \qquad (24)$$

denote a maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$. Moreover, let $P_n^+ = |\psi_n^+\rangle \langle \psi_n^+|$ denote the corresponding rank-1 projector.

Proposition 3 (Choi [14]) Φ is CP if and only if $[\mathbb{1}_n \otimes \Phi](P_n^+) \ge 0$.

This beautiful result states that in order to prove that Φ is CP, or equivalently that $\mathbb{1}_n \otimes \Phi$ is positive, it is enough to check wether $\mathbb{1}_n \otimes \Phi$ is positive on one particular projector P_n^+ . Positivity of $[\mathbb{1}_n \otimes \Phi](P_n^+)$ guaranties that $[\mathbb{1}_n \otimes \Phi](X) \ge 0$ for all positive $X \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$.

Corollary 1 If Φ_1 and Φ_2 are CP maps, then $\Phi_1 \otimes \Phi_2$ is always CP as well.

This analysis shows that the motivation to use CP maps is deeply rooted in physics and it is not just mathematical trick! It is the very presence of quantum entangled states enforces us to deal with maps which are completely positive. The following result provides the most important characterization of CP maps.

Theorem 1 ([11, 12, 14, 15]) A map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is CP if and only if

$$\Phi(X) = \sum_{\alpha} K_{\alpha} X K_{\alpha}^{\dagger} , \qquad (25)$$

for $X \in M_n(\mathbb{C})$.

Formula (25) is usually called Kraus or Operator Sum Representation of Φ and K_{α} are called Kraus operators. Actually, the above formula appeared already in the Sudarshan et. al. paper [16]. It should be stressed that the Kraus representation is highly non unique. Completely positive trace preserving map (CPTP) is called a *quantum channel*. A CP map possessing Kraus representation is trace-preserving iff

$$\sum_{\alpha} K_{\alpha}^{\dagger} K_{\alpha} = \mathbb{I}_n .$$
 (26)

The following result shows what is the origin of a genuine quantum channel.

Theorem 2 (Unitary dilation) Any quantum channel Φ may be constructed as follows

$$\Phi(\rho) = \operatorname{tr}_E \left[U(\rho \otimes \omega) U^{\dagger} \right] \,, \tag{27}$$

where U is a unitary operator in $\mathcal{H} \otimes \mathcal{H}_E$ and ω is a density operator in \mathcal{H}_E , and Tr_E denotes partial trace over \mathcal{H}_E .

One usually interprets \mathcal{H}_E as a Hilbert space of the environment and ω as its fixed state. Let

$$\omega |E_k\rangle = \lambda_k |E_k\rangle \; ,$$

with $\lambda_k \geq 0$. Moreover, let $U = \sum_{k,l} U_{kl} \otimes |E_k\rangle \langle E_l|$. Formula (27) implies

$$\Phi(\rho) = \sum_{m,n} \sum_{i,j} \sum_{k} \lambda_k \operatorname{Tr}_E \left[(U_{ij} \otimes |E_i\rangle \langle E_j|) (\rho \otimes |E_k\rangle \langle E_k|) (U_{mn}^{\dagger} \otimes |E_n\rangle \langle E_m|) \right]$$
$$= \sum_{m,n} \sum_{i,j} \sum_{k} \lambda_k \operatorname{Tr}[|E_i\rangle \langle E_j|E_k\rangle \langle E_k|E_n\rangle \langle E_m|] U_{ij}\rho U_{mn}^{\dagger}.$$

Using $\text{Tr}[|E_i\rangle\langle E_j|E_k\rangle\langle E_k|E_n\rangle\langle E_m|] = \delta_{im}\delta_{jk}\delta_{kn}$ and introducing $K_{\alpha} := K_{mn} = \sqrt{\lambda_n} U_{mn}$ one arrives at the Kraus representation $\Phi(\rho) = \sum_{\alpha} K_{\alpha}\rho K_{\alpha}^{\dagger}$ which proves that Φ defined *via* formula (27) is completely positive. One easily proves that Φ is also trace preserving and hence defines a quantum channel.

3. How to describe quantum evolution

If ρ is an initial state of *n*-level quantum system, then by its evolution we mean a trajectory ρ_t for $t \ge 0$ starting at ρ . The simplest example of quantum evolution is provided by the von Neumann equation

$$i\dot{\rho}_t = [H, \rho_t] , \qquad (28)$$

with the corresponding solution

$$\rho_t = \mathcal{U}_t(\rho) \ , \tag{29}$$

where the map $\mathcal{U}_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is defined by

$$\mathcal{U}_t(\rho) := U_t \rho U_t^{\dagger} , \qquad (30)$$

with $U_t = e^{-iHt}$. Note that (30) defines a family of quantum channels. Let us observe that 1-parameter unitary group U_t implies the following composition law

$$\mathcal{U}_t \, \mathcal{U}_s = \mathcal{U}_{t+s} \,\,, \tag{31}$$

for all $t, s \in \mathbb{R}$. Hence \mathcal{U}_t defines 1-parameter group of CP maps. Equation (28) may be transformed into the following equation for \mathcal{U}_t

$$\dot{\mathcal{U}}_t = L \,\mathcal{U}_t \;, \quad \mathcal{U}_0 = \mathbb{1} \;, \tag{32}$$

where the generator $L: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is defined by

$$L(X) = -i[H, X] , \qquad (33)$$

for any $X \in M_n(\mathbb{C})$. Now come natural questions:

1. how to generalize the unitary evolution defined by (30), and

2. how to generalize the corresponding equation of motion (32)?

Definition 1 By a general quantum evolution we mean a dynamical map, i.e. a family of quantum channels $\Lambda_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ for $t \ge 0$ such that $\Lambda_0 = \mathbb{1}_n$.

A dynamical map Λ_t maps an initial state ρ into a current state $\rho_t := \Lambda_t(\rho)$ and hence provides natural generalization of the unitary evolution $\rho_t = \mathcal{U}_t(\rho)$. Assuming that ρ_t satisfies a linear equation and that the initial state ρ provides all necessary information to uniquely determine ρ_t we expect that ρ_t satisfies the following equation

$$\dot{\rho}_t = L_t(\rho_t) , \qquad (34)$$

or equivalently

$$\Lambda_t = L_t \Lambda_t , \quad \Lambda_0 = \mathbb{1}_n , \qquad (35)$$

where $L_t: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ denotes time-dependent generator. This equation provides a natural generalization of (32). The formal solution of (35) may be written as follows

$$\Lambda_t = \mathrm{T} \exp\left(\int_0^t L_\tau \, d\tau\right) \,, \tag{36}$$

where T denotes the chronological product. The above formula is defined by the following Dyson series

$$\Lambda_t = \mathbb{1}_n + \int_0^t dt_1 \, L_{t_1} + \int_0^t dt_1 \int_0^{t_1} dt_2 \, L_{t_1} \, L_{t_2} + \dots , \qquad (37)$$

provided that it converges. In this paper we address the following

Problem 1 What are the properties of local time-dependent generator L_t which guarantee that Λ_t defined by T-product exponential formula (36) defines a legitimate dynamical map?

The formulation of our problem is pretty simple, however, in general the answer is not known. Let us observe that if we knew a dynamical map Λ_t which was invertible, i.e. there exists $\Lambda_t^{-1} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $\Lambda_t^{-1} \Lambda_t = \Lambda_t \Lambda_t^{-1} = \mathbb{1}_n$, then

$$\dot{\Lambda}_t = \dot{\Lambda}_t \Lambda_t^{-1} \Lambda_t = L_t \Lambda_t , \qquad (38)$$

where we defined

$$L_t := \dot{\Lambda}_t \Lambda_t^{-1} . \tag{39}$$

It should be stressed that the inverse of Λ_t needs not be CP. One may prove that if Λ_t is CP then Λ_t^{-1} is CP if and only if $\Lambda_t(\rho) = U_t \rho U_t^{\dagger}$ with unitary U_t .

Example 4 Consider a unitary dynamical map \mathcal{U}_t defined in (30). It is clear that \mathcal{U}_t is invertible and $\mathcal{U}_t^{-1} = \mathcal{U}_{-t}$ is CP. One finds for the corresponding generator

$$L_t(\rho) = [\dot{\mathcal{U}}_t \mathcal{U}_{-t}](\rho) = \dot{U}_t (U_t^{\dagger} \rho U_t) U_t^{\dagger} + U_t (U_t^{\dagger} \rho U_t) \dot{U}_t^{\dagger} , \qquad (40)$$

and hence recalling that U_t satisfies Schrödinger equation $\dot{U}_t = -iHU_t$, one obtains $L_t(\rho) = -i[H, \rho]$.

In this paper instead of analyzing this problem in full generality we restrict ourselves to study special classes of dynamical maps and corresponding local generators. In what follows we analyze 3 important classes of generators

- C_1 a class of time independent generators giving rise to Markovian semigroup,
- C_2 a class of time dependent generators giving rise to commutative dynamics
- C_3 a class of time dependent generators giving rise to the so-called divisible dynamical maps.

Our analysis proves the following relations between these classes: $C_1 \subset C_2 \cap C_3$.

4. Markovian semigroups

In this section we recall the celebrated result derived independently by Gorini, Kossakowski and Sudarshan [18] and Lindblad [19]. Consider the simplest case of a time independent generator L

$$\Lambda_t = L\Lambda_t , \quad \Lambda_0 = \mathbb{1}_n . \tag{41}$$

The formal solution is given by $\Lambda_t = e^{tL}$ for $t \ge 0$ and hence satisfies the following composition law

$$\Lambda_t \Lambda_s = \Lambda_{t+s} , \qquad (42)$$

for $t, s \geq 0$. It proves that Λ_t provides a semigroup of linear maps (it is a semigroup since the inverse Λ_t^{-1} needs not be CP). The properties of L are summarized in the following

Theorem 3 ([18, 19]) A linear map $L: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ generates legitimate dynamical semigroup if and only if

$$L(\rho) = -i[H,\rho] + \Phi(\rho) - \frac{1}{2} \{ \Phi^*(\mathbb{I}_n), \rho \} , \qquad (43)$$

where $\Phi: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is CP, Φ^* denotes the dual map, and $H = H^{\dagger} \in M_n(\mathbb{C})$.

In what follows we call L satisfying (43) a GKSL generator.

Remark 2 If Φ is only a positive map, then L generate a semigroup of positive maps Λ_t which needs not be CP.

Let

$$\Phi(\rho) = \sum_{k} V_k \rho V_k^{\dagger} , \qquad (44)$$

be a Kraus representation of Φ . Its dual is represented by $\Phi^*(X) = \sum_k V_k^{\dagger} X V_k$.

Corollary 2 A generator of a dynamical semigroup can be written in the following form

$$L(\rho) = -i[H,\rho] + \sum_{k} \left(V_{k}\rho V_{k}^{\dagger} - \frac{1}{2} \{ V_{k}^{\dagger} V_{k}, \rho \} \right) , \qquad (45)$$

or equivalently

$$L(\rho) = -i[H,\rho] + \frac{1}{2} \sum_{k} \left([V_k, \rho V_k^{\dagger}] + [V_k \rho, V_k^{\dagger}] \right) .$$
 (46)

Remark 3 Note, that if Φ is CP and trace preserving (i.e. a quantum channel), then $\Phi^*(\mathbb{I}_n) = \mathbb{I}_n$ and hence the formula (43) simplifies to

$$L(\rho) = -i[H,\rho] + \Phi(\rho) - \rho .$$
(47)

Note that the dual map L^* defines the generator of quantum evolution in the Heisenberg picture

$$L^{*}(X) = i[H, X] + \Phi^{*}(X) - \frac{1}{2} \{ \Phi^{*}(\mathbb{I}_{n}), X \} , \qquad (48)$$

or using Kraus representation (44)

$$L^{*}(X) = i[H, X] + \frac{1}{2} \sum_{k} \left([V_{k}^{\dagger}, XV_{k}] + [V_{k}^{\dagger}X, V_{k}] \right) .$$
(49)

Let us consider simple examples of Markovian semigroups Λ_t and corresponding generators L.

Example 5 Suppose that $\Phi: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a quantum channel such that Φ is a CP projector, i.e. $\Phi^2 = \Phi$, and consider

$$L = \gamma (\Phi - \mathbb{1}_n) , \qquad (50)$$

where $\gamma > 0$ and we assumed H = 0. One finds the corresponding solution

$$\Lambda_t = e^{tL} = e^{-\gamma t} e^{\gamma t \Phi} = e^{-\gamma t} \left(\mathbb{1}_n + \gamma t \Phi + \frac{1}{2} (\gamma t)^2 \Phi + \dots \right)$$
$$= e^{-\gamma t} \mathbb{1}_n + (1 - e^{-\gamma t}) \Phi , \qquad (51)$$

that is, Λ_t is a convex combination of two quantum channels: $\mathbb{1}_n$ and Φ . A typical example of a completely positive projector is given by

$$\Phi(\rho) = \sum_{k=1}^{n} P_k \rho P_k , \qquad (52)$$

where $P_k = |k\rangle\langle k|$, i.e. Φ projects ρ onto the diagonal: $\Phi(\rho) = \sum_k \rho_{kk} P_k$. Hence

$$\rho_t = e^{-\gamma t} \rho + (1 - e^{-\gamma t}) \sum_{k=1}^n P_k \rho P_k , \qquad (53)$$

which shows that diagonal elements ρ_{kk} remain invariant and off-diagonal ρ_{kl} are multiplied by the damping factor $e^{-\gamma t}$. It is, therefore, clear that this dynamics describes pure decoherence with respect to an orthonormal basis $\{|1\rangle, \ldots, |n\rangle\}$.

Example 6 Suppose that $\Phi: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a quantum channel such that $\Phi^2 = \mathbb{1}_n$ and consider L defined by (50). One finds the corresponding solution

$$\Lambda_t = e^{tL} = e^{-\gamma t} e^{\gamma t \Phi} = e^{-\gamma t} \left(\mathbb{1}_n + \gamma t \Phi + \frac{1}{2} (\gamma t)^2 \mathbb{1}_n + \frac{1}{3!} (\gamma t)^3 \Phi + \dots \right)$$
$$= e^{-\gamma t} \left[\cosh(\gamma t) \mathbb{1}_n + \sinh(\gamma t) \Phi \right],$$
(54)

or equivalently

$$\Lambda_t = \frac{1}{2} (1 + e^{-2\gamma t}) \mathbb{1}_n + \frac{1}{2} (1 - e^{-2\gamma t}) \Phi , \qquad (55)$$

which is another convex combination of $\mathbb{1}_n$ and Φ . To illustrate this class consider n = 2 and let $\Phi(\rho) = \sigma_z \rho \sigma_z$ which gives rise to the following generator

$$L(\rho) = \gamma(\sigma_z \rho \sigma_z - \rho) .$$
(56)

One finds $\Phi^2(\rho) = \sigma_z(\sigma_z \rho \sigma_z) \sigma_z = \rho$ and hence the corresponding evolution is given by

$$\rho_t = \Lambda_t(\rho) = \frac{1}{2} (1 + e^{-2\gamma t}) \rho + \frac{1}{2} (1 - e^{-2\gamma t}) \sigma_z \rho \sigma_z .$$
 (57)

Again this corresponds to a pure decoherence: $\rho_{12} \rightarrow e^{-2\gamma t}\rho_{12}$ while the diagonal elements ρ_{11} and ρ_{22} remain invariant.

Example 7 Let us consider a qubit generator defined by $H = \frac{\omega}{2}\sigma_z$ and the following CP map

$$\Phi(\rho) = \gamma_1 \sigma_+ \rho \,\sigma_+^\dagger + \gamma_2 \sigma_- \rho \,\sigma_-^\dagger + \gamma \sigma_z \rho \,\sigma_z \,\,, \tag{58}$$

where $\sigma_+ = |2\rangle\langle 1|$ and $\sigma_- = |1\rangle\langle 2| = \sigma_+^{\dagger}$ are standard qubit raising and lowering operators. The corresponding generator reads $L(\rho) = -i[H,\rho] + L_D(\rho)$ with the dissipative part

$$L_D = \frac{\gamma_1}{2} L_1 + \frac{\gamma_2}{2} L_2 + \frac{\gamma}{2} L_z , \qquad (59)$$

where

$$L_{1}(\rho) = [\sigma_{+}, \rho\sigma_{-}] + [\sigma_{+}\rho, \sigma_{-}] ,$$

$$L_{2}(\rho) = [\sigma_{-}, \rho\sigma_{+}] + [\sigma_{-}\rho, \sigma_{+}] ,$$

$$L_{3}(\rho) = \sigma_{z}\rho\sigma_{z} - \rho .$$
(60)

 L_1 corresponds to pumping (heating) process, L_2 corresponds to relaxation (cooling), and L_3 is responsible for pure decoherence. To solve the master equation $\dot{\rho}_t = L\rho_t$ let us parameterize ρ_t as follows

$$\rho_t = p_1(t)P_1 + p_2(t)P_2 + \alpha(t)\sigma_+ + \alpha(t)\sigma_- , \qquad (61)$$

with $P_k = |k\rangle \langle k|$. Using the following relations

$$L(P_1) = \gamma_1(P_2 - P_1) = -\gamma_1 \sigma_3 ,$$

$$L(P_2) = \gamma_2(P_1 - P_2) = \gamma_2 \sigma_3 ,$$

$$L(\sigma_+) = (i\omega - \eta) \sigma_+ ,$$

$$L(\sigma_-) = (-i\omega - \eta) \sigma_- ,$$

where $\eta = \frac{\gamma_1 + \gamma_2}{2} + \gamma$, one finds the following Pauli master equations for the probability distribution $(p_1(t), p_2(t))$

$$\dot{p}_1(t) = -\gamma_1 \, p_1(t) + \gamma_2 \, p_2(t) , \qquad (62)$$

$$\dot{p}_2(t) = \gamma_1 p_1(t) - \gamma_2 p_2(t) ,$$
 (63)

together with $\alpha(t) = e^{(i\omega - \eta)t}\alpha(0)$. The corresponding solution reads

$$p_1(t) = p_1(0) e^{-(\gamma_1 + \gamma_2)t} + p_1^* \left[1 - e^{-(\gamma_1 + \gamma_1)t} \right], \qquad (64)$$

$$p_2(t) = p_2(0) e^{-(\gamma_1 + \gamma_2)t} + p_2^* \left[1 - e^{-(\gamma_2 + \gamma_2)t} \right], \qquad (65)$$

where we introduced

$$p_1^* = \frac{\gamma_1}{\gamma_1 + \gamma_2} , \qquad p_2^* = \frac{\gamma_2}{\gamma_1 + \gamma_2} .$$
 (66)

Hence, we have purely classical evolution of probability vector $(p_1(t), p_2(t))$ on the diagonal of ρ_t and very simple evolution of the off-diagonal element $\alpha(t)$. Note, that asymptotically one obtains completely decohered density operator

$$\rho_t \longrightarrow \begin{pmatrix} p_1^* & 0 \\ 0 & p_2^* \end{pmatrix}.$$

In particular if $\gamma_1 = \gamma_2$ a state ρ_t relaxes to the maximally mixed state (a state becomes completely depolarized).

5. Commutative dynamics

We call a dynamical map Λ_t commutative if $[\Lambda_t, \Lambda_u] = 0$ for all $t, u \ge 0$. It means that for each $A \in M_n(\mathbb{C})$ one has

$$\Lambda_t(\Lambda_u(A)) = \Lambda_u(\Lambda_t(A)) .$$
(67)

It is easy to show that commutativity of Λ_t is equivalent to commutativity of the local generator

$$[L_t, L_u] = 0 {,} {(68)}$$

for any $t, u \ge 0$. Note that in this case the formula (37) considerably simplifies: the 'T' product drops out and the solution is fully controlled by the integral $\int_0^t L_u du$:

$$\Lambda_t = \exp\left(\int_0^t L_u du\right) = \mathbb{1}_n + \int_0^t L_u du + \frac{1}{2}\left(\int_0^t L_u du\right)^2 + \dots$$
(69)

Now, it follows from Theorem 3 that if $\Lambda = e^M$, then Λ is a quantum channel if M is GKSL generator. Therefore, one has the following

Theorem 4 If L_t satisfies (68), then L_t is a legitimate generator if $\int_0^t L_\tau d\tau$ is a GKSL generator for all $t \ge 0$.

Note, that if $L_t = L$ is time independent, then $\int_0^t L_u du = tL$ and the above theorem reproduces Theorem 3. It is clear that if L is a legitimate GKSL generator and $f : \mathbb{R}_+ \to \mathbb{R}$ an arbitrary function, then $L_t = f(t)L$ generates a commutative dynamical map Λ_t iff $\int_0^t f(u) du \ge 0$ for all $t \ge 0$. A typical example of commutative dynamics is provided by

$$L_t = \omega(t)L_0 + a_1(t)L_1 + \ldots + a_N(t)L_N , \qquad (70)$$

where $[L_{\alpha}, L_{\beta}] = 0$ with $L_0(\rho) = -i[H, \rho]$, and for $\alpha > 0$ the generators L_{α} are purely dissipative, that is, $L_{\alpha}(\rho) = \Phi_{\alpha}(\rho) - \frac{1}{2} \{\Phi_{\alpha}^*(\mathbb{I}), \rho\}$. One has for the corresponding dynamical map

$$\Lambda_t = e^{\Omega(t)L_0} \cdot e^{A_1(t)L_1} \cdot \ldots \cdot e^{A_N(t)L_N} , \qquad (71)$$

with

$$\Omega(t) = \int_0^t \omega(u) du ; \quad A_\alpha(t) = \int_0^t a_\alpha(u) du .$$

It is clear that Λ_t is CP iff $A_{\alpha}(t) \ge 0$ for all $\alpha = 1, \ldots, N$.

Example 8 Consider qubit generator $L_0(\rho) = -i[\sigma_3, \rho]$ together with L_1, L_2, L_3 defined in (60). One easily proves

$$[L_0, L_\alpha] = [L_3, L_\alpha] = 0 ; \quad \alpha = 1, 2, 3 , \qquad (72)$$

and

$$[L_1, L_2] = L_1 - L_2 . (73)$$

Define the time-dependent commutative generator

$$L_t = \frac{\omega(t)}{2}L_0 + \frac{\delta(t)}{2}(\mu_1 L_1 + \mu_2 L_2) + \frac{\gamma(t)}{2}L_z , \qquad (74)$$

with $\mu_1, \mu_2 \geq 0$ and $\mu_1 + \mu_2 = 1$. Defining

$$\Omega(t) = \int_0^t \omega(u) du \; ; \quad \Delta(t) = \int_0^t \delta(u) du \; ; \quad \Gamma(t) = \int_0^t \gamma(u) du \; , \qquad (75)$$

one finds that L_t is a legitimate generator iff $\Delta(t) \geq 0$ and $\Gamma(t) \geq 0$. The following evolution of ρ has the following form: the off-diagonal elements evolve according to

$$\rho_{12} \rightarrow e^{\Omega(t) + \frac{1}{2}\Delta(t) + \Gamma(t)} \rho_{12}$$
,

and diagonal elements

$$\rho_{11} \rightarrow \rho_{11} e^{-\Delta(t)} + \mu_1 \Big[1 - e^{-\Delta(t)} \Big] ,$$

$$\rho_{22} \rightarrow \rho_{22} e^{-\Delta(t)} + \mu_2 \Big[1 - e^{-\Delta(t)} \Big] .$$

If $\Delta(t) \to \infty$ for $t \to \infty$, then dynamics possesses an equilibrium state

$$\rho_t \longrightarrow \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

Example 9 (Random unitary qubit dynamics) Consider the following time-dependent generator

$$L_t(\rho) = \frac{1}{2} \sum_{k=1}^{3} \gamma_k(t) (\sigma_k \rho \, \sigma_k - \rho) \,\,, \tag{76}$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ are Pauli matrices. It is easy to prove that $[L_t, L_u] = 0$ and hence L_t generates a legitimate dynamical map iff

$$\Gamma_1(t) \ge 0$$
; $\Gamma_2(t) \ge 0$; $\Gamma_3(t) \ge 0$,

where $\Gamma_k(t) = \int_0^t \gamma_k(u) du$. One finds [21] that the corresponding dynamical map Λ_t is given by

$$\Lambda_t(\rho) = \sum_{\alpha=0}^3 p_\alpha(t) \sigma_\alpha \rho \,\sigma_\alpha \,\,, \tag{77}$$

where $\sigma_0 = \mathbb{I}_2$ and

$$p_{0}(t) = \frac{1}{4} [1 + \lambda_{3}(t) + \lambda_{2}(t) + \lambda_{1}(t)] ,$$

$$p_{1}(t) = \frac{1}{4} [1 - \lambda_{3}(t) - \lambda_{2}(t) + \lambda_{1}(t)] ,$$

$$p_{2}(t) = \frac{1}{4} [1 - \lambda_{3}(t) + \lambda_{2}(t) - \lambda_{1}(t)] ,$$

$$p_{3}(t) = \frac{1}{4} [1 + \lambda_{3}(t) - \lambda_{2}(t) - \lambda_{1}(t)] ,$$

with

$$\lambda_1(t) = e^{-\Gamma_2(t) - \Gamma_3(t)} ,$$

and similarly for $\lambda_2(t)$ and $\lambda_3(t)$. Interestingly $\Lambda_t(\sigma_k) = \lambda_k(t)\sigma_k$. The formula (77) defines so-called random unitary dynamics. Note that

$$p_0(t) + p_1(t) + p_2(t) + p_3(t) = 1$$
.

Moreover, $p_{\alpha}(t) \geq 0$ for $\alpha = 0, 1, 2, 3$ iff $\Gamma_k(t) \geq 0$ for k = 1, 2, 3. Note that Λ_t is unital. Actually, in the case of qubit any unital dynamical map is random unitary, i.e.

$$\Lambda_t(\rho) = \sum_k p_k(t) U_k(t) \rho U_k^{\dagger}(t) , \qquad (78)$$

where $p_k(t)$ defines time-dependent probability distribution and $U_k(t)$ is a family of time-dependent unitary matrices. It is no longer true for n-level systems with n > 2.

6. Markovian evolution — divisible dynamical maps

We call a dynamical map Λ_t divisible if for any $t \ge s \ge 0$ one has the following decomposition

$$\Lambda_t = V_{t,s} \Lambda_s , \qquad (79)$$

with a completely positive propagator $V_{t,s}$. Note, that if Λ_t is invertible then

$$V_{t,s} = \Lambda_t \Lambda_s^{-1} , \qquad (80)$$

and hence $V_{t,s}$ satisfies an inhomogeneous composition law

$$V_{t,s}V_{s,u} = V_{t,u} av{81}$$

for any $t \ge s \ge u$. The above formula provides a generalization of the semigroup composition low (42). In this paper we accept the following definition of Markovian evolution: a dynamical map Λ_t corresponds to Markovian evolution if and only if it is divisible. Interestingly, the property of being Markovian (or divisible) is fully characterized in terms of the local generator L_t . Note, that if Λ_t satisfies (35) then $V_{t,s}$ satisfies

$$\frac{d}{dt}V_{t,s} = L_t V_{t,s} , \quad V_{s,s} = 1 ,$$
 (82)

and the corresponding solution reads

$$V_{t,s} = \mathcal{T} \exp\left(\int_{s}^{t} L_{u} du\right) .$$
(83)

It is clear that $\Lambda_t = V_{t,0}$ which shows that divisibility puts very strong requirements upon the dynamical map Λ_t . One proves [4] the following

Theorem 5 The map Λ_t is divisible if and only if L_t is a GKSL generator for all t, that is,

$$L_t(\rho) = -i[H(t),\rho] + \frac{1}{2} \sum_k \left([V_k(t),\rho V_k^{\dagger}(t)] + [V_k(t)\rho, V_k^{\dagger}(t)] \right) , \qquad (84)$$

with time-dependent Hamiltonian H(t) and noise operators $V_k(t)$.

Remark 4 If

$$L_t = \omega(t)L_0 + a_1(t)L_1 + \dots + a_N(t)L_N , \qquad (85)$$

where $L_0(\rho) = -i[H,\rho]$, and for $\alpha > 0$ the generators L_α are purely dissipative and linearly independent, then L_t generates Markovian evolution if and only if $a_1(t), \ldots, a_N(t) \ge 0$.

Example 10 Consider a qubit generator

$$L_t = -i[H(t), \cdot] + \frac{\gamma_1(t)}{2}L_1 + \frac{\gamma_2(t)}{2}L_2 + \frac{\gamma(t)}{2}L_3 , \qquad (86)$$

where $\{L_1, L_2, L_3\}$ are defined in (60). L_t gives rise to Markovian evolution if and only if $\gamma_1(t), \gamma_2(t), \gamma(t) \ge 0$. We stress that due to the non-commutativity of L_t it is not easy to find the corresponding solution defined in terms of the T-product formula (37) (cf. Section 7.). In particular if we consider the special case corresponding to

$$L_t(\rho) = \frac{1}{2}\gamma(t)L_3(\rho) = \frac{1}{2}\gamma(t)(\sigma_z \rho \sigma_z - \rho) , \qquad (87)$$

and introduce $\Gamma(t) = \int_0^t \gamma(\tau) d\tau$, then it is clear that

$$\Lambda_t(\rho) = \frac{1}{2} \left[1 + e^{-\Gamma(t)} \right] \rho + \frac{1}{2} \left[1 - e^{-\Gamma(t)} \right] \sigma_z \rho \sigma_z , \qquad (88)$$

and hence

- 1. L_t is a legitimate generator iff $\Gamma(t) \geq 0$,
- 2. L_t generates Markovian evolution iff $\gamma(t) \geq 0$,
- 3. L_t generates Markovian semigroup iff $\gamma(t) = \text{const.} > 0$.

Divisible maps possess several important properties [23]. Note that Proposition 1 implies the following

Proposition 4 If $\Lambda_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a dynamical map, then

$$\frac{d}{dt} ||[\mathbb{1}_n \otimes \Lambda_t](X)||_1 \le 0 , \qquad (89)$$

for any Hermitian operator $X \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$.

Proof: one has

$$\frac{d}{dt} ||[\mathbb{1}_n \otimes \Lambda_t](X)||_1 = \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \left[||[\mathbb{1}_n \otimes \Lambda_{t+\epsilon}](X)||_1 - ||[\mathbb{1}_n \otimes \Lambda_t](X)||_1 \right]
= \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \left[||[\mathbb{1}_n \otimes V_{t+\epsilon,t}\Lambda_t](X)||_1 - ||[\mathbb{1}_n \otimes \Lambda_t](X)||_1 \right]
\leq \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \left[||[\mathbb{1}_n \otimes \Lambda_t](X)||_1 - ||[\mathbb{1}_n \otimes \Lambda_t](X)||_1 \right] = 0,$$

where we have used the divisibility property $\Lambda_{t+\epsilon} = V_{t+\epsilon,t}\Lambda_t$ and Proposition 1 for the map $\Phi = \mathbb{1}_n \otimes V_{t+\epsilon,t}$.

Remark 5 Note that taking $X = \mathbb{I}_n \otimes x$ one immediately gets

$$\frac{d}{dt} ||\Lambda_t(x)||_1 \le 0 , \qquad (90)$$

for any Hermitian operator $x \in M_n(\mathbb{C})$. In particular if $X = \rho - \sigma$ one arrives at

$$\frac{d}{dt} ||\Lambda_t(\rho - \sigma)||_1 \le 0 , \qquad (91)$$

for any pair of density matrices ρ and σ .

Remark 6 Actually, the formula (91) was used in [24] as a definition of Markovian dynamics: the dynamics Λ_t is Markovian iff

$$\frac{d}{dt} D[\Lambda_t(\rho), \Lambda_t(\sigma)] \le 0 , \qquad (92)$$

for all initial states $\rho, \sigma \in \mathfrak{S}_n$. It is clear that if Λ_t is divisible, then (92) is satisfied. Note, however, that the converse needs not be true. Consider the following example: let the dynamics Λ_t be governed by the local in time generator

$$L_t(\rho) = \gamma(t) \left(\omega_t \operatorname{Tr} \rho - \rho\right) , \qquad (93)$$

where ω_t is a family of Hermitian operators satisfying Tr $\omega_t = 1$. The above generator gives rise to Markovian evolution iff L_t has the standard GKSL form [18, 19] for all $t \ge 0$, that is, iff $\gamma(t) \ge 0$ and ω_t defines a legitimate state, i.e. $\omega_t \ge 0$. The corresponding solution of the master equation $\dot{\rho}_t = L_t \rho_t$ with an initial condition ρ reads as follows

$$\rho_t = e^{-\Gamma(t)} \widetilde{\rho}_t \ ,$$

where as usual $\Gamma(t) = \int_0^t \gamma(u) du$ and $\tilde{\rho}_t$ satisfies

$$\partial_t \widetilde{\rho}_t = \gamma(t) \omega_t \operatorname{Tr} \widetilde{\rho}_t \; .$$

One has

$$\partial_t [\operatorname{Tr} \widetilde{\rho}_t] = \gamma(t) \operatorname{Tr} \widetilde{\rho}_t ,$$

which implies $\operatorname{Tr} \tilde{\rho}_t = e^{\Gamma(t)} \operatorname{Tr} \tilde{\rho} = e^{\Gamma(t)} \operatorname{Tr} \rho$ due to $\Gamma(0) = 1$. Therefore, one arrives at the following equation

$$\partial_t \widetilde{\rho}_t = \gamma(t) e^{\Gamma(t)} \omega_t \operatorname{Tr} \rho ,$$

with the corresponding solution

$$\widetilde{\rho}_t = \widetilde{\rho} + \left[\int_0^t \gamma(u) e^{\Gamma(u)} \omega_u \, du \right] \operatorname{Tr} \rho \; .$$

$$\rho_t = \Lambda_t(\rho) = e^{-\Gamma(t)}\rho + [1 - e^{-\Gamma(t)}]\Omega_t \operatorname{Tr} \rho , \qquad (94)$$

where

$$\Omega_t = \frac{1}{e^{\Gamma(t)} - 1} \int_0^t \gamma(\tau) e^{\Gamma(\tau)} \,\omega_\tau d\tau \; .$$

It is therefore clear that L_t generates a legitimate quantum evolution iff $\Gamma(t) \geq 0$ and $\Omega_t \geq 0$, that is, Ω_t defines a legitimate state (note, that $\operatorname{Tr} \Omega_t = 1$). In particular, if $\omega_t = \omega$ is time-independent, then $\Omega_t = \omega$ and the solution simplifies to a convex combination of the initial state ρ and the asymptotic invariant state ω : $\rho_t = e^{-\Gamma(t)}\rho + [1 - e^{-\Gamma(t)}]\omega$. One easily shows that the evolution is Markovian iff $\gamma(t) \geq 0$ and ω_t is a legitimate density operator. Consider now condition (92). One has $\rho_t - \sigma_t = e^{-\Gamma(t)}(\rho - \sigma)$ and hence

$$\frac{d}{dt} ||\rho_t - \sigma_t||_1 = -\gamma(t) e^{-\Gamma(t)} ||\rho - \sigma||_1 \le 0 ,$$

implies only $\gamma(t) \geq 0$ but says nothing about ω_t . It shows that any ω_t which gives rise to $\Omega_t \geq 0$ leads to the evolution satisfying condition (92) but only $\omega_t \geq 0$ gives rise to Markovian dynamics. Hence, we may have non-Markovian dynamics which satisfies condition (92) for all $t \geq 0$. In this case ω_t , contrary to Ω_t , is no longer a state. The interested reader will find other examples showing that (92) may differ from divisibility property in [25, 26, 27].

7. The general structure of local generators

Consider now a dynamical map Λ_t and suppose that it can be represented as follows

$$\Lambda_t = e^{Z_t} , \qquad (95)$$

where $Z_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a legitimate GKSL generator for all $t \ge 0$ and $Z_0 = 0$ which guarantees that $\Lambda_0 = \mathbb{1}_n$. It is clear that such representation enforces Λ_t to be a family of quantum channels for $t \ge 0$. Let us assume that Z_t is differentiable and let

$$Z_t = \int_0^t X_\tau d\tau \ . \tag{96}$$

Assume now that Λ_t satisfies local in time Master Equation (35). The corresponding local generator is given by

$$L_t = \dot{\Lambda}_t \Lambda_t^{-1} , \qquad (97)$$

provided that the inverse map Λ_t^{-1} does exist. Note, however, that the representation (95) implies that $\Lambda_t^{-1} = e^{-Z_t}$ and hence L_t is well defined. To find L_t one needs to compute $\partial_t \Lambda_t$. In order to do it we use the following Wilcox formula [28]

$$\partial_t e^{Z_t} = \int_0^1 e^{sZ_t} \dot{Z}_t e^{(1-s)Z_t} ds , \qquad (98)$$

and hence

$$L_t = \dot{\Lambda}_t \Lambda_t^{-1} = \int_0^1 e^{sZ_t} X_t \, e^{-sZ_t} \, ds \,, \tag{99}$$

where we have used $X_t = Z_t$. The formula (99) establishes very general form of the local generator. The construction of a legitimate generator is pretty simple, nevertheless, the formula (99) is highly nontrivial and the computation of L_t out of Z_t might be highly complicated. This is the price we pay for the simple representation of evolution (95). Hence, we have a kind of complementarity: either one uses T-product formula (37) with a relatively simple generator or one avoids the T-product through (95) but uses highly nontrivial generator (99). The advantage of our approach is that one knows how to construct the generator (in practice it might be complicated) giving rise to the legitimate quantum dynamics.

Remark 7 Note, that if Z_t is a commutative family, then

$$0 = \partial_t [Z_t, Z_u] = [X_t, Z_u] , \qquad (100)$$

and hence e^{sZ_t} and X_t commute. In this case formula (99) simplifies to

$$L_t = \int_0^1 e^{sZ_t} X_t e^{-sZ_t} ds = \int_0^1 X_t ds = X_t , \qquad (101)$$

which shows that L_t coincides with X_t . In the noncommutative case this simple relation is no longer true.

Remark 8 It should be clear why we use the special representation $\Lambda_t = e^{Z_t}$ for the dynamical map. Using for example the Kraus representation

$$\Lambda_t(\rho) = \sum_{\alpha} K_{\alpha}(t) \rho \, K_{\alpha}^{\dagger}(t) \; ,$$

there is no simple way to calculate the inverse Λ_t^{-1} . If $\Lambda_t = e^{Z_t}$ we have the inverse $\Lambda_t^{-1} = e^{-Z_t}$ for free!

To illustrate this approach let us consider the following instructive example of a qubit dynamics: let

$$X_t = a_1(t)L_1 + a_2(t)L_2 , (102)$$

where L_1, L_2 are defined in (60) and $a_1(t), a_2(t)$ are real functions of time. One has therefore

$$Z_t = A_1(t)L_1 + A_2(t)L_2 , (103)$$

where

$$A_1(t) = \int_0^t a_1(u) \, du \; ; \quad A_2(t) = \int_0^t a_2(u) \, du \; . \tag{104}$$

It is clear that $\Lambda_t = e^{Z_t}$ defines a legitimate qubit dynamics if and only if $A_k(t) \geq 0$ for k = 1, 2. Our goal now is to find the corresponding local generator L_t using our basic formula (99). Note that a family $X_t : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ provides a noncommutative family of maps due to the following commutation relation (cf. formula (73))

$$[L_1, L_2] = L_1 - L_2 . (105)$$

Hence, L_t is different from X_t . To compute L_t via Wilcox formula (99) let us observe that L_1 and L_2 close a Lie algebra and hence we may use well-known Lie algebraic methods. One has the following quite involved expression for L_t

$$L_t = \int_0^t e^{s[A_1(t)L_1 + A_2(t)L_2]} [a_1(t)L_1 + a_2(t)L_2] e^{-s[A_1(t)L_1 + A_2(t)L_2]} ds .$$
(106)

To deal with it we shall use the well-known Baker-Campbell-Hausdorff (BCH) formula

$$e^{X}Ye^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{X}^{k}Y$$
, (107)

where $\operatorname{ad}_X Y := [X, Y]$. Using the simple commutation rule (105) one easily proves for $k \ge 1$

where we introduced

$$A = A_1 + A_2$$

Therefore, one gets

$$e^{sZ}L_1e^{-sZ} = \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_Z^k L_1 = L_1 + \sum_{k=1}^{\infty} \frac{1}{k!} (-s)^k A^{k-1} A_2 (L_2 - L_1)$$

= $L_1 + (L_2 - L_1) \frac{A_2}{A} \sum_{k=1}^{\infty} \frac{1}{k!} (-sA)^k = L_1 + \frac{A_2}{A} (1 - e^{-sA}) (L_2 - L_1) ,$

and similarly

$$e^{sZ}L_2e^{-sZ} = L_2 + \frac{A_1}{A}(1 - e^{-sA})(L_1 - L_2) ,$$

where to simplify the notation we omit the time dependence of A_1 and A_2 . Inserting

$$e^{sZ}L_1e^{-sZ} = \left(1 - \frac{A_2}{A}\left[1 - e^{-sA}\right]\right)L_1 + \frac{A_2}{A}\left[1 - e^{-sA}\right]L_2 , \qquad (108)$$

and

$$e^{sZ}L_2e^{-sZ} = \left(1 - \frac{A_1}{A}[1 - e^{-sA}]\right)L_2 + \frac{A_1}{A}[1 - e^{-sA}]L_1 , \qquad (109)$$

into (106) one obtains finally

$$L_t = b_1(t)L_1 + b_2(t)L_2 , (110)$$

where the functions $b_1(t)$ and $b_2(t)$ are defined by

$$b_1 = a_1 - f , (111)$$

$$b_2 = a_2 + f , \qquad (112)$$

and the function f(t) reads

$$f = e^{-A} \frac{W}{A} , \qquad (113)$$

with $W = a_1A_2 - a_2A_1$ being the Wronskian of a pair $\{A_1(t), A_2(t)\}$. Hence, the local generator L_t has exactly the same structure as X_t but with functions $a_k(t)$ replaced by $b_k(t)$. Note, that if $a_1(t)$ and $a_2(t)$ are not linearly independent, i.e. $a_2(t) = \lambda a_1(t)$, then the Wronskian vanishes and $b_k = a_k$ which implies $L_t = X_t$. This result should be clear since in this case X_t defines a commutative family $X_t = a_1(t)[L_1 + \lambda L_2]$.

To summarize: the local generator L_t is related with X_t by the following simple relation

$$L_t = X_t - f(t)[L_1 - L_2] . (114)$$

Roughly speaking the function f(t) measures the non-commutativity of the dynamical map Λ_t . Note that we have two representations of Λ_t : as a solution of the Master Equation $\Lambda_t = \text{T} \exp\left(\int_0^t L_u du\right)$ and $\Lambda_t = \exp\left(\int_0^t X_u du\right)$. Using the formula (114) one arrives at the following interesting observation

$$\operatorname{Texp}\left(\int_0^t (X_u - f(u)[L_1 - L_2])du\right) = \exp\left(\int_0^t X_u du\right) , \qquad (115)$$

which shows that the role of T-product is just to kill the unwanted term $f(u)[L_1 - L_2]!$

Note, that the representation $\Lambda_t = \exp(\int_0^t X_u du)$ is legitimate if and only if $A_1(t), A_2(t) \ge 0$. However, this condition says nothing about positivity of $b_1(t)$ and $b_2(t)$. Even the integrals

$$B_1 = A_1 - F ; \quad B_2 = A_2 + F , \tag{116}$$

with $F(t) = \int_0^t f(u) du$ need not be positive. Only the sum

$$B_1(t) + B_2(t) = A_1(t) + A_2(t) \ge 0 , \qquad (117)$$

is fully controlled by A_1 and A_2 . Interestingly, one can prove the following relations between $\{a_1, a_2\}$ and $\{b_1, b_2\}$:

Proposition 5 If $b_1(t), b_2(t) \ge 0$, then $A_1(t), A_2(t) \ge 0$.

This statement is obvious: if $b_1(t), b_2(t) \ge 0$, then L_t generates divisible dynamical map and hence $\Lambda_t = e^{Z_t}$ implies that Z_t is a legitimate GKSL generator for $t \ge 0$ which in turn is equivalent to $A_1(t), A_2(t) \ge 0$. One has also the *dual* result

Proposition 6 If $a_1(t), a_2(t) \ge 0$, then $B_1(t), B_2(t) \ge 0$.

The proof is very easy: due to (116) it is enough to show that

$$-A_2(t) \le F(t) \le A_1(t) , \qquad (118)$$

for all $t \ge 0$. One has

$$F(t) = \int_0^t du \, e^{-A(u)} \Big[a_1(u) A_2(u) - a_2(u) A_1(u) \Big] / A(u)$$

= $\int_0^t du \Big[e^{-A(u)} a_1(u) - e^{-A(u)} A_1(u) [a_1(u) + a_2(u)] / A(u) \Big]$
 $\leq \int_0^t du \, e^{-A(u)} a_1(u) \leq \int_0^t du \, a_1(u) = A_1(t) ,$

and in the same way one proves that $-F(t) \leq A_2(t)$.

Remark 9 Actually, the Master Equation with $L_t = b_1(t)L_1 + b_2(t)L_2$ may be easily solved. Defining

$$\omega_t = p_1(t)|1\rangle\langle 1| + p_2(t)|2\rangle\langle 2| , \qquad (119)$$

with $p_1(t) = b_2(t)/b(t)$ and $p_2(t) = b_1(t)/b(t)$ one easily finds

$$b_1(t)L_1 + b_2(t)L_2 = b(t)[\omega_t \operatorname{Tr} \rho - \rho] - \frac{1}{4}b(t)L_z . \qquad (120)$$

Note that generator $b(t)[\omega_t \operatorname{Tr} \rho - \rho]$ was already considered in Remark 6 (see formula (93). Moreover, one easily checks that this generator commutes with L_z and hence one finds the following formula for Λ_t

$$\Lambda_t(\rho) = \frac{1}{2} e^{-B(t)} \left[(1 + e^{B(t)/2})\rho + (1 - e^{B(t)/2})\sigma_z \rho \sigma_z \right] + (1 - e^{-B(t)})\Omega_t ,$$

where

$$\Omega_t = \frac{1}{e^{B(t)} - 1} \int_0^t b(\tau) e^{B(\tau)} \left[p_1(\tau) |1\rangle \langle 1| + p_2(\tau) |2\rangle \langle 2| \right] d\tau \ .$$

To check for complete positivity of Λ_t one computes $[\mathbb{1}_2 \otimes \Lambda_t](P_2^+)$:

$$\begin{pmatrix} e^{-B} + (1 - e^{-B})\Omega_{11} & \cdot & e^{-B/2} \\ \cdot & (1 - e^{-B})\Omega_{22} & \cdot & \cdot \\ \hline \cdot & \cdot & (1 - e^{-B})\Omega_{11} & \cdot \\ e^{-B/2} & \cdot & e^{-B} + (1 - e^{-B})\Omega_{22} \end{pmatrix} ,$$

where we skiped the time dependence. Now, $[\mathbb{1}_2 \otimes \Lambda_t](P_2^+) \ge 0$ iff $B(t) \ge 0$ and $\Omega_t \ge 0$. The last condition is equivalent to

$$0 \le \int_0^t b_k(\tau) e^{B(\tau)} d\tau \le e^{B(t)} - 1 , \qquad (121)$$

for k = 1, 2. This way we derived conditions for $b_1(t)$ and $b_2(t)$. Note, however, that in order to do this we have to solve the original Master Equation! Only the condition $B(t) \ge 0$ was obtained via the Lie algebraic method.

Corollary 3 If X_t is GKSL generator, then $\int_0^t L_u du$ is GKSL generator. If L_t is GKSL generator (the dynamics is Markovian), then $\int_0^t X_u du$ is GKSL generator.

Proposition 7 The corresponding solution e^{Z_t} may be represented as follows

$$e^{Z_t} = e^{\nu_1(t) L_1} e^{\nu_2(t) L_2} , \qquad (122)$$

$$\nu_1 = \ln\left(\frac{A}{A_1e^{-A} + A_2}\right) , \quad \nu_2 = \ln\left(\frac{A_1 + A_2e^A}{A}\right) , \quad (123)$$

and hence $\nu_1 + \nu_2 = A$.

Proof: Since $\{L_1, L_2\}$ form a Lie algebra one has

$$e^{s(A_1L_1+A_2L_2)} = e^{u_1(s)L_1}e^{u_2(s)L_2} , \qquad (124)$$

where $u_1(s), u_2(s)$ are real functions of a parameter s satisfying $u_1(0) = u_2(0) = 0$. Differentiating both sides with respect to s one gets

$$e^{s(A_1L_1+A_2L_2)}(A_1L_1+A_2L_2)$$

= $u'_1(s)L_1e^{u_1(s)L_1}e^{u_2(s)L_2}+u'_2(s)e^{u_1(s)L_1}e^{u_2(s)L_2}L_2$
= $u'_1(s)L_1e^{s(A_1L_1+A_2L_2)}+u'_2(s)e^{s(A_1L_1+A_2L_2)}L_2$,

where u'(s) = du(s)/ds. Multiplying both sides by $e^{-s(A_1L_1+A_2L_2)}$ leads to

$$A_1L_1 + A_2L_2 = u_1'(s)L_1 + u_2'(s)e^{s(A_1L_1 + A_2L_2)}L_2e^{-s(A_1L_1 + A_2L_2)}$$

and hence using (109) one obtains the following equations for unknown functions u_1 and u_2 :

$$A_1 = u'_1(s) \left[1 - \frac{A_2}{A} (1 - e^{sA}) \right], \qquad (125)$$

$$A_2 = u'_1(s)\frac{A_2}{A}(1-e^{sA}) + u'_2(s) .$$
(126)

Using

$$\int_0^1 \frac{dx}{1+ae^{bx}} = \frac{1}{b} \ln \frac{1+a}{e^{-b}+a} , \qquad (127)$$

one finds

$$u_1(1) = A \int_0^1 \frac{ds}{1 + (A_2/A_1)e^{sA}} = \ln\left(\frac{A}{A_1e^{-A} + A_2}\right) , \qquad (128)$$

and

$$u_2(1) = \ln\left(\frac{A_1 + A_2 e^A}{A}\right)$$
 (129)

Finally, one takes $\nu_1 := u_1(1)$ and $\nu_2 := u_2(1)$. It is clear that $\nu_1, \nu_2 \ge 0$ which proves that $e^{\nu_1(t) L_1}$ and $e^{\nu_2(t) L_2}$ are CPTP maps. \Box

27

Remark 10 This example may be easily generalized: suppose that $\{L_1, \ldots, L_N\}$ is a set of GKSL generators closing a Lie algebra

$$[L_i, L_j] = \sum_{k=1}^{N} c_{ij}^k L_k , \qquad (130)$$

and consider

$$X_t = a_1(t)L_1 + \ldots + a_N(t)L_N , \qquad (131)$$

such that

$$A_1(t) = \int_0^t a_1(u) du \ge 0 \; ; \; \dots \; ; A_N(t) = \int_0^t a_N(u) du \ge 0 \; . \tag{132}$$

It should be clear that Wilcox formula leads to

$$L_t = b_1(t)L_1 + \ldots + b_N(t)L_N , \qquad (133)$$

where the functions $\{b_1(t), \ldots, b_N(t)\}$ are fully determined by $\{a_1(t), \ldots, a_N(t)\}$ and the structure constants c_{ij}^k of the above Lie algebra. There are welldeveloped algebraic methods (like Wei-Norman or Magnus expansions [29, 30] see also the recent review [31]) which may help to deal with such dynamical problems.

8. Conclusions

We provided a basic introduction to the time-local description of open quantum systems. The full characterization of Markovian semigroups, commutative evolutions and dynamics corresponding to divisible maps (Markovian evolution) is provided. In the non-commutative case the Wilcox formula shows the intricate structure of time-local generator L_t and partially explains the problem of finding necessary and sufficient conditions which do guarantee a legitimate quantum evolution generated by L_t . Recently, there is an increasing interest in the non-Markovian quantum evolutions and quantum memory effects. For recent papers devoted to both theoretical and experimental aspects of quantum evolution with memory see e.g. a collection of papers in [32] and references therein. See also Haikka and Maniscalco [33] in this volume.

Acknowledgments

This work was partially supported by the National Science Center project DEC-2011/03/B/ST2/00136. Many thanks to Andrzej Kossakowski for sharing his passion for open quantum systems.

Bibliography

- H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford Univ. Press, Oxford, 2007).
- 2. U. Weiss, Quantum Dissipative Systems, (World Scientific, Singapore, 2000).
- 3. E.B. Davies, Quantum Theory of Open Systems, Academic Press, London, 1976.
- R. Alicki and K. Lendi, Quantum Dynamical Semigroups and Applications (Springer, Berlin, 1987).
- F. Benatti and R. Floreanini, *Irreversible Quantum Dynamics*, Lecture Notes in Physics vol. 622, Springer, Berlin, 2003.
- 6. A. Rivas and S. F. Huelga, Open Quantum Systems: An Introduction, Springer, 2011.
- 7. F. Benatti and R. Floreanini, Int. J. Mod. Phys. B 19, 3063 (2005).
- 8. H.-P. Breuer, J. Phys. B: At. Mol. Opt. Phys. 45, 154001 (2012).
- 9. K. Życzkowski and I. Bengtsson, *Geometry of quantum states*, Cambridge University Press, 2006.
- D. Chruściński and A. Jamiołkowski, Geometric Phases in Classical and Quantum Mechanics, Birkhäuser, Boston 2004.
- V. Paulsen, Completely Bounded Maps and Operator Algebras, Cambridge University Press, 2003.
- R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras, vol I & II. Graduate Studies in Mathematics, American Mathematical Society, 1997.
- 13. R. Bhatia, *Positive Definite Matrices*, Princeton University Press, 2006.
- 14. M.-D. Choi, Lin. Alg. Appl. 10, 285 (1975).
- K. Kraus, States, Effects and Operations: Fundamental Notions of Quantum Theory, Springer Verlag, 1983.
- 16. E. C. G. Sudarshan, P. Mathews, and J. Rau, Phys. Rev. 121, 920 (1961).
- G. Lindblad, Comm. Math. Phys. 39, 111 (1974); G. Lindblad, Non-Equilibrium Entropy and Irreversability, R. Reidel Publishing Company, Dordrecht 2001.
- 18. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. 17, 821 (1976).
- 19. G. Lindblad, Comm. Math. Phys. 48, 119 (1976).
- 20. M.M. Wolf, J. Eisert, T.S. Cubitt, and J.I. Cirac, Phys. Rev. Lett. 101, 150402 (2008).
- 21. D. Chruściński and F. A. Wudarski, Phys. Lett. A 377, 1425 (2013).
- 22. Á. Rivas, S.F. Huelga, and M.B. Plenio, Phys. Rev. Lett. 105, 050403 (2010).
- 23. D. Chruściński and A. Kossakowski, J. Phys. B: At. Mol. Opt. Phys. 45, 154002 (2012).
- 24. H.-P. Breuer, E.-M. Laine, J. Piilo, Phys. Rev. Lett. 103, 210401 (2009).
- 25. P. Haikka, J. D. Cresser, and S. Maniscalco, Phys. Rev. A 83, 012112 (2011).
- B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer, New J. Phys. 13, 093004 (2011).
- 27. D. Chruściński, A. Kossakowski and Á. Rivas, Phys. Rev. A 83, 052128 (2011).
- 28. R. Wilcox, J. Math. Phys. 8, 962 (1967).
- 29. W. Magnus, Comm. Pure and Appl. Math. VII, 649 (1954).

- 30. J. Wei and E. Norman, J. Math. Phys., 4, 575 (1963).
- 31. S. Blanesa, F. Casasb, J.A. Oteoc, and J. Ros, Phys. Rep. 470, 151 (2009).
- F. Benatti, R. Floreanini, and G. Scholes eds., Special issue on loss of coherence and memory effects in quantum dynamics, J. Phys. B 45, No. 15 (2012).
- 33. P. Haikka and A. Maniscalco, Non-Markovian quantum probes (this volume).