# On time-local generators of quantum evolution* 

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#### Abstract

We present a basic introduction to the dynamics of open quantum systems based on local-in-time master equations. We characterize the properties of time-local generators giving rise to legitimate completely positive trace preserving quantum evolutions. The analysis of Markovian and non-Markovian quantum dynamics is presented as well. The whole discussion is illustrated by the family of many instructive examples.


## 1. Introduction

The dynamics of open quantum systems attracts nowadays considerable attention. It is relevant not only for a better understanding of quantum theory but it is fundamental in various modern applications of quantum mechanics such as quantum communication, cryptography, computation and quantum metrology. Any realistic quantum system is an open system since it always interacts with its environment or "rest of the world". The proper descriptions of such systems is therefore of fundamental importance. There are several excellent books $[1,2,3,4,5,6]$ and review articles $[7,8]$ devoted to this subject. In this paper we provide an introduction to the dynamics of open quantum systems based on local-in-time master equation. There are several approaches to the dynamics of open quantum systems. It is clear that each description presents a departure from the standard von Neumann equation

$$
\begin{equation*}
i \dot{\rho}_{t}=\left[H, \rho_{t}\right] \text {, } \tag{1}
\end{equation*}
$$

which governs the evolution of the closed system fully characterized by the system Hamiltonian $H$ (by $\rho_{t}$ we denote the density operator (density matrix) at time $t$ - see the next section for the notation) and $\hbar=1$ throughout the paper. Introducing a linear map acting on the space of density operators

[^0](sometimes called a superoperator) $L(\rho)=-i[H, \rho]$ the von Neumann equation may be rewritten in a more compact way as $\dot{\rho}_{t}=L\left(\rho_{t}\right) . L$ is called a generator of the evolution. A natural way to depart from the world of closed systems is to change the corresponding equation of motion by changing the generator. The aim of this paper is to provide basic introduction to so called time-local master equation
\[

$$
\begin{equation*}
\dot{\rho}_{t}=L_{t}\left(\rho_{t}\right), \tag{2}
\end{equation*}
$$

\]

which is fully characterized by the time-dependent generator $L_{t}$. Clearly, von Neumann equation is a special case of this general scheme.

There is an alternative approach (see e.g. [1]) which is based on the following non-local equation

$$
\begin{equation*}
\dot{\rho}_{t}=\int_{0}^{t} \mathcal{K}_{t-\tau}\left(\rho_{\tau}\right) d \tau \tag{3}
\end{equation*}
$$

fully characterized by the so called memory $\operatorname{kernel} \mathcal{K}_{t}$. Usually, the non-local character of (3) is attributed to the presence of quantum memory effects: this simply means that the rate of change of the state $\rho_{t}$ at time $t$ depends on its history (starting at $t=0$ ). Note that time-local master equation (2) with constant generator $L$ is reobtained when $\mathcal{K}_{t}=2 \delta(t) L$.

One of the fundamental problems in the theory of open quantum systems is to find conditions on $L_{t}$ and $\mathcal{K}_{t}$ that ensure that solutions to (2) and (3) are physically legitimate. In the present paper we deal with conditions for time-local generator $L_{t}$ only. Surprisingly, it turns out that the problem of necessary and sufficient conditions for $L_{t}$ is open. Our aim is to show why this problem is difficult, what is already known and what are the perspectives.

The paper is organized as follows: in the next Section we introduce basic notation and recall basic notions we use in this paper like states, linear positive maps, quantum channels and all that. Section 3. provides the description of quantum evolution in terms of time-local master equation. Sections 4., 5. and 6. characterize classes of local generators for which the corresponding conditions for $L_{t}$ may be easily formulated. We analyze time independent generators characterizing Markovian semigroups in Section 4., so called commutative dynamics in Section 5., and Markovian evolution in Section 6.. Section 7. provides a discussion on the general structure of time-local generator: we analyze a simple qubit dynamics to show an intricate structure of the corresponding local generator. Final conclusions are collected in the last section.

## 2. Preliminaries: quantum states and quantum channels

We begin by introducing basic notation and terminology.

### 2.1. The structure of quantum states

In this paper we consider a quantum system living in finite-dimensional Hilbert space $\mathcal{H}$ isomorphic to $\mathbb{C}^{n}$. Fixing an orthonormal basis $\left\{e_{1}, \ldots, e_{n}\right\}$ in $\mathcal{H}$ any linear operator in $\mathcal{H}$ may be identified with an $n \times n$ complex matrix, i.e. an element from $M_{n}(\mathbb{C})$. A mixed state of such system is represented by a density matrix, i.e. a matrix $\rho$ from $M_{n}(\mathbb{C})$ such that $\rho \geq 0$ and $\operatorname{Tr} \rho=1$. A space of states $\mathfrak{S}_{n}$ of $n$-level quantum system defines an $\left(n^{2}-1\right)$-dimensional convex set. Pure states correspond to rank-1 projectors $|\psi\rangle\langle\psi|$ and define extremal elements of $\mathfrak{S}_{n}$. Any density matrix may be therefore decomposed as follows

$$
\begin{equation*}
\rho=\sum_{k} w_{k}\left|\psi_{k}\right\rangle\left\langle\psi_{k}\right|, \tag{4}
\end{equation*}
$$

with $w_{k}>0$ and $\sum_{k} w_{k}=1$, i.e. $w_{k}$ provides a probability distribution. It should be stressed that the above decomposition is highly non-unique.

To illustrate a concept of density operators let us consider the following
Example 1 A 2-level system (qubit) living in $\mathbb{C}^{2}$. Any hermitian operator $\rho$ may be decomposed as follows

$$
\begin{equation*}
\rho=\frac{1}{2}\left(\mathbb{I}_{2}+\sum_{k=1}^{3} x_{k} \sigma_{k}\right), \tag{5}
\end{equation*}
$$

where $\mathbf{x}=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}$ and $\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}\right\}$ are Pauli matrices. As usual $\mathbb{I}_{n}$ denotes a unit matrix in $M_{n}(\mathbb{C})$. It is, therefore, clear that $\rho$ is entirely characterized by the Bloch vector $\mathbf{x}$. This representation already guaranties that $\operatorname{Tr} \rho=1$. Hence, $\rho$ represents density operator if and only if the corresponding eigenvalues $\left\{\lambda_{-}, \lambda_{+}\right\}$are non-negative. One easily finds

$$
\begin{equation*}
\lambda_{-}=\frac{1}{2}(1-|\mathbf{x}|), \quad \lambda_{+}=\frac{1}{2}(1+|\mathbf{x}|), \tag{6}
\end{equation*}
$$

and hence $\rho \geq 0$ if and only if $|\mathbf{x}|=\sqrt{x_{1}^{2}+x_{2}^{2}+x_{3}^{2}} \leq 1$. This condition defines a unit ball in $\mathbb{R}^{3}$ known as a Bloch ball. A state is pure if $\rho$ defines rank-1 projector, i.e. $\lambda_{-}=0$ and $\lambda_{+}=1$. It shows that pure states belong to Bloch sphere corresponding to $|\mathbf{x}|=1$. Unfortunately, this simple geometric picture is much more complicated if $n>2$.

For any $A \in M_{n}(\mathbb{C})$ we denote by $\|A\|_{1}:=\operatorname{Tr}|A|=\operatorname{Tr} \sqrt{A A^{\dagger}}$ the tracenorm of $A$. If $\lambda_{1}, \ldots \lambda_{n}$ are (necessarily nonnegative) eigenvalues of $A A^{\dagger}$, then

$$
\|A\|_{1}=\sqrt{\lambda_{1}}+\ldots+\sqrt{\lambda_{n}} .
$$

The space of states $\mathfrak{S}_{n}$ is equipped with a natural metric structure: given two states $\rho, \sigma \in \mathfrak{S}_{n}$ one defines the corresponding distance

$$
\begin{equation*}
D[\rho, \sigma]=\frac{1}{2}\|\rho-\sigma\|_{1} . \tag{7}
\end{equation*}
$$

This quantity measurers distinguishability between $\rho$ and $\sigma$. It is clear that $D[\rho, \sigma]=0$, i.e. $\rho$ and $\sigma$ are indistinguishable, if and only if $\rho=\sigma$. Note, that if $\rho$ and $\sigma$ are orthogonally supported, then

$$
D[\rho, \sigma]=\frac{1}{2}\left(\|\rho\|_{1}+\|\sigma\|_{1}\right)=1,
$$

since $\|\rho\|_{1}=1$ for any density matrix $\rho$. In this case $\rho$ and $\sigma$ are perfectly distinguishable. Hence

$$
\begin{equation*}
0 \leq D[\rho, \sigma] \leq 1 \tag{8}
\end{equation*}
$$

In particular, if $\rho$ and $\sigma$ are two states of a qubit and $\mathbf{x}$ and $\mathbf{y}$ are corresponding Bloch vectors then

$$
\begin{equation*}
D[\rho, \sigma]=\frac{1}{2}|\mathbf{x}-\mathbf{y}|, \tag{9}
\end{equation*}
$$

reproduces standard Euclidean distance in $\mathbb{R}^{3}$. For more information about the structure of quantum states see $[9,10]$.

### 2.2. Positive and completely positive maps [11, 12, 13]

Consider now a linear map $\Phi: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ and let $M_{n}^{+}(\mathbb{C})=\{A \in$ $\left.M_{n}(\mathbb{C}) \mid A \geq 0\right\} \subset M_{n}(\mathbb{C})$ be a convex subset of positive matrices. One calls a linear map $\Phi$

- Hermicity-preserving iff $\Phi\left(A^{\dagger}\right)=[\Phi(A)]^{\dagger}$,
- positive iff $\Phi\left(M_{n}^{+}(\mathbb{C})\right) \subset M_{n}^{+}(\mathbb{C})$,
- trace-preserving iff $\operatorname{tr} \Phi(A)=\operatorname{tr} A$,
- unital iff $\Phi\left(\mathbb{I}_{n}\right)=\mathbb{I}_{n}$.

It is easy to show that positive map is necessarily Hermicity-preserving. Moreover, observing that $\mathfrak{S}_{n}=\left\{A \in M_{n}^{+}(\mathbb{C}) \mid \operatorname{tr} A=1\right\}$ it is clear that if $\Phi$ is positive and trace preserving than it maps density matrices into density matrices, i.e. $\Phi\left(\mathfrak{S}_{n}\right) \subset \mathfrak{S}_{n}$. If $\Phi$ is a linear map then one defines a dual map $\Phi^{*}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ by

$$
\begin{equation*}
\operatorname{Tr}\left[A \Phi^{*}(B)\right]=\operatorname{Tr}[\Phi(A) B] \tag{10}
\end{equation*}
$$

for all $A, B \in M_{n}(\mathbb{C}) . \Phi$ is trace-preserving iff $\Phi^{*}$ is unital.

Example 2 Consider a transposition $T_{n}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$, i.e. $T_{n}(\rho)=$ $\rho^{T}$. Since transposition does not affect the spectrum of $A$ it clear that $A^{T} \geq 0$ whenever $A \geq 0$. Note that $T_{n}$ is trace-preserving and unital. As another example consider $R_{n}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ defined by

$$
\begin{equation*}
R_{n}(A)=\frac{1}{n-1}\left(\mathbb{I}_{n} \operatorname{Tr} A-A\right) \tag{11}
\end{equation*}
$$

One calls $R_{n}$ a reduction map. Let $|x\rangle \in \mathbb{C}^{n}$ and $P_{x}:=|x\rangle\langle x|$ be the corresponding rank-1 projector (we assume that $\langle x \mid x\rangle=1$ ). One finds $R_{n}\left(P_{x}\right)=$ $\frac{1}{n-1} P_{x}^{\perp}$, where $P_{x}^{\perp}$ is a projector complementary to $P_{x}$, i.e. $P_{x}+P_{x}^{\perp}=\mathbb{I}_{n}$. Since any $A \geq 0$ is a convex combination of rank-1 projectors it proves the positivity of $R_{n}$. Moreover, $R_{n}$ is trace-preserving and unital.

Remark 1 Note that fixing an orthonormal basis $\left\{e_{k}\right\}$ in $\mathcal{H}$ and defining $P_{k}=\left|e_{k}\right\rangle\left\langle e_{k}\right|$ one easily shows that if $\Phi$ is a positive trace-preserving map, then the following $n \times n$ matrix

$$
\begin{equation*}
T_{i j}=\operatorname{Tr}\left(P_{i} \Phi\left(P_{j}\right)\right), \tag{12}
\end{equation*}
$$

is stochastic. Recall, that $A \in M_{n}(\mathbb{R})$ is a stochastic matrix iff $A_{i j} \geq 0$ and $\sum_{i=1}^{n} A_{i j}=1$ for all $j=1, \ldots, n$. Indeed, $T_{i j} \geq 0$ and

$$
\begin{equation*}
\sum_{i=1}^{n} T_{i j}=\operatorname{Tr}\left(\left[\sum_{i=1}^{n} P_{i}\right] \Phi\left(P_{j}\right)\right)=\operatorname{Tr}\left[\Phi\left(P_{j}\right)\right]=1 \tag{13}
\end{equation*}
$$

for all $j=1, \ldots, n$. It is clear, that if $\Phi$ is also unital, then $T_{i j}$ is doubly stochastic.

Positive trace-preserving maps possess the following fundamental property
Proposition 1 ([11, 12, 13]) If $\Phi$ is positive and trace-preserving, then

$$
\begin{equation*}
\|\Phi(X)\|_{1} \leq\|X\|_{1}, \tag{14}
\end{equation*}
$$

for all $X \in M_{n}(\mathbb{C})$, that is, $\Phi$ is a contraction in trace-norm. Hence

$$
\begin{equation*}
D[\Phi(\rho), \Phi(\sigma)] \leq D[\rho, \sigma] \tag{15}
\end{equation*}
$$

which means that distinguishability of $\rho$ and $\sigma$ never increases under the action of positive and trace-preserving map.

It turns out that positivity property is not sufficient for quantum physics. It is connected with the notion of composed systems. Composing two systems
living in $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$, respectively, one obtains a system living in $\mathcal{H}=\mathcal{H}_{1} \otimes \mathcal{H}_{2}$. Let $\operatorname{dim} \mathcal{H}_{1}=n, \operatorname{dim} \mathcal{H}_{2}=m$ and consider two linear maps

$$
\Phi_{1}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C}), \quad \Phi_{2}: M_{m}(\mathbb{C}) \rightarrow M_{m}(\mathbb{C})
$$

Recalling that $M_{n \times m}(\mathbb{C})=M_{n}(\mathbb{C}) \otimes M_{n}(\mathbb{C})$ one defines a tensor product

$$
\Phi_{1} \otimes \Phi_{2}: M_{n \times m}(\mathbb{C}) \rightarrow M_{n \times m}(\mathbb{C})
$$

as follows: for a fixed orthonormal basis $\left\{e_{1}, \ldots, e_{n}\right\}$ in $\mathcal{H}_{1}$ let us define $e_{i j}:=$ $\left|e_{i}\right\rangle\left\langle e_{j}\right| \in M_{n}(\mathbb{C})$. Elements $\left\{e_{i j}\right\}$ for $i, j=1, \ldots, n$ define an orthonormal basis in $M_{n}(\mathbb{C})$ with respect to the standard inner product $(A, B)=\operatorname{tr}\left(A^{\dagger} B\right)$. Now, any matrix $A \in M_{n \times m}(\mathbb{C})$ may be represented in the following block form

$$
\begin{equation*}
A=\sum_{i, j=1}^{n} e_{i j} \otimes A_{i j} \tag{16}
\end{equation*}
$$

with $A_{i j} \in M_{m}(\mathbb{C})$. For example if $n=2$ one has

$$
A=\sum_{i, j=1}^{2} e_{i j} \otimes A_{i j}=\left(\begin{array}{c|c}
A_{11} & A_{12}  \tag{17}\\
\hline A_{21} & A_{22}
\end{array}\right)
$$

Hence the action of $\Phi_{1} \otimes \Phi_{2}$ is given by

$$
\begin{equation*}
\left[\Phi_{1} \otimes \Phi_{2}\right](A):=\sum_{i, j=1}^{n} \Phi_{1}\left(e_{i j}\right) \otimes \Phi_{2}\left(A_{i j}\right) \tag{18}
\end{equation*}
$$

In particular if $n=2$ and $\Phi_{1}=\mathbb{1}_{2}$, where $\mathbb{1}_{n}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ denotes an identity map defined by $\mathbb{1}_{n}(X)=X$, then

$$
\left[\mathbb{1}_{2} \otimes \Phi\right](A)=\sum_{i, j=1}^{2} e_{i j} \otimes \Phi\left(A_{i j}\right)=\left(\begin{array}{l|l}
\Phi\left(A_{11}\right) & \Phi\left(A_{12}\right)  \tag{19}\\
\hline \Phi\left(A_{21}\right) & \Phi\left(A_{22}\right)
\end{array}\right)
$$

Now comes a surprise: even if $\Phi_{1}$ and $\Phi_{2}$ are positive $\Phi_{1} \otimes \Phi_{2}$ needs not be a positive map.

Example 3 Interestingly, both maps considered in Example 2 loose their positivity when tensoring with other positive maps. This map is evidently positive and trace-preserving. Consider $\mathbb{1}_{2} \otimes T_{2}$ and $\mathbb{1}_{2} \otimes R_{2}$. It turns out that these maps are not positive in $M_{4}(\mathbb{C})$. Indeed, let

$$
P_{2}^{+}=\frac{1}{2} \sum_{i, j=1}^{2} e_{i j} \otimes e_{i j}=\frac{1}{2}\left(\begin{array}{cc|cc}
1 & 0 & 0 & 1  \tag{20}\\
0 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{array}\right)
$$

be a state in $\mathbb{C}^{2} \otimes \mathbb{C}^{2}$. Note that $P_{2}^{+}=\left|\psi_{2}^{+}\right\rangle\left\langle\psi_{2}^{+}\right|$with $\psi_{2}^{+}=\left(e_{1} \otimes e_{1}+\right.$ $\left.e_{2} \otimes e_{2}\right) / \sqrt{2}$ being one of the well-known Bell states of two qubits. One finds

$$
\left[\mathbb{1}_{2} \otimes T_{2}\right]\left(P_{2}^{+}\right)=\frac{1}{2} \sum_{i, j=1}^{2} e_{i j} \otimes e_{j i}=\frac{1}{2}\left(\begin{array}{cc|cc}
1 & 0 & 0 & 0  \tag{21}\\
0 & 0 & 1 & 0 \\
\hline 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

and

$$
\left[\mathbb{1}_{2} \otimes R_{2}\right]\left(P_{2}^{+}\right)=\frac{1}{2} \sum_{i, j=1}^{2} e_{i j} \otimes R_{2}\left(e_{i j}\right)=\frac{1}{2}\left(\begin{array}{cc|cc}
0 & 0 & 0 & -1  \tag{22}\\
0 & 1 & 0 & 0 \\
\hline 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0
\end{array}\right)
$$

Note that both matrices $\left[\mathbb{1}_{2} \otimes T_{2}\right]\left(P_{2}^{+}\right)$and $\left[\mathbb{1}_{2} \otimes R_{2}\right]\left(P_{2}^{+}\right)$have one negative eigenvalue and hence neither $\mathbb{1}_{2} \otimes T_{2}$ nor $\mathbb{1}_{2} \otimes R_{2}$ is a positive map.

This example proves that quantum physics of composed systems needs a more refined notion of positivity. Consider again a linear map $\Phi: M_{n}(\mathbb{C}) \rightarrow$ $M_{n}(\mathbb{C})$. One calls $\Phi k$-positive if

$$
\begin{equation*}
\mathbb{1}_{k} \otimes \Phi: M_{k}(\mathbb{C}) \otimes M_{n}(\mathbb{C}) \rightarrow M_{k}(\mathbb{C}) \otimes M_{n}(\mathbb{C}) \tag{23}
\end{equation*}
$$

is positive. Clearly 1-positive is just positive and $k$-positivity implies $\ell$ positivity for $\ell<k$. Finally, $\Phi$ is called completely positive ( CP ) if it is $k$-positive for $k=1,2, \ldots$. Interestingly, one has the following characterization

Proposition 2 (Choi [14]) If $\operatorname{dim} \mathcal{H}=n$, then $\Phi$ is $C P$ if and only if $\Phi$ is $n$-positive.

Denoting by $\mathcal{P}_{k}$ a convex set of $k$-positive maps one has the following chain of inclusions

$$
\mathrm{CP} \equiv \mathcal{P}_{n} \subset \ldots \subset \mathcal{P}_{2} \subset \mathcal{P}_{1} \equiv \text { Positive maps }
$$

Let $\left\{e_{1}, \ldots, e_{n}\right\}$ be a fixed orthonormal basis in $\mathcal{H}$ and let

$$
\begin{equation*}
\left|\psi_{n}^{+}\right\rangle=\frac{1}{\sqrt{n}} \sum_{k=1}^{n} e_{k} \otimes e_{k} \tag{24}
\end{equation*}
$$

denote a maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$. Moreover, let $P_{n}^{+}=\left|\psi_{n}^{+}\right\rangle\left\langle\psi_{n}^{+}\right|$ denote the corresponding rank-1 projector.

Proposition 3 (Choi [14]) $\Phi$ is $C P$ if and only if $\left[\mathbb{1}_{n} \otimes \Phi\right]\left(P_{n}^{+}\right) \geq 0$.
This beautiful result states that in order to prove that $\Phi$ is CP , or equivalently that $\mathbb{1}_{n} \otimes \Phi$ is positive, it is enough to check wether $\mathbb{1}_{n} \otimes \Phi$ is positive on one particular projector $P_{n}^{+}$. Positivity of $\left[\mathbb{1}_{n} \otimes \Phi\right]\left(P_{n}^{+}\right)$guaranties that $\left[\mathbb{1}_{n} \otimes \Phi\right](X) \geq 0$ for all positive $X \in M_{n}(\mathbb{C}) \otimes M_{n}(\mathbb{C})$.

Corollary 1 If $\Phi_{1}$ and $\Phi_{2}$ are CP maps, then $\Phi_{1} \otimes \Phi_{2}$ is always $C P$ as well.
This analysis shows that the motivation to use CP maps is deeply rooted in physics and it is not just mathematical trick! It is the very presence of quantum entangled states enforces us to deal with maps which are completely positive. The following result provides the most important characterization of CP maps.

Theorem $1([11,12,14,15])$ A map $\Phi: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ is CP if and only if

$$
\begin{equation*}
\Phi(X)=\sum_{\alpha} K_{\alpha} X K_{\alpha}^{\dagger}, \tag{25}
\end{equation*}
$$

for $X \in M_{n}(\mathbb{C})$.
Formula (25) is usually called Kraus or Operator Sum Representation of $\Phi$ and $K_{\alpha}$ are called Kraus operators. Actually, the above formula appeared already in the Sudarshan et. al. paper [16]. It should be stressed that the Kraus representation is highly non unique. Completely positive trace preserving map (CPTP) is called a quantum channel. A CP map possessing Kraus representation is trace-preserving iff

$$
\begin{equation*}
\sum_{\alpha} K_{\alpha}^{\dagger} K_{\alpha}=\mathbb{I}_{n} . \tag{26}
\end{equation*}
$$

The following result shows what is the origin of a genuine quantum channel.
Theorem 2 (Unitary dilation) Any quantum channel $\Phi$ may be constructed as follows

$$
\begin{equation*}
\Phi(\rho)=\operatorname{tr}_{E}\left[U(\rho \otimes \omega) U^{\dagger}\right], \tag{27}
\end{equation*}
$$

where $U$ is a unitary operator in $\mathcal{H} \otimes \mathcal{H}_{E}$ and $\omega$ is a density operator in $\mathcal{H}_{E}$, and $\operatorname{Tr}_{E}$ denotes partial trace over $\mathcal{H}_{E}$.

One usually interprets $\mathcal{H}_{E}$ as a Hilbert space of the environment and $\omega$ as its fixed state. Let

$$
\omega\left|E_{k}\right\rangle=\lambda_{k}\left|E_{k}\right\rangle
$$

with $\lambda_{k} \geq 0$. Moreover, let $U=\sum_{k, l} U_{k l} \otimes\left|E_{k}\right\rangle\left\langle E_{l}\right|$. Formula (27) implies

$$
\begin{aligned}
\Phi(\rho) & =\sum_{m, n} \sum_{i, j} \sum_{k} \lambda_{k} \operatorname{Tr}_{E}\left[\left(U_{i j} \otimes\left|E_{i}\right\rangle\left\langle E_{j}\right|\right)\left(\rho \otimes\left|E_{k}\right\rangle\left\langle E_{k}\right|\right)\left(U_{m n}^{\dagger} \otimes\left|E_{n}\right\rangle\left\langle E_{m}\right|\right)\right] \\
& =\sum_{m, n} \sum_{i, j} \sum_{k} \lambda_{k} \operatorname{Tr}\left[\left|E_{i}\right\rangle\left\langle E_{j} \mid E_{k}\right\rangle\left\langle E_{k} \mid E_{n}\right\rangle\left\langle E_{m}\right|\right] U_{i j} \rho U_{m n}^{\dagger} .
\end{aligned}
$$

Using $\operatorname{Tr}\left[\left|E_{i}\right\rangle\left\langle E_{j} \mid E_{k}\right\rangle\left\langle E_{k} \mid E_{n}\right\rangle\left\langle E_{m}\right|\right]=\delta_{i m} \delta_{j k} \delta_{k n}$ and introducing $K_{\alpha}:=K_{m n}=$ $\sqrt{\lambda_{n}} U_{m n}$ one arrives at the Kraus representation $\Phi(\rho)=\sum_{\alpha} K_{\alpha} \rho K_{\alpha}^{\dagger}$ which proves that $\Phi$ defined via formula (27) is completely positive. One easily proves that $\Phi$ is also trace preserving and hence defines a quantum channel.

## 3. How to describe quantum evolution

If $\rho$ is an initial state of $n$-level quantum system, then by its evolution we mean a trajectory $\rho_{t}$ for $t \geq 0$ starting at $\rho$. The simplest example of quantum evolution is provided by the von Neumann equation

$$
\begin{equation*}
i \dot{\rho}_{t}=\left[H, \rho_{t}\right] \text {, } \tag{28}
\end{equation*}
$$

with the corresponding solution

$$
\begin{equation*}
\rho_{t}=\mathcal{U}_{t}(\rho), \tag{29}
\end{equation*}
$$

where the map $\mathcal{U}_{t}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ is defined by

$$
\begin{equation*}
\mathcal{U}_{t}(\rho):=U_{t} \rho U_{t}^{\dagger}, \tag{30}
\end{equation*}
$$

with $U_{t}=e^{-i H t}$. Note that (30) defines a family of quantum channels. Let us observe that 1-parameter unitary group $U_{t}$ implies the following composition law

$$
\begin{equation*}
\mathcal{U}_{t} \mathcal{U}_{s}=\mathcal{U}_{t+s} \tag{31}
\end{equation*}
$$

for all $t, s \in \mathbb{R}$. Hence $\mathcal{U}_{t}$ defines 1-parameter group of CP maps. Equation (28) may be transformed into the following equation for $\mathcal{U}_{t}$

$$
\begin{equation*}
\dot{\mathcal{U}}_{t}=L \mathcal{U}_{t}, \quad \mathcal{U}_{0}=\mathbb{1} \tag{32}
\end{equation*}
$$

where the generator $L: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ is defined by

$$
\begin{equation*}
L(X)=-i[H, X], \tag{33}
\end{equation*}
$$

for any $X \in M_{n}(\mathbb{C})$. Now come natural questions:

1. how to generalize the unitary evolution defined by (30), and
2. how to generalize the corresponding equation of motion (32)?

Definition 1 By a general quantum evolution we mean a dynamical map, i.e. a family of quantum channels $\Lambda_{t}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ for $t \geq 0$ such that $\Lambda_{0}=\mathbb{1}_{n}$.

A dynamical map $\Lambda_{t}$ maps an initial state $\rho$ into a current state $\rho_{t}:=\Lambda_{t}(\rho)$ and hence provides natural generalization of the unitary evolution $\rho_{t}=\mathcal{U}_{t}(\rho)$. Assuming that $\rho_{t}$ satisfies a linear equation and that the initial state $\rho$ provides all necessary information to uniquely determine $\rho_{t}$ we expect that $\rho_{t}$ satisfies the following equation

$$
\begin{equation*}
\dot{\rho}_{t}=L_{t}\left(\rho_{t}\right), \tag{34}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
\dot{\Lambda}_{t}=L_{t} \Lambda_{t}, \quad \Lambda_{0}=\mathbb{1}_{n}, \tag{35}
\end{equation*}
$$

where $L_{t}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ denotes time-dependent generator. This equation provides a natural generalization of (32). The formal solution of (35) may be written as follows

$$
\begin{equation*}
\Lambda_{t}=\mathrm{T} \exp \left(\int_{0}^{t} L_{\tau} d \tau\right) \tag{36}
\end{equation*}
$$

where T denotes the chronological product. The above formula is defined by the following Dyson series

$$
\begin{equation*}
\Lambda_{t}=\mathbb{1}_{n}+\int_{0}^{t} d t_{1} L_{t_{1}}+\int_{0}^{t} d t_{1} \int_{0}^{t_{1}} d t_{2} L_{t_{1}} L_{t_{2}}+\ldots \tag{37}
\end{equation*}
$$

provided that it converges. In this paper we address the following
Problem 1 What are the properties of local time-dependent generator $L_{t}$ which guarantee that $\Lambda_{t}$ defined by T-product exponential formula (36) defines a legitimate dynamical map?

The formulation of our problem is pretty simple, however, in general the answer is not known. Let us observe that if we knew a dynamical map $\Lambda_{t}$ which was invertible, i.e. there exists $\Lambda_{t}^{-1}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ such that $\Lambda_{t}^{-1} \Lambda_{t}=\Lambda_{t} \Lambda_{t}^{-1}=\mathbb{1}_{n}$, then

$$
\begin{equation*}
\dot{\Lambda}_{t}=\dot{\Lambda}_{t} \Lambda_{t}^{-1} \Lambda_{t}=L_{t} \Lambda_{t}, \tag{38}
\end{equation*}
$$

where we defined

$$
\begin{equation*}
L_{t}:=\dot{\Lambda}_{t} \Lambda_{t}^{-1} \tag{39}
\end{equation*}
$$

It should be stressed that the inverse of $\Lambda_{t}$ needs not be CP. One may prove that if $\Lambda_{t}$ is CP then $\Lambda_{t}^{-1}$ is CP if and only if $\Lambda_{t}(\rho)=U_{t} \rho U_{t}^{\dagger}$ with unitary $U_{t}$.

Example 4 Consider a unitary dynamical map $\mathcal{U}_{t}$ defined in (30). It is clear that $\mathcal{U}_{t}$ is invertible and $\mathcal{U}_{t}^{-1}=\mathcal{U}_{-t}$ is $C P$. One finds for the corresponding generator

$$
\begin{equation*}
L_{t}(\rho)=\left[\dot{\mathcal{U}}_{t} \mathcal{U}_{-t}\right](\rho)=\dot{U}_{t}\left(U_{t}^{\dagger} \rho U_{t}\right) U_{t}^{\dagger}+U_{t}\left(U_{t}^{\dagger} \rho U_{t}\right) \dot{U}_{t}^{\dagger} \tag{40}
\end{equation*}
$$

and hence recalling that $U_{t}$ satisfies Schrödinger equation $\dot{U}_{t}=-i H U_{t}$, one obtains $L_{t}(\rho)=-i[H, \rho]$.

In this paper instead of analyzing this problem in full generality we restrict ourselves to study special classes of dynamical maps and corresponding local generators. In what follows we analyze 3 important classes of generators

- $C_{1}$ - a class of time independent generators giving rise to Markovian semigroup,
- $C_{2}$ - a class of time dependent generators giving rise to commutative dynamics
- $C_{3}$ - a class of time dependent generators giving rise to the so-called divisible dynamical maps.

Our analysis proves the following relations between these classes: $C_{1} \subset$ $C_{2} \cap C_{3}$.

## 4. Markovian semigroups

In this section we recall the celebrated result derived independently by Gorini, Kossakowski and Sudarshan [18] and Lindblad [19]. Consider the simplest case of a time independent generator $L$

$$
\begin{equation*}
\dot{\Lambda}_{t}=L \Lambda_{t}, \quad \Lambda_{0}=\mathbb{1}_{n} \tag{41}
\end{equation*}
$$

The formal solution is given by $\Lambda_{t}=e^{t L}$ for $t \geq 0$ and hence satisfies the following composition law

$$
\begin{equation*}
\Lambda_{t} \Lambda_{s}=\Lambda_{t+s} \tag{42}
\end{equation*}
$$

for $t, s \geq 0$. It proves that $\Lambda_{t}$ provides a semigroup of linear maps (it is a semigroup since the inverse $\Lambda_{t}^{-1}$ needs not be CP). The properties of $L$ are summarized in the following

Theorem $3([18,19])$ A linear map $L: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ generates legitimate dynamical semigroup if and only if

$$
\begin{equation*}
L(\rho)=-i[H, \rho]+\Phi(\rho)-\frac{1}{2}\left\{\Phi^{*}\left(\mathbb{I}_{n}\right), \rho\right\}, \tag{43}
\end{equation*}
$$

where $\Phi: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ is $C P, \Phi^{*}$ denotes the dual map, and $H=H^{\dagger} \in$ $M_{n}(\mathbb{C})$.

In what follows we call $L$ satisfying (43) a GKSL generator.
Remark 2 If $\Phi$ is only a positive map, then $L$ generate a semigroup of positive maps $\Lambda_{t}$ which needs not be $C P$.

Let

$$
\begin{equation*}
\Phi(\rho)=\sum_{k} V_{k} \rho V_{k}^{\dagger} \tag{44}
\end{equation*}
$$

be a Kraus representation of $\Phi$. Its dual is represented by $\Phi^{*}(X)=\sum_{k} V_{k}^{\dagger} X V_{k}$.
Corollary 2 A generator of a dynamical semigroup can be written in the following form

$$
\begin{equation*}
L(\rho)=-i[H, \rho]+\sum_{k}\left(V_{k} \rho V_{k}^{\dagger}-\frac{1}{2}\left\{V_{k}^{\dagger} V_{k}, \rho\right\}\right) \tag{45}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
L(\rho)=-i[H, \rho]+\frac{1}{2} \sum_{k}\left(\left[V_{k}, \rho V_{k}^{\dagger}\right]+\left[V_{k} \rho, V_{k}^{\dagger}\right]\right) \tag{46}
\end{equation*}
$$

Remark 3 Note, that if $\Phi$ is CP and trace preserving (i.e. a quantum channel), then $\Phi^{*}\left(\mathbb{I}_{n}\right)=\mathbb{I}_{n}$ and hence the formula (43) simplifies to

$$
\begin{equation*}
L(\rho)=-i[H, \rho]+\Phi(\rho)-\rho \tag{47}
\end{equation*}
$$

Note that the dual map $L^{*}$ defines the generator of quantum evolution in the Heisenberg picture

$$
\begin{equation*}
L^{*}(X)=i[H, X]+\Phi^{*}(X)-\frac{1}{2}\left\{\Phi^{*}\left(\mathbb{I}_{n}\right), X\right\} \tag{48}
\end{equation*}
$$

or using Kraus representation (44)

$$
\begin{equation*}
L^{*}(X)=i[H, X]+\frac{1}{2} \sum_{k}\left(\left[V_{k}^{\dagger}, X V_{k}\right]+\left[V_{k}^{\dagger} X, V_{k}\right]\right) \tag{49}
\end{equation*}
$$

Let us consider simple examples of Markovian semigroups $\Lambda_{t}$ and corresponding generators $L$.

Example 5 Suppose that $\Phi: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ is a quantum channel such that $\Phi$ is a $C P$ projector, i.e. $\Phi^{2}=\Phi$, and consider

$$
\begin{equation*}
L=\gamma\left(\Phi-\mathbb{1}_{n}\right) \tag{50}
\end{equation*}
$$

where $\gamma>0$ and we assumed $H=0$. One finds the corresponding solution

$$
\begin{align*}
\Lambda_{t} & =e^{t L}=e^{-\gamma t} e^{\gamma t \Phi}=e^{-\gamma t}\left(\mathbb{1}_{n}+\gamma t \Phi+\frac{1}{2}(\gamma t)^{2} \Phi+\ldots\right) \\
& =e^{-\gamma t} \mathbb{1}_{n}+\left(1-e^{-\gamma t}\right) \Phi \tag{51}
\end{align*}
$$

that is, $\Lambda_{t}$ is a convex combination of two quantum channels: $\mathbb{1}_{n}$ and $\Phi . A$ typical example of a completely positive projector is given by

$$
\begin{equation*}
\Phi(\rho)=\sum_{k=1}^{n} P_{k} \rho P_{k} \tag{52}
\end{equation*}
$$

where $P_{k}=|k\rangle\langle k|$, i.e. $\Phi$ projects $\rho$ onto the diagonal: $\Phi(\rho)=\sum_{k} \rho_{k k} P_{k}$. Hence

$$
\begin{equation*}
\rho_{t}=e^{-\gamma t} \rho+\left(1-e^{-\gamma t}\right) \sum_{k=1}^{n} P_{k} \rho P_{k}, \tag{53}
\end{equation*}
$$

which shows that diagonal elements $\rho_{k k}$ remain invariant and off-diagonal $\rho_{k l}$ are multiplied by the damping factor $e^{-\gamma t}$. It is, therefore, clear that this dynamics describes pure decoherence with respect to an orthonormal basis $\{|1\rangle, \ldots,|n\rangle\}$.

Example 6 Suppose that $\Phi: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ is a quantum channel such that $\Phi^{2}=\mathbb{1}_{n}$ and consider $L$ defined by (50). One finds the corresponding solution

$$
\begin{align*}
\Lambda_{t} & =e^{t L}=e^{-\gamma t} e^{\gamma t \Phi}=e^{-\gamma t}\left(\mathbb{1}_{n}+\gamma t \Phi+\frac{1}{2}(\gamma t)^{2} \mathbb{1}_{n}+\frac{1}{3!}(\gamma t)^{3} \Phi+\ldots\right) \\
& =e^{-\gamma t}\left[\cosh (\gamma t) \mathbb{1}_{n}+\sinh (\gamma t) \Phi\right] \tag{54}
\end{align*}
$$

or equivalently

$$
\begin{equation*}
\Lambda_{t}=\frac{1}{2}\left(1+e^{-2 \gamma t}\right) \mathbb{1}_{n}+\frac{1}{2}\left(1-e^{-2 \gamma t}\right) \Phi, \tag{55}
\end{equation*}
$$

which is another convex combination of $\mathbb{1}_{n}$ and $\Phi$. To illustrate this class consider $n=2$ and let $\Phi(\rho)=\sigma_{z} \rho \sigma_{z}$ which gives rise to the following generator

$$
\begin{equation*}
L(\rho)=\gamma\left(\sigma_{z} \rho \sigma_{z}-\rho\right) . \tag{56}
\end{equation*}
$$

One finds $\Phi^{2}(\rho)=\sigma_{z}\left(\sigma_{z} \rho \sigma_{z}\right) \sigma_{z}=\rho$ and hence the corresponding evolution is given by

$$
\begin{equation*}
\rho_{t}=\Lambda_{t}(\rho)=\frac{1}{2}\left(1+e^{-2 \gamma t}\right) \rho+\frac{1}{2}\left(1-e^{-2 \gamma t}\right) \sigma_{z} \rho \sigma_{z} . \tag{57}
\end{equation*}
$$

Again this corresponds to a pure decoherence: $\rho_{12} \rightarrow e^{-2 \gamma t} \rho_{12}$ while the diagonal elements $\rho_{11}$ and $\rho_{22}$ remain invariant.

Example 7 Let us consider a qubit generator defined by $H=\frac{\omega}{2} \sigma_{z}$ and the following CP map

$$
\begin{equation*}
\Phi(\rho)=\gamma_{1} \sigma_{+} \rho \sigma_{+}^{\dagger}+\gamma_{2} \sigma_{-} \rho \sigma_{-}^{\dagger}+\gamma \sigma_{z} \rho \sigma_{z} \tag{58}
\end{equation*}
$$

where $\sigma_{+}=|2\rangle\langle 1|$ and $\sigma_{-}=|1\rangle\langle 2|=\sigma_{+}^{\dagger}$ are standard qubit raising and lowering operators. The corresponding generator reads $L(\rho)=-i[H, \rho]+$ $L_{D}(\rho)$ with the dissipative part

$$
\begin{equation*}
L_{D}=\frac{\gamma_{1}}{2} L_{1}+\frac{\gamma_{2}}{2} L_{2}+\frac{\gamma}{2} L_{z}, \tag{59}
\end{equation*}
$$

where

$$
\begin{align*}
& L_{1}(\rho)=\left[\sigma_{+}, \rho \sigma_{-}\right]+\left[\sigma_{+} \rho, \sigma_{-}\right], \\
& L_{2}(\rho)=\left[\sigma_{-}, \rho \sigma_{+}\right]+\left[\sigma_{-} \rho, \sigma_{+}\right],  \tag{60}\\
& L_{3}(\rho)=\sigma_{z} \rho \sigma_{z}-\rho .
\end{align*}
$$

$L_{1}$ corresponds to pumping (heating) process, $L_{2}$ corresponds to relaxation (cooling), and $L_{3}$ is responsible for pure decoherence. To solve the master equation $\dot{\rho}_{t}=L \rho_{t}$ let us parameterize $\rho_{t}$ as follows

$$
\begin{equation*}
\rho_{t}=p_{1}(t) P_{1}+p_{2}(t) P_{2}+\alpha(t) \sigma_{+}+\overline{\alpha(t)} \sigma_{-}, \tag{61}
\end{equation*}
$$

with $P_{k}=|k\rangle\langle k|$. Using the following relations

$$
\begin{aligned}
L\left(P_{1}\right) & =\gamma_{1}\left(P_{2}-P_{1}\right)=-\gamma_{1} \sigma_{3}, \\
L\left(P_{2}\right) & =\gamma_{2}\left(P_{1}-P_{2}\right)=\gamma_{2} \sigma_{3}, \\
L\left(\sigma_{+}\right) & =(i \omega-\eta) \sigma_{+}, \\
L\left(\sigma_{-}\right) & =(-i \omega-\eta) \sigma_{-},
\end{aligned}
$$

where $\eta=\frac{\gamma_{1}+\gamma_{2}}{2}+\gamma$, one finds the following Pauli master equations for the probability distribution $\left(p_{1}(t), p_{2}(t)\right)$

$$
\begin{align*}
\dot{p}_{1}(t) & =-\gamma_{1} p_{1}(t)+\gamma_{2} p_{2}(t),  \tag{62}\\
\dot{p}_{2}(t) & =\gamma_{1} p_{1}(t)-\gamma_{2} p_{2}(t), \tag{63}
\end{align*}
$$

together with $\alpha(t)=e^{(i \omega-\eta) t} \alpha(0)$. The corresponding solution reads

$$
\begin{align*}
& p_{1}(t)=p_{1}(0) e^{-\left(\gamma_{1}+\gamma_{2}\right) t}+p_{1}^{*}\left[1-e^{-\left(\gamma_{1}+\gamma_{1}\right) t}\right],  \tag{64}\\
& p_{2}(t)=p_{2}(0) e^{-\left(\gamma_{1}+\gamma_{2}\right) t}+p_{2}^{*}\left[1-e^{-\left(\gamma_{2}+\gamma_{2}\right) t}\right], \tag{65}
\end{align*}
$$

where we introduced

$$
\begin{equation*}
p_{1}^{*}=\frac{\gamma_{1}}{\gamma_{1}+\gamma_{2}}, \quad p_{2}^{*}=\frac{\gamma_{2}}{\gamma_{1}+\gamma_{2}} \tag{66}
\end{equation*}
$$

Hence, we have purely classical evolution of probability vector $\left(p_{1}(t), p_{2}(t)\right)$ on the diagonal of $\rho_{t}$ and very simple evolution of the off-diagonal element $\alpha(t)$. Note, that asymptotically one obtains completely decohered density operator

$$
\rho_{t} \longrightarrow\left(\begin{array}{cc}
p_{1}^{*} & 0 \\
0 & p_{2}^{*}
\end{array}\right)
$$

In particular if $\gamma_{1}=\gamma_{2}$ a state $\rho_{t}$ relaxes to the maximally mixed state ( $a$ state becomes completely depolarized).

## 5. Commutative dynamics

We call a dynamical map $\Lambda_{t}$ commutative if $\left[\Lambda_{t}, \Lambda_{u}\right]=0$ for all $t, u \geq 0$. It means that for each $A \in M_{n}(\mathbb{C})$ one has

$$
\begin{equation*}
\Lambda_{t}\left(\Lambda_{u}(A)\right)=\Lambda_{u}\left(\Lambda_{t}(A)\right) \tag{67}
\end{equation*}
$$

It is easy to show that commutativity of $\Lambda_{t}$ is equivalent to commutativity of the local generator

$$
\begin{equation*}
\left[L_{t}, L_{u}\right]=0 \tag{68}
\end{equation*}
$$

for any $t, u \geq 0$. Note that in this case the formula (37) considerably simplifies: the ' T ' product drops out and the solution is fully controlled by the integral $\int_{0}^{t} L_{u} d u$ :

$$
\begin{equation*}
\Lambda_{t}=\exp \left(\int_{0}^{t} L_{u} d u\right)=\mathbb{1}_{n}+\int_{0}^{t} L_{u} d u+\frac{1}{2}\left(\int_{0}^{t} L_{u} d u\right)^{2}+\ldots \tag{69}
\end{equation*}
$$

Now, it follows from Theorem 3 that if $\Lambda=e^{M}$, then $\Lambda$ is a quantum channel if $M$ is GKSL generator. Therefore, one has the following

Theorem 4 If $L_{t}$ satisfies (68), then $L_{t}$ is a legitimate generator if $\int_{0}^{t} L_{\tau} d \tau$ is a GKSL generator for all $t \geq 0$.

Note, that if $L_{t}=L$ is time independent, then $\int_{0}^{t} L_{u} d u=t L$ and the above theorem reproduces Theorem 3. It is clear that if $L$ is a legitimate GKSL generator and $f: \mathbb{R}_{+} \rightarrow \mathbb{R}$ an arbitrary function, then $L_{t}=f(t) L$ generates a commutative dynamical map $\Lambda_{t}$ iff $\int_{0}^{t} f(u) d u \geq 0$ for all $t \geq 0$. A typical example of commutative dynamics is provided by

$$
\begin{equation*}
L_{t}=\omega(t) L_{0}+a_{1}(t) L_{1}+\ldots+a_{N}(t) L_{N} \tag{70}
\end{equation*}
$$

where $\left[L_{\alpha}, L_{\beta}\right]=0$ with $L_{0}(\rho)=-i[H, \rho]$, and for $\alpha>0$ the generators $L_{\alpha}$ are purely dissipative, that is, $L_{\alpha}(\rho)=\Phi_{\alpha}(\rho)-\frac{1}{2}\left\{\Phi_{\alpha}^{*}(\mathbb{I}), \rho\right\}$. One has for the corresponding dynamical map

$$
\begin{equation*}
\Lambda_{t}=e^{\Omega(t) L_{0}} \cdot e^{A_{1}(t) L_{1}} \cdot \ldots \cdot e^{A_{N}(t) L_{N}}, \tag{71}
\end{equation*}
$$

with

$$
\Omega(t)=\int_{0}^{t} \omega(u) d u ; \quad A_{\alpha}(t)=\int_{0}^{t} a_{\alpha}(u) d u .
$$

It is clear that $\Lambda_{t}$ is CP iff $A_{\alpha}(t) \geq 0$ for all $\alpha=1, \ldots, N$.
Example 8 Consider qubit generator $L_{0}(\rho)=-i\left[\sigma_{3}, \rho\right]$ together with $L_{1}, L_{2}, L_{3}$ defined in (60). One easily proves

$$
\begin{equation*}
\left[L_{0}, L_{\alpha}\right]=\left[L_{3}, L_{\alpha}\right]=0 ; \quad \alpha=1,2,3, \tag{72}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[L_{1}, L_{2}\right]=L_{1}-L_{2} \tag{73}
\end{equation*}
$$

Define the time-dependent commutative generator

$$
\begin{equation*}
L_{t}=\frac{\omega(t)}{2} L_{0}+\frac{\delta(t)}{2}\left(\mu_{1} L_{1}+\mu_{2} L_{2}\right)+\frac{\gamma(t)}{2} L_{z}, \tag{74}
\end{equation*}
$$

with $\mu_{1}, \mu_{2} \geq 0$ and $\mu_{1}+\mu_{2}=1$. Defining

$$
\begin{equation*}
\Omega(t)=\int_{0}^{t} \omega(u) d u ; \quad \Delta(t)=\int_{0}^{t} \delta(u) d u ; \quad \Gamma(t)=\int_{0}^{t} \gamma(u) d u \tag{75}
\end{equation*}
$$

one finds that $L_{t}$ is a legitimate generator iff $\Delta(t) \geq 0$ and $\Gamma(t) \geq 0$. The following evolution of $\rho$ has the following form: the off-diagonal elements evolve according to

$$
\rho_{12} \rightarrow e^{\Omega(t)+\frac{1}{2} \Delta(t)+\Gamma(t)} \rho_{12},
$$

and diagonal elements

$$
\begin{aligned}
& \rho_{11} \rightarrow \rho_{11} e^{-\Delta(t)}+\mu_{1}\left[1-e^{-\Delta(t)}\right], \\
& \rho_{22} \rightarrow \rho_{22} e^{-\Delta(t)}+\mu_{2}\left[1-e^{-\Delta(t)}\right] .
\end{aligned}
$$

If $\Delta(t) \rightarrow \infty$ for $t \rightarrow \infty$, then dynamics possesses an equilibrium state

$$
\rho_{t} \longrightarrow\left(\begin{array}{cc}
\mu_{1} & 0 \\
0 & \mu_{2}
\end{array}\right) .
$$

Example 9 (Random unitary qubit dynamics) Consider the following time-dependent generator

$$
\begin{equation*}
L_{t}(\rho)=\frac{1}{2} \sum_{k=1}^{3} \gamma_{k}(t)\left(\sigma_{k} \rho \sigma_{k}-\rho\right) \tag{76}
\end{equation*}
$$

where $\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}\right\}$ are Pauli matrices. It is easy to prove that $\left[L_{t}, L_{u}\right]=0$ and hence $L_{t}$ generates a legitimate dynamical map iff

$$
\Gamma_{1}(t) \geq 0 ; \quad \Gamma_{2}(t) \geq 0 ; \quad \Gamma_{3}(t) \geq 0,
$$

where $\Gamma_{k}(t)=\int_{0}^{t} \gamma_{k}(u) d u$. One finds [21] that the corresponding dynamical map $\Lambda_{t}$ is given by

$$
\begin{equation*}
\Lambda_{t}(\rho)=\sum_{\alpha=0}^{3} p_{\alpha}(t) \sigma_{\alpha} \rho \sigma_{\alpha} \tag{77}
\end{equation*}
$$

where $\sigma_{0}=\mathbb{I}_{2}$ and

$$
\begin{aligned}
& p_{0}(t)=\frac{1}{4}\left[1+\lambda_{3}(t)+\lambda_{2}(t)+\lambda_{1}(t)\right], \\
& p_{1}(t)=\frac{1}{4}\left[1-\lambda_{3}(t)-\lambda_{2}(t)+\lambda_{1}(t)\right], \\
& p_{2}(t)=\frac{1}{4}\left[1-\lambda_{3}(t)+\lambda_{2}(t)-\lambda_{1}(t)\right], \\
& p_{3}(t)=\frac{1}{4}\left[1+\lambda_{3}(t)-\lambda_{2}(t)-\lambda_{1}(t)\right],
\end{aligned}
$$

with

$$
\lambda_{1}(t)=e^{-\Gamma_{2}(t)-\Gamma_{3}(t)},
$$

and similarly for $\lambda_{2}(t)$ and $\lambda_{3}(t)$. Interestingly $\Lambda_{t}\left(\sigma_{k}\right)=\lambda_{k}(t) \sigma_{k}$. The formula (77) defines so-called random unitary dynamics. Note that

$$
p_{0}(t)+p_{1}(t)+p_{2}(t)+p_{3}(t)=1 .
$$

Moreover, $p_{\alpha}(t) \geq 0$ for $\alpha=0,1,2,3$ iff $\Gamma_{k}(t) \geq 0$ for $k=1,2,3$. Note that $\Lambda_{t}$ is unital. Actually, in the case of qubit any unital dynamical map is random unitary, i.e.

$$
\begin{equation*}
\Lambda_{t}(\rho)=\sum_{k} p_{k}(t) U_{k}(t) \rho U_{k}^{\dagger}(t) \tag{78}
\end{equation*}
$$

where $p_{k}(t)$ defines time-dependent probability distribution and $U_{k}(t)$ is a family of time-dependent unitary matrices. It is no longer true for $n$-level systems with $n>2$.

## 6. Markovian evolution - divisible dynamical maps

We call a dynamical map $\Lambda_{t}$ divisible if for any $t \geq s \geq 0$ one has the following decomposition

$$
\begin{equation*}
\Lambda_{t}=V_{t, s} \Lambda_{s}, \tag{79}
\end{equation*}
$$

with a completely positive propagator $V_{t, s}$. Note, that if $\Lambda_{t}$ is invertible then

$$
\begin{equation*}
V_{t, s}=\Lambda_{t} \Lambda_{s}^{-1} \tag{80}
\end{equation*}
$$

and hence $V_{t, s}$ satisfies an inhomogeneous composition law

$$
\begin{equation*}
V_{t, s} V_{s, u}=V_{t, u}, \tag{81}
\end{equation*}
$$

for any $t \geq s \geq u$. The above formula provides a generalization of the semigroup composition low (42). In this paper we accept the following definition of Markovian evolution: a dynamical map $\Lambda_{t}$ corresponds to Markovian evolution if and only if it is divisible. Interestingly, the property of being Markovian (or divisible) is fully characterized in terms of the local generator $L_{t}$. Note, that if $\Lambda_{t}$ satisfies (35) then $V_{t, s}$ satisfies

$$
\begin{equation*}
\frac{d}{d t} V_{t, s}=L_{t} V_{t, s}, \quad V_{s, s}=\mathbb{1}, \tag{82}
\end{equation*}
$$

and the corresponding solution reads

$$
\begin{equation*}
V_{t, s}=\mathrm{T} \exp \left(\int_{s}^{t} L_{u} d u\right) . \tag{83}
\end{equation*}
$$

It is clear that $\Lambda_{t}=V_{t, 0}$ which shows that divisibility puts very strong requirements upon the dynamical map $\Lambda_{t}$. One proves [4] the following

Theorem 5 The map $\Lambda_{t}$ is divisible if and only if $L_{t}$ is a GKSL generator for all $t$, that is,

$$
\begin{equation*}
L_{t}(\rho)=-i[H(t), \rho]+\frac{1}{2} \sum_{k}\left(\left[V_{k}(t), \rho V_{k}^{\dagger}(t)\right]+\left[V_{k}(t) \rho, V_{k}^{\dagger}(t)\right]\right), \tag{84}
\end{equation*}
$$

with time-dependent Hamiltonian $H(t)$ and noise operators $V_{k}(t)$.

## Remark 4 If

$$
\begin{equation*}
L_{t}=\omega(t) L_{0}+a_{1}(t) L_{1}+\ldots+a_{N}(t) L_{N}, \tag{85}
\end{equation*}
$$

where $L_{0}(\rho)=-i[H, \rho]$, and for $\alpha>0$ the generators $L_{\alpha}$ are purely dissipative and linearly independent, then $L_{t}$ generates Markovian evolution if and only if $a_{1}(t), \ldots, a_{N}(t) \geq 0$.

Example 10 Consider a qubit generator

$$
\begin{equation*}
L_{t}=-i[H(t), \cdot]+\frac{\gamma_{1}(t)}{2} L_{1}+\frac{\gamma_{2}(t)}{2} L_{2}+\frac{\gamma(t)}{2} L_{3}, \tag{86}
\end{equation*}
$$

where $\left\{L_{1}, L_{2}, L_{3}\right\}$ are defined in (60). $L_{t}$ gives rise to Markovian evolution if and only if $\gamma_{1}(t), \gamma_{2}(t), \gamma(t) \geq 0$. We stress that due to the non-commutativity of $L_{t}$ it is not easy to find the corresponding solution defined in terms of the T-product formula (37) (cf. Section 7.). In particular if we consider the special case corresponding to

$$
\begin{equation*}
L_{t}(\rho)=\frac{1}{2} \gamma(t) L_{3}(\rho)=\frac{1}{2} \gamma(t)\left(\sigma_{z} \rho \sigma_{z}-\rho\right), \tag{87}
\end{equation*}
$$

and introduce $\Gamma(t)=\int_{0}^{t} \gamma(\tau) d \tau$, then it is clear that

$$
\begin{equation*}
\Lambda_{t}(\rho)=\frac{1}{2}\left[1+e^{-\Gamma(t)}\right] \rho+\frac{1}{2}\left[1-e^{-\Gamma(t)}\right] \sigma_{z} \rho \sigma_{z}, \tag{88}
\end{equation*}
$$

and hence

1. $L_{t}$ is a legitimate generator iff $\Gamma(t) \geq 0$,
2. $L_{t}$ generates Markovian evolution iff $\gamma(t) \geq 0$,
3. $L_{t}$ generates Markovian semigroup iff $\gamma(t)=$ const. $>0$.

Divisible maps possess several important properties [23]. Note that Proposition 1 implies the following

Proposition 4 If $\Lambda_{t}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ is a dynamical map, then

$$
\begin{equation*}
\frac{d}{d t}\left\|\left[\mathbb{1}_{n} \otimes \Lambda_{t}\right](X)\right\|_{1} \leq 0 \tag{89}
\end{equation*}
$$

for any Hermitian operator $X \in M_{n}(\mathbb{C}) \otimes M_{n}(\mathbb{C})$.
Proof: one has

$$
\begin{aligned}
\frac{d}{d t}\left\|\left[\mathbb{1}_{n} \otimes \Lambda_{t}\right](X)\right\|_{1} & =\lim _{\epsilon \rightarrow 0+} \frac{1}{\epsilon}\left[\left\|\left[\mathbb{1}_{n} \otimes \Lambda_{t+\epsilon}\right](X)\right\|\left\|_{1}-\right\|\left[\mathbb{1}_{n} \otimes \Lambda_{t}\right](X) \|_{1}\right] \\
& =\lim _{\epsilon \rightarrow 0+} \frac{1}{\epsilon}\left[\left\|\left[\mathbb{1}_{n} \otimes V_{t+\epsilon, t} \Lambda_{t}\right](X)\right\|_{1}-\left\|\left[\mathbb{1}_{n} \otimes \Lambda_{t}\right](X)\right\|_{1}\right] \\
& \leq \lim _{\epsilon \rightarrow 0+} \frac{1}{\epsilon}\left[\left\|\left[\mathbb{1}_{n} \otimes \Lambda_{t}\right](X)\right\|_{1}-\left\|\left[\mathbb{1}_{n} \otimes \Lambda_{t}\right](X)\right\|_{1}\right]=0,
\end{aligned}
$$

where we have used the divisibility property $\Lambda_{t+\epsilon}=V_{t+\epsilon, t} \Lambda_{t}$ and Proposition 1 for the map $\Phi=\mathbb{1}_{n} \otimes V_{t+\epsilon, t}$.

Remark 5 Note that taking $X=\mathbb{I}_{n} \otimes x$ one immediately gets

$$
\begin{equation*}
\frac{d}{d t}\left\|\Lambda_{t}(x)\right\|_{1} \leq 0 \tag{90}
\end{equation*}
$$

for any Hermitian operator $x \in M_{n}(\mathbb{C})$. In particular if $X=\rho-\sigma$ one arrives at

$$
\begin{equation*}
\frac{d}{d t}\left\|\Lambda_{t}(\rho-\sigma)\right\|_{1} \leq 0 \tag{91}
\end{equation*}
$$

for any pair of density matrices $\rho$ and $\sigma$.
Remark 6 Actually, the formula (91) was used in [24] as a definition of Markovian dynamics: the dynamics $\Lambda_{t}$ is Markovian iff

$$
\begin{equation*}
\frac{d}{d t} D\left[\Lambda_{t}(\rho), \Lambda_{t}(\sigma)\right] \leq 0, \tag{92}
\end{equation*}
$$

for all initial states $\rho, \sigma \in \mathfrak{S}_{n}$. It is clear that if $\Lambda_{t}$ is divisible, then (92) is satisfied. Note, however, that the converse needs not be true. Consider the following example: let the dynamics $\Lambda_{t}$ be governed by the local in time generator

$$
\begin{equation*}
L_{t}(\rho)=\gamma(t)\left(\omega_{t} \operatorname{Tr} \rho-\rho\right), \tag{93}
\end{equation*}
$$

where $\omega_{t}$ is a family of Hermitian operators satisfying $\operatorname{Tr} \omega_{t}=1$. The above generator gives rise to Markovian evolution iff $L_{t}$ has the standard GKSL form [18, 19] for all $t \geq 0$, that is, iff $\gamma(t) \geq 0$ and $\omega_{t}$ defines a legitimate state, i.e. $\omega_{t} \geq 0$. The corresponding solution of the master equation $\dot{\rho}_{t}=$ $L_{t} \rho_{t}$ with an initial condition $\rho$ reads as follows

$$
\rho_{t}=e^{-\Gamma(t)} \widetilde{\rho}_{t},
$$

where as usual $\Gamma(t)=\int_{0}^{t} \gamma(u) d u$ and $\widetilde{\rho}_{t}$ satisfies

$$
\partial_{t} \widetilde{\rho}_{t}=\gamma(t) \omega_{t} \operatorname{Tr} \widetilde{\rho}_{t}
$$

One has

$$
\partial_{t}\left[\operatorname{Tr} \widetilde{\rho}_{t}\right]=\gamma(t) \operatorname{Tr} \widetilde{\rho}_{t},
$$

which implies $\operatorname{Tr} \widetilde{\rho}_{t}=e^{\Gamma(t)} \operatorname{Tr} \widetilde{\rho}=e^{\Gamma(t)} \operatorname{Tr} \rho$ due to $\Gamma(0)=1$. Therefore, one arrives at the following equation

$$
\partial_{t} \widetilde{\rho}_{t}=\gamma(t) e^{\Gamma(t)} \omega_{t} \operatorname{Tr} \rho,
$$

with the corresponding solution

$$
\widetilde{\rho}_{t}=\widetilde{\rho}+\left[\int_{0}^{t} \gamma(u) e^{\Gamma(u)} \omega_{u} d u\right] \operatorname{Tr} \rho .
$$

Finally,

$$
\begin{equation*}
\rho_{t}=\Lambda_{t}(\rho)=e^{-\Gamma(t)} \rho+\left[1-e^{-\Gamma(t)}\right] \Omega_{t} \operatorname{Tr} \rho \tag{94}
\end{equation*}
$$

where

$$
\Omega_{t}=\frac{1}{e^{\Gamma(t)}-1} \int_{0}^{t} \gamma(\tau) e^{\Gamma(\tau)} \omega_{\tau} d \tau
$$

It is therefore clear that $L_{t}$ generates a legitimate quantum evolution iff $\Gamma(t) \geq$ 0 and $\Omega_{t} \geq 0$, that is, $\Omega_{t}$ defines a legitimate state (note, that $\operatorname{Tr} \Omega_{t}=1$ ). In particular, if $\omega_{t}=\omega$ is time-independent, then $\Omega_{t}=\omega$ and the solution simplifies to a convex combination of the initial state $\rho$ and the asymptotic invariant state $\omega: \rho_{t}=e^{-\Gamma(t)} \rho+\left[1-e^{-\Gamma(t)}\right] \omega$. One easily shows that the evolution is Markovian iff $\gamma(t) \geq 0$ and $\omega_{t}$ is a legitimate density operator. Consider now condition (92). One has $\rho_{t}-\sigma_{t}=e^{-\Gamma(t)}(\rho-\sigma)$ and hence

$$
\frac{d}{d t}\left\|\rho_{t}-\sigma_{t}\right\|_{1}=-\gamma(t) e^{-\Gamma(t)}\|\rho-\sigma\|_{1} \leq 0
$$

implies only $\gamma(t) \geq 0$ but says nothing about $\omega_{t}$. It shows that any $\omega_{t}$ which gives rise to $\Omega_{t} \geq 0$ leads to the evolution satisfying condition (92) but only $\omega_{t} \geq 0$ gives rise to Markovian dynamics. Hence, we may have non-Markovian dynamics which satisfies condition (92) for all $t \geq 0$. In this case $\omega_{t}$, contrary to $\Omega_{t}$, is no longer a state. The interested reader will find other examples showing that (92) may differ from divisibility property in [25, 26, 27].

## 7. The general structure of local generators

Consider now a dynamical map $\Lambda_{t}$ and suppose that it can be represented as follows

$$
\begin{equation*}
\Lambda_{t}=e^{Z_{t}} \tag{95}
\end{equation*}
$$

where $Z_{t}: M_{n}(\mathbb{C}) \rightarrow M_{n}(\mathbb{C})$ is a legitimate GKSL generator for all $t \geq 0$ and $Z_{0}=0$ which guarantees that $\Lambda_{0}=\mathbb{1}_{n}$. It is clear that such representation enforces $\Lambda_{t}$ to be a family of quantum channels for $t \geq 0$. Let us assume that $Z_{t}$ is differentiable and let

$$
\begin{equation*}
Z_{t}=\int_{0}^{t} X_{\tau} d \tau \tag{96}
\end{equation*}
$$

Assume now that $\Lambda_{t}$ satisfies local in time Master Equation (35). The corresponding local generator is given by

$$
\begin{equation*}
L_{t}=\dot{\Lambda}_{t} \Lambda_{t}^{-1} \tag{97}
\end{equation*}
$$

provided that the inverse map $\Lambda_{t}^{-1}$ does exist. Note, however, that the representation (95) implies that $\Lambda_{t}^{-1}=e^{-Z_{t}}$ and hence $L_{t}$ is well defined. To find $L_{t}$ one needs to compute $\partial_{t} \Lambda_{t}$. In order to do it we use the following Wilcox formula [28]

$$
\begin{equation*}
\partial_{t} e^{Z_{t}}=\int_{0}^{1} e^{s Z_{t}} \dot{Z}_{t} e^{(1-s) Z_{t}} d s \tag{98}
\end{equation*}
$$

and hence

$$
\begin{equation*}
L_{t}=\dot{\Lambda}_{t} \Lambda_{t}^{-1}=\int_{0}^{1} e^{s Z_{t}} X_{t} e^{-s Z_{t}} d s \tag{99}
\end{equation*}
$$

where we have used $X_{t}=\dot{Z}_{t}$. The formula (99) establishes very general form of the local generator. The construction of a legitimate generator is pretty simple, nevertheless, the formula (99) is highly nontrivial and the computation of $L_{t}$ out of $Z_{t}$ might be highly complicated. This is the price we pay for the simple representation of evolution (95). Hence, we have a kind of complementarity: either one uses T-product formula (37) with a relatively simple generator or one avoids the T-product through (95) but uses highly nontrivial generator (99). The advantage of our approach is that one knows how to construct the generator (in practice it might be complicated) giving rise to the legitimate quantum dynamics.

Remark 7 Note, that if $Z_{t}$ is a commutative family, then

$$
\begin{equation*}
0=\partial_{t}\left[Z_{t}, Z_{u}\right]=\left[X_{t}, Z_{u}\right] \tag{100}
\end{equation*}
$$

and hence $e^{s Z_{t}}$ and $X_{t}$ commute. In this case formula (99) simplifies to

$$
\begin{equation*}
L_{t}=\int_{0}^{1} e^{s Z_{t}} X_{t} e^{-s Z_{t}} d s=\int_{0}^{1} X_{t} d s=X_{t} \tag{101}
\end{equation*}
$$

which shows that $L_{t}$ coincides with $X_{t}$. In the noncommutative case this simple relation is no longer true.

Remark 8 It should be clear why we use the special representation $\Lambda_{t}=e^{Z_{t}}$ for the dynamical map. Using for example the Kraus representation

$$
\Lambda_{t}(\rho)=\sum_{\alpha} K_{\alpha}(t) \rho K_{\alpha}^{\dagger}(t)
$$

there is no simple way to calculate the inverse $\Lambda_{t}^{-1}$. If $\Lambda_{t}=e^{Z_{t}}$ we have the inverse $\Lambda_{t}^{-1}=e^{-Z_{t}}$ for free!

To illustrate this approach let us consider the following instructive example of a qubit dynamics: let

$$
\begin{equation*}
X_{t}=a_{1}(t) L_{1}+a_{2}(t) L_{2}, \tag{102}
\end{equation*}
$$

where $L_{1}, L_{2}$ are defined in (60) and $a_{1}(t), a_{2}(t)$ are real functions of time. One has therefore

$$
\begin{equation*}
Z_{t}=A_{1}(t) L_{1}+A_{2}(t) L_{2} \tag{103}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{1}(t)=\int_{0}^{t} a_{1}(u) d u ; \quad A_{2}(t)=\int_{0}^{t} a_{2}(u) d u \tag{104}
\end{equation*}
$$

It is clear that $\Lambda_{t}=e^{Z_{t}}$ defines a legitimate qubit dynamics if and only if $A_{k}(t) \geq 0$ for $k=1,2$. Our goal now is to find the corresponding local generator $L_{t}$ using our basic formula (99). Note that a family $X_{t}: M_{2}(\mathbb{C}) \rightarrow M_{2}(\mathbb{C})$ provides a noncommutative family of maps due to the following commutation relation (cf. formula (73))

$$
\begin{equation*}
\left[L_{1}, L_{2}\right]=L_{1}-L_{2} \tag{105}
\end{equation*}
$$

Hence, $L_{t}$ is different from $X_{t}$. To compute $L_{t}$ via Wilcox formula (99) let us observe that $L_{1}$ and $L_{2}$ close a Lie algebra and hence we may use well-known Lie algebraic methods. One has the following quite involved expression for $L_{t}$

$$
\begin{equation*}
L_{t}=\int_{0}^{t} e^{s\left[A_{1}(t) L_{1}+A_{2}(t) L_{2}\right]}\left[a_{1}(t) L_{1}+a_{2}(t) L_{2}\right] e^{-s\left[A_{1}(t) L_{1}+A_{2}(t) L_{2}\right]} d s \tag{106}
\end{equation*}
$$

To deal with it we shall use the well-known Baker-Campbell-Hausdorff (BCH) formula

$$
\begin{equation*}
e^{X} Y e^{-X}=Y+[X, Y]+\frac{1}{2!}[X,[X, Y]]+\ldots=\sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{X}^{k} Y \tag{107}
\end{equation*}
$$

where $\operatorname{ad}_{X} Y:=[X, Y]$. Using the simple commutation rule (105) one easily proves for $k \geq 1$

$$
\begin{aligned}
\operatorname{ad}_{Z}^{k} L_{1} & =(-1)^{k} A^{k-1} A_{2}\left(L_{2}-L_{1}\right), \\
\operatorname{ad}_{Z}^{k} L_{2} & =(-1)^{k} A^{k-1} A_{1}\left(L_{1}-L_{2}\right),
\end{aligned}
$$

where we introduced

$$
A=A_{1}+A_{2}
$$

Therefore, one gets

$$
\begin{aligned}
& e^{s Z} L_{1} e^{-s Z}=\sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{Z}^{k} L_{1}=L_{1}+\sum_{k=1}^{\infty} \frac{1}{k!}(-s)^{k} A^{k-1} A_{2}\left(L_{2}-L_{1}\right) \\
= & L_{1}+\left(L_{2}-L_{1}\right) \frac{A_{2}}{A} \sum_{k=1}^{\infty} \frac{1}{k!}(-s A)^{k}=L_{1}+\frac{A_{2}}{A}\left(1-e^{-s A}\right)\left(L_{2}-L_{1}\right),
\end{aligned}
$$

and similarly

$$
e^{s Z} L_{2} e^{-s Z}=L_{2}+\frac{A_{1}}{A}\left(1-e^{-s A}\right)\left(L_{1}-L_{2}\right),
$$

where to simplify the notation we omit the time dependence of $A_{1}$ and $A_{2}$. Inserting

$$
\begin{equation*}
e^{s Z} L_{1} e^{-s Z}=\left(1-\frac{A_{2}}{A}\left[1-e^{-s A}\right]\right) L_{1}+\frac{A_{2}}{A}\left[1-e^{-s A}\right] L_{2}, \tag{108}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{s Z} L_{2} e^{-s Z}=\left(1-\frac{A_{1}}{A}\left[1-e^{-s A}\right]\right) L_{2}+\frac{A_{1}}{A}\left[1-e^{-s A}\right] L_{1}, \tag{109}
\end{equation*}
$$

into (106) one obtains finally

$$
\begin{equation*}
L_{t}=b_{1}(t) L_{1}+b_{2}(t) L_{2}, \tag{110}
\end{equation*}
$$

where the functions $b_{1}(t)$ and $b_{2}(t)$ are defined by

$$
\begin{align*}
& b_{1}=a_{1}-f,  \tag{111}\\
& b_{2}=a_{2}+f, \tag{112}
\end{align*}
$$

and the function $f(t)$ reads

$$
\begin{equation*}
f=e^{-A} \frac{W}{A}, \tag{113}
\end{equation*}
$$

with $W=a_{1} A_{2}-a_{2} A_{1}$ being the Wronskian of a pair $\left\{A_{1}(t), A_{2}(t)\right\}$. Hence, the local generator $L_{t}$ has exactly the same structure as $X_{t}$ but with functions $a_{k}(t)$ replaced by $b_{k}(t)$. Note, that if $a_{1}(t)$ and $a_{2}(t)$ are not linearly independent, i.e. $a_{2}(t)=\lambda a_{1}(t)$, then the Wronskian vanishes and $b_{k}=a_{k}$ which implies $L_{t}=X_{t}$. This result should be clear since in this case $X_{t}$ defines a commutative family $X_{t}=a_{1}(t)\left[L_{1}+\lambda L_{2}\right]$.

To summarize: the local generator $L_{t}$ is related with $X_{t}$ by the following simple relation

$$
\begin{equation*}
L_{t}=X_{t}-f(t)\left[L_{1}-L_{2}\right] . \tag{114}
\end{equation*}
$$

Roughly speaking the function $f(t)$ measures the non-commutativity of the dynamical map $\Lambda_{t}$. Note that we have two representations of $\Lambda_{t}$ : as a solution of the Master Equation $\Lambda_{t}=\mathrm{T} \exp \left(\int_{0}^{t} L_{u} d u\right)$ and $\Lambda_{t}=\exp \left(\int_{0}^{t} X_{u} d u\right)$. Using the formula (114) one arrives at the following interesting observation

$$
\begin{equation*}
\mathrm{T} \exp \left(\int_{0}^{t}\left(X_{u}-f(u)\left[L_{1}-L_{2}\right]\right) d u\right)=\exp \left(\int_{0}^{t} X_{u} d u\right) \tag{115}
\end{equation*}
$$

which shows that the role of T -product is just to kill the unwanted term $f(u)\left[L_{1}-L_{2}\right]$ !

Note, that the representation $\Lambda_{t}=\exp \left(\int_{0}^{t} X_{u} d u\right)$ is legitimate if and only if $A_{1}(t), A_{2}(t) \geq 0$. However, this condition says nothing about positivity of $b_{1}(t)$ and $b_{2}(t)$. Even the integrals

$$
\begin{equation*}
B_{1}=A_{1}-F ; \quad B_{2}=A_{2}+F \tag{116}
\end{equation*}
$$

with $F(t)=\int_{0}^{t} f(u) d u$ need not be positive. Only the sum

$$
\begin{equation*}
B_{1}(t)+B_{2}(t)=A_{1}(t)+A_{2}(t) \geq 0 \tag{117}
\end{equation*}
$$

is fully controlled by $A_{1}$ and $A_{2}$. Interestingly, one can prove the following relations between $\left\{a_{1}, a_{2}\right\}$ and $\left\{b_{1}, b_{2}\right\}$ :

Proposition 5 If $b_{1}(t), b_{2}(t) \geq 0$, then $A_{1}(t), A_{2}(t) \geq 0$.
This statement is obvious: if $b_{1}(t), b_{2}(t) \geq 0$, then $L_{t}$ generates divisible dynamical map and hence $\Lambda_{t}=e^{Z_{t}}$ implies that $Z_{t}$ is a legitimate GKSL generator for $t \geq 0$ which in turn is equivalent to $A_{1}(t), A_{2}(t) \geq 0$. One has also the dual result

Proposition 6 If $a_{1}(t), a_{2}(t) \geq 0$, then $B_{1}(t), B_{2}(t) \geq 0$.
The proof is very easy: due to (116) it is enough to show that

$$
\begin{equation*}
-A_{2}(t) \leq F(t) \leq A_{1}(t) \tag{118}
\end{equation*}
$$

for all $t \geq 0$. One has

$$
\begin{aligned}
F(t) & =\int_{0}^{t} d u e^{-A(u)}\left[a_{1}(u) A_{2}(u)-a_{2}(u) A_{1}(u)\right] / A(u) \\
& =\int_{0}^{t} d u\left[e^{-A(u)} a_{1}(u)-e^{-A(u)} A_{1}(u)\left[a_{1}(u)+a_{2}(u)\right] / A(u)\right] \\
& \leq \int_{0}^{t} d u e^{-A(u)} a_{1}(u) \leq \int_{0}^{t} d u a_{1}(u)=A_{1}(t)
\end{aligned}
$$

and in the same way one proves that $-F(t) \leq A_{2}(t)$.

Remark 9 Actually, the Master Equation with $L_{t}=b_{1}(t) L_{1}+b_{2}(t) L_{2}$ may be easily solved. Defining

$$
\begin{equation*}
\omega_{t}=p_{1}(t)|1\rangle\langle 1|+p_{2}(t)|2\rangle\langle 2|, \tag{119}
\end{equation*}
$$

with $p_{1}(t)=b_{2}(t) / b(t)$ and $p_{2}(t)=b_{1}(t) / b(t)$ one easily finds

$$
\begin{equation*}
b_{1}(t) L_{1}+b_{2}(t) L_{2}=b(t)\left[\omega_{t} \operatorname{Tr} \rho-\rho\right]-\frac{1}{4} b(t) L_{z} . \tag{120}
\end{equation*}
$$

Note that generator $b(t)\left[\omega_{t} \operatorname{Tr} \rho-\rho\right]$ was already considered in Remark 6 (see formula (93). Moreover, one easily checks that this generator commutes with $L_{z}$ and hence one finds the following formula for $\Lambda_{t}$

$$
\Lambda_{t}(\rho)=\frac{1}{2} e^{-B(t)}\left[\left(1+e^{B(t) / 2}\right) \rho+\left(1-e^{B(t) / 2}\right) \sigma_{z} \rho \sigma_{z}\right]+\left(1-e^{-B(t)}\right) \Omega_{t},
$$

where

$$
\Omega_{t}=\frac{1}{e^{B(t)}-1} \int_{0}^{t} b(\tau) e^{B(\tau)}\left[p_{1}(\tau)|1\rangle\langle 1|+p_{2}(\tau)|2\rangle\langle 2|\right] d \tau .
$$

To check for complete positivity of $\Lambda_{t}$ one computes $\left[\mathbb{1}_{2} \otimes \Lambda_{t}\right]\left(P_{2}^{+}\right)$:

$$
\left(\begin{array}{cc|cc}
e^{-B}+\left(1-e^{-B}\right) \Omega_{11} & \cdot & \cdot & e^{-B / 2} \\
\cdot & \left(1-e^{-B}\right) \Omega_{22} & \cdot & \cdot \\
\hline \cdot & \cdot & \left(1-e^{-B}\right) \Omega_{11} & \cdot \\
e^{-B / 2} & \cdot & \cdot & e^{-B}+\left(1-e^{-B}\right) \Omega_{22}
\end{array}\right)
$$

where we skiped the time dependence. Now, $\left[\mathbb{1}_{2} \otimes \Lambda_{t}\right]\left(P_{2}^{+}\right) \geq 0$ iff $B(t) \geq 0$ and $\Omega_{t} \geq 0$. The last condition is equivalent to

$$
\begin{equation*}
0 \leq \int_{0}^{t} b_{k}(\tau) e^{B(\tau)} d \tau \leq e^{B(t)}-1 \tag{121}
\end{equation*}
$$

for $k=1,2$. This way we derived conditions for $b_{1}(t)$ and $b_{2}(t)$. Note, however, that in order to do this we have to solve the original Master Equation! Only the condition $B(t) \geq 0$ was obtained via the Lie algebraic method.

Corollary 3 If $X_{t}$ is GKSL generator, then $\int_{0}^{t} L_{u} d u$ is GKSL generator. If $L_{t}$ is GKSL generator (the dynamics is Markovian), then $\int_{0}^{t} X_{u} d u$ is GKSL generator.

Proposition 7 The corresponding solution $e^{Z_{t}}$ may be represented as follows

$$
\begin{equation*}
e^{Z_{t}}=e^{\nu_{1}(t) L_{1}} e^{\nu_{2}(t) L_{2}} \tag{122}
\end{equation*}
$$

where

$$
\begin{equation*}
\nu_{1}=\ln \left(\frac{A}{A_{1} e^{-A}+A_{2}}\right), \quad \nu_{2}=\ln \left(\frac{A_{1}+A_{2} e^{A}}{A}\right) \tag{123}
\end{equation*}
$$

and hence $\nu_{1}+\nu_{2}=A$.
Proof: Since $\left\{L_{1}, L_{2}\right\}$ form a Lie algebra one has

$$
\begin{equation*}
e^{s\left(A_{1} L_{1}+A_{2} L_{2}\right)}=e^{u_{1}(s) L_{1}} e^{u_{2}(s) L_{2}} \tag{124}
\end{equation*}
$$

where $u_{1}(s), u_{2}(s)$ are real functions of a parameter $s$ satisfying $u_{1}(0)=$ $u_{2}(0)=0$. Differentiating both sides with respect to $s$ one gets

$$
\begin{aligned}
& e^{s\left(A_{1} L_{1}+A_{2} L_{2}\right)}\left(A_{1} L_{1}+A_{2} L_{2}\right) \\
= & u_{1}^{\prime}(s) L_{1} e^{u_{1}(s) L_{1}} e^{u_{2}(s) L_{2}}+u_{2}^{\prime}(s) e^{u_{1}(s) L_{1}} e^{u_{2}(s) L_{2}} L_{2} \\
= & u_{1}^{\prime}(s) L_{1} e^{s\left(A_{1} L_{1}+A_{2} L_{2}\right)}+u_{2}^{\prime}(s) e^{s\left(A_{1} L_{1}+A_{2} L_{2}\right)} L_{2}
\end{aligned}
$$

where $u^{\prime}(s)=d u(s) / d s$. Multiplying both sides by $e^{-s\left(A_{1} L_{1}+A_{2} L_{2}\right)}$ leads to

$$
A_{1} L_{1}+A_{2} L_{2}=u_{1}^{\prime}(s) L_{1}+u_{2}^{\prime}(s) e^{s\left(A_{1} L_{1}+A_{2} L_{2}\right)} L_{2} e^{-s\left(A_{1} L_{1}+A_{2} L_{2}\right)}
$$

and hence using (109) one obtains the following equations for unknown functions $u_{1}$ and $u_{2}$ :

$$
\begin{align*}
& A_{1}=u_{1}^{\prime}(s)\left[1-\frac{A_{2}}{A}\left(1-e^{s A}\right)\right]  \tag{125}\\
& A_{2}=u_{1}^{\prime}(s) \frac{A_{2}}{A}\left(1-e^{s A}\right)+u_{2}^{\prime}(s) \tag{126}
\end{align*}
$$

Using

$$
\begin{equation*}
\int_{0}^{1} \frac{d x}{1+a e^{b x}}=\frac{1}{b} \ln \frac{1+a}{e^{-b}+a} \tag{127}
\end{equation*}
$$

one finds

$$
\begin{equation*}
u_{1}(1)=A \int_{0}^{1} \frac{d s}{1+\left(A_{2} / A_{1}\right) e^{s A}}=\ln \left(\frac{A}{A_{1} e^{-A}+A_{2}}\right) \tag{128}
\end{equation*}
$$

and

$$
\begin{equation*}
u_{2}(1)=\ln \left(\frac{A_{1}+A_{2} e^{A}}{A}\right) . \tag{129}
\end{equation*}
$$

Finally, one takes $\nu_{1}:=u_{1}(1)$ and $\nu_{2}:=u_{2}(1)$. It is clear that $\nu_{1}, \nu_{2} \geq 0$ which proves that $e^{\nu_{1}(t) L_{1}}$ and $e^{\nu_{2}(t) L_{2}}$ are CPTP maps.

Remark 10 This example may be easily generalized: suppose that $\left\{L_{1}, \ldots, L_{N}\right\}$ is a set of GKSL generators closing a Lie algebra

$$
\begin{equation*}
\left[L_{i}, L_{j}\right]=\sum_{k=1}^{N} c_{i j}^{k} L_{k}, \tag{130}
\end{equation*}
$$

and consider

$$
\begin{equation*}
X_{t}=a_{1}(t) L_{1}+\ldots+a_{N}(t) L_{N}, \tag{131}
\end{equation*}
$$

such that

$$
\begin{equation*}
A_{1}(t)=\int_{0}^{t} a_{1}(u) d u \geq 0 ; \quad \ldots \quad ; A_{N}(t)=\int_{0}^{t} a_{N}(u) d u \geq 0 \tag{132}
\end{equation*}
$$

It should be clear that Wilcox formula leads to

$$
\begin{equation*}
L_{t}=b_{1}(t) L_{1}+\ldots+b_{N}(t) L_{N} \tag{133}
\end{equation*}
$$

where the functions $\left\{b_{1}(t), \ldots, b_{N}(t)\right\}$ are fully determined by $\left\{a_{1}(t), \ldots, a_{N}(t)\right\}$ and the structure constants $c_{i j}^{k}$ of the above Lie algebra. There are welldeveloped algebraic methods (like Wei-Norman or Magnus expansions [29, 30] see also the recent review [31]) which may help to deal with such dynamical problems.

## 8. Conclusions

We provided a basic introduction to the time-local description of open quantum systems. The full characterization of Markovian semigroups, commutative evolutions and dynamics corresponding to divisible maps (Markovian evolution) is provided. In the non-commutative case the Wilcox formula shows the intricate structure of time-local generator $L_{t}$ and partially explains the problem of finding necessary and sufficient conditions which do guarantee a legitimate quantum evolution generated by $L_{t}$. Recently, there is an increasing interest in the non-Markovian quantum evolutions and quantum memory effects. For recent papers devoted to both theoretical and experimental aspects of quantum evolution with memory see e.g. a collection of papers in [32] and references therein. See also Haikka and Maniscalco [33] in this volume.

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