On Kossakowski construction of positive maps in matrix algebras

Dariusz Chruściński
Institute of Physics, Faculty of Physics, Astronomy and Informatics
Nicolaus Copernicus University, Grudziadzka 5, 87–100 Toruń, Poland

Abstract

We provide a further analysis of the class of positive maps proposed ten years ago by Kossakowski. In particular we propose a new parametrization which reveals an elegant geometric structure and an interesting interplay between group theory and a certain class of positive maps.

Dedicated to Andrzej Kossakowski on his 75th birthday

1 Introduction – a diagonal type positive maps

Ten year ago in a remarkable paper [1] Kossakowski provided a construction of a family of positive maps in matrix algebras $M_n(\mathbb{C})$. This construction reproduces many examples of positive maps already known in the literature. The maps from [1] belong to the following class: let $\{e_0, \ldots, e_{n-1}\}$ denotes an orthonormal basis in $\mathbb{C}^n$ and let $E_{ij} := |e_i\rangle\langle e_j|$. Consider the linear map $\Lambda : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined as follows

$$\Lambda(E_{ii}) = \sum_{j=0}^{n-1} a_{ij} E_{jj} \ , \quad \Lambda(E_{ij}) = -E_{ij} \ , \quad i \neq j \ . \tag{1}$$

where a_{ij} provides a set of complex parameters. In what follows we call the above maps *diagonal type maps*, since only diagonal elements E_{ii} are transformed in a non-trivial way. A map Λ is Hermitian, i.e. $[\Lambda(X)]^\dagger = \Lambda(X^\dagger)$ iff $a_{ij} \in \mathbb{R}$. The basic question one poses is:

what are conditions for a_{ij} which guarantee that Λ is a positive map.

It is clear that a necessary condition is that all matrix elements $a_{ij} \geq 0$. Observe, that $n \times n$ matrix $A = [a_{ij}]$ with matrix elements $[a_{ij}] \geq 0$ may be considered as a "classical" positive linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Therefore, formula (1) provides a construction of a "quantum" positive map Λ out of the "classical" map A if "classical" conditions $a_{ij} \geq 0$ are completed by a set of suitable "quantum" conditions. This problem is easily solvable for $n = 2$. One proves the following

Proposition 1. *If $n = 2$, then Λ is positive if and only if $a_{ij} \geq 0$ and*

$$\sqrt{a_{00}a_{11}} + \sqrt{a_{01}a_{10}} \geq 1 \ . \tag{2}$$

Moreover, Λ is completely positive if and only if $a_{ij} \geq 0$ and $a_{00}a_{11} \geq 1$.

The prescription (1) for Λ is so simple that it seems that for $n > 2$ the corresponding additional conditions for a_{ij} are easy to find. Surprisingly, it is not the case and starting with $n = 3$ the general problem is open. We stress that there is an essential difference between $n = 2$ and $n > 2$. For $n = 2$ all positive maps are decomposable. It is no longer true for $n > 2$. And there are well known examples of indecomposable maps belonging to a general family (1).

Let us recall that a map Λ is positive iff for all rank-1 projectors P and Q

$$\mathrm{tr}[P\Lambda(Q)] \geq 0 \ . \tag{3}$$

Taking $P = |x\rangle\langle x|$ and $Q = |y\rangle\langle y|$ one has $\langle x|\Lambda(|y\rangle\langle y|)|x\rangle \geq 0$ for all $x, y \in \mathbb{C}^n$. Using this definition one may prove the following

Theorem 1 ([5])**.** *A map Λ defined in (1) is positive if and only if $a_{ij} \geq 0$ and for all vectors $x \in \mathbb{C}^n$*

$$\sum_{i=0}^{n-1} \frac{|x_i|^2}{B_i(x)} \leq 1 \ , \tag{4}$$

where

$$B_i(x) = |x_i|^2 + \sum_{j=0}^{n-1} a_{ij}|x_j|^2 \ . \tag{5}$$

Moreover, Λ is completely positive if and only if the matrix $D = [d_{ij}]$ such that $d_{ij} = -1$ for $i \neq j$ and $d_{ii} = a_{ii}$ is positive semi-definite.

We stress that an inequality (4) does not provide a solution to our problem. It is just a reformulation of the original definition of positivity for the special class of maps! One may easily check that for $n = 2$ an inequality (4) reproduces condition (2). However, for $n > 2$ we do not know how to translate the above inequality into the closed set of conditions upon the matrix elements a_{ij}.

2 Circulant matrices

Consider now a special case when a_{ij} defines a circulant matrix, i.e. $a_{ij} = \alpha_{i-j}$ (mod n). Actually, many well known examples of positive maps belongs to such class (e.g. reduction map, Choi map and its generalizations). We assume that $\alpha_k \geq 0$ for $k = 0, \ldots, n-1$ and we denote the corresponding map by $\Lambda[\alpha_0, \ldots, \alpha_{n-1}]$.

Example 1. *For $n = 2$ denoting $a_{00} = a_{11} = \alpha_0 =: a$ and $a_{01} = a_{10} = \alpha_1 =: b$ formula (2) reduces to*

$$a + b \geq 1 \ . \tag{6}$$

Recall, that $a = 0$ and $b = 1$ corresponds to the reduction map $R_2(X) = \mathbb{I}_2 \mathrm{tr}\, X - X$.

For a circulant matrix Theorem 1 reduces to the following

Proposition 2. *A map $\Lambda[\alpha_0, \ldots, \alpha_{n-1}]$ defined in (1) is positive if and only if for all vectors $x \in \mathbb{C}^n$*

$$\sum_{i=0}^{n-1} \frac{|x_i|^2}{(\alpha_0 + 1)|x_i|^2 + \sum_{k=1}^{n-1} \alpha_k |x_{i+k}|^2} \leq 1 \ . \tag{7}$$

Moreover, Λ is completely positive if and only if $\alpha_0 \geq n - 1$.

An inequality (7) is known as *circulant inequlity* [6]. In particular taking $|x_0| = \ldots = |x_{n-1}|$ one finds the following necessary condition for positivity of Λ

$$\alpha_0 + \alpha_1 + \ldots + \alpha_{n-1} \geq n - 1 \ . \tag{8}$$

Note, that the above condition is necessary but not sufficient. Actually, it is sufficient only for $n = 2$ (see Example 1). For $n = 3$ a full class of parameters $\alpha_0 = a$, $\alpha_1 = b$ and $\alpha_2 = c$ satisfying circulant inequality (7) was derived in [4].

Theorem 2. *([4]) For $n = 3$ a map $\Lambda[a, b, c]$ is positive if and only if*

1. $a + b + c \geq 2$,
2. *if* $a \leq 1$, *then* $bc \geq (1 - a)^2$.

Moreover, being a positive map it is indecomposable if and only if

$$4bc < (2 - a)^2 \ . \tag{9}$$

Λ *is completely positive if and only if* $a \geq 2$.

Hence, for $n = 3$ a necessary condition $a + b + c \geq 2$ is supplemented by an extra condition *3*.

Corollary 1. *If $a > 1$, then condition (8) is necessary and sufficient for positivity of $\Lambda[a, b, c]$.*

Remark 1. *For $n > 3$ a full set of necessary and sufficient conditions for positivity of $\Lambda[\alpha_0, \ldots, \alpha_{n-1}]$ is not known.*

3 Kossakowski construction

Let us define a set of Hermitian diagonal traceless matrices

$$F_\ell = \frac{1}{\sqrt{\ell(\ell+1)}} \Big(\sum_{k=0}^{\ell-1} E_{kk} - \ell E_{\ell\ell} \Big) \ , \qquad \ell = 1, \ldots, n-1 \ . \tag{10}$$

These matrices span the Cartan subalgebra of $su(n-1)$. Moreover, $\mathrm{tr}(F_\alpha F_\beta) = \delta_{\alpha\beta}$. Define a real $n \times n$ matrix

$$a_{ij} := \frac{n-1}{n} + \sum_{\alpha,\beta=1}^{n-1} \langle e_i | F_\alpha | e_i \rangle R_{\alpha\beta} \langle e_j | F_\beta | e_j \rangle \ , \tag{11}$$

where $R_{\alpha\beta}$ is an $(n-1) \times (n-1)$ orthogonal matrix. Consider now a linear map Λ defined by (1) with a_{ij} defined by (11).

Theorem 3 ([1])**.** *For any orthogonal matrix $R_{\alpha\beta}$ a linear map Λ is positive.*

Remark 2. *Actually Kossakowski provided more general construction [1]. However, in this paper we restrict our analysis to the special class of diagonal type maps corresponding to (11).*

Due to the fact that F_α is traceless for $\alpha = 1, \ldots, n-1$, one finds

$$\sum_{i=1}^{n-1} a_{ij} = \sum_{j=1}^{n-1} a_{ij} = n - 1 \ . \tag{12}$$

Moreover, since matrix elements $a_{ij} \geq 0$ (it follows from Theorem 3) one finds that

$$\widetilde{a}_{ij} := \frac{1}{n-1} \, a_{ij} \ , \tag{13}$$

defines a doubly stochastic matrix.

Remark 3. *A map $\widetilde{\Lambda} := \frac{1}{n-1} \Lambda$ is unital trace preserving.*

Consider now an inverse problem: suppose we are given a $n \times n$ matrix $[a_{ij}]$ such that $[\widetilde{a}_{ij}]$ is doubly stochastic. How to check whether a_{ij} is defined *via* (11)? The answer is given by the following

Proposition 3 ([5])**.** *A matrix* $[a_{ij}]$ *can be represented by (11) if and only if*

$$\sum_{k=0}^{n-1} a_{ik}a_{jk} = \delta_{ij} + n - 2 \ , \tag{14}$$

for $i,j = 0,\ldots,n-1$*.*

Define

$$b_{ij} := a_{ij} - 1 \ , \tag{15}$$

that is,

$$b_{ij} = \sum_{\alpha,\beta=1}^{n-1} \langle e_i|F_\alpha|e_i\rangle R_{\alpha\beta}\langle e_j|F_\beta|e_j\rangle - \frac{1}{n} \ , \tag{16}$$

One easily proves

Proposition 4. *A matrix* $[a_{ij}]$ *satisfies (17) if and only if matrix* $[b_{ij}]$ *satisfies*

$$\sum_{k=0}^{n-1} b_{ik}b_{jk} = \delta_{ij} \ , \tag{17}$$

for $i,j = 0,\ldots,n-1$*, i.e.* $[b_{ij}]$ *is an orthogonal matrix.*

Note, that if $[b_{ij}]$ defines an orthogonal matrix, then $|b_{ij}| \leq 1$ and hence $a_{ij} = b_{ij} + 1 \geq 0$.

Corollary 2. *A map* Λ *defined in (1) is positive if the corresponding* b_{ij} *defines* $n \times n$ *orthogonal matrix such that*

$$\sum_{i=0}^{n-1} b_{ij} = \sum_{j=0}^{n-1} b_{ij} = -1 \ . \tag{18}$$

It is clear that formula (16) provides an embedding of $O(n-1)$ into $O(n)$, i.e. an orthogonal matrix $R_{\alpha\beta}$ from $O(n-1)$ is mapped into an orthogonal matrix b_{ij} from $O(n)$.

Now, we provide a geometric interpretation of Kossakowski construction. Let $\{\mathbf{b}^{(0)},\ldots,\mathbf{b}^{(n-1)}\}$ be an orthonormal basis in $\mathbb{R}^n$ such that

$$(\mathbf{b}^{(i)},\mathbf{e}) = -\frac{1}{\sqrt{n}} \ , \tag{19}$$

where $(\mathbf{a},\mathbf{b})$ denotes the canonical inner product in $\mathbb{R}^n$ and

$$\mathbf{e} = \frac{1}{\sqrt{n}}(1,\ldots,1) \ . \tag{20}$$

Let us define

$$b_{ij} := \mathbf{b}^{(i)}_j \ . \tag{21}$$

Clearly, $[b_{ij}]$ defines an orthogonal matrix. Moreover, (19) guarantees (18).

Corollary 3. *Any Kossakowski map is uniquely defined by an arbitrary orthonormal basis* $\{\mathbf{b}^{(0)},\ldots,\mathbf{b}^{(n-1)}\}$ *satisfying (19).*

Corollary 4. *If* $[b_{ij}]$ *defines a Kossakowski map, then* $[b_{i\pi(j)}]$ *defines another Kossakowski map for an arbitrary permutation* $\pi \in S_n$*.*

Let $\Sigma_{\mathbf{e}}$ denote an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ orthogonal to vector $\mathbf{e}$. Let $\{\mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(n-1)}\}$ be an arbitrary orthonormal basis in $\Sigma_{\mathbf{e}}$. An example of such a basis is provided by

$$\mathbf{f}_i^{(\alpha)} = \langle e_i | F_\alpha | e_i \rangle \ , \tag{22}$$

where F_α are defined in (10). Clearly, $\{\mathbf{f}^{(0)} := \mathbf{e}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(n-1)}\}$ defines an orthonormal basis in $\mathbb{R}^n$. Consider now an orthogonal operator $\mathbf{R}$ such that its matrix representation in the basis $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(n-1)}\}$ has the following form

$$\mathbf{R}_{00} = -1 \ , \quad \mathbf{R}_{0k} = \mathbf{R}_{k0} = 0 \ , \quad \mathbf{R}_{ij} = R_{ij} \ . \tag{23}$$

It is clear that $\mathbf{R}$ represents rotation (or pseudo-rotation) around $\mathbf{e}$.

Proposition 5. *Let $[b_{ij}]$ be the matrix representation of $\mathbf{R}$ in the canonical basis in $\mathbb{R}^n$. Then b_{ij} satisfy (18).*

Proof: denote by $\{\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}\}$ the canonical basis and let

$$\mathbf{f}^{(i)} = \sum_{j=0}^{n-1} S_{ij} \mathbf{e}_j \ . \tag{24}$$

One has

$$b = S^{\mathrm{T}} \mathbf{R} S \ . \tag{25}$$

and hence

$$\sum_{i=0}^{n-1} b_{ij} \ = \ \sum_{k,l=0}^{n-1} \sum_{i=0}^{n-1} S_{ki} \mathbf{R}_{kl} S_{lj} = \sum_{i=0}^{n-1} S_{0i} \mathbf{R}_{00} S_{0j} + \sum_{\alpha,\beta=1}^{n-1} \sum_{i=0}^{n-1} S_{\alpha i} R_{\alpha\beta} S_{j\beta} = -1 \ , \tag{26}$$

due to

$$S_{0i} = \frac{1}{\sqrt{n}} \ , \quad \sum_{i=0}^{n-1} S_{\alpha i} = 0 \ , \quad i = 0, 1, \ldots, n-1 \ ; \quad \alpha = 1, \ldots, n-1 \ . \tag{27}$$

In particular if $\mathbf{f}^{(i)}$ are defined *via* (22), then (25) reproduces (16).

Consider now the following symmetric set of n vectors $\{\mathbf{g}^{(0)}, \ldots, \mathbf{g}^{(n-1)}\}$ in $\Sigma_{\mathbf{e}}$ defined by:

1. they have the same length,
2. the angle 'ϕ_n' between arbitrary two vectors is the same.

One proves that

$$\cos \phi_n = -\frac{1}{n-1} \ . \tag{28}$$

Remark 4. *Actually, a set of n vectors $\{\mathbf{g}^{(0)}, \ldots, \mathbf{g}^{(n-1)}\}$ in $\mathbb{R}^{n-1}$ satisfying the above conditions is called an equiangular frame [7].*

Proposition 6. *Vectors*

$$\mathbf{b}^{(i)} := \mathbf{g}^{(i)} - \frac{1}{\sqrt{n}} \mathbf{e} \ ,$$

such that $|\mathbf{g}^{(i)}|^2 = 1 - \frac{1}{n}$, define an orthonormal basis in $\mathbb{R}^n$ and satisfy (19).

Consider now a special case when the matrix $[a_{ij}]$ defined in (11) is circulant. Formula (12) implies

$$\alpha_0 + \ldots + \alpha_{n-1} = n - 1 \ . \tag{29}$$

In this case Proposition 4 reduces to

Proposition 7. *A circulant matrix $a_{ij} = \alpha_{i-j}$ satisfies (17) if and only if*

$$\sum_{k=0}^{n-1} \alpha_{i-k}\alpha_{j-k} = \delta_{ij} + n - 2 \ , \tag{30}$$

for $i,j = 0,\ldots,n-1$.

Introducing

$$\beta_i = \alpha_i - 1 \ , \tag{31}$$

one finds

$$\beta_0 + \ldots + \beta_{n-1} = -1 \ , \tag{32}$$

together with

$$\sum_{k=0}^{n-1} \beta_{i-k}\beta_{j-k} = \delta_{ij} \ , \tag{33}$$

for $i,j = 0,\ldots,n-1$. Clearly, $b_{ij} = \beta_{i-j}$ defines a circulant orthogonal matrix satisfying an additional constraint (32).

4 Examples

Example 2. *For $n = 2$ one has $F_1 = \frac{1}{\sqrt{2}}\sigma_z$ and $R = \pm 1$, and hence one easily finds*

$$R = 1 \ \rightarrow \ [a_{ij}] = \mathbb{I}_2 \ ; \quad R = -1 \ \rightarrow \ [a_{ij}] = \sigma_x \ . \tag{34}$$

Example 3. *For $n = 3$*

$$F_1 = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \ ; \quad F_2 = \frac{1}{\sqrt{6}} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{array} \right) \ , \tag{35}$$

and

$$[R_{\alpha\beta}] = \left(\begin{array}{cc} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{array} \right) \ . \tag{36}$$

Interestingly, in this case one finds that the matrix $[a_{ij}]$ is circulant. Denoting $a := a_{00}, b := a_{01}$ and $c := a_{02}$ one obtains

$$\begin{aligned} a &= \frac{2}{3}(1 + \cos\phi) \ , \\ b &= \frac{1}{3}(2 - \cos\phi - \sqrt{3}\sin\phi) \ , \\ c &= \frac{1}{3}(2 - \cos\phi + \sqrt{3}\sin\phi) \ . \end{aligned} \tag{37}$$

Let us observe that introducing $\widetilde{a} = a - 1, \widetilde{b} = b - 1$ and $\widetilde{c} = c - 1$ the above family of maps is uniquely characterized by a circulant orthogonal matrix

$$[b_{ij}] = \left(\begin{array}{ccc} \widetilde{a} & \widetilde{b} & \widetilde{c} \\ \widetilde{c} & \widetilde{a} & \widetilde{b} \\ \widetilde{b} & \widetilde{c} & \widetilde{a} \end{array} \right) \ , \tag{38}$$

with $\widetilde{a}+\widetilde{b}+\widetilde{c}=-1$. Interestingly, the well known maps: Choi maps $\Lambda[1,1,0]$, $\Lambda[1,0,1]$ and the reduction map $\Lambda[0,1,1]$ have the following representation in terms of the matrix $[b_{ij}]$:

$$\left(\begin{array}{ccc} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right) \ ; \ \left(\begin{array}{ccc} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{array}\right) \ ; \ \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right) \ , \tag{39}$$

that is, up to a sign they correspond to circulant permutation matrices.

Remark 5. *For $n=2$ and $n=3$ all Kossakowski maps are characterized by a circulant matrix $[a_{ij}]$. It is no longre true for $n>3$.*

Remark 6. *Let us observe that parameters a,b,c defined in (37) are compatible with Theorem 2. Note, that maps defined via (37) belong to the boundary of a set of positive maps defined by two equalities in conditions 1. and 2. of Theorem 2, that is,*

$$a+b+c=2 \ ; \quad bc=(1-a)^2 \ . \tag{40}$$

Detailed analysis of the structure of these maps was performed in [2].

Example 4. *For $n=4$ one has the following circulant orthogonal $[b_{ij}]$ matrix: $\widetilde{a}=b_{00}$, $\widetilde{b}=b_{01}$, $\widetilde{c}=b_{02}$ and $\widetilde{d}=b_{03}$ satisfying*

$$\widetilde{a}+\widetilde{b}+\widetilde{c}+\widetilde{d}=-1 \ . \tag{41}$$

Orthogonality conditions imply

$$\widetilde{a}^2+\widetilde{b}^2+\widetilde{c}^2+\widetilde{d}^2=1 \ , \quad \widetilde{a}\widetilde{c}+\widetilde{b}\widetilde{d}=0, \quad (\widetilde{a}+\widetilde{c})(\widetilde{b}+\widetilde{d})=0 \ . \tag{42}$$

Therefore, we have two classes of admissible parameters $\{\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}\}$ constrained by

$$\widetilde{a}+\widetilde{b}+\widetilde{c}+\widetilde{d}=-1 \ , \ \ \widetilde{a}^2+\widetilde{b}^2+\widetilde{c}^2+\widetilde{d}^2=1 \ , \ \ \widetilde{b}+\widetilde{d}=0 \ , \tag{43}$$

and

$$\widetilde{a}+\widetilde{b}+\widetilde{c}+\widetilde{d}=-1 \ , \ \ \widetilde{a}^2+\widetilde{b}^2+\widetilde{c}^2+\widetilde{d}^2=1 \ , \ \ \widetilde{a}+\widetilde{c}=0 \ , \tag{44}$$

Equivalently, the above conditions may be rewritten as follows

$$\widetilde{a}^2+\widetilde{b}^2+\widetilde{c}^2+\widetilde{d}^2=1 \ , \ \ \widetilde{a}+\widetilde{c}=-1 \ , \ \ \widetilde{b}+\widetilde{d}=0 \ , \tag{45}$$

and

$$\widetilde{a}^2+\widetilde{b}^2+\widetilde{c}^2+\widetilde{d}^2=1 \ , \ \ \widetilde{a}+\widetilde{c}=0 \ , \ \ \widetilde{b}+\widetilde{d}=-1 \ . \tag{46}$$

They describe two circles: the intersection of 3D sphere with two planes. Again, characteristic well known maps $\Lambda[1,1,1,0], \Lambda[1,1,0,1], \Lambda[1,0,1,1]$ and $\Lambda[0,1,1,1]$ (up to a sign) correspond to circulant permutation matrices:

$$\left(\begin{array}{cccc} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) , \left(\begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) , \left(\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{array}\right) , \left(\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) .$$

Example 5. *For $n=5$ one has the following circulant orthogonal $[b_{ij}]$ matrix: $\widetilde{a}=b_{00}$, $\widetilde{b}=b_{01}$, $\widetilde{c}=b_{02}, \widetilde{d}=b_{03}$ and $\widetilde{e}=b_{04}$ satisfying*

$$\widetilde{a}+\widetilde{b}+\widetilde{c}+\widetilde{d}+\widetilde{e}=-1 \ . \tag{47}$$

Orthogonality conditions imply

$$\widetilde{a}^2+\widetilde{b}^2+\widetilde{c}^2+\widetilde{d}^2+\widetilde{e}^2=1 \ , \ \ \widetilde{a}\,\widetilde{e}+\widetilde{b}\,\widetilde{a}+\widetilde{c}\,\widetilde{b}+\widetilde{d}\,\widetilde{c}+\widetilde{e}\,\widetilde{d}=0 \ . \tag{48}$$

One easily checks that the remaining orthogonality conditions are not independent from (47) and (48). The corresponding set of admissible parameters $\{\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d},\widetilde{e}\}$ is 2-dimensional but its shape is not very transparent (47) and (48).

5 Circulant case — a complementary parametrization

Let us recall that if $a_{ij} = \alpha_{i-j}$ defines a circulant matrix, then its eigenvalues are given by

$$\lambda_k = \sum_{l=0}^{n-1} \omega^{-kl} \alpha_l \ , \quad k = 0, \ldots, n-1 \ , \tag{49}$$

and the corresponding eigenvectors read

$$\mathbf{x}_k = (1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k})^{\mathrm{T}} \ , \tag{50}$$

where $\omega = e^{2\pi i/n}$. Two sets $\{\alpha_0, \ldots, \alpha_{n-1}\}$ and $\{\lambda_0, \ldots, \lambda_{n-1}\}$ are related by the discrete Fourier transform. Note that

$$\lambda_0 = \alpha_0 + \ldots + \alpha_{n-1} = n-1 \ . \tag{51}$$

Consider now a circulant orthogonal matrix $b_{ij} = \beta_{i-j}$ with $\beta_k = \alpha_k - 1$. The corresponding eigenvalues μ_k of $[b_{ij}]$ are defined by

$$\mu_0 = \lambda_0 - n = -1 \ , \quad \mu_\alpha = \lambda_\alpha \ , \quad \alpha = 1, \ldots, n-1 \ . \tag{52}$$

Now, since $[b_{ij}]$ is orthogonal one has $|\mu_k| = 1$ and hence

Proposition 8. *Real parameters $\{\alpha_0, \ldots, \alpha_{n-1}\}$ satisfy (30) if and only if $|\lambda_\alpha| = 1$ for $\alpha = 1, \ldots, n-1$.*

This way we obtain a new parametrization of a set of admissible circulant matrices $[a_{ij}]$ by phases of $\lambda_\alpha = e^{i\phi_\alpha}$. Due to $\lambda_k = \lambda^*_{n-k}$ one has two cases:

1. if $n = 2m+1$, then we have m independent phases $\lambda_1 = e^{i\phi_1}, \ldots, \lambda_m = e^{i\phi_m}$.
2. if $n = 2m+2$, then we have m independent phases $\lambda_1 = e^{i\phi_1}, \ldots, \lambda_m = e^{i\phi_m}$ and one real parameter $\lambda_{m+1} = \pm 1$.

Example 6. *For $n = 3$ putting $\lambda_1 = e^{i\phi} = \lambda_2^*$ one finds*

$$\begin{aligned} a &= \frac{1}{3}(2 + \lambda_1 + \lambda_1^*) = \frac{2}{3}(1 + \cos\phi) \ , \\ b &= \frac{1}{3}(2 + \omega\lambda_1 + \omega^*\lambda_1^*) = \frac{1}{3}(2 - \cos\phi - \sqrt{3}\sin\phi) \ , \\ c &= \frac{1}{3}(2 + \omega^*\lambda_1 + \omega\lambda_1^*) = \frac{1}{3}(2 - \cos\phi + \sqrt{3}\sin\phi) \ , \end{aligned} \tag{53}$$

due to $\omega = e^{2\pi i/3} = \frac{1}{2}(-1 + i\sqrt{3})$. This reproduces result of Example 3.

Example 7. *For $n = 4$ if $\lambda_1 = e^{i\phi} = \lambda_3^*$ and $\lambda_2 = 1$ one finds*

$$a = \frac{1}{2}(2 + \cos\phi) \ , \quad b = \frac{1}{2}(1 - \sin\phi) \ , \quad c = \frac{1}{2}(2 - \cos\phi) \ , \quad d = \frac{1}{2}(1 + \sin\phi) \ , \tag{54}$$

and similarly if $\lambda_1 = e^{i\psi} = \lambda_3^*$ and $\lambda_2 = -1$ one has

$$a = \frac{1}{2}(1 + \cos\psi) \ , \quad b = \frac{1}{2}(2 - \sin\psi) \ , \quad c = \frac{1}{2}(1 - \cos\psi) \ , \quad d = \frac{1}{2}(2 + \sin\psi) \ . \tag{55}$$

Note, that for $\lambda_2 = 1$ one has $b + d = 1$, whereas for $\lambda_2 = -1$ one has $b + d = 2$. This way we reproduced two classes from Example 4.

Corollary 5. *It is therefore clear that*

1. *if $n = 2m+1$, then a set of admissible parameters defines m-dimensional torus $\mathbb{T}_m$. Note that $O(n-1) = O(2m)$ and a single torus $\mathbb{T}_m$ corresponds to a maximal commutative subgroup of $SO(2m)$.*

2. *if $n = 2m+2$, we have two m-dimensional tori $\mathbb{T}_m$ and $\mathbb{T}'_m$. Torus $\mathbb{T}_m$ corresponds to a maximal commutative subgroup of $SO(2m+1)$ whereas $\mathbb{T}'_m$ is defined by composing $\mathbb{T}_m$ with reflection, that is, $g \in \mathbb{T}'_m$ iff $-g \in \mathbb{T}_m$ (cf. [8]).*

Corollary 6. *Positive maps $\Lambda[\alpha_0, \ldots, \alpha_{n-1}]$ are invertible. It follows from the fact that*

$$|\det[a_{ij}]| = |\lambda_0 \ldots \lambda_{n-1}| = n - 1 \neq 0 \ . \tag{56}$$

Note, however, that the inverse $\Lambda^{-1}[\alpha_0, \ldots, \alpha_{n-1}]$ is no longer positive.

6 Conclusions

We analyzed a class of positive maps introduced by Kossakowski [1]. It turns out that these maps display interesting geometric features. In particular its maximal commutative subset — $\Lambda[\alpha_0, \ldots, \alpha_{n-1}]$ — corresponding to circulant matrices $[a_{ij}]$ is parameterized by tori which defines maximal commutative subgroups of the orthogonal group. For further properties of these maps like (in)decomposability and/or optimality see also [8]. It is clear that *via* Choi-Jamiołkowski isomorphism can one provide a similar analysis in terms of entanglement witnesses (see [9] for the recent review).

Acknowledgements

This paper was partially supported by the National Science Center project DEC- 2011/03/B/ST2/00136. I thank Andrzej Kossakowski for the fruitful and inspiring discussions and Andrzej Jamiołkowski for pointing out Ref. [7].

References

[1] A. Kossakowski, Open Sys. Information Dyn. **10**, 213 (2003).

[2] D. Chruściński and F. A. Wudarski, Open Syst. Inf. Dyn. **18**, 387 (2011).

[3] D. Chruściński and F. A. Wudarski, Open Syst. Inf. Dyn. **19**, 1250020 (2012).

[4] S. J. Cho, S.-H. Kye, and S. G. Lee, Linear Alg. Appl. **171**, 213 (1992).

[5] D. Chruściński and A. Kossakowski, Open Systems and Inf. Dynamics, **14**, 275 (2007).

[6] S. Yamagami, Proc. Amer. Math. Soc. **118**, 521 (1993).

[7] P. G. Casazza, G. Kutyniok, *Introduction to finite frame theory*, in Finite Frames: Theory and Applications, Eds. P. G. Casazza and G. Kutyniok, Birkhäuser, Boston (2012).

[8] D. Chruściński, J. Phys. A: Math. Theor. **47**, 424033 (2014).

[9] D. Chruściński and G. Sarbicki, J. Phys. A: Math. Theor. **47**, 483001 (2014).